On the Equality of Solutions of Max-Min and Min-Max Systems of Variational Inequalities with Interconnected Bilateral Obstacles.

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August 20, 2014

Abstract

In this paper we deal with the solutions of systems of PDEs with bilateral inter-connected obstacles of min-max and max-min types. These systems arise naturally in stochastic switching zero-sum game problems. We show that when the switching costs of one side are regular, the solutions of the min-max and max-min systems coincide. Furthermore, this solution is identified as the value function of a zero-sum switching game.

AMS Classification subjects: 49N70, 49L25, 60H30, 90C39, 93E20.

Keywords: Switching zero-sum game; variational inequalities; PDEs with obstacles; oblique reflection; reflected backward stochastic differential equation; Hamilton-Jacobi-Bellman-Isaacs equation; Perron's method.

1 Introduction

Let us consider the following two systems of partial differential equations (PDEs) with bilateral inter-connected obstacles (i.e., the obstacles depend on the solution) of min-max and max-min types: for any $(i,j) \in \Gamma^1 \times \Gamma^2$, $(t,x) \in [0,T] \times \mathbb{R}^k$,

$$\begin{cases} \min\Big\{v^{ij}(t,x) - L^{ij}(\vec{v})(t,x)\,;\, \max\Big\{v^{ij}(t,x) - U^{ij}(\vec{v})(t,x); \\ \qquad -\partial_t v^{ij} - \mathcal{L}^X(v^{ij})(t,x) - f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma(t,x)^\top D_x v^{ij}(t,x))\Big\}\Big\} = 0\,; \\ v^{ij}(T,x) = h^{ij}(x) \end{cases} \tag{1.1}$$

and

$$\begin{cases} \max\Big\{\check{v}^{ij}(t,x) - U^{ij}(\vec{\check{v}})(t,x)\,;\, \min\Big\{\check{v}^{ij}(t,x) - L^{ij}(\vec{\check{v}})(t,x) \\ \qquad -\partial_t \check{v}^{ij} - \mathcal{L}^X(\check{v}^{ij})(t,x) - f^{ij}(t,x,(\check{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma(t,x)^\top D_x \check{v}^{ij}(t,x))\Big\}\Big\} = 0\,; \\ \check{v}^{ij}(T,x) = h^{ij}(x) \end{cases} \tag{1.2}$$

where

(i) Γ^1 and Γ^2 are finite sets (possibly different);

(ii) For any $(t,x) \in [0,T] \times \mathbb{R}^k$, $\vec{v}(t,x) = (v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}$ and for any $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$L^{ij}(\vec{v})(t,x) = \max_{k\in\Gamma^1, k\neq i}\{v^{kj}(t,x) - \underline{g}_{ik}(t,x)\},\; U^{ij}(\vec{v})(t,x) = \min_{p\in\Gamma^2, p\neq j}\{v^{ip}(t,x) + \bar{g}_{jp}(t,x)\}.$$

*This work was completed while the first author was visiting the Department of Mathematics of Université du Maine. Financial support from GEANPYL and Svensk Export Kredit (SEK) is gratefully acknowledged.

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(iii) $\mathcal{L}^X$ is a second order generator associated with a diffusion process described below.

The systems (1.1) and (1.2) are of min-max and max-min types respectively. The barriers $L^{ij}(\vec{v}), U^{ij}(\vec{v})$ and $L^{ij}(\vec{\underline{v}}), U^{ij}(\vec{\underline{v}})$ depend on the solution $\vec{v}$ and $\vec{\underline{v}}$ of (1.1) and (1.2) respectively. They are related to zero-sum switching game problems since actually, specific cases of these systems, stand for the Hamilton-Jacobi-Bellman-Isaacs equations associated with those games.

Switching problems have recently attracted a lot of research activities, especially in connection with mathematical finance, commodities, and in particular energy, markets, etc (see e.g. [3, 25, 4, 5, 11, 1, 9, 8, 10, 16, 17, 19, 21, 24, 26, 27, 28, 22, 31, 29, 30] and the references therein). Several points of view, mainly dealing with control problems have been considered (theoritical and applied [3, 25, 5, 9, 10, 16, 19, 28], numerics [4, 16], filtering and partial information [24]). However, except [20, 21], problems related to games did not attract that much interest in the literature.

In [8], by means of systems of reflected backward stochastic differential equations (BSDEs) with interconnected obstacles in combination with Perron's method, Djehiche et al. ([8]) have shown that each of the systems (1.1) and (1.2) has a unique continuous solution with polynomial growth, under classical assumptions on the data f^{ij}, $\bar{g}_{ij}$, $\underline{g}_{ij}$, h^{ij}. The question of whether or not these solutions coincide was conjectured as an open problem, leaving a possible connection of the solution of system (1.1) and (1.2) with zero-sum switching games unanswered. The main objective of this paper is two-fold: (i) to investigate under which additional assumptions on the data of these problems, the unique solutions of systems (1.1) and (1.2) coincide; (ii) to make a connection between this solution and the value function of the associated zero-sum switching game. Indeed, we show that if the switching costs of one side, i.e. either $(\bar{g}_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ or $(\underline{g}_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$, are regular enough, then the solutions of the systems (1.1) and (1.2) coincide. Furthermore, we show that this solution has a representation as a value function of a zero-sum switching game. To the best of our knowledge these issues have not been addressed in the literature yet. The main strategy to obtain these results is to show that the barriers, which depend on the solution, are comparable and then to make use of Theorem 6.1 (whose proof in an appendix at the end of the paper) to conclude that the solutions of the min-max and max-min systems coincide. This comparison is obtained under a regularity assumption on $(\bar{g}_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ or $(\underline{g}_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$. Theorem 6.1 extends a result derived in [14] on min-max and max-min PDEs with fixed obstacles, where we relax the condition of strict separation between the obstacles.

The paper is organized as follows. In Section 2, we fix some notation and, for sake of completeness, recall accurately under which conditions each of the systems (1.1) and (1.2) has a unique solution. These results are already given in [8]. In Section 3, we show that if mainly the switching costs $\bar{g}_{ij}$, $(i,j)\in\Gamma^1\times\Gamma^2$, are $\mathcal{C}^{1,2}$ then the unique solutions of (1.1) and (1.2) coincide. In Section 4, we describe the zero-sum switching game problem and show that it has a value which is given by the unique solution of (1.1) and (1.2). A proof of Theorem 6.1 and related double barriers reflected BSDEs together with their connection with zero-sum Dynkin games, is displayed in an appendix at the end of the paper. ■

2 Notations and first results

Let T (resp. k, d) be a fixed positive constant (resp. two integers) and Γ^1 (resp. Γ^2) denote the set of switching modes for player 1 (resp. 2). For later use, we shall denote by Λ the cardinal of the product set $\Gamma^1\times\Gamma^2$ and for $(i,j)\in\Gamma^1\times\Gamma^2$, $(\Gamma^1)^{-i} := \Gamma^1 - \{i\}$ and $(\Gamma^2)^{-j} := \Gamma^2 - \{j\}$. For $\vec{y} = (y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2} \in \mathbb{R}^\Lambda$, $(i,j)\in\Gamma^1\times\Gamma^2$, and $\underline{y}\in\mathbb{R}$, we denote by $[(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2-\{i,j\}}, \underline{y}]$ the matrix obtained from the matrix $\vec{y} = (y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$ by replacing the element y^{ij} with $\underline{y}$.

For any $(i,j) \in \Gamma^1 \times \Gamma^2$, let

$$\begin{aligned}
&b:\ (t,x) \in [0,T] \times \mathbb{R}^k \mapsto b(t,x) \in \mathbb{R}^k;\\
&\sigma:\ (t,x) \in [0,T] \times \mathbb{R}^k \mapsto \sigma(t,x) \in \mathbb{R}^{k\times d};\\
&f^{ij}:\ (t,x,\vec{y},z) \in [0,T] \times \mathbb{R}^{k+\Lambda+d} \mapsto f^{ij}(t,x,\vec{y},z) \ \in \mathbb{R};\\
&\underline{g}_{ik}:\ (t,x) \in [0,T] \times \mathbb{R}^k \mapsto \underline{g}_{ik}(t,x) \in \mathbb{R} \quad (k \in (\Gamma^1)^{-i});\\
&\bar{g}_{jl}:\ (t,x) \in [0,T] \times \mathbb{R}^k \mapsto \bar{g}_{jl}(t,x) \in \mathbb{R} \quad (l \in (\Gamma^2)^{-j});\\
&h^{ij}:\ x \in \mathbb{R}^k \mapsto h^{ij}(x) \in \mathbb{R}.
\end{aligned}$$

A function $\Phi:\ (t,x) \in [0,T] \times \mathbb{R}^k \mapsto \Phi(t,x) \in \mathbb{R}$ is called of *polynomial growth* if there exist two non-negative real constants C and γ such that

$$|\Phi(t,x)| \le C(1+|x|^\gamma), \quad (t,x) \in [0,T] \times \mathbb{R}^k.$$

Hereafter, this class of functions is denoted by Π_g. Let $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^k)$ (or simply $\mathcal{C}^{1,2}$) denote the set of real-valued functions defined on $[0,T]\times\mathbb{R}^k$, which are once (resp. twice) differentiable w.r.t. t (resp. x) and with continuous derivatives.

The following assumptions on the data of the systems (1.1) and (1.2) are in force throughout the paper.

(**H0**) The functions b and σ are jointly continuous in (t,x) and Lipschitz continuous w.r.t. x uniformly in t, meaning that there exists a non-negative constant C such that for any $(t,x,x') \in [0,T]\times\mathbb{R}^{k+k}$ we have

$$|\sigma(t,x)-\sigma(t,x')| + |b(t,x)-b(t,x')| \le C|x-x'|.$$

Therefore, they are also of linear growth w.r.t. x, i.e., there exists a constant C such that for any $(t,x) \in [0,T]\times\mathbb{R}^k$,

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|).$$

(**H1**) Each function f^{ij}

(i) is continuous in (t,x) uniformly w.r.t. the other variables $(\vec{y},z)$ and, for any (t,x), the mapping $(t,x) \to f^{ij}(t,x,0,0)$ is of polynomial growth.

(ii) is Lipschitz continuous with respect to the variables $(\vec{y} := (y^{ij})_{(i,j)\in\Gamma_1\times\Gamma_2}, z)$ uniformly in (t,x), i.e. $\forall\ (t,x) \in [0,T]\times\mathbb{R}^k$, $\forall\ (\vec{y}_1,\vec{y}_2) \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$, $(z^1,z^2) \in \mathbb{R}^d\times\mathbb{R}^d$,

$$|f^{ij}(t,x,\vec{y}_1,z_1) - f^{ij}(t,x,\vec{y}_2,z_2)| \le C\left(|\vec{y}_1-\vec{y}_2| + |z_1-z_2|\right),$$

where, $|\vec{y}|$ stands for the standard Euclidean norm of $\vec{y}$ in $\mathbb{R}^\Lambda$.

(**H2**) Monotonicity: Let $\vec{y} = (y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$. For any $(i,j) \in \Gamma^1\times\Gamma^2$ and any $(k,l) \ne (i,j)$ the mapping $y^{kl} \to f^{ij}(s,\vec{y},z)$ is non-decreasing.

(**H3**) The functions h^{ij}, which are the terminal conditions in the systems (1.1) and (1.2), are continuous with respect to x, belong to class Π_g and satisfy

$\forall\ (i,j) \in \Gamma^1\times\Gamma^2$ and $x \in \mathbb{R}^k$, $\max_{k\in(\Gamma^1)^{-i}}\big(h^{kj}(x) - \underline{g}_{ik}(T,x)\big) \le h^{ij}(x) \le \min_{l\in(\Gamma^2)^{-j}}\big(h^{il}(x) + \bar{g}_{jl}(T,x)\big)$.

(**H4**) The no free loop property: The switching costs $\underline{g}_{ik}$ and $\bar{g}_{jl}$ are non-negative, jointly continuous in (t,x), belong to Π_g and satisfy the following condition:

For any loop in $\Gamma^1\times\Gamma^2$, i.e., any sequence of pairs $(i_1,j_1),\ldots,(i_N,j_N)$ of $\Gamma^1\times\Gamma^2$ such that $(i_N,j_N)=(i_1,j_1)$, $\mathrm{card}\{(i_1,j_1),\ldots,(i_N,j_N)\}=N-1$ and any $q=1,\ldots,N-1$, either $i_{q+1}=i_q$ or $j_{q+1}=j_q$, we have $\forall(t,x)\in[0,T]\times\mathbb{R}^k$,

$$\sum_{q=1,N-1}\varphi_{i_q i_{q+1}}(t,x)\neq 0, \tag{2.1}$$

where, $\forall\ q=1,\ldots,N-1,\ \varphi_{i_q i_{q+1}}(t,x)=-\underline{g}_{i_q i_{q+1}}(t,x)1\!\!1_{i_q\neq i_{q+1}}+\bar{g}_{j_q i_{q+1}}(t,x)1\!\!1_{j_q\neq j_{q+1}}$.

This assumption implies in particular that

$$\forall\ (i_1,\ldots,i_N)\in(\Gamma^1)^N \text{ such that } i_N=i_1 \text{ and } \mathrm{card}\{i_1,\ldots,i_N\}=N-1,\ \sum_{p=1}^{N-1}\underline{g}_{i_k,i_{k+1}}>0 \tag{2.2}$$

and

$$\forall\ (j_1,\ldots,j_N)\in(\Gamma^2)^N \text{ such that } j_N=j_1 \text{ and } \mathrm{card}\{j_1,\ldots,j_N\}=N-1,\ \sum_{p=1}^{N-1}\bar{g}_{j_k,j_{k+1}}>0. \tag{2.3}$$

By convention we set $\bar{g}_{j,j}=\underline{g}_{i,i}=0$.

Conditions (2.2) and (2.3) are classical in the literature of switching problems and usually referred to as the *no free loop property.* ■

We now introduce the probabilistic tools we need later. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a fixed probability space on which is defined a standard d-dimensional Brownian motion $B=(B_t)_{0\leq t\leq T}$ whose natural filtration is $(\mathcal{F}_t^0:=\sigma\{B_s,s\leq t\})_{0\leq t\leq T}$. Let $\mathbf{F}=(\mathcal{F}_t)_{0\leq t\leq T}$ be the completed filtration of $(\mathcal{F}_t^0)_{0\leq t\leq T}$ with the $\mathbb{P}$-null sets of $\mathcal{F}$, hence $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfies the usual conditions, i.e., it is right continuous and complete. On the other hand let $\mathcal{P}$ be the σ-algebra on $[0,T]\times\Omega$ of $\mathbf{F}$-progressively measurable sets.

Next, let

(i) $\mathcal{H}^{2,\ell}$ ($\ell\geq 1$) be the set of $\mathcal{P}$-measurable and $\mathbb{R}^\ell$-valued processes $w=(w_t)_{t\leq T}$ such that $\mathbb{E}[\int_0^T|w_s|^2ds]<\infty$;

(ii) $\mathcal{S}^2$ (resp. $\mathcal{S}^2_d$) be the set of $\mathcal{P}$-measurable continuous (resp. RCLL) processes such that $\mathbb{E}[\sup_{t\leq T}|w_t|^2]<\infty$.

(iii) $\mathcal{A}^2_i$ be the subset of $\mathcal{S}^2$ of non-decreasing processes $K=(K_t)_{t\leq T}$ such that $K_0=0$.

For $(t,x)\in[0,T]\times\mathbb{R}^k$, let $X^{t,x}$ be the diffusion process solution of the following standard SDE:

$$\forall s\in[t,T],\ \ X_s^{t,x}=x+\int_t^s b(r,X_r^{t,x})dr+\int_t^s\sigma(r,X_r^{t,x})dB_r;\ \ X_s^{t,x}=x,\ \ s\in[0,t]. \tag{2.4}$$

Under Assumption (H0) on b and σ, the process $X^{t,x}$ exists and is unique. Moreover, it satisfies the following estimates: For all $p\geq 1$,

$$\mathbb{E}[\sup_{s\leq T}|X_s^{t,x}|^p]\leq C(1+|x|^p). \tag{2.5}$$

Its infinitesimal generator $\mathcal{L}^X$ is given, for every $(t,x)\in[0,T]\times\mathbb{R}^k$ and $\phi\in\mathcal{C}^{1,2}$, by

$$\begin{aligned}\mathcal{L}^X\phi(t,x)\ &:=\ \tfrac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma^*(t,x))_{i,j}\partial^2_{x_ix_j}\phi(t,x)+\textstyle\sum_{i=1,k}b_i(t,x)\partial_{x_i}\phi(t,x)\\ &=\ \tfrac{1}{2}Tr[\sigma\sigma^\top(t,x)D^2_{xx}\phi(t,x)]+b(t,x)^\top D_x\phi(t,x).\quad\blacksquare\end{aligned} \tag{2.6}$$

Under Assumptions (H0)-(H4), we have

Theorem 2.1. *([8], Theorems 5.4 and 5.5) There exists a unique continuous viscosity solution in the class* Π_g $(\bar{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *(resp.* $(\underline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$*) of the following system:* $\forall (i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases} min\Big\{(\bar{v}^{ij}-L^{ij}(\vec{\bar{v}}))(t,x), max\Big\{(\bar{v}^{ij}-U^{ij}(\vec{\bar{v}}))(t,x), \\ \qquad -\partial_t\bar{v}^{ij}(t,x)-\mathcal{L}^X(\bar{v}^{ij})(t,x)-f^{ij}(t,x,(\bar{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(t,x)D_x\bar{v}^{ij}(t,x))\Big\}\Big\}=0, \\ \bar{v}^{ij}(T,x)=h^{ij}(x) \end{cases} \tag{2.7}$$

(resp.

$$\begin{cases} max\Big\{(\underline{v}^{ij}-U^{ij}(\vec{\underline{v}}))(t,x); min\Big\{(\underline{v}^{ij}-L^{ij}(\vec{\underline{v}}))(t,x); \\ \qquad -\partial_t\underline{v}^{ij}-\mathcal{L}^X(\underline{v}^{ij})(t,x)-f^{ij}(t,x,(\underline{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma(t,x)^\top D_x\underline{v}^{ij}(t,x))\Big\}\Big\}=0, \\ \underline{v}^{ij}(T,x)=h^{ij}(x)). \end{cases} \tag{2.8}$$

In order to obtain the solutions of the systems (2.7) and (2.8) Djehiche et al. ([8]) introduce the following sequences of backward reflected BSDEs with inter-connected obstacles: $\forall m,n\ge 0$, $\forall (i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases} \hat{Y}^{ij,m}\in\mathcal{S}^2,\ \hat{Z}^{ij,m}\in\mathcal{H}^{2,d} \text{ and } \hat{K}^{ij,m}\in\mathcal{A}_i^2\ ; \\ \hat{Y}_s^{ij,m}=h^{ij}(X_T^{t,x})+\int_s^T \hat{f}^{ij,m}(r,X_r^{t,x},(\hat{Y}_r^{kl,m})_{(k,l)\in\Gamma^1\times\Gamma^2},\hat{Z}_r^{ij,m})dr+\int_s^T d\hat{K}_s^{ij,m}-\int_s^T \hat{Z}_r^{ij,m}dB_r,\ s\le T; \\ \hat{Y}_s^{ij,m}\ge \max_{k\in(\Gamma^1)^{-i}}\{\hat{Y}_s^{kj,m}-\underline{g}_{ik}(s,X_s^{t,x})\},\ s\le T; \\ \int_0^T(\hat{Y}_s^{ij,m}-\max_{k\in(\Gamma^1)^{-i}}\{\hat{Y}_s^{kj,m}-\underline{g}_{ik}(s,X_s^{t,x})\})d\hat{K}_s^{ij,m}=0 \end{cases} \tag{2.9}$$

and

$$\begin{cases} Y^{ij,n}\in\mathcal{S}^2,\ Z^{ij,n}\in\mathcal{H}^{2,d} \text{ and } K^{ij,n}\in\mathcal{A}_i^2\ ; \\ Y_s^{ij,n}=h^{ij}(X_T^{t,x})+\int_s^T f^{ij,n}(r,X_r^{t,x},(Y_r^{kl,n})_{(k,l)\in\Gamma^1\times\Gamma^2},Z_r^{ij,n})dr-\int_s^T Z_r^{ij,n}dB_r-\int_s^T dK_r^{ij,n},\ s\le T; \\ Y_s^{ij,n}\le \min_{l\in(\Gamma^2)^{-j}}\big(Y_s^{il,n}+\bar{g}_{jl}(s,X_s^{t,x})\big),\ s\le T; \\ \int_0^T(Y_s^{ij,n}-\min_{l\in(\Gamma^2)^{-j}}\big(Y_s^{il,n}+\bar{g}_{jl}(s,X_s^{t,x})\big))dK_s^{ij,n}=0 \end{cases} \tag{2.10}$$

where, for any $(i,j)\in\Gamma^1\times\Gamma^2$, $n,m\ge 0$ and $(s,x,\vec{y},z^{ij})$,

$$\hat{f}^{ij,m}(s,x,\vec{y},z^{ij}):=f^{ij}(s,x,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z^{ij})-m\big(y^{ij}-\min_{l\in(\Gamma^2)^{-j}}(y^{il}+\bar{g}_{jl}(s,x))\big)^+ \tag{2.11}$$

and

$$f^{ij,n}(s,x,\vec{y},z^{ij}):=f^{ij}(s,x,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z^{ij})+n\big(y^{ij}-\max_{k\in(\Gamma^1)^{-i}}(y^{kj}-\underline{g}_{ik}(s,x))\big)^-. \tag{2.12}$$

Under Assumptions (H0)-(H4) it is shown in [17] (see also [5] or [18]) that each one of the systems (2.9) and (2.10) has a unique solution $(\hat{Y}^{ij,m},\hat{Z}^{ij,m},\hat{K}^{ij,m})$ and $(Y^{ij,m},Z^{ij,m},K^{ij,m})$ respectively. In addition, they enjoy the following properties:

(i) For any $m,n\ge 0$ and $(i,j)\in\Gamma^1\times\Gamma^2$

$$\hat{Y}^{ij,m}\ge\hat{Y}^{ij,m+1}\ge Y^{ij,n+1}\ge Y^{ij,n}. \tag{2.13}$$

(ii) For any $n,m\ge 0$ and $(i,j)\in\Gamma^1\times\Gamma^2$ there exist deterministic continuous functions $\hat{v}^{ij,m}$ and $v^{ij,n}$ such that for any $(t,x)\in[0,T]\times\mathbb{R}^k$ and $s\in[t,T]$, we have

$$\hat{Y}_s^{ij,m}=\hat{v}^{ij,m}(s,X_s^{t,x}) \quad\text{and}\quad Y_s^{ij,n}=v^{ij,n}(s,X_s^{t,x}).$$

Moreover, from (2.13) we easily deduce that, for any $n,m\ge 0$ and $(i,j)\in\Gamma^1\times\Gamma^2$,

$$\hat{v}^{ij,m}\ge\hat{v}^{ij,m+1}\ge v^{ij,n+1}\ge v^{ij,n}. \tag{2.14}$$

Finally, for any $m \geq 0$ (resp. $n \geq 0$), $\hat{v}_m := (\hat{v}^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ (resp. $v_n := (v^{ij,n})_{(i,j)\in\Gamma^1\times\Gamma^2}$) is the unique continuous viscosity solution, in the class Π_g, of the following system of PDEs with inter-connected obstacles: $\forall(i,j) \in \Gamma^1 \times \Gamma^2$, $\forall(t,x) \in [0,T] \times \mathbb{R}^k$,

$$\begin{cases} \min\Big\{(\hat{v}^{ij,m} - L^{ij}(\vec{\hat{v}}_m)(t,x); -\partial_t \hat{v}^{ij,m} - \mathcal{L}^X(\hat{v}^{ij,m})(t,x) - \hat{f}^{ij,m}(t,x,(\hat{v}^{kl,m}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma(t,x)^\top D_x \hat{v}^{ij,m}(t,x))\Big\} = 0, \\ \hat{v}^{ij,m}(T,x) = h^{ij}(x) \end{cases}$$

(resp.

$$\begin{cases} \max\Big\{(v^{ij,n} - U^{ij}(\vec{v}_n))(t,x); -\partial_t v^{ij,n}(t,x) - \mathcal{L}^X(v^{ij,n})(t,x) - f^{ij,n}(t,x,(v^{kl,n}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma^\top(t,x) D_x v^{ij,n}(t,x))\Big\} = 0 \\ v^{ij,n}(T,x) = h^{ij}(x)). \end{cases}$$

(iii) For $(i,j) \in \Gamma^1 \times \Gamma^2$ and $(t,x) \in [0,T] \times \mathbb{R}^k$, let us set

$$\bar{v}^{ij}(t,x) := \lim_{m\to\infty} \searrow \hat{v}^{ij,m}(t,x) \quad \text{and} \quad \underline{v}^{ij}(t,x) := \lim_{n\to\infty} \nearrow v^{ij,n}(t,x).$$

Then, using Perron's method, it is shown that $(\bar{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ (resp. $(\underline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$) is continuous, belongs to Π_g and is the unique viscosity solution, in class Π_g, of system (2.7) (resp. (2.8)). Finally, by construction and in view of (2.14), it holds that, for any $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$\underline{v}^{ij} \leq \bar{v}^{ij}. \tag{2.15}$$

3 Equality of min-max and max-min solutions

In [8], the question whether or not for any $(i,j) \in \Gamma^1 \times \Gamma^2$, $\underline{v}^{ij} \equiv \bar{v}^{ij}$ was left open. This was mainly due to the fact we have not been able to compare the inter-connected obstacles neither in (2.7) nor in (2.8).

Actually, had we known that

$$\begin{array}{ll} \text{(i)} & \forall(i,j) \in \Gamma^1 \times \Gamma^2, \quad L^{ij}(\vec{\bar{v}}) \leq U^{ij}(\vec{\bar{v}}) \\ \text{or} & \\ \text{(ii)} & \forall(i,j) \in \Gamma^1 \times \Gamma^2, \quad L^{ij}(\vec{\underline{v}}) \leq U^{ij}(\vec{\underline{v}}) \end{array} \tag{3.1}$$

then we would have deduced from Theorem 6.1 in Appendix and the uniqueness of the solution of (2.7) or (2.8) that for any $(i,j) \in \Gamma^1 \times \Gamma^2$, $\bar{v}^{ij} = \underline{v}^{ij}$. In this section we are going to investigate under which additional regularity assumptions on the data of the problem, one of the inequalities in (3.1) is satisfied to be able to conclude that $\bar{v}^{ij} = \underline{v}^{ij}$, for any $(i,j) \in \Gamma^1 \times \Gamma^2$, i.e., the solutions of (2.7) and (2.8) are the same.

For that let us introduce the following assumption.
(H5):

(i) For any $(i,j) \in \Gamma^1 \times \Gamma^2$, the functions $\bar{g}_{ij}$ are $\mathcal{C}^{1,2}$. Moreover, $D_x \bar{g}_{ij}$ and $D^2_{xx}\bar{g}_{ij}$ belong to Π_g. Furthermore, for any $j_1, j_2, j_3 \in \Gamma_2$ such that $|\{j_1, j_2, j_3\}| = 3$,

$$\bar{g}_{j_1 j_3}(t,x) < \bar{g}_{j_1 j_2}(t,x) + \bar{g}_{j_2 j_3}(t,x), \ \forall(t,x) \in [0,T] \times \mathbb{R}^k.$$

(ii) For any $(i,j) \in \Gamma^1 \times \Gamma^2$, the function f^{ij} verifies the following estimate:

$$|f^{ij}(t,x,\vec{y},z^{ij})| \leq C(1 + |x|^p)$$

for some real constants C and p.

Remark 3.1. *By Itô's formula, for ant $(i,j) \in \Gamma^1 \times \Gamma^2$,*

$$\begin{array}{l} \bar{g}_{ij}(s, X_s^{t,x}) = \bar{g}_{ij}(t,x) + \int_t^S \mathcal{L}^X(\bar{g}_{ij})(s, X_s^{t,x})ds + \int_t^s D_x\bar{g}_{ij}(s, X_s^{t,x})\sigma(s, X_s^{t,x})dB_s, \quad s \in [t,T] \\ \bar{g}_{ij}(s, X_s^{t,x}) = \bar{g}_{ij}(s,x), \quad s \le t. \end{array}$$

Hereafter, we denote by

$$a^{ij}(s) := \mathcal{L}^X(\bar{g}_{ij})(s, X_s^{t,x}), \quad b^{ij}(s) := D_x\bar{g}_{ij}(s, X_s^{t,x})\sigma(s, X_s^{t,x}), \quad s \le T.$$

Proposition 3.1. *Under Assumptions (H0)-(H5) we have, for every $(i,j) \in \Gamma^1 \times \Gamma^2$,*

$$L^{ij}(\vec{\bar{v}}) \le \bar{v}^{ij} \le U^{ij}(\vec{\bar{v}}).$$

Proof. We derive this inequality through the following three steps.

<u>**Step 1**</u>: For any $m \ge 0$, $(i,j) \in \Gamma^1 \times \Gamma^2$ and $(t,x) \in [0,T] \times \mathbb{R}^k$, let us consider the system of reflected BSDEs with one inter-connected obstacles:

$$\left\{ \begin{array}{l} \check{Y}^{ij,m} \in \mathcal{S}^2, \; \check{Z}^{ij,m} \in \mathcal{H}^{2,d} \text{ and } \check{K}^{ij,m} \in \mathcal{A}_i^2 \; ; \\ \check{Y}_s^{ij,m} = h^{ij}(X_T^{t,x}) + \int_s^T \check{f}^{ij,m}(r, X_r^{t,x}, (\check{Y}_r^{kl,m})_{(k,l)\in\Gamma^1\times\Gamma^2}, \check{Z}_r^{ij,m})dr + \int_s^T d\check{K}_s^{ij,m} - \int_s^T \check{Z}_r^{ij,m}dB_r, \; s \le T; \\ \check{Y}_s^{ij,m} \ge \max_{k\in(\Gamma^1)^{-i}}\{\check{Y}_s^{kj,m} - \underline{g}_{ik}(s, X_s^{t,x})\}, s \le T; \\ \int_0^T(\check{Y}_s^{ij,m} - \max_{k\in(\Gamma^1)^{-i}}\{\hat{Y}_s^{kj,m} - \underline{g}_{ik}(s, X_s^{t,x})\})d\check{K}_s^{ij,m} = 0, \end{array} \right. \tag{3.2}$$

where,

$$\check{f}^{ij,m}(s,x,\vec{y},z^{ij}) := f^{ij}(s,x,\vec{y},z^{ij}) - m \sum_{l\in(\Gamma^2)^{-j}} (y^{ij} - y^{il} - \bar{g}_{jl}(s,x))^+. \tag{3.3}$$

By Corollary 2, in [17], the solution of this system exists and is unique and there exist deterministic continuous functions $(\check{v}^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$, which belong also to Π_g such that, for any i,j and $m \ge 0$, it holds that

$$\forall s \in [t,T], \quad \check{Y}_s^{ij,m} = \check{v}^{ij,m}(s, X_s^{t,x}).$$

Moreover, the family of function $\check{v}_m := (\check{v}^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is the unique continuous solution in viscosity sense in Π_g of the following system of PDEs with obstacles:

$$\left\{ \begin{array}{l} \min\Big\{(\check{v}^{i,j,m} - L^{i,j}(\vec{\check{v}}_m))(t,x); \\ \qquad -\partial_t\check{v}^{i,j,m}(t,x) - \mathcal{L}^X(\check{v}^{i,j,m})(t,x) - \check{f}^{ij,m}(t,x,(\check{v}^{kl,m}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma(t,x)^\top D_x\check{v}^{i,j,m}(t,x))\Big\} = 0, \\ \check{v}^{ij,m}(T,x) = h^{ij}(x). \end{array} \right.$$

Finally, by the Comparison Theorem (see [17], Remark 1), since $\check{f}^{ij,m+1} \le \check{f}^{ij,m}$ and $\hat{f}^{ij,|\Gamma_2|m} \le \check{f}^{ij,m} \le \hat{f}^{ij,m}$, we have, $\forall (i,j) \in \Gamma^1 \times \Gamma^2$ and $m \ge 0$,

$$\check{Y}^{ij,m+1} \le \check{Y}^{ij,m} \quad \text{and} \quad \hat{Y}^{ij,|\Gamma_2|m} \le \check{Y}^{ij,m} \le \hat{Y}^{ij,m},$$

which implies that, for any $(i,j) \in \Gamma^1 \times \Gamma^2$ and $m \ge 0$,

$$\check{v}^{ij,m+1} \le \check{v}^{ij,m} \text{ and } \hat{v}^{ij,|\Gamma_2|m} \le \check{v}^{ij,m} \le \hat{v}^{ij,m}.$$

Then, for any $(i,j) \in \Gamma^1 \times \Gamma^2$, the sequence $(\check{v}^{ij,m})_{m\ge0}$ is decreasing and converges, uniformly on compact subsets of $[0,T] \times \mathbb{R}^k$, to $\bar{v}^{ij}$ since $\lim_{m\to\infty} \hat{v}^{ij,m}(t,x) = \bar{v}^{ij}(t,x)$, for any $(t,x) \in [0,T] \times \mathbb{R}^k$.

<u>**Step 2**</u>: The following estimate holds: For every $(i,j) \in \Gamma^1 \times \Gamma^2$ and $m \ge 0$,

$$\mathbb{E}\Big\{m\int_0^T \sum_{l\in(\Gamma^2)^{-j}} \{\check{Y}_s^{ij,m} - \check{Y}_s^{il,m} - \bar{g}_{jl}(s, X_s^{t,x})\}^+ds\Big\} \le C(1 + |x|^p), \tag{3.4}$$

where, the constant C is independent of m and x.

We first give a representation of $\check{Y}^{ij,m}$ as the optimal payoff of a switching problem. Indeed, let $\delta := (\tau_n, \zeta_n)_{n\geq 0}$ be an admissible strategy of switching, i.e.,

(a) $(\tau_n)_{n\geq 0}$ is an increasing sequence of stopping times such that $\mathbb{P}[\tau_n < T, \forall n \geq 0] = 0$;

(b) $\forall n \geq 0$, ζ_n is a random variable with values in Γ^1 and $\mathcal{F}_{\tau_n}$-measurable;

(c) Let $(A^\delta_s)_{s\leq T}$ be the RCLL $\mathcal{F}_t$-adapted process defined by

$$\forall s \in [0,T), \quad A^\delta_s = \sum_{n\geq 1} \underline{g}_{\zeta_{n-1}\zeta_n}(\tau_n, X^{t,x}_{\tau_n}) 1\!\!1_{\{\tau_n \leq s\}} \quad \text{and} \quad A^a_T = \lim_{s\to T} A^a_s.$$

Then, $\mathbb{E}[(A^a_T)^2] < \infty$. The quantity A^δ_T stands for the switching cost at terminal time T when the strategy δ is implemented.

Next, with an admissible strategy $\delta := (\tau_n, \zeta_n)_{n\geq 0}$ we associate a piecewise constant process $a = (a_s)_{s\in[0,T]}$ defined by

$$a_s := \zeta_0 1\!\!1_{\{\tau_0\}}(s) + \sum_{j=1}^{\infty} \zeta_{j-1} 1\!\!1_{]\tau_{j-1},\tau_j]}(s), \quad s \leq T. \tag{3.5}$$

For any $s \geq \tau_0$, a_s is the mode indicator at time s. Note that there is a bijection between the processes a and the admissible strategies δ, therefore hereafter A^a is nothing else but A^δ.

Finally, for any fixed $i \in \Gamma^1$ and a real constant $\theta \in [0,T]$, we denote by $\mathcal{A}^i_\theta$ the following set:

$$\mathcal{A}^i_\theta := \Big\{\delta = (\tau_n, \zeta_n)_{n\geq 0} \text{ admissible strategy such that } \tau_0 = \theta \text{ and } \zeta_0 = i\Big\}.$$

Now, for an admissible strategy $\delta = (\tau_n, \alpha_n)_{n\geq 0}$, or equivalently a, let us define the pair of processes $(\check{U}^{aj,m}, \check{V}^{aj,m})$ which belongs to $\mathcal{S}^2_d \times \mathcal{H}^{2,d}$ solution of the following BSDE (which is of non standard form): For every $s \leq T$,

$$\check{U}^{aj,m}_s = h^{a(T)j}(X_T) + \int_s^T 1_{\{r\geq\tau_0\}} \check{f}^{aj,m}(r, X^{t,x}_r, \check{U}^{aj,m}_r, \check{V}^{aj,m}_r)dr - \int_s^T \check{V}^{aj,m}_r dB_r - \big(A^a_T - A^a_s\big), \tag{3.6}$$

where, for any $s \geq \tau_0$ and $(\bar{y}, \bar{z}) \in \mathbb{R}^{1+d}$, $\check{f}^{aj,m}(s, X^{t,x}_s, \bar{y}, \bar{z})$ (resp. $\check{f}^{aj}(s, X^{t,x}_s, \bar{y}, \bar{z})$) is equal to

$$\check{f}^{\ell j,m}(s, X^{t,x}_s, [(\check{v}^{kl,m}(s, X^{t,x}_s))_{(k,l)\in\Gamma^1\times\Gamma^2-\{(\ell,j)\}}, \bar{y}], \bar{z})$$

(resp.

$$\check{f}^{\ell j}(s, X^{t,x}_s, [(\check{v}^{kl,m}(s, X^{t,x}_s))_{(k,l)\in\Gamma^1\times\Gamma^2-\{(\ell,j)\}}, \bar{y}], \bar{z}))$$

if at time s, $a(s) = \ell$. Let us point out that since a is admissible, the solution of equation (3.6) exists and is unique. Furthermore, we have the following representation of $\check{Y}^{ij,m}$ (see e.g. [17, 19] for more details on this representation):

$$\check{Y}^{ij,m}_\theta = \text{ess sup}_{a\in\mathcal{A}^i_\theta}\{\check{U}^{a,j,m}_\theta - A^a_\theta\}, \quad \theta \leq T. \tag{3.7}$$

Note that even though the function $\check{f}^{aj,m}$ depend on y^{kl}, $(k,l) \neq (i,j)$, the representation (3.7) still holds since the solution of system of reflected BSDEs (3.2) is unique. It follows that, for any $j, l \in \Gamma^2$ and $\theta \leq T$,

$$(\check{Y}^{ij,m}_\theta - \check{Y}^{il,m}_\theta - \bar{g}_{jl}(\theta, X^{t,x}_\theta))^+ \leq \text{ess sup}_{a\in\mathcal{A}^i_\theta}(\check{U}^{aj,m}_\theta - \check{U}^{al,m}_\theta - \bar{g}_{jl}(\theta, X^{t,x}_\theta))^+. \tag{3.8}$$

We now examine $(\check{U}^{aj,m}_\theta - \check{U}^{al,m}_\theta - \bar{g}_{jl}(\theta, X^{t,x}_\theta))^+$. Define the set $\mathcal{B}_{jl}$ as follows:

$$\mathcal{B}_{jl} = \{(s,\omega) \in [0,T] \times \Omega, \quad \text{such that } \check{U}^{aj,m}_s - \check{U}^{al,m}_s - \bar{g}_{jl}(s, X^{t,x}_s) > 0\}$$

and, for any $s \in [0,T]$,

$$W^{a,jl,m}_s := \check{U}^{aj,m}_s - \check{U}^{al,m}_s - \bar{g}_{jl}(s, X^{t,x}_s).$$

Then, by Itô-Tanaka's formula, we have, for every $s \in [\theta, T]$,

$$(W_s^{a,jl,m})^+ + \frac{1}{2}\int_s^T dL_r^{a,jl,m} + m\int_s^T dr\{\sum_{j''\neq j} \mathbf{1}_{\mathcal{B}_{j,l}(r)}(W_r^{a,jj'',m})^+ - \sum_{j''\neq l} \mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,lj'',m})^+\}$$

$$= \int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)(\check{f}^{aj}(r, X_r^{t,x}, \check{U}_r^{aj,m}) - \check{f}^{al}(r, X_r^{t,x}, U_r^{al,m}) - a_r^{jl})dr$$

$$- \int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)\big(V_r^{aj,m} - V_r^{al,m} - b_r^{jl}\big)dB_r$$

where, $L^{a,jl,m}$ is the local time at 0 of the semimartingale $W^{a,jl,m}$. Splitting the difference

$$\Delta_{a,j,l,m}(r) := m\sum_{j''\neq j} \mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,jj'',m})^+ - m\sum_{j''\neq l} \mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,lj'',m})^+$$

as

$$\Delta_{j,l,m}(r) = m\mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,jl,m})^+ - \mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,lj,m})^+ + m\sum_{j''\neq j, j''\neq l} \mathbf{1}_{\mathcal{B}_{j,l}}(r)\big((W_r^{a,jj'',m})^+ - (W_r^{a,lj'',m})^+\big),$$

the previous formula can be rewritten as follows: $\forall s \in [\theta, T]$,

$$(W_s^{a,jl,m})^+ + \frac{1}{2}\int_s^T dL_r^{a,jl,m} + m\int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,jl,m})^+dr$$

$$= \int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)(\check{f}^{aj}(r, X_r^{t,x}, \check{U}_r^{aj,m}, \check{V}_r^{aj,m}) - \check{f}^{al}(r, X_r^{t,x}, \check{U}_r^{al,m}, \check{V}_r^{al,m}) - a_r^{jl})dr + m\int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,lj,m})^+dr$$

$$- \int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)\big(\check{V}_r^{aj,m} - \check{V}_r^{al,m} - b_r^{jl}\big)dB_r - m\int_s^T dr\{\sum_{j''\neq j,l} \mathbf{1}_{\mathcal{B}_{j,l}(r)}[(W_r^{a,j'',m})^+ - (W_r^{a,lj",m})^+]\} \quad (3.9)$$

But, $\bar{g}_{jl}(t,x) + \bar{g}_{lj}(t,x) > \bar{g}_{jj}(t,x) = 0$. Thus, we obtain that, for every $(t,x) \in [0,T] \times \mathbb{R}^k$,

$$\{y \in \mathbb{R}^m,\ y_j - y_l - \bar{g}_{j,l}(t,x) \geq 0\} \cap \{y \in \mathbb{R}^m,\ y_l - y_j - \bar{g}_{l,j}(t,x) \geq 0\} = \emptyset,$$

from which we deduce that

$$\mathbf{1}_{\mathcal{B}_{j,l}}(r)(W_r^{a,lj,m})^+ = 0,\ \forall r \in [\theta, T]. \quad (3.10)$$

Relying next on the elementary inequality $a^+ - b^+ \leq (a-b)^+$, it holds

$$\mathbf{1}_{\mathcal{B}_{j,l}(r)}[(W_r^{a,jj",m})^+ - (W_r^{a,lj",m})^+] \leq \mathbf{1}_{\mathcal{B}_{j,l}}(r)\left(\check{U}_r^{al,m} - \check{U}_r^{aj,m} - \bar{g}_{lj''}(r, X_r^{t,x}) + \bar{g}_{jj''}(r, X_r^{t,x})\right)^+. \quad (3.11)$$

Using here that the family of penalty costs satisfies $\bar{g}_{j,j"} < \bar{g}_{jl} + \bar{g}_{lj"}$ we deduce that

$$\{y \in \mathbb{R}^m,\ y_j - y_l - \bar{g}_{jl}(t,x) \geq 0\} \cap \{y \in \mathbb{R}^m,\ y_l - y_j - \bar{g}_{lj''}(t,x) + \bar{g}_{jj''}(t,x) \geq 0\} = \emptyset$$

which therefore yields

$$\forall r \in [\theta, T],\ \mathbf{1}_{\mathcal{B}_{j,l}}(r)\left(\check{U}_r^{al,m} - \check{U}_r^{aj,m} - \bar{g}_{lj''}(r, X_r^{t,x}) + \bar{g}_{jj''}(r, X_r^{t,x})\right)^+ = 0. \quad (3.12)$$

Going back now to (3.9), applying Itô's formula to $e^{-ms}(W_s^{a,jl,m})^+$ and taking into account of (3.10), (3.11) and (3.12) to obtain: $\forall s \in [\theta, T]$,

$$(W_s^{a,jl,m})^+ \leq \int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)e^{-m(r-s)}(\check{f}^{aj}(r, X_r^{t,x}, \check{U}_r^{aj,m}, \check{V}_r^{aj,m}) - \check{f}^{al}(r, X_r^{t,x}, \check{U}_r^{al,m}, \check{V}_r^{al,m}) - u_r^{jl})dr$$

$$- \int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)e^{-m(r-s)}\big(\check{V}_r^{aj,m} - \check{V}_r^{al,m} - v_r^{jl}\big)dB_r.$$

Now in taking the conditional expectation and making use of estimates of Assumptions (H0)-(H5) (namely the polynomial growth of the functions) we obtain: $\forall s \in [\theta, T]$,

$$
\begin{aligned}
(W_s^{a,jl,m})^+ &\le C\mathbb{E}[\int_s^T \mathbf{1}_{\mathcal{B}_{j,l}}(r)e^{-m(r-s)}(1+|X_r^{t,x}|^p)dr|\mathcal{F}_s] \\
&\le \tfrac{C}{m}\mathbb{E}[(1+\sup_{r\le T}|X_r^{t,x}|^p)|\mathcal{F}_s].
\end{aligned}
$$

Recall now (3.8) to obtain

$$
m(\check{Y}_\theta^{ij,m} - \check{Y}_\theta^{il,m} - \bar{g}_{j,l}(\theta, X_\theta^{t,x}))^+ \le C\mathbb{E}[(1+\sup_{r\le T}|X_r^{t,x}|^p)|\mathcal{F}_\theta] \tag{3.13}
$$

and then in taking into account estimate (2.5) on $X^{t,x}$ we obtain

$$
m\mathbb{E}\Big\{\sum_{l\neq j}(\check{Y}_\theta^{ij,m} - \check{Y}_\theta^{il,m} - \bar{g}_{j,l}(\theta, X_\theta^{t,x}))^+\Big\} \le C(1+|x|^p), \quad \forall \theta \le T.
$$

As θ is arbitrary in $[0,T]$ then by integration with respect to $d\theta$ in the previous inequality we obtain (3.4).

Step 3 : For any $(t_0, x_0) \in [0,T]\times\mathbb{R}^k$ and $(i,j)\in\Gamma^1\times\Gamma^2$,

$$
L^{ij}(\vec{v})(t_0,x_0) \le \bar{v}^{ij}(t_0,x_0) \le U^{ij}(\vec{v})(t_0,x_0).
$$

We first claim that $\bar{v}^{ij}(t_0,x_0) \ge L^{ij}(\vec{v})(t_0,x_0)$ holds. Indeed, by construction of $\hat{v}^m := (\hat{v}^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ one has $\hat{v}^{ij,m}(t_0,x_0) \ge L^{ij}(\vec{\hat{v}}_m)(t_0,x_0)$. Therefore, taking the limit w.r.t. m, we obtain $\bar{v}^{ij}(t_0,x_0) \ge L^{ij}(\vec{v})(t_0,x_0)$.

We now show that $\bar{v}^{ij}(t_0,x_0) \le U^{ij}(\vec{v})(t_0,x_0)$. First, assume that $\bar{v}^{ij}(t_0,x_0) > L^{ij}(\vec{v})(t_0,x_0)$. Then, relying on the viscosity subsolution property of $\bar{v}^{ij}$ yields

$$
\begin{aligned}
\min\Big\{&(\bar{v}^{ij} - L^{ij}(\vec{v}))(t_0,x_0); \max\Big\{(\bar{v}^{ij} - U^{ij}(\vec{v}))(t_0,x_0); \\
&-\partial_t\bar{v}^{ij}(t_0,x_0) - \mathcal{L}^X(\bar{v}^{ij})(t_0,x_0) - f^{ij}(t_0,x_0,(\bar{v}^{kl}(t_0,x_0))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma(t_0,x_0)^\top D_x\bar{v}^{ij}(t_0,x_0))\Big\}\Big\} \le 0,
\end{aligned}
$$

which implies that

$$
\max\Big\{(\bar{v}^{ij} - U^{ij})(t_0,x_0); -\partial_t\bar{v}^{ij}(t_0,x_0) - \mathcal{L}^X(\bar{v}^{ij}) - f^{ij}(t_0,x_0,(\bar{v}^{kl}(t_0,x_0))_{(k,l)\in\Gamma^1\times\Gamma^2}))\Big\} \le 0.
$$

Hence, $(\bar{v}^{ij} - U^{ij}(\vec{v}))(t_0,x_0) \le 0$.

Suppose now that at (t_0,x_0) we have $\bar{v}^{ij}(t_0,x_0) = L^{ij}(\vec{v})(t_0,x_0)$. Proceeding by contradiction we suppose in addition that

$$
\exists\, \epsilon > 0, \quad (\bar{v}^{ij} - U^{ij}(\vec{v}))(t_0,x_0) > \epsilon. \tag{3.14}
$$

Using both the continuity of $(t,x)\mapsto \bar{v}^{i,j}(t,x)$ and $(t,x)\mapsto U^{ij}(\vec{v})(t,x)$ and the uniform convergence on compact subsets of $(\check{v}^{ij,m})_m$ to $\bar{v}^{ij}$ we claim that for some strictly positive ρ and for m_0 large enough it holds that

$$
\forall m \ge m_0,\ \forall (t,x)\in\mathcal{B}((t_0,x_0),\rho), \quad (\check{v}^{ij,m} - U^{ij}(\vec{\check{v}}_m))(t,x) \ge \frac{\epsilon}{2},
$$

with $\mathcal{B}((t_0,x_0),\rho) = \{(t,x)\in[0,T]\times\mathbb{R}^k \text{ s.t. } |t-t_0|\le\rho,\ |x-x_0|\le\rho\}$.
Without loss of generality we can now assume $[t_0,t_0+\rho]\subset[t_0,T]$. By the definition of $U^{ij}(\vec{\check{v}}_m)$, there exists one index $l_0\neq j$ such that the inequalities

$$
\check{v}^{ij,m} - (\check{v}^{il_0,m} + \bar{g}_{jl_0}) \ge \frac{\epsilon}{2}
$$

and

$$\sum_{l\in(\Gamma^2)^{-j}} \big(\check{v}^{ij,m} - \check{v}^{il,m} - \bar{g}_{jl}\big)^+ \geq \frac{\epsilon}{2}, \tag{3.15}$$

hold on the ball $\mathcal{B}((t_0,x_0),\rho)$.

Let us now introduce the following stopping time τ_X:

$$\tau_X = \inf\{s \geq t_0,\ X_s^{t_0,x_0} \notin \mathcal{B}((t_0,x_0),\rho)\} \wedge \big(t_0+\rho\big).$$

We then have, for all $m \geq m_0$,

$$m\mathbb{E}\left(\int_{t_0}^{\tau_X} \sum_{l\neq j}(\check{v}^{ij,m}(s,X_s^{t_0,x_0}) - \big(\check{v}^{il,m}(s,X_s^{t_0,x_0}) + \bar{g}_{jl}(s,X_s^{t_0,x_0})\big)^+ ds\right) \geq m\frac{\epsilon}{2}\mathbb{E}(\tau_X - t_0) \to \infty, \tag{3.16}$$

as $m \to \infty$. But, this is contradictory to (3.4). Then $\bar{v}^{ij}(t_0,x_0) \leq U^{ij}(\vec{\bar{v}}))(t_0,x_0)$ and the proof is complete. ■

As a by product of Proposition 3.1 and Theorem 6.2 (displayed in the appendix), we have:

Theorem 3.2. *Under Assumptions (H0)-(H5), for any $(i,j)\in\Gamma^1\times\Gamma^2$, it holds that*

$$\bar{v}^{ij} = \underline{v}^{ij}.$$

Remark 3.3. *(i) The result of Theorem 6.2 (see the appendix) is still valid if (H0)-(H4) are in force and the functions $(\underline{g}_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ verify (H5).*
(ii) From (3.13) and Doob's inequality we have, for every $(t,x)\in[0,T]\times\mathbb{R}^k$,

$$m^2\mathbb{E}\Big\{\sum_{l\neq j}[(\check{Y}_s^{ij,m} - \check{Y}_s^{il,m} - \bar{g}_{jl}(s,X_s^{t,x}))^+]^2\Big\} \leq C(1+|x|^{2p}), \quad s\leq T, \tag{3.17}$$

where, C is a constant. ■

4 The min-max solution as a the value of the zero-sum switching game

Let us consider now the following assumption which is used later:

(H6):

(**i**) For any $(i,j)\in\Gamma^1\times\Gamma^2$, the function f^{ij} does not depend on z^{ij}.

(**ii**) For any $(i,j)\in\Gamma^1\times\Gamma^2$, the function f^{ij} does not depend on $(\vec{y},z^{ij})$. ■

Once for all, in this section we suppose that Assumptions (H0)-(H5) hold.

Set

$$Y_s^{ij} = v^{ij}(s,X_s^{t,x}), \quad s\in[t,T] \text{ and } (i,j)\in\Gamma^1\times\Gamma^2.$$

We then have the following representation of Y^{ij} as the value function of a Dynkin game. This is a by-product of Theorems 2.1, 6.2 and Propositions 3.1 and 6.10 (displayed in the appendix) since the barriers

$$L^{ij}(\vec{v}) = \max_{k\neq i}\big(v^{kj} - \underline{g}_{ik}\big) \text{ and } U^{ij}(\vec{v}) = \min_{l\neq j}\big(v^{il} + \bar{g}_{jl}\big)$$

are comparable, i.e., $L^{ij}(\vec{v}) \leq U^{ij}(\vec{v})$ for any i,j.

Proposition 4.1. *Assume that Assumptions (H0)-(H5) and (H6)-(i) are fulfilled. For any $(i,j) \in \Gamma^1 \times \Gamma^2$ and $s \in [t,T]$ we have,*

$$\begin{aligned}
v^{ij}(s, X_s^{t,x}) &= Y_s^{ij} \\
&= ess\ sup_{\sigma \geq s} ess\ inf_{\tau \geq s} \mathbb{E}\Big\{ \int_s^{\sigma\wedge\tau} f^{ij}(r, X_r^{t,x}, (v^{kl}(r, X_r^{t,x}))_{(k,l)\in\Gamma^1\times\Gamma^2})dr \\
&+\{max_{k\in(\Gamma^1)^{-i}}\{v^{kj}(\sigma, X_\sigma^{t,x}) - \underline{g}_{ik}(\sigma, X_\sigma^{t,x})\}\}1\!\!1_{[\sigma<\tau]} + min_{l\in(\Gamma^2)^{-j}}\{v^{il}(\tau, X_\tau^{t,x}) + \bar{g}_{jl}(\tau, X_\tau^{t,x})\}1\!\!1_{[\tau\leq\sigma<T]} \\
&\quad +h_{ij}(X_T^{t,x})1\!\!1_{[\tau=\sigma=T]}|\mathcal{F}_s\Big\} \\
&= ess\ inf_{\tau \geq s} ess\ sup_{\sigma \geq s} \mathbb{E}\Big\{ \int_s^{\sigma\wedge\tau} f^{ij}(r, X_r^{t,x}, (v^{kl}(r, X_r^{t,x}))_{(k,l)\in\Gamma^1\times\Gamma^2})dr \\
&+\{max_{k\in(\Gamma^1)^{-i}}\{v^{kj}(\sigma, X_\sigma^{t,x}) - \underline{g}_{ik}(\sigma, X_\sigma^{t,x})\}\}1\!\!1_{[\sigma<\tau]} + min_{l\in(\Gamma^2)^{-j}}\{v^{il}(\tau, X_\tau^{t,x}) + \bar{g}_{jl}(\tau, X_\tau^{t,x})\}1\!\!1_{[\tau\leq\sigma<T]} \\
&\quad +h_{ij}(X_T^{t,x})1\!\!1_{[\tau=\sigma=T]}|\mathcal{F}_s\Big\}. \qquad \blacksquare
\end{aligned} \tag{4.1}$$

On the other hand, it is shown in ([15], Theorem 3.1) , that Y^{ij} is the unique local solution of the two barriers reflected BSDEs associated with $(f^{ij}(s, X_s^{t,x}, \vec{y}), h_{ij}(X_T^{t,x}), L^{ij}(\vec{v})(s, X_s^{t,x}), U^{ij}(\vec{v})(s, X_s^{t,x}))$. Precisely we have:

Proposition 4.2. *Let $(i,j) \in \Gamma^1 \times \Gamma^2$ be fixed. For any stopping time $\tau \geq t$, there exists another stopping time $\delta_\tau \geq \tau$, $\mathbb{P}-a.s.$ (δ_τ depends also on i,j but we omit it as far as there is no confusion) and three processes $Z^{ij,\tau}$, $K^{ij,\pm,\tau}$ such that:*
(i) $Y_T^{ij} = h^{ij}(X_T^{t,x})$;
(ii)

$$\begin{cases}
Z^{ij,\tau} \in \mathcal{H}^{2,d},\ K^{ij,\pm,\tau} \in \mathcal{A}_i^2 \text{ and non-decreasing} ; \\
\forall s \in [\tau, \delta_\tau], Y_s^{ij} = Y_{\delta_\tau}^{ij} + \int_s^{\delta_\tau} f^{ij}(r, X_r^{t,x}, (Y_r^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2})dr - \int_s^{\delta_\tau} Z_r^{ij} dB_r + \int_s^{\delta_\tau} dK_r^{ij,+,\tau} - \int_s^{\delta_\tau} dK_r^{ij,-,\tau} \\
L^{ij}(\vec{v})(s, X_s^{t,x}) \leq Y_s^{ij} \leq U^{ij}(\vec{v})(s, X_s^{t,x}),\ \forall s \in [t,T] ; \\
\int_\tau^{\delta_\tau} \big(Y_r^{ij} - L^{i,j}(\vec{v})(r, X_r^{t,x})\big)dK_r^{ij,+,\tau} = 0 \text{ and } \int_\tau^{\delta_\tau} \big(Y_r^{ij} - U^{ij}(\vec{v})(r, X_r^{t,x})\big)dK_r^{ij,-,\tau} = 0;
\end{cases} \tag{4.2}$$

(iii) Let γ_τ and θ_τ be the following two stopping times:

$$\gamma_\tau := \inf\{s \geq \tau,\ \ Y_s^{ij} = L^{ij}(\vec{v})(s, X_s^{t,x})\} \wedge T \text{ and } \theta_\tau := \inf\{s \geq \tau,\ \ Y_s^{ij} = U^{ij}(\vec{v})(s, X_s^{t,x})\} \wedge T.$$

Then, $\mathbb{P}-a.s.$, $\gamma_\tau \vee \theta_\tau \leq \delta_\tau$. $\blacksquare$

4.1 Description of the zero-sum switching game

We now address the issue of the relationship between the value function of a zero-sum switching game and the functions $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ solution of system (2.7). We first suppose that Assumption (H6)-(ii) is satisfied, i.e., f^{ij} does not depend on $(\vec{y}, z^{ij})$, for any $(i,j) \in \Gamma^1 \times \Gamma^2$.

To begin with let us describe briefly the zero-sum switching game. Assume we have two players π_1 and π_2 who intervene on a system with the help of switching strategies. An admissible switching strategy for π_1 (resp. π_2) is a sequence $\delta := (\sigma_n, \xi_n)_{n\geq 0}$ (resp. $\nu := (\tau_n, \zeta_n)_{n\geq 0}$) where for any $n \geq 0$,

(i) σ_n (resp. τ_n) is an $\mathbf{F}$-stopping times such that $\mathbb{P}$-a.s., $\sigma_n \leq \sigma_{n+1} \leq T$ (resp. $\tau_n \leq \tau_{n+1} \leq T$) ;

(ii) ξ_n (resp. ζ_n) is a random variable with values in Γ^1 (resp. Γ^2) which is $\mathcal{F}_{\sigma_n}$ (resp. $\mathcal{F}_{\tau_n}$)-measurable ;

(iii) $\mathbb{P}[\sigma_n < T, \forall n \geq 0] = \mathbb{P}[\tau_n < T, \forall n \geq 0] = 0$;

(iv) If $(A^\delta_s)_{s\leq T}$ and $(B^\nu_s)_{s\leq T}$ are the $\mathbf{F}$-adapted RCLL processes defined by:

$$\forall\ s \in [t,T), \quad A^\delta_s = \sum_{n\geq 1} \underline{g}_{\xi_{n-1}\xi_n}(\sigma_n, X^{t,x}_{\sigma_n})1_{[\sigma_n\leq s]} \quad \text{and} \quad A^\delta_T = \lim_{s\to T} A^\delta_s,$$

and

$$\forall\ s \in [t,T), \quad B^\nu_s = \sum_{n\geq 1} \bar{g}_{\zeta_{n-1}\zeta_n}(\tau_n, X^{t,x}_{\tau_n})1_{[\tau_n\leq s]} \quad \text{and} \quad B^\nu_T = \lim_{s\to T} B^\nu_s.$$

Then, $\mathbb{E}[(A^\delta_T)^2 + (B^\nu_T)^2] < \infty$. For any $s \leq T$, A^δ_s (resp. B^ν_s) is the cumulative switching cost at time s for π_1 (resp. π_2) when she implements the strategy δ (resp. ν).

Next, for $t \in \mathbb{R}$, $i \in \Gamma^1$ (resp. $j \in \Gamma^2$), we say that the admissible strategy $\delta := (\sigma_n, \xi_n)_{n\geq 0}$ (resp. $\nu := (\tau_n, \zeta_n)_{n\geq 0}$) belongs $\mathcal{A}^i_{\pi_1}(t)$ (resp. $\mathcal{A}^i_{\pi_2}(t)$) if

$$\sigma_0 = t,\ \xi_0 = i,\ \mathbb{E}[(A^\delta_T)^2] < \infty \quad (\text{resp. } \tau_0 = t,\ \zeta_0 = j,\ \mathbb{E}[(B^\nu_T)^2] < \infty).$$

Given an admissible strategy δ (resp. ν) of π_1 (resp. π_2) one associates a stochastic process $(u_s)_{s\leq T}$ (resp. $(v_s)_{s\leq T}$) which indicates along with time the current mode of π_1 (resp. π_2) and which is defined by:

$$\forall s \leq T,\ u_s = \xi_0 1_{\{\sigma_0\}}(s) + \sum_{n\geq 1} \xi_{n-1}1_{]\sigma_{n-1},\sigma_n]}(s) \ (\text{resp. } v_s = \zeta_0 1_{\{\tau_0\}}(s) + \sum_{n\geq 1} \zeta_{n-1}1_{]\tau_{n-1},\tau_n]}(s)). \tag{4.3}$$

Let now $\delta = (\sigma_n, \xi_n)_{n\geq 0}$ (resp. $\nu = (\tau_n, \zeta_n)_{n\geq 0}$) be a strategy for π_1 (resp. π_2) which belongs to $\mathcal{A}^i_{\pi_1}(t)$ (resp. $\mathcal{A}^j_{\pi_2}(t)$). The interventions of the players are not free and generate a payoff which is a reward (resp. cost) for π_1 (resp. π_2) and whose expression is given by

$$J_t(\delta,\nu) := \mathbb{E}[h^{u_T v_T}(X_T) + \int_t^T f(r, X^{t,x}_r, u_r, v_r)dr - A^\delta_T + B^\nu_T|\mathcal{F}_t], \tag{4.4}$$

where, for any $(k,l) \in \Gamma^1 \times \Gamma^2$, we set $f(s,x,k,l) = f^{kl}(s,x)$, since f^{kl} is assumed to not depend on $(\vec{y}, z^{ij})$.

As usual in the literature of zero-sum games, we are interested in the following issue:
Does this zero-sum switching game have a value, that is, does the following equality hold?

$$\text{ess inf}_{\nu\in\mathcal{A}^j_{\pi_2}(t)}\text{ess sup}_{\delta\in\mathcal{A}^i_{\pi_1}(t)} J_t(\delta,\nu) = \text{ess sup}_{\delta\in\mathcal{A}^i_{\pi_1}(t)}\text{ess inf}_{\nu\in\mathcal{A}^j_{\pi_2}(t)} J_t(\delta,\nu)$$

In the remaining part of this section, we focus on this issue.

For later use, let us introduce two new families of auxiliary processes $(\hat{U}^{\delta,j})_{j\in\Gamma^2}$ (resp. $(\hat{U}^{i,\nu})_{i\in\Gamma^1}$) associated with a given admissible strategy δ (resp. ν) of π_1 (resp π_2). They are defined by: $\forall j \in \Gamma^2$,

$$\begin{cases} \hat{U}^{\delta,j} \in \mathcal{S}^2_d,\ \hat{Z}^{\delta,j} \in \mathcal{H}^{2,d},\ K^{-,\delta,j} \in \mathcal{A}^2_i; \\ \hat{U}^{\delta,j}_s = \ h^{u(T)j}(X^{t,x}_T) + \int_s^T f(r, X^{t,x}_r, u_r, j)dr - \int_s^T Z^{\delta,j}_r dB_r - (A^\delta_T - A^\delta_r) - (K^{-,\delta,j}_T - K^{-,\delta,j}_s),\ s \in [t,T]; \\ \forall s \in [t,T],\ \hat{U}^{\delta,j}_s \leq \min_{l\neq j}\left(\hat{U}^{\delta,l}_s + \bar{g}_{jl}(s, X^{t,x}_s)\right) \text{ and } \int_t^T \{\hat{U}^{\delta,j}_r - \min_{l\neq j}\{\hat{U}^{\delta,l}_r + \bar{g}_{jl}(r, X^{t,x}_r)\}\}dK^{-,\delta,j}_r = 0. \end{cases} \tag{4.5}$$

and for any $i \in \Gamma^1$

$$\begin{cases} U^{i,\nu} \in \mathcal{S}^2_d,\ Z^{i,\nu} \in \mathcal{H}^{2,d},\ K^{+,i,\nu} \in \mathcal{A}^2_i; \\ U^{i,\nu}_s = h^{iv(T)} + \int_s^T f(r, X^{t,x}_r, i, v_r)dr - \int_s^T Z^{i,\nu}_u dB_u + (B^\nu_T - B^\nu_r) + (K^{+,i,\nu}_T - K^{+,i,\nu}_r),\ s \in [t,T]\,; \\ \forall s \in [t,T],\ U^{i,\nu}_s \geq \max_{k\neq i}\{U^{k,\nu}_s - \underline{g}_{ik}(s, X^{t,x}_s)\} \text{ and } \int_t^T \left(\hat{U}^{\delta,j}_r - \max_{k\neq i}\{U^{k,\nu}_r - \underline{g}_{ik}(r, X^{t,x}_r)\}\right) dK^{+,i,\nu}_r = 0. \end{cases} \tag{4.6}$$

These equations are actually not of standard form, but by an obvious change of variables one can easily show that they have unique solutions. On the other hand, let us point out that thanks to the connection between the standard switching problem and multidimensional RBSDE with a lower interconnected obstacle (see e.g. [9] or [19]) the family $(\hat{U}^{\delta,j} - A^\delta)_{j\in\Gamma^2}$ (resp. $(U^{i,\nu} + B^\nu)_{i\in\Gamma^1}$) of processes verifies:

$$\hat{U}^{\delta,j}_t = \text{ess inf}_{\nu\in\mathcal{A}^j_{\pi_2}(t)}\{J_t(\delta,\nu) + A^\delta_t\} \text{ and } U^{i,\nu}_t = \text{ess sup}_{\delta\in\mathcal{A}^i_{\pi_1}(t)}\{J_t(\delta,\nu) - B^\nu_t\}. \tag{4.7}$$

We now give the main result of this section. It relates $(Y^{ij}_s)_{s\leq T} = (v^{ij}(s, X^{t,x}_s))_{s\leq T}$, $(i,j) \in \Gamma^1 \times \Gamma^2$, with the value of the zero-sum switching game described above.

Theorem 4.1. *Suppose Assumptions (H0)-(H5) and (H6)-(ii) are satisfied. Then, for any $(i_0, j_0) \in \Gamma^1 \times \Gamma^2$,*

$$v^{i_0j_0}(t,x) = Y^{i_0j_0}_t = \textit{ess sup}_{\delta\in\mathcal{A}^{i_0}_{\pi_1}(t)}\textit{ess inf}_{\nu\in\mathcal{A}^{j_0}_{\pi_2}(t)} J_t(\delta,\nu) = \textit{ess inf}_{\nu\in\mathcal{A}^{j_0}_{\pi_2}(t)}\textit{ess sup}_{\delta\in\mathcal{A}^{i_0}_{\pi_1}(t)} J_t(\delta,\nu). \tag{4.8}$$

Proof: Recall the definition of $(\check{Y}^{ij,m}, \check{Z}^{ij,m}, \check{K}^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$, $m \geq 0$, given in (3.2). In order to alleviate notations, we denote it simply by $(Y^{ij,m}, Z^{ij,m}, K^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$, $m \geq 0$: $\forall (i,j) \in \Gamma^1 \times \Gamma^2$,

$$\begin{cases} Y^{ij,m} \in \mathcal{S}^2,\ Z^{i,j,m} \in \mathcal{H}^{2,d} \text{ and } K^{ij,m} \in \mathcal{A}^2_i\ ; \\ Y^{ij,m}_s = h^{ij}(X^{t,x}_T) + \int_s^T f^{ij,m}(r, X^{t,x}_r, (Y^{kl,m}_r)_{(k,l)\in\Gamma^1\times\Gamma^2})dr + (K^{ij,m}_T - K^{ij,m}_s) - \int_s^T Z^{ij,m}_r dB_r,\ \forall s \in [t,T]; \\ Y^{ij,m}_s \geq \max_{k\in(\Gamma^2)^{-i}}\{Y^{kj,m}_s - \underline{g}_{ik}(s, X^{t,x}_s)\},\ \forall s \in [t,T]; \\ \int_t^T (Y^{ij,m}_s - \max_{k\in(\Gamma^1)^{-i}}\{Y^{kj,m}_s - \underline{g}_{ik}(s, X^{t,x}_s)\})dK^{ij,m}_s = 0 \end{cases} \tag{4.9}$$

where, we recall that, for any $s \in [t,T]$, $m \geq 0$ and $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$f^{ij,m}(s, X^{t,x}_s, \vec{y}) = f^{ij}(s, X^{t,x}_s) - m \sum_{l\in(\Gamma^2)^{-j}} \left(y^{ij} - (y^{il} + \bar{g}_{jl}(s, X^{t,x}_s))\right)^+.$$

As already mentioned above, we know that, for any $(i,j) \in \Gamma^1 \times \Gamma^2$, $Y^{ij,m} \to_m Y^{ij}$ in $\mathcal{S}^2$. For sake of clarity, we divide the proof into two steps

Step 1: For any $(i_0, j_0) \in \Gamma^1 \times \Gamma^2$,

$$Y^{i_0j_0}_t = \text{ess sup}_{\delta\in\mathcal{A}^{i_0}_{\pi_1}(t)}\{\hat{U}^{\delta,j_0}_t - A^\delta_t\}. \tag{4.10}$$

Let $\delta = (\sigma_l, \xi_l)_{l\geq 0}$ be a strategy of $\mathcal{A}^{i_0}_{\pi_1}(t)$. We are going first to show that $Y^{i_0j_0}_t \geq \hat{U}^{\delta,j_0}_t - A^\delta_t$. So let us define the processes $(Y^{\delta j,m})_{j\in\Gamma^2}$ and $(\hat{U}^{\delta,j,m})_{j\in\Gamma^2}$ as follows:

(i) $\forall j \in \Gamma^2$,

$$\forall s \in [t,T),\quad Y^{\delta,j,m}_s = \sum_{l\geq 0} Y^{\xi_l j,m}_s 1\!\!1_{[\sigma_l\leq s<\sigma_{l+1}]} \quad \text{and} \quad Y^{\delta,j,m}_T = h^{u(T)j}(X^{t,x}_T),$$

where,

$$\forall s \in [t,T],\quad Y^{\xi_l j,m}_s = \sum_{q\in\Gamma^1} Y^{qj,m}_s 1\!\!1_{[\xi_l=q]}. \tag{4.11}$$

The process $Y^{\delta,j,m}$ is well defined since the sum contains only finite many terms since the stratgey δ is admissible and then $\mathbb{P}[\sigma_l < T, \forall l \geq 0] = 0$. On the other hand, at time $0 < \sigma_l < T$, $Y^{\delta,j,m}$ has a jump which is equal to $Y^{\xi_l j,m}_{\sigma_l} - Y^{\xi_{l-1} j,m}_{\sigma_l}$.

(ii) The processes $(\hat{U}^{\delta,j,m})_{j\in\Gamma^2}$ are defined as the solution of the following non standard multi-dimensional BSDE: $\forall j \in \Gamma^2$,

$$\begin{aligned}\hat{U}^{\delta,j,m}_s = \quad & h^{u(T)j}(X^{t,x}_T) + \int_s^T \Big\{ f(r, X^{t,x}_r, u_r, j) - m \sum_{l\neq j} (\hat{U}^{\delta,j,m}_r - \hat{U}^{\delta,l,m}_r - \bar{g}_{jl})^+ \Big\} dr \\ & -(A^\delta_T - A^\delta_s) - \int_s^T \hat{V}^{\delta,j,m}_u dB_u, \ s \in [t,T].\end{aligned} \tag{4.12}$$

Note that $(\hat{U}^{\delta,j,m} + A^\delta)_{j\in\Gamma^2}$ is a solution of a standard multidimensional BSDE whose coefficient is Lipschitz. As those latter processes exist, then so are $(\hat{U}^{\delta,j,m})_{j\in\Gamma^2}$. On the other hand, as for the system given in (2.10), the sequence of processes $((\hat{U}^{\delta,j,m})_{j\in\Gamma^2})_{m\geq 0}$ converges in $\mathcal{S}^2_d$ toward $(\hat{U}^{\delta,j})_{j\in\Gamma^2}$. We now prove the following: for any $m \geq 0$, $j \in \Gamma^2$,

$$Y^{\delta,j,m}_0 \geq \hat{U}^{\delta,j,m}_0. \tag{4.13}$$

For any $j \in \Gamma^2$ let us define $K^{\delta,j,m}$ and $Z^{\delta,j,m}$ as follows: $\forall s \in [t,T]$,

$$Z^{\delta,j,m}_s := \sum_{l\geq 0} Z^{\xi_l j,m}_s \mathbb{1}_{[\sigma_l \leq s < \sigma_{l+1}[} \quad \text{and} \quad K^{\delta,j,m}_s = \sum_{l\geq 0} \int_{s\wedge\sigma_l}^{s\wedge\sigma_{l+1}} dK^{\xi_l j,m}_s,$$

where, $Z^{\xi_l j,m}_s$ and $K^{\xi_l j,m}_s$ are defined in the same way as in (4.11). Once more there is no definition problem of those processes since δ is admissible. Therefore the triple of processes $(Y^{\delta,j,m}, Z^{\delta,j,m}, K^{\delta,j,m})_{j\in\Gamma^2}$ verify: $\forall s \in [t,T)$,

$$\begin{aligned} Y^{\delta,j,m}_s &= Y^{\delta,j,m}_t - \int_t^s \Big\{ f^{u_r j}(r, X^{t,x}_r) dr + m \sum_{l\neq j} \big(Y^{\delta,j,m}_r - Y^{\delta,l,m}_r - \bar{g}_{jl}(r, X^{t,x}_r) \big)^+ dr + Z^{\delta,j,m}_r dB_r - dK^{\delta,j,m}_r \Big\} \\ &\qquad + \sum_{l\geq 1} (Y^{\xi_l j,m}_{\sigma_l} - Y^{\xi_{l-1} j,m}_{\sigma_l}) \mathbb{1}_{[\sigma_l \leq s]} \\ &= Y^{\delta,j,m}_t - \int_t^s \Big\{ f^{u_r j}(r, X^{t,x}_r) dr + m \sum_{l\neq j} \big(Y^{\delta,j,m}_r - Y^{\delta,l,m}_r - \bar{g}_{jl}(r, X^{t,x}_r) \big)^+ dr + Z^{\delta,j,m}_r dB_r - dK^{\delta,j,m}_r \Big\} \\ &\qquad - \sum_{l\geq 1} (Y^{\xi_{l-1} j,m}_{\sigma_l} - Y^{\xi_l j,m}_{\sigma_l} + \underline{g}_{\xi_{l-1}\xi_l}(\sigma_l, X^{t,x}_{\sigma_l})) \mathbb{1}_{[\sigma_l \leq s]} + A^\delta_s. \end{aligned}$$

Next, let us define $\tilde{A}^{\delta,j,m}$ by:

$$\tilde{A}^{\delta,j,m}_s := \sum_{l\geq 1} (Y^{\xi_{l-1} j,m}_{\sigma_l} - Y^{\xi_l j,m}_{\sigma_l} + \underline{g}_{\xi_{l-1}\xi_l}(\sigma_l, X^{t,x}_{\sigma_l})) \mathbb{1}_{[\sigma_l \leq s]} \quad \text{for} \quad s \in [t,T) \quad \text{and} \quad \tilde{A}^{\delta,j}_T = \lim_{s\to T} \tilde{A}^{\delta,j}_s$$

which is an $\mathbf{F}$-adapted non-decreasing process. As the stragey δ is admissible, then writing backwardly between $s \wedge \sigma_k$ and $\sigma_k \vee s$ the equation for the process $Y^{\delta,j,m}$ and take the limit as $k \to \infty$ to obtain: $\forall j \in \Gamma^2$,

$$\begin{aligned} Y^{\delta,j,m}_s &= h^{u(T)j}(X^{t,x}_T) + \int_s^T \Big\{ f^{u_r j}(r, X^{t,x}_r) dr - m \sum_{l\neq j} \big(Y^{\delta,j,m}_r - Y^{\delta,l,m}_r - \bar{g}_{jl}(r, X^{t,x}_r) \big)^+ dr \\ &\qquad - Z^{\delta,j,m}_r dB_r + dK^{\delta,j,m}_r \Big\} - (A^\delta_T - A^\delta_s) + (\tilde{A}^{\delta,j,m}_T - \tilde{A}^{\delta,j,m}_s), \forall s \in [t,T]. \end{aligned} \tag{4.14}$$

This equation implies also that $\mathbb{E}[(\tilde{A}^{\delta,j,m}_T)^2] < \infty$. Comparing now equation (4.14) for $(Y^{\delta,j,m}_s, Z^{\delta,j,m}_s)_{s\in[t,T]}$ and the one satisfied by $(\hat{U}^{\delta,j,m}_s, \hat{V}^{\delta,j,m}_s)_{s\in[t,T]}$ we have, by uniqueness of the solution of the multi-dimensional BSDE (4.12), that

$$Y^{\delta,j,m}_s - \mathbb{E}[(\tilde{A}^{\delta,j,m}_T - \tilde{A}^{\delta,j,m}_s) + (K^{\delta,j,m}_T - K^{\delta,j,m}_s) | \mathcal{F}_s] = \hat{U}^{\delta,j,m}_s, \ \ \forall s \in [t,T] \quad \text{and} \quad j \in \Gamma^2. \tag{4.15}$$

As the processes $\tilde{A}^{\delta,j,m}$ and $K^{\delta,j,m}$ are non-decreasing then

$$Y^{\delta,j,m}_s \geq \hat{U}^{\delta,j,m}_s, \quad \forall s \in [t,T].$$

Taking now the limit w.r.t. m, we obtain that

$$Y^{i_0 j}_t = \lim_{m\to\infty} Y^{i_0 j,m}_t \geq \lim_{m\to\infty} \{Y^{\delta,j,m}_t - A^\delta_t\} \geq \lim_{m\to\infty} \{\hat{U}^{\delta,j,m}_t - A^\delta_t\} = \hat{U}^{\delta,j}_t - A^\delta_t, \quad \forall j \in \Gamma^2.$$

Step 2: In order to complete the proof of the claim we construct a strategy $\bar{\delta}$ of $\mathcal{A}^{i_0}_{\pi_1}(t)$ such that $Y^{i_0,j_0}_t = \hat{U}^{\bar{\delta},j_0}_t$.

Let us first define the strategy $\bar{\delta}= (\xi^*_l, \sigma^*_l)_{l\geq 0}$ as follows:
(i) $\xi^*_0 = i$, $\sigma^*_0 = t$.
(ii) Next, for any $l \geq 1$, we define σ^*_l and ξ^*_l by:

$$\left\{\begin{array}{l} \sigma^*_l = \inf\left\{s \geq \sigma^*_{l-1}, \quad Y^{\xi^*_{l-1}j_0}_s = \max_{k\neq\xi^*_{l-1}}\left(Y^{kj_0}_s - \underline{g}_{\xi^*_{l-1}k}(s, X^{t,x}_s)\right)\right\} \wedge T, \\ \\ \xi^*_l \in \ \mathrm{argmax}_{k,\ k\neq\xi^*_{l-1}}\left\{Y^{kj_0}_{\sigma^*_l} - \underline{g}_{\xi^*_{l-1}k}(\sigma^*_l, X^{t,x}_{\sigma^*_l})\right\}. \end{array}\right. \tag{4.16}$$

We first prove that $\bar{\delta}$ verifies

$$\mathbb{P}\left(\{\omega, \ \forall\, l \geq 0, \ \ \sigma^*_l(\omega) < T\}\right) = 0. \tag{4.17}$$

We proceed by contradiction. Assume that the last property does not hold. As the set Γ^1 is finite then one can find a loop $(i_1, i_2, \cdots, i_l = i_1)$ of exactly $l-1$ ($l \geq 2$) indices and a subsequence $(l_p)_{p\geq 0}$ (which may depend on ω) satisfying $l_{p+1} - l_p \geq l$ and such that:

$$\mathbb{P}\left(Y^{i_1j_0}_{\sigma^*_{l_p}} = Y^{i_2j_0}_{\sigma^*_{l_p}} - \underline{g}_{i_1i_2}(\sigma^*_{l_p}, X^{t,x}_{\sigma^*_{l_p}}), \cdots, Y^{i_{l-1}j_0}_{\sigma^*_{l_p+l-1}} = Y^{i_lj_0}_{\sigma^*_{l_p+l-1}} - \underline{g}_{i_{l-1}i_l}(\sigma^*_{l_p+l-1}, X^{t,x}_{\sigma^*_{l_p+l-1}}), \quad \forall p \geq 0\right) > 0.$$

Next, let us set τ^* by $\tau^*(\omega) := \lim_p \sigma^*_{l_p}(\omega)$, then by taking the limit in the previous equalities we obtain

$$\mathbb{P}\left(Y^{i_1j}_{\tau^*} = Y^{i_2j_0}_{\tau^*} - \underline{g}_{i_1i_2}(\tau^*, X^{t,x}_{\tau^*}), \cdots, Y^{i_{l-1}j_0}_{\tau^*} = Y^{i_lj_0}_{\tau^*} - \underline{g}_{i_{l-1}i_l}(\tau^*, X^{t,x}_{\tau^*})\right) > 0$$

Since $i_1 = i_l$, we obtain

$$\mathbb{P}\left(\sum_{k=1}^{l-1} \underline{g}_{i_ki_{k+1}}(\tau^*, X^{t,x}_{\tau^*}) = 0\right) > 0$$

which contradicts to the so called non free loop property and then $\bar{\delta}$ satisfies (4.17).

Let us show that $\mathbb{E}[(A^{\bar{\delta}}_T)^2] < \infty$. First note that due to the non-free loop property $\mathbb{E}[(A^{\bar{\delta}}_t)^2] < \infty$. Next let us introduce the process $Y^{\bar{\delta},j_0}$ by setting

$$\forall\ s \in [t, T), \quad Y^{\bar{\delta},j_0}_s = \sum_{l\geq 0} Y^{\xi^*_l j_0}_s \mathbf{1}_{\{\sigma^*_l \leq s < \sigma^*_{l+1}\}} \quad \text{ and } \quad Y^{\bar{\delta},j_0}_T = h^{u^{\bar{\delta}}(T)j_0}, \tag{4.18}$$

where, $(u^{\bar{\delta}}(s))_{s\in[t,T]}$, as in (4.12), is the RCLL process associated with $\bar{\delta}$ which indicates the mode of π_1 at time s when the strategy $\bar{\delta}$ is implemented. Next, by the local solution property of Proposition 4.2, for any $l \geq 0$, we have

$$\left\{\begin{array}{l} Y^{\xi^*_l j_0}_s = Y^{\xi^*_l j_0}_{\sigma^*_{l+1}} + \int_s^{\sigma^*_{l+1}} f^{\xi^*_l j_0}(r, X^{t,x}_r)dr - (K^{\xi^*_l j_0,-}_{\sigma^*_{l+1}} - K^{\xi^*_l j_0,-}_s) - \int_s^{\sigma^*_{l+1}} Z^{\xi^*_l j_0}_r dB_r, \ \ \forall s \in [\sigma^*_l, \sigma^*_{l+1}]; \\ Y^{\xi^*_l j_0}_s \leq \min_{p\in(\Gamma^2)^{-j_0}}\{Y^{\xi^*_l j_0}_s + \bar{g}_{j_0p}(s, X^{t,x}_s)\}, \ \ \forall s \in [\sigma^*_l, \sigma^*_{l+1}]; \\ \int_{\sigma^*_l}^{\sigma^*_{l+1}}(Y^{\xi^*_l j_0}_s - \min_{p\in(\Gamma^2)^{-j_0}}\{Y^{\xi^*_l p}_s + \bar{g}_{j_0p}(s, X^{t,x}_s)\})dK^{\xi^*_l j_0,-}_u = 0, \end{array}\right. \tag{4.19}$$

where, $Z^{\xi^*_l j_0}$ and $K^{\xi^*_l j_0,-}$ are fixed processes which depend actually on σ^*_l, for all $l \geq 0$. Let us now define $Z^{\bar{\delta},j_0}$ and $K^{\bar{\delta},j_0,-}$ by:

$$Z^{\bar{\delta},j_0}_s := \sum_{l\geq 0} Z^{\xi^*_l j_0}_s \mathbb{1}_{[\sigma^*_l \leq s < \sigma^*_{l+1}[} \quad \text{ and } \quad K^{\bar{\delta},j_0,-}_s := \sum_{l\geq 0} \int_{s\wedge\sigma^*_l}^{s\wedge\sigma^*_{l+1}} dK^{\xi^*_l j_0,-}_s, \quad s \in [t, T].$$

We note that, by definition, we have, for any $l \geq 0$,

$$\{Y^{\xi_l^* j}_{\sigma_{l+1}^*} - Y^{\xi_{l+1}^* j}_{\sigma_{l+1}} + \underline{g}_{\xi_l^* \xi_{l+1}^*}(\sigma_{l+1}^*, X^{t,x}_{\sigma_{l+1}^*})\}\mathbf{1}_{\{\sigma_{l+1}^* < T\}} = 0.$$

Then, taking into account the jump of $Y^{\bar{\delta},j_0}$ at σ_{l+1}^* (when smaller than T) which is equal to $Y^{\xi_{l+1}^* j_0}_{\sigma_{l+1}^*} - Y^{\xi_l^* j_0}_{\sigma_{l+1}^*}$ and by (4.17), we have, for every $s \in [t,T]$,

$$Y_s^{\bar{\delta},j_0} = h^{u^{\bar{\delta}}(T) j_0}(X_T^{t,x}) - (A_T^{\bar{\delta}} - A_s^{\bar{\delta}}) + \int_s^T f(r, X_r^{t,x}, u_r^{\bar{\delta}}, j_0)dr - (K_T^{\bar{\delta},j_0,-} - K_s^{\bar{\delta},j_0,-}) - \int_s^T Z_r^{\bar{\delta},j_0} dB_r, \qquad (4.20)$$

which implies that

$$Y_s^{\bar{\delta},j_0} = Y_t^{\bar{\delta},j_0} + (A_s^{\bar{\delta}} - A_t^{\bar{\delta}}) + K_s^{\bar{\delta},j_0,-} - \int_t^s f(r, X_r^{t,x}, u_r^{\bar{\delta}}, j_0)dr + \int_t^s Z_r^{\bar{\delta},j_0} dB_r, \quad \forall s \in [t,T].$$

As $Y^{\bar{\delta},j_0}$ belongs to $\mathcal{S}_d^2$ then a localization procedure and Fatou's Lemma permit to deduce that

$$\mathbb{E}[A_T^{\bar{\delta}} + K_T^{\bar{\delta},j_0,-}] < \infty.$$

Thus, for any $s \in [t,T]$,

$$\begin{aligned}\mathbb{E}[(A_T^{\bar{\delta}} - A_s^{\bar{\delta}}) \quad &+ (K_T^{\bar{\delta},j_0,-} - K_s^{\bar{\delta},j_0,-})|\mathcal{F}_s] \\ &= \mathbb{E}[h^{u^{\bar{\delta}}(T),j_0}(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, u_r^{\bar{\delta}}, j_0)dr|\mathcal{F}_s] - Y_s^{\bar{\delta},j_0} \\ &= \mathbb{E}[h^{u^{\bar{\delta}}(T),j_0}(X_T^{t,x}) + \int_t^T f(r, X_r^{t,x}, u_r^{\bar{\delta}}, j_0)dr|\mathcal{F}_s] - \{Y_s^{\bar{\delta},j_0} + \int_t^s f(r, X_r^{t,x}, u_r^{\bar{\delta}}, j_0)dr\}.\end{aligned} \qquad (4.21)$$

Therefore, the right-hand side of the previous equality is a supermartingale which moreover, by Doob's inequality, belongs to $\mathcal{S}_d^2$. Hence, using a result by Dellacherie-Meyer ([7], pp. 220-221) we deduce that

$$\mathbb{E}[\{A_T^{\bar{\delta}} + K_T^{\bar{\delta},j_0}\}^2] < \infty,$$

since the right-hand side of (4.21) belongs to $\mathcal{S}_d^2$. Thus, the strategy $\bar{\delta}$ is admissible.

It remains now to show that $Y_t^{i_0 j_0} = \hat{U}_t^{\bar{\delta},j_0} - A_t^{\bar{\delta}}$.

The equality (4.15) applied with $\bar{\delta}$ reads as:

$$Y_s^{\bar{\delta},j_0,m} - \mathbb{E}[(\tilde{A}_T^{\bar{\delta},j_0,m} - \tilde{A}_s^{\bar{\delta},j_0,m}) + (K_T^{\bar{\delta},j_0,m} - K_s^{\bar{\delta},j_0,m})|\mathcal{F}_s] = \hat{U}_s^{\bar{\delta},j_0,m}, \forall s \in [t,T]. \qquad (4.22)$$

Therefore, the process $(Y_s^{\bar{\delta},j_0,m} - \hat{U}_s^{\bar{\delta},j_0,m})_{s \leq T}$ is a supermartingale which satisfies

$$\sup_{m \geq 0} \mathbb{E}[\sup_{s \leq T} |Y_s^{\bar{\delta},j_0,m} - \hat{U}_s^{\bar{\delta},j_0,m}|^2] < \infty,$$

since, for any $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$\sup_{m \geq 0} \mathbb{E}[\sup_{s \leq T} \{|Y_s^{ij,m}| + |\hat{U}_s^{ij,m}|\}^2] < \infty.$$

Once more, by a Dellacherie-Meyer's result ([7], pp. 220-221), we obtain

$$\sup_{m \geq 0} \mathbb{E}[\{\tilde{A}_T^{\bar{\delta},j_0,m} + K_T^{\bar{\delta},j_0,m}\}^2] < \infty. \qquad (4.23)$$

But,

$$\tilde{A}_s^{\bar{\delta},j_0,m} := \sum_{l \geq 1} (Y^{\xi_{l-1}^* j_0,m}_{\sigma_l^*} - Y^{\xi_l^* j_0,m}_{\sigma_l^*} + \underline{g}_{\xi_{l-1}^* \xi_l^*}(\sigma_l^*, X^{t,x}_{\sigma_l^*}))\mathbb{1}_{[\sigma_l^* \leq s]} \quad \text{for} \quad s \in [t,T) \quad \text{and} \quad \tilde{A}^{\bar{\delta},j_0,m} = \lim_{s \to T} \tilde{A}_s^{\bar{\delta},j_0,m},$$

and, by definition of the strategy $\bar{\delta}$, for any $l \geq 0$, it holds

$$\{Y^{\xi_l^* j_0}_{\sigma_{l+1}^*} - Y^{\xi_{l+1}^* j_0}_{\sigma_{l+1}} + \underline{g}_{\xi_l^* \xi_{l+1}^*}(\sigma_{l+1}^*, X_{\sigma_{l+1}^*})\}\mathbf{1}_{\sigma_{l+1}^* < T} = 0.$$

As the strategy $\bar{\delta}$ is admissible (i.e. for ω fixed there is only a finite many σ_l^* such that $\sigma_l^* < T$) and $Y^{ij_0,m} \searrow Y^{ij_0}$ as $m \to \infty$ in $\mathcal{S}^2$ then $\mathbb{P} - a.s.$, $\tilde{A}_s^{\bar{\delta},j_0,m} \to 0$ as $m \to \infty$, for any $s \in [t,T]$. Therefore, with (4.23) we deduce that $\tilde{A}_T^{\bar{\delta},j_0,m} \to 0$ in $L^1(d\mathbb{P})$.

Next, we shall show that there exists a subsequence of $\{m\}$ which we still denote by $\{m\}$ such that for any $l \geq 0$, the random variable

$$\sum_{p=0}^{p=l} \int_{\sigma_p^*}^{\sigma_{p+1}^*} dK_s^{\xi_p^* j_0,m} \to_m 0 \text{ weakly in } L^2(d\mathbb{P}).$$

To begin with, by using (3.17), let $\{m\}$ be a subsequence such that for any $(i,j) \in \Gamma^1 \times \Gamma^2$

$$(m \sum_{l \neq j} (Y_r^{ij,m} - Y_r^{il,m} - \bar{g}_{jl}(r, X_r^{t,x}))^+)_{r \in [t,T]}$$

converges weakly in $\mathcal{H}^{2,d}$ to a process $(\alpha_r^{ij})_{r \in [t,T]}$.

We only consider the sequence $(\int_{\sigma_0^*}^{\sigma_1^*} dK_s^{\xi_0^* j_0,m})_{m \geq 0} = (\int_t^{\sigma_1^*} dK_s^{ij_0,m})_{m \geq 0}$ since for the other cases a similar procedure applies (keep in mind that we should have $\mathbb{P}[\sigma_1^* > \sigma_0^*] > 0$, otherwise this case is irrelevant and then one should begin with the next case, i.e, taking $p = 1$). For $s \in [t, \sigma_1^*]$ and from (4.9) we have

$$Y_s^{ij_0,m} = Y_{\sigma_1^*}^{ij_0,m} + \int_s^{\sigma_1^*} f^{ij_0}(r, X_r^{t,x}) dr - m \int_s^{\sigma_1^*} \sum_{l \neq j_0} (Y_r^{ij_0,m} - Y_r^{il,m} - \bar{g}_{j_0 l}(r, X_r^{t,x}))^+ dr - \int_s^{\sigma_1^*} Z_r^{ij_0,m} dB_r + K_{\sigma_1^*}^{ij_0,m} - K_s^{ij_0,m}.$$

As $Y^{ij_0,m}$ converges to Y^{ij_0} in $\mathcal{S}^2$, by Itô's formula, we have:

(i) $\sup_{m \geq 0} \mathbb{E}[(K_{\sigma_1^*}^{ij_0,m})^2] < \infty$;

(ii) the sequence $(Z_s^{ij_0,m} 1\!\!1_{[s \leq \sigma_1^*]})$ converges in $\mathcal{H}^{2,d}$ to some process $\bar{Z}^{ij_0}$.

Now, for $s \leq \sigma_1^*$, define

$$K_s^{ij_0} = Y_t^{ij_0} - Y_s^{ij_0} + \int_t^s \bar{Z}_r^{ij_0} dB_r + \int_t^s \alpha_r^{ij_0} dr - \int_t^s f^{ij_0}(r, X_r^{t,x}) dr.$$

Then, the process K^{ij_0} is continuous on $[t, \sigma_1^*]$. Moreover, using the weak convergence pointed out previously, for any any stopping time $\tau \in [t, \sigma_1^*]$, $K_\tau^{ij_0,m} \to_m K_\tau^{ij_0}$ weakly in $L^2(d\mathbb{P})$.

Next, let τ be a stopping time such that $t \leq \tau < \sigma_1^*$. The properties of $K^{ij_0,m}$ (especially the Skorokhod condition) combined with the uniform convergence of $(Y^{ij,m})_m$ to Y^{ij} and the definition of σ_1^*, i.e.,

$$\forall s < \sigma_1^*, \quad Y_s^{ij_0} > \max_{k \neq i} \{Y_s^{kj_0} - \underline{g}_{ik}(s, X_s^{t,x})\}$$

imply the existence of some $m_0(\omega)$ such that if $m \geq m_0$ then $K_\tau^{ij_0,m} = 0$. Therefore, the sequence $(K_\tau^{ij_0,m})_m$ converges $\mathbb{P} - a.s.$ to 0 and by (i) above it converges also in $L^{2-\varepsilon}(d\mathbb{P})$ to 0 and then $K_\tau^{ij_0} = 0$. Finally by continuity we have $K_s^{ij_0} = 0$ for any $s \in [t, \sigma_1^*]$ and then the sequence $(\int_t^{\sigma_1^*} dK_s^{i,j_0,m})_{m \geq 0}$ converges weakly in $L^2(d\mathbb{P})$ to 0. As we can do the same for the other sequences, the claim holds.

Let l be fixed. By using (4.22) between t and σ_l^* one obtains:

$$Y_t^{\bar{\delta},j_0,m} - \mathbb{E}[Y_{\sigma_l^*}^{\bar{\delta},j_0,m} + (\tilde{A}_{\sigma_l^*}^{\bar{\delta},j_0,m} - \tilde{A}_t^{\bar{\delta},j_0,m}) + K_{\sigma_l^*}^{\bar{\delta},j_0,m} | \mathcal{F}_t] = \hat{U}_t^{\bar{\delta},j_0,m} - \mathbb{E}[U_{\sigma_l^*}^{\bar{\delta},j_0,m} | \mathcal{F}_t].$$

Taking now the weak limit w.r.t. m (at least through the subsequence constructed above) we obtain that

$$Y_t^{\bar{\delta},j_0} - \mathbb{E}[Y_{\sigma_l^*}^{\bar{\delta},j_0} | \mathcal{F}_t] = \hat{U}_t^{\bar{\delta},j_0} - \mathbb{E}[U_{\sigma_l^*}^{\bar{\delta},j_0} | \mathcal{F}_t].$$

Finally, taking the limit as $l \to \infty$, noting that $Y_T^{\bar{\delta},j_0} = U_T^{\bar{\delta},j_0} = h^{u^{\bar{\delta}}(T) j_0}$, we obtain

$$Y_t^{\bar{\delta},j_0} = \hat{U}_t^{\bar{\delta},j_0}.$$

Thus, in view of the definition of $\bar{\delta}$, we have

$$Y_t^{i_0,j_0} = Y_t^{\bar{\delta},j_0} - A_t^{\bar{\delta}} = \hat{U}_t^{\bar{\delta},j_0} - A_t^{\bar{\delta}}.$$

Now, taking into account of (4.7), the first equality holds.

Finally, in order to obtain the second equality of (4.8), it is enough to consider the approximating increasing scheme (which is the opposite of (4.9)) and which can be transformed into a decreasing scheme by taking its opposite sign. Then, from the result of Step 1, we have

$$-v^{i_0j_0}(t,x) = -Y_t^{i_0j_0} = \text{ess sup}_{\nu\in\mathcal{A}_{\pi_2}^{j_0}} \text{ess inf}_{\delta\in\mathcal{A}_{\pi_1}^{i_0}} - J_t(\delta,\nu)$$

and the proof is finished. ■

As a by product of Theorem (4.1) and the uniqueness of the solution of system (2.7) we have the following result in the case when the functions f^{ij} depend also on $\vec{y}$.

Corollary 1. *Suppose Assumptions (H0)-(H5) and (H6)-(i) are satisfied and let* $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *be the unique solution of system (2.7) and (2.8). Then for any* $(t,x)\in[0,T]\times\mathbb{R}^k$ *and* $(i_0,j_0)\in\Gamma^1\times\Gamma^2$*,*

$$v^{i_0j_0}(t,x) = ess\ sup_{\delta\in\mathcal{A}_{\pi_1}^{i_0}}\, ess\ inf_{\nu\in\mathcal{A}_{\pi_2}^{j_0}}\, \bar{J}_t(\delta,\nu) = ess\ inf_{\nu\in\mathcal{A}_{\pi_2}^{j_0}}\, ess\ sup_{\delta\in\mathcal{A}_{\pi_1}^{i_0}}\, \bar{J}_t(\delta,\nu). \tag{4.24}$$

where,

$$\bar{J}_t(\delta,\nu) := \mathbb{E}[h^{u_Tv_T}(X_T) + \int_t^T f^{u_rv_r}(r, X_r^{t,x}, (v^{kl}(r,X_r^{t,x}))_{(k,l)\in\Gamma^1\times\Gamma^2})dr - A_T^{\delta} + B_T^{\nu}|\mathcal{F}_t]. \tag{4.25}$$

Proof: Let $(w^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ be the unique solution in viscosity sense of the following system of PDEs with inter-connected obstacles: $\forall(i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases} \min\Big\{(w^{ij} - L^{ij}(\vec{w}))(t,x); \max\Big\{(w^{ij} - U^{ij}(\vec{w}))(t,x); \\ \qquad -\partial_t w^{ij}(t,x) - \mathcal{L}^X(w^{ij})(t,x) - f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2})\Big\}\Big\} = 0; \\ w^{ij}(T,x) = h^{ij}(x). \end{cases} \tag{4.26}$$

Then, by Theorem 4.1 we have, for any $(t,x)\in[0,T]\times\mathbb{R}^k$ and $(i_0,j_0)\in\Gamma^1\times\Gamma^2$,

$$w^{i_0j_0}(t,x) = \text{ess sup}_{\delta\in\mathcal{A}_{\pi_1}^{i_0}} \text{ess inf}_{\nu\in\mathcal{A}_{\pi_2}^{j_0}} \bar{J}_t(\delta,\nu) = \text{ess inf}_{\nu\in\mathcal{A}_{\pi_2}^{j_0}} \text{ess sup}_{\delta\in\mathcal{A}_{\pi_1}^{i_0}} \bar{J}_t(\delta,\nu). \tag{4.27}$$

But $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is also solution of the system (4.26), then by uniqueness for any $(i,j)\in\Gamma^1\times\Gamma^2$, $v^{ij}=w^{ij}$. Plug now this equality in (4.27) to obtain the desired result. ■

Remark 4.2. *We have also the following relation:* $\forall(i_0,j_0)\in\Gamma^1\times\Gamma^2$*,*

$$v^{i_0j_0}(t,x) = \sup_{\delta\in\mathcal{A}_{\pi_1}^{i_0}(t)} \inf_{\nu\in\mathcal{A}_{\pi_2}^{j_0}(t)} \mathbb{E}[\bar{J}_t(\delta,\nu)] = \inf_{\nu\in\mathcal{A}_{\pi_2}^{j_0}(t)} \sup_{\delta\in\mathcal{A}_{\pi_1}^{i_0}(t)} \mathbb{E}[\bar{J}_t(\delta,\nu)]. \tag{4.28}$$

5 Conclusion

In this paper, we have given appropriate conditions on the data of both the min-max and max-min systems so that their respective unique viscosity solutions coincide. These unique continuous viscosity solution have been constructed by means of a penalization procedure in the recent paper [8]. The main difficulty faced in that paper is that the two obstacles are interconnected and therefore not comparable. For this reason and without the separation of the two barriers, we cannot apply the classical relationship between doubly reflected BSDEs, system of PDEs with lower and upper obstacles and the underlying game obtained e.g. in [14]). By providing appropriate regularity conditions so that comparison holds, we establish in the present paper that the solutions of the Min-Max and Max-Min systems coincide. Finally, under further conditions on the drivers, this solution can be interpreted as the value function of a switching game.

We note that to obtain the required condition of comparison, we rely on the regularity of penalty costs. We also need to get precise estimates of penalized terms which can be obtained by controlling the growth of the driver. Our analysis deeply relies on the Markovian setting, therefore it seems quite natural to ask whether one can study the switching game in the general non-Markovian case. We leave this question for future research. ■

6 Appendix

Let $(t,x) \in [0,T] \times \mathbb{R}^k$ and $(X_s^{t,x})_{s\le T}$ be the solution of the standard SDE given in (2.4) where the functions b and σ satisfy Assumption (H0). Let us now consider the following functions:

$$\begin{array}{ll} g: & x \in \mathbb{R}^k \longmapsto g(x)\,\mathbb{R} \\ f: & (t,x,y,z) \in [0,T] \times \mathbb{R}^{k+1+d} \longmapsto f(t,x,y,z) \in \mathbb{R} \\ H: & (t,x) \in [0,T] \times \mathbb{R}^k \longmapsto H(t,x) \in \mathbb{R} \\ L: & (t,x) \in [0,T] \times \mathbb{R}^k \longmapsto L(t,x) \in \mathbb{R} \end{array}$$

We assume that all those functions are continuous and satisfy the following assumptions (A1)-(A2).

(A1): $\forall\ t \in [0,T]$, $x \in \mathbb{R}^k$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$,

$$\begin{cases} \text{(i) } |g(x)| + |f(t,x,0,0)| + |H(t,x)| + |L(t,x)| \le C(1+|x|^p), \\ \text{(ii) } |f(t,x,y,z) - f(t,x,y',z')| \le C(|y-y'| + |z-z'|), \\ \text{(iii) } L(t,x) \le H(t,x) \text{ and } L(T,x) \le g(x) \le H(T,x), \end{cases}$$

where C and p are some positive constants.

(A2): For each $R > 0$, there is a continuous function φ_R such that $\varphi_R(0) = 0$ and

$$|f(t,x,y,z) - f(t,x',y,z)| \le \varphi_R((1+|z|)|x-x'|)$$

for all $t \in (0,T)$, $|x|$, $|x'|$, $|y| \le R$ and $z \in \mathbb{R}^d$.

Next for $n \ge 0$, let $({}^nY_s^{t,x})_{s\le T}$ (resp.$({}^n\bar{Y}_s^{t,x})_{s\le T}$) be the first component of the unique solution of the BSDE with on reflecting lower (resp. upper) barrier associated with the quadruple
$(f(s,X_s^{t,x},y) - n(H(s,X_s^{t,x}) - y)^-, g(X_T^{t,x}), L(s,X_s^{t,x}))$
(resp. $(f(s,X_s^{t,x},y,z) + n(L(s,X_s^{t,x}) - y)^+, g(X_T^{t,x}), H(s,X_s^{t,x})))$, which exists and is unique (see e.g. [12]). It has been shown in [12] that, under Assumptions (H0) and (A1)-(A2), for any $n \ge 0$ there exist deterministic functions ${}^nu(t,x)$ and ${}^n\bar{u}$, $(t,x) \in [0,T] \times \mathbb{R}^k$, such that

$$\forall s \in [t,T], \quad {}^nY_s^{t,x} = {}^n u(s,X_s^{t,x}) \textit{ and } {}^n\bar{Y}_s^{t,x} = {}^n \bar{u}(s,X_s^{t,x}),$$

where ${}^nu$ (resp. ${}^n\bar{u}$) is continuous with uniform polynomial growth i.e. there exist two non negative real constants C and p such that

$$|{}^nu(t,x)| \text{ (resp. } |{}^n\bar{u}(t,x)|) \le C(1+|x|^p), \ \forall (t,x) \in [0,T] \times \mathbb{R}^k.$$

Moreover it is a unique viscosity solution, in the classe Π_g, of the following PDE with obstacle:

$$\begin{cases} \min\Big\{v(t,x) - L(t,x)\ ; -\partial_t v(t,x) - \mathcal{L}^X v(t,x) \\ \qquad\qquad -f(t,x,v(t,x),\sigma(t,x)^\top D_x\phi(t,x)) + n(H(t,x) - v(t,x))^-\Big\} = 0, \\ v(T,x) = g(x), \end{cases} \tag{6.1}$$

(resp.

$$\begin{cases} \max\Big\{v(t,x) - H(t,x)\ ; -\partial_t v(t,x) - \mathcal{L}^X v(t,x) \\ \qquad\qquad -f(t,x,v(t,x),\sigma(t,x)^\top D_x v(t,x)) + n(L(t,x) - v(t,x))^-\Big\} = 0, \\ v(T,x) = g(x)). \end{cases} \tag{6.2}$$

By comparison (see e.g. [14]) we easily deduce that the sequence of processes $({}^nY^{t,x})_{n\geq 0}$ (resp. $({}^n\bar{Y}^{t,x})_{n\geq 0}$) is decreasing (resp. increasing), moreover they converge in $\mathcal{S}^2$ to a same processus $(Y_s^{t,x})_{s\leq T}$ which satisfies

$$L(s,X_s^{t,x}) \leq Y_s^{t,x} \leq H(s,X_s^{t,x}),\ \forall s \leq T.$$

Therefore for any $(t,x) \in [0,T]\times\mathbb{R}^k$, the sequence $({}^nu(t,x))_{n\geq 0}$ (resp. $({}^n\bar{u}(t,x))_{n\geq 0}$) converges decreasingly (resp. increasingly) to the same limit

$$u(t,x) := Y_t^{t,x} \tag{6.3}$$

which verifies

$$u(T,x) = g(x) \text{ and } L(t,x) \leq u(t,x) \leq H(t,x), \forall (t,x) \in [0,T]\times\mathbb{R}^k.$$

Next as ${}^nu$ and ${}^n\bar{u}$ are continuous and belong to Π_g, then the function u belongs also to Π_g and is also continuous since it is both lsc and usc. By Dini's Theorem we deduce that the convergence of the sequence $({}^nu)_{n\geq 0}$ (resp. $({}^n\bar{u})_{n\geq 0}$) is uniform on compact subsets of $[0,T]\times\mathbb{R}^k$.

Next let us consider the following PDE with two obstacle of min-max type:

$$\begin{cases} \min\Big\{v(t,x) - L(t,x)\,;\ \max\Big[v(t,x) - H(t,x); \\ \qquad\qquad -\partial_t v(t,x) - \mathcal{L}^X v(t,x) - f(t,x,v(t,x),\sigma(t,x)^\top D_x v(t,x))\Big]\Big\} = 0\,; \\ v(T,x) = g(x). \end{cases} \tag{6.4}$$

To begin with, we are going to give the notion of viscosity solution of (6.4).

Definition 1. *Let v be a function which belongs to $\mathcal{C}([0,T]\times\mathbb{R}^k)$. It is called a viscosity:*
(i) subsolution of (6.4) if $v(T,x) \leq g(x)$ and for any $\phi \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R}^k)$ and any local maximum point $(t,x) \in (0,T)\times\mathbb{R}^k$ of $v-\phi$, we have

$$min\Big\{v(t,x) - L(t,x); max\Big[v(t,x) - H(t,x); -\partial_t\phi(t,x) - \mathcal{L}^X\phi(t,x) - f(t,x,v(t,x),\sigma(t,x)^\top D_x\phi(t,x))\Big]\Big\} \leq 0.$$

(ii) supersolution of (6.4) if $v(T,x) \geq g(x)$ and for any $\phi \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R}^k)$ and any local minimum point $(t,x) \in (0,T)\times\mathbb{R}^k$ of $v-\phi$, we have

$$min\Big\{v(t,x) - L(t,x); max\Big[v(t,x) - H(t,x); -\partial_t\phi(t,x) - \mathcal{L}^X\phi(t,x) - f(t,x,v(t,x),\sigma(t,x)^\top D_x\phi(t,x))\Big]\Big\} \geq 0.$$

(iii) solution of (6.4) if it is both a viscosity subsolution and supersolution. ■

Theorem 6.1. *The function u defined in (6.3) is a viscosity solution of (6.4).*

Proof. First we show that u is a viscosity subsolution of (6.4). Note that $u(T,x) = g(x)$ and $L(t,x) \le u(t,x) \le H(t,x)$. Let now $\phi \in \mathcal{C}^{1,2}$ and $(t,x) \in (0,T) \times \mathbb{R}^k$ be a local maximum of $u - \phi$ in $[0,T] \times \mathbb{R}^k$ such that $u(t,x) > L(t,x)$. Let (t_n, x_n) be a sequence of local maximum points of ${}^n u - \phi$ such that (t_n, x_n) converges to (t,x) (such a sequence exists because of the uniform convergence of ${}^n u$ to u on compact subsets (see e.g. [23], pp.117). For n large enough we have ${}^n u(t_n, x_n) > L(t_n, x_n)$ and since ${}^n u$ is a viscosity solution of (6.1) then

$$\begin{aligned}
&-\partial_t \phi(t_n, x_n) - \mathcal{L}^X \phi(t_n, x_n) - f(t_n, x_n, {}^n u(t_n, x_n), \sigma(t_n, x_n)^\top D_x \phi(t_n, x_n)) \\
&\quad \le -n(H(t_n, x_n) - {}^n u(t_n, x_n))^- \le 0.
\end{aligned}$$

Now by the continuity of the functions and the uniform convergence, we have

$$-\partial_t \phi(t,x) - \mathcal{L}^X \phi(t,x) - f(t,x,u(t,x), \sigma(t,x)^\top D_x \phi(t,x)) \le 0.$$

Thus u is a viscosity subsolution of (6.4). In a similar way we can show that u is also a viscosity supersolution. ■

The following result is of comparison type between sub. and supersolutions of (6.4). Namely we have:

Proposition 6.1. *Assume that Assumptions (H0), (A1)-(A2) are in force. Then if v (resp. u) is a viscosity supersolution (resp. subsolution) of (6.4) with polynomial growth, then for all $(t,x) \in [0,T] \times \mathbb{R}^k$ we have $u(t,x) \le v(t,x)$.*

Proof. Step (i): First we proof that $v \ge L$ and $u \le H$.

By the definition of supersolution, it is clear that $v \ge L$. Let us now show that $u(t,x) \le H(t,x)$. Suppose that for $t \le T$, $u(t,x) > H(t,x)$. Therefore we have $t < T$ and $u(t,x) - L(t,x) > 0$ since $H \ge L$. Now if ϕ is a test function for u at $(t,x) \in (0,T) \times \mathbb{R}^k$ then

$$\min\Big\{u(t,x) - L(t,x), \max\Big[u(t,x) - H(t,x), -\partial_t \phi(t,x) - \mathcal{L}\phi(t,x) - f(t,x,u(t,x), D_x\phi\sigma(t,x))\Big]\Big\} > 0,$$

which is contradictory. Thus $u \le H$.

Step (ii): Let us define $v' := v \wedge H$ and $u' = u \vee L$, then v' is a viscosity supersolution and u' is a viscosity subsolution of (6.4).

In fact, since $H(T,x) \ge g(x)$, then $v'(T,x) = v(T,x) \wedge H(T,x) \ge g(x)$. Let now $(t,x) \in (0,T) \times \mathbb{R}^k$ and $\phi \in \mathcal{C}^{1,2}$ such that (t,x) is a local minimum point of $v' - \phi$ in $[0,T] \times \mathbb{R}^k$. If $v(t,x) < H(t,x)$, then $v'(t,x) = v(t,x)$ and by continuity, (t,x) is also a local minimum point of $v - \phi$. Since v is a supersolution of (6.4), then v' verifies

$$\min\Big\{v'(t,x,) - L(t,x); \max\Big[v'(t,x) - H(t,x); -\partial_t \phi(t,x) - \mathcal{L}\phi(t,x) - f(t,x,v'(t,x), \sigma(t,x)^\top D_x \phi(t,x))\Big]\Big\} \ge 0.$$

Next if $v(t,x) \ge H(t,x)$, then $v'(t,x) = H(t,x)$. Since $H(t,x) - L(t,x) \ge 0$, then we have

$$\min\Big\{H(t,x) - L(t,x); \max\Big[H(t,x) - H(t,x); -\partial_t \phi(t,x) - \mathcal{L}\phi(t,x) - f(t,x,H(t,x), D_x\phi\sigma(t,x))\Big]\Big\} \ge 0.$$

Thus v' is a viscosity supersolution of (6.4). In the same way we can prove u' is a subsolution of (6.4).

Step (iii): Modification of the problem.

Let $\lambda \in \mathbb{R}$ and ξ, η and κ be the functions defined on $\mathbb{R}^k$ as

$$\begin{aligned}
\xi(x) &:= (1 + |x|^2)^{\frac{p}{2}}, \\
\eta(x) &:= \xi(x)^{-1} D_x \xi(x) = px(1 + |x|^2)^{-1} \\
\kappa(x) &:= \xi(x)^{-1} D^2_{xx} \xi(x) = p(1 + |x|^2)^{-1} I_k - p(p-2)(1 + |x|^2)^{-2} x \otimes x
\end{aligned}$$

where p is choosen in such a way that $\bar{u}$ and $\bar{v}$ below are bounded and converge uniformly to 0 as $\|x\| \to \infty$. It exists since u and v are both in Π_g. Next let us consider the followings

$$\begin{array}{l}
\bar{u}(t,x) := e^{\lambda t}\xi^{-1}(x)u'(t,x),\ \bar{v}(t,x) := e^{\lambda t}\xi^{-1}(x)v'(t,x), \\
\bar{L}(t,x) := e^{\lambda t}\xi^{-1}(x)L(t,x),\ \hat{H}(t,x) := e^{\lambda t}\xi^{-1}(x)H(t,x), \\
\bar{g}(x) := e^{\lambda T}\xi^{-1}(x)g(x), \\
\bar{\mathcal{L}}\varphi := \mathcal{L}^X\varphi + < \sigma\sigma^\top\eta, D_x\varphi > + \{\frac{1}{2}\mathrm{Tr}((\sigma\sigma^\top)\kappa) + < b,\eta > -\lambda\}\varphi \\
\bar{f}(t,x,y,z) := e^{\lambda t}\xi^{-1}(x)f(t,x,e^{-\lambda t}\xi(x)y, e^{-\lambda t}\xi(x)z + e^{-\lambda t}D_x\xi(x)\sigma(t,x)y).
\end{array}$$

Therefore one can easily check that $\bar{u}$ (resp. $\bar{v}$) is a viscosity subsolution (resp. supersolution) of

$$\begin{cases}
\min\Big\{\tilde{u}(t,x) - \bar{L}(t,x); \max\Big[\tilde{u}(t,x) - \bar{H}(t,x); \\
\qquad -\partial_t\tilde{u}(t,x) - \bar{\mathcal{L}}\tilde{u}(t,x) - \bar{f}(t,x,\tilde{u}(t,x),\sigma(t,x)^\top D_x\tilde{u}(t,x))\Big]\Big\} = 0; \\
\tilde{u}(T,x) = \bar{g}(x).
\end{cases} \tag{6.5}$$

Let now F be the function from $[0,T]\times\mathbb{R}^{k+1+d}\times\mathbb{S}_k$ ($\mathbb{S}_k$ is the space of symmetric real matrices of dimension k) which with (t,x,y,z,M) associates $F(t,x,y,z,M)\in\mathbb{R}$ and verifying

$$F(t,x,\tilde{u}(t,x),D_x\tilde{u}(t,x),D^2_{xx}\tilde{u}(t,x)) = \bar{\mathcal{L}}\tilde{u}(t,x) + \bar{f}(t,x,\tilde{u}(t,x),\sigma(t,x)^\top D_x\tilde{u}(t,x)).$$

We choose λ great enough in such a way that the mapping $y\in\mathbb{R}\longmapsto F(t,x,y,z,M)\in\mathbb{R}$ is strictly decreasing for all $(t,x,z,M)\in[0,T]\times\mathbb{R}^{k+1}\times\mathbb{S}_k$. Finally note that for all $\varepsilon>0$ the function $\bar{v}+\frac{\varepsilon}{t}$ is also a supersolution solution of (6.5). Therefore in order to obtain the comparison result it is enough to show that $\bar{u}\le\bar{v}+\frac{\varepsilon}{t}$ and then to take the limit as $\varepsilon\to 0$.

<u>Step (iv)</u>: Last part of the proof.

We are going to show by contradiction that: $\forall R>0$

$$\sup_{t\in[0,T],|x|\le R}(u'(t,x)-v'(t,x)-\frac{\varepsilon}{t})^+ \le \sup_{t\in[0,T],|x|=R}(u'(t,x)-v'(t,x)-\frac{\varepsilon}{t})^+ \tag{6.6}$$

where now L (resp. H) is $\bar{L}$ (resp. $\bar{H}$), $u'=\bar{u}\vee\bar{L}$, $v'=\bar{v}\wedge\bar{H}$, and finally f is $\bar{f}$ which is defined previously. Note that from Steps (i)-(ii), u' (resp. v') is a viscosity subsolution (resp. supersolution) of (6.5) and due to assumption (A1), $u'\le H$ and $v'\ge L$.

So suppose that for some $R>0$

$$\delta := \sup_{t\in[0,T],|x|\le R}(u'(t,x)-v'(t,x)-\frac{\varepsilon}{t})^+ > \sup_{t\in[0,T],|x|=R}(u'(t,x)-v'(t,x)-\frac{\varepsilon}{t})^+ \ge 0.$$

For each $n>0$, let (t_n,x_n,y_n) be a point in the compact set $[0,T]\times\bar{B}_R\times\bar{B}_R$ where $B_R\triangleq\{x\in\mathbb{R}^k;|x|<R\}$, and the continuous function

$$\Phi_n(t,x,y) = u'(t,x) - v'(t,y) - \frac{\varepsilon}{t} - n|x-y|^2$$

achieves its maximum. As u' and v' are bounded this maximum belongs to $(0,T)\times B_R\times B_R$. By Lemma 8.7 in [12], there exists $(p_n,X_n,Y_n)\in\mathbb{R}\times\mathbb{S}_k\times\mathbb{S}_k$ such that:

(i) $n|x_n-y_n|^2\to 0$ as $n\to\infty$;

(ii) $u'(t_n,x_n)\ge v'(t_n,y_n)+\frac{\varepsilon}{t_n}+\delta$;

(iii) $(p_n, n(x_n - y_n), X_n) \in \bar{J}^{2,+}(u'(t_n, x_n))$;

(iv) $(p_n, n(x_n - y_n), Y_n) \in \bar{J}^{2,-}(v'(t_n, y_n) + \frac{\varepsilon}{t_n})$;

(v) $\begin{pmatrix} X_n & 0 \\ 0 & -Y_n \end{pmatrix} \leq 3n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$

where $\bar{J}^{2,+}(u'(t,x))$ (resp. $\bar{J}^{2,-}(v'(t,x) + \frac{\varepsilon}{t})$) is respectively the limiting superjet (resp. subjet) of u' (resp. $v'(t,x) + \frac{\varepsilon}{t}$) (see e.g. [6], pp. 728 or [13], pp. 210).

Next as v' is continuous, combined with point (i), we deduce the existence of $N > 0$ such that for any $n \geq N$, $|v'(t_n, x_n) - v'(t_n, y_n)| < \frac{\delta}{2}$. Therefore for $n \geq N_0$,

$$\begin{aligned} &u'(t_n, x_n) \geq v'(t_n, y_n) + \tfrac{\varepsilon}{t_n} + \delta \\ &\geq v'(t_n, x_n) - |v'(t_n, x_n) - v'(t_n, y_n)| + \tfrac{\varepsilon}{t_n} + \delta \\ &> v'(t_n, x_n) + \tfrac{\varepsilon}{t_n} + \tfrac{\delta}{2} \end{aligned}$$

which implies that for any $n \geq N_0$, $u'(t_n, x_n) > v'(t_n, x_n)$. On the other hand, by the results obtained in Steps (i)-(ii),

$$H(t_n, x_n) \geq u(t_n, x_n) \vee L(t_n, x_n) = u'(t_n, x_n) > v'(t_n, x_n) = v(t_n, x_n) \wedge H(t_n, x_n) \geq L(t_n, x_n)$$

and then $v'(t_n, x_n) < H(t_n, x_n)$ and $u'(t_n, x_n) > L(t_n, x_n)$. As u' (resp. $v' + \frac{\varepsilon}{t}$) is a sub (resp. super) solution of ((6.4) modified), we then have

$$-p_n - \frac{1}{2} Tr(\sigma\sigma^\top(t_n, x_n) X_n) - < b, n(x_n - y_n) > - f(t_n, x_n, u'(t_n, x_n), n(x_n - y_n)) \leq 0,$$

and

$$-p_n - \frac{1}{2} Tr(\sigma\sigma^\top(t_n, y_n) Y_n) - < b, n(x_n - y_n) > - f(t_n, y_n, v'(t_n, y_n) + \frac{\varepsilon}{t_n}, n(x_n - y_n)) \geq \frac{\varepsilon}{t_n^2}.$$

then

$$\begin{aligned} \tfrac{\varepsilon}{t_n^2} \leq \Lambda_n := &\tfrac{1}{2} Tr(\sigma\sigma^\top(t_n, x_n) X_n - \sigma\sigma^\top(t_n, x_n) Y_n) \\ &+ f(t_n, x_n, u'(t_n, x_n), n(x_n - y_n)) - f(t_n, y_n, v'(t_n, y_n) + + \tfrac{\varepsilon}{t_n}, n(x_n - y_n)). \end{aligned}$$

With the same argument as in ([12], pp. 734), under (H0),(A1)-(A2), we obtain that $\underline{\lim}_{n\to\infty} \Lambda_n \leq 0$ and then $\varepsilon \leq 0$ which is contradictory. Finally taking the limits in (6.6), first when $R \to \infty$ then $\varepsilon \to \infty$, we obtain $u' \leq v'$ and then $u \leq v$. ■

As a by-product we have:

Theorem 6.2. *Under (H0),(A1) and (A2) we have:*

(i) There is a unique continuous viscosity solution of (6.4) with polynomial growth ;

(ii) The function u is also a unique viscosity solution, in the class Π_g, for the following max-min problem:

$$\begin{cases} max\Big\{v(t,x) - H(t,x) \; ; \; min\Big[v(t,x) - L(t,x); \\ \qquad -\partial_t v(t,x) - \mathcal{L}^X v(t,x) - f(t,x,v(t,x),\sigma(t,x)^\top D_x v(t,x))\Big]\Big\} = 0; \\ v(T,x) = g(x). \end{cases} \quad (6.7)$$

Proof. (i) The existence follows from Theorem 6.1 and uniqueness follows from Proposition 6.1.

(ii) The construction of the function u implies that $w = -u$ is the unique viscosity solution in the class Π_g of the following system:

$$\begin{cases} \min\Big\{w(t,x) + H(t,x), \min\Big[w(t,x) + L(t,x), \\ \qquad -\partial_t w(t,x) - \mathcal{L}w(t,x) + f(t,x,-w(t,x),-\sigma(t,x)^\top D_x w(t,x))\Big]\Big\} = 0; \\ w(T,x) = -g(x). \end{cases} \tag{6.8}$$

Thus $-w = u$ is the unique solution in the class Π_g of system (6.8) (see e.g. [2], pp.18). ■

In terms of BSDEs the process $Y^{t,x}$ defined in (6.3) is a local solution for the two barriers reflected BSDE associated with $(f(s, X_s^{t,x}, y, z), g(X_T^{t,x}), L(s, X_s^{t,x}), H(s, X_s^{t,x}))$. Namely we have the following result:

Proposition 6.2. *([15], Theorem 3.1) For any stopping time τ, there exists another stopping time $\delta_\tau \geq \tau$, $\mathbb{P}-a.s.$ and three processes Z^τ, $K^{\pm,\tau}$ such that:*
(i) $Y_T^{t,x} = g(X_T^{t,x})$;
(ii)

$$\begin{cases} Z^\tau \in \mathcal{H}^{2,d},\ K^{\pm,\tau} \in \mathcal{S}^2 \text{ and non-decreasing };\\ \forall s \in [\tau,\delta_\tau], Y_s^{t,x} = Y_{\delta_\tau}^{t,x} + \int_s^{\delta_\tau} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^\tau)dr - \int_s^{\delta_\tau} Z_r^\tau dB_r + \int_s^{\delta_\tau} dK_r^{+,\tau} - \int_s^{\delta_\tau} dK_r^{-,\tau} \\ L(s, X_s^{t,x}) \leq Y_s^{t,x} \leq H(s, X_s^{t,x}),\ \forall s \in [t,T] ;\\ \int_\tau^{\delta_\tau} \big(Y_r^\tau - L(r, X_r^{t,x})\big)dK_r^{+,\tau} = 0 \text{ and } \int_\tau^{\delta_\tau} \big(Y_r^\tau - U(r, X_r^{t,x})\big)dK_r^{-,\tau} = 0; \end{cases} \tag{6.9}$$

(iii) Let γ_τ and θ_τ be the following two stopping times:

$$\gamma_\tau := \inf\{s \geq \tau, Y_s^{t,x} = L^{i,j}(s, X_s^{t,x})\} \wedge T \text{ and } \theta_\tau := \inf\{s \geq \tau, Y_s^{t,x} = U(s, X_s^{t,x})\} \wedge T.$$

Then $\mathbb{P}-a.s.$, $\gamma_\tau \vee \theta_\tau \leq \delta_\tau$.
The process $Y^{t,x}$ is unique to satisfy (i)-(iii). ■

Finally in the case when f does not depend on z we have the following charaterization of $Y^{t,x}$ as the value function of a zero-sum Dynkin game.

Proposition 6.3. *([15], pp.894) The process $Y^{t,x}$ verifies: for any stopping time $\theta \geq t$,*

$$\begin{aligned} Y_\theta^{t,x} &= ess\ sup_{\sigma\geq\theta} ess\ inf_{\tau\geq\theta} \mathbb{E}\Big\{ \int_\theta^{\sigma\wedge\tau} f(r, X_r^{t,x}, Y_r^{t,x})dr \\ &\qquad + L(\sigma, X_\sigma^{t,x})1\!\!1_{[\sigma<\tau]} + H(\tau, X_\tau^{t,x})1\!\!1_{[\tau\leq\sigma<T]} + g(X_T^{t,x})1\!\!1_{[\tau=\sigma=T]}|\mathcal{F}_\theta\Big\} \\ &= ess\ inf_{\tau\geq\theta} ess\ sup_{\sigma\geq\theta} \mathbb{E}\Big\{ \int_\theta^{\sigma\wedge\tau} f(r, X_r^{t,x}, Y_r^{t,x})dr \\ &\qquad + L(\sigma, X_\sigma^{t,x})1\!\!1_{[\sigma<\tau]} + H(\tau, X_\tau^{t,x})1\!\!1_{[\tau\leq\sigma<T]} + g(X_T^{t,x})1\!\!1_{[\tau=\sigma=T]}|\mathcal{F}_\theta\Big\}. \end{aligned} \tag{6.10}$$

■

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