Thèse de Doctorat de l'Université du Maine

Spécialité : Mathématiques
Option : Statistique

Sujet de la thèse

Problèmes Statistiques pour les EDS et les EDS Rétrogrades

Présentée par

Li Zhou

Pour obtenir le grade de l'Université du Maine

Soutenue le 28 mars 2013

Composition du Jury :

Dehay Dominique	(Université Rennes 2)	Rapporteur
Hamadene Said	(Université du Maine)	Examinateur
Ji Shaolin	(Shandong University)	Rapporteur
Kabanov Youri	(Université de Franche-Comté)	Examinateur
Kleptsyna Marina	(Université du Maine)	Examinateur
Kutoyants Yury	(Université du Maine)	Directeur de thèse
Nikulin Mikhail	(Université Victor Segalen Bordeaux 2)	Examinateur

Remerciements

Mes premiers remerciements s'adressent à mon directeur de thèse, Youri Koutoyants, sans qui cette thèse n'aurait jamais pu voir le jour.

Je suis infiniment reconnaissante envers Youri Koutoyants pour la qualité exceptionnelle de son encadrement. Sa grande disponibilité, ses conseils avisés et son soutien de chaque instant m'ont été très précieux. Je n'oublierai pas la gentillesse avec laquelle il m'a accueillie dès le début de ma thèse ainsi que toute l'attention qu'il m'a accordée. Il m'a en effet guidé, se rendant toujours disponible et partageant avec moi ses expériences et connaissances précieuses, toujours avec une immense générosité. Je lui exprime toute mon admiration, tant sur le plan humain que sur le plan professionnel.

Je voudrais témoigner toute ma gratitude à Dominique Dehay et Shaolin Ji qui ont accepté d'être les rapporteurs de cette thèse. Leur lecture attentive et leurs remarques précieuses m'ont permis d'améliorer ce travail. Je remercie très vivement Youri Kabanov, Mikhail Nikulin, Marina Kleptsyna et Saïd Hamadène qui ont accepté de faire partie du jury lors de la soutenance de ma thèse.

Je voudrais remercier tout les membres de l'équipe «Probabilité et Statistique» de l'université du Maine. Je garderai un excellent souvenir de la bonne ambiance qui règne au sein du laboratoire. Un grand merci à tous mes amis doctorants pour les bons moments que nous avons partagés. Ces années de thèse n'auraient certainement pas été aussi agréables sans mes amis hors laboratoire. Je les remercie tous pour leur convivialité et leurs aides. Je pense tout spécialement à Jing Zhang pour son amitié, je n'oublierai jamais ces instants formidables que nous avons partagés.

Enfin, je ne peux conclure qu'en ayant une pensée toute particulière à ma famille : mon père, ma mère et ma sœur qui, malgré la distance, ont toujours été présents.

Résumé

Nous considérons deux problèmes. Le premier est la construction des tests d'ajustement (goodness-of-fit) pour les modèles de processus de diffusion ergodique. Nous considérons d'abord le cas où le processus sous l'hypothèse nulle appartient à une famille paramétrique. Nous étudions les tests de type Cramer-von Mises et Kolmogorov-Smirnov. Le paramètre inconnu est estimé par l'estimateur de maximum de vraisemblance ou l'estimateur de distance minimale. Nous construisons alors les tests basés sur l'estimateur du temps local de la densité invariante, et sur la fonction de répartition empirique. Nous montrons alors que les statistiques de ces deux types de test convergent tous vers des limites qui ne dépendent pas du paramètre inconnu. Par conséquent, ces tests sont appelés *asymptotically parameter free*. Ensuite, nous considérons l'hypothèse simple. Nous étudions donc le test du khi-deux. Nous montrons que la limite de la statistique ne dépend pas de la dérive, ainsi on dit que le test est *asymptotically distribution free*. Par ailleurs, nous étudions également la puissance du test du khi-deux. En outre, ces tests sont consistants.

Nous traitons ensuite le deuxième problème : l'approximation des équations différentielles stochastiques rétrogrades. Supposons que l'on observe un processus de diffusion satisfaisant à une équation différentielle stochastique, où la dérive dépend du paramètre inconnu. Nous estimons premièrement le paramètre inconnu et après nous construisons un couple de processus tel que la valeur finale de l'un est une fonction de la valeur finale du processus de diffusion donné. Par la suite, nous montrons que, lorsque le coefficient de diffusion est petit, le couple de processus se rapproche de la solution d'une équations différentielles stochastiques rétrograde. A la fin, nous prouvons que cette approximation est asymptotiquement efficace.

Abstract

We consider two problems in this work. The first one is the goodness of fit test for the model of ergodic diffusion process. We consider firstly the case where the process under the null hypothesis belongs to a given parametric family. We study the Cramer-von Mises type and the Kolmogorov-Smirnov type tests in different cases. The unknown parameter is estimated via the maximum likelihood estimator or the minimum distance estimator, then we construct the tests in using the local time estimator for the invariant density function, or the empirical distribution function. We show that both the Cramer-von Mises type and the Kolmogorov-Smirnov type statistics converge to some limits which do not depend on the unknown parameter, thus the tests are asymptotically parameter free. The alternatives as usual are nonparametric and we show the consistency of all these tests. Then we study the chi-square test. The basic hypothesis is now simple The chi-square test is asymptotically distribution free. Moreover, we study also power function of the chi-square test to compare with the others.

The other problem is the approximation of the forward-backward stochastic differential equations. Suppose that we observe a diffusion process satisfying some stochastic differential equation, where the trend coefficient depends on some unknown parameter. We try to construct a couple of processes such that the final value of one is a function of the final value of the given diffusion process. We show that when the diffusion coefficient is small, the couple of processes approximates well the solution of a backward stochastic differential equation. Moreover, we present that this approximation is asymptotically efficient.

Table of contents

Chapitre 1

Introduction

La problématique générale de cette thèse porte sur l'étude des tests d'ajustement de processus de diffusion ergodique et de l'approximation des équations différentielles stochastiques rétrogrades. Dans le chapitre 2, nous étudions le test d'ajustement (goodness-of-fit) pour les modèles de processus de diffusion ergodique. Nous introduisons trois types de test : le test de Cramér-von Mises, le test de Kolmogorov-Smirnov et le test du khi-deux. L'objectif de nos travaux dans ce chapitre est de construire les tests consistants, et qui ne dépend asymptotiquement pas soit du paramètre ou soit de la distribution. Une partie des résultats de ce chapitre est issue d'un travail réalisé en collaboration avec Ilia Negri. Le chapitre 3, quant à lui, est consacré à l'étude de l'approximation des équations différentielles stochastiques rétrogrades. Nous construisons un couple de processus dont la valeur finale est une fonction de la valeur finale d'un processus de diffusion donné, dans lequel la dérive dépend d'un paramètre inconnu. Nous montrons ensuite que, lorsque le coefficient de diffusion est petit, le couple de processus se rapproche de la solution d'une équation différentielle stochastique rétrograde. Les résultats de ce chapitre sont issus d'un travail en collaboration avec Yury A. Kutoyants. Tous nos résultats sont illustrés numériquement par la méthode numérique.

1.1 Test d'Ajustement

1.1.1 Les cas des v.a. i.i.d. et des processus de diffusion

Nous rappelons dans un premier temps le problème de test d'ajustement pour le cas d'observations $X^n = (X_1, \ldots, X_n)$ de variables aléatoires indépendantes et identiquement distribuées (i.i.d.), dont la fonction de répartition est $F(x)$. On teste l'hypothèse $\mathcal{H}_0$ contre l'alternative $\mathcal{H}_1$

$$\mathcal{H}_0 : \quad F(x) = F_*(x), \qquad \mathcal{H}_1 : \quad F(x) \neq F_*(x).$$

Ce genre de problème a été introduit au début de $20^{\text{ème}}$ siècle, et a été bien étudié pendant les années 50. Nous citons ici les livres de Cramér [7] et de Lehmann & Romano [32], qui ont introduit différents types de test pour le cas i.i.d.

Cramér [6] et Smirnov [46] ont considéré le test ci-dessous, que l'on appelle maintenant *test de Cramer-von Mises* (test de C-vM) :

$$\widehat{\psi}_{n,1}\left(X^n\right) = 1\!\!1_{\left\{\omega_{n,1}^2 > e_{\varepsilon,1}\right\}}, \qquad \omega_{n,1}^2 = n \int_{-\infty}^{\infty} \left[\widehat{F}_n\left(x\right) - F_*\left(x\right)\right]^2 \mathrm{d}F_*\left(x\right),$$

où $\widehat{F}_n(\cdot)$ est la fonction de répartition empirique, $e_{\varepsilon,1}$ est le $(1-\varepsilon)$-quantile de cette distribution, c'est à dire la solution de l'équation suivante :

$$\mathbf{P}\left\{\omega_1^2 > e_{\varepsilon,1}\right\} = \varepsilon. \tag{1.1}$$

Ils ont donné la limite ω_1^2 de la statistique $\omega_{n,1}^2$ sous l'hypothèse nulle $\mathcal{H}_0$. Ils ont vérifié que cette limite ne dépend pas de la distribution, le test n'en dépend pas non plus. On dit alors que ce test est *asymptotically distribution free* (ADF).

Par la suite, dans Kolmogorov [26], le test que l'on appelle maintenant *test de Kolmogorov-Smirnov* (test de K-S) a été introduit. It a été dévloppé ensuite, par exemple par Smirnov [47], Fasano & Franceschini [16], etc. Ils ont considéré la statistique de test ci-dessous

$$\omega_{n,2} = \sqrt{n} \sup_x \left|\widehat{F}_n\left(x\right) - F_*\left(x\right)\right|.$$

Un résultat similaire à celui de C-vM a été présenté : la statistique $\omega_{n,2}$ converge en loi vers une variable aléatoire ω_2. Alors, le test de K-S a été défini comme :

$$\widehat{\psi}_{n,2}\left(X^n\right) = 1\!\!1_{\{\omega_{n,2} > e_{\varepsilon,2}\}},$$

où $e_{\varepsilon,2}$ est le $(1-\varepsilon)$-quantile de la distribution de ω_2. Etant donné que ce test ne dépend pas de la distribution, le test est ADF.

Par la suite, le test du khi-deux a été étudié. Nous citons par exemple, Cramér [7], Cochran [5], Dahiya & Gurland [9], Watson [49] et Greenwood & Nikulin [21]. On partitionne $\mathbb{R}$ en r intervalles $I_1 = (a_0, a_1],\ I_2 = (a_1, a_2], ..., I_r = (a_{r-1}, a_r)$, où $-\infty = a_0 < a_1 < \cdots < a_r = +\infty$. On définit $p_i > 0$, la probabilité que X_1 prenne les valeurs dans I_i. Alors, $p_i = F_*(a_i) - F_*(a_{i-1}) > 0$ et $\sum\limits_{i=1}^{r} p_i = 1$. La statistique est définie comme

$$\omega_{n,3} = \sum_{i=0}^{r} \frac{(v_i - np_i)^2}{np_i},$$

où v_i est le nombre des valeurs de l'échantillon qui appartiennent à I_i. Cramer [7] a montré que, quand $n \to \infty$, la limite de la distribution de $\omega_{n,3}$ est la loi du khi-deux avec $(r-1)$-degré de liberté, que l'on note $\chi^2(r-1)$. Par conséquent, le test

$$\widehat{\psi}_{n,3}\left(X^n\right) = 1\!\!1_{\{\omega_{n,3} > e_{\varepsilon,3}\}},$$

où $e_{\varepsilon,3}$ est le $(1-\varepsilon)$-quantile de la loi $\chi^2(r-1)$, est ADF.

Ensuite, les modèles avec paramètres inconnus ont été considérés. Kac al. [23], Durbin [12], Martynov [37] et [38] ont étudié le test d'hypothèse suivant :

$$\mathcal{H}_0 : \quad F(x) = F_*(x, \vartheta),$$

où ϑ est un paramètre inconnu. Darling [10] a défini les tests de C-vM et de K-S sous les formes suivantes :

$$\widehat{\psi}_{n,1}\left(X^n\right) = 1\!\!1_{\left\{\omega_{n,1}^2 > e_{\varepsilon,1}\right\}}, \qquad \omega_{n,1}^2 = n\int_{-\infty}^{\infty} \left[\widehat{F}_n\left(x\right) - F_*\left(x, \widehat{\vartheta}_n\right)\right]^2 \mathrm{d}F_*\left(x - \widehat{\vartheta}_n\right),$$

et

$$\widehat{\psi}_{n,2}\left(X^n\right) = 1\!\!1_{\{\omega_{n,2} > e_{\varepsilon,2}\}}, \qquad \omega_{n,2} = \sqrt{n}\sup_x \left|\widehat{F}_n\left(x\right) - F_*\left(x, \widehat{\vartheta}_n\right)\right|,$$

où $\widehat{\vartheta}_n$ est un certain estimateur du paramètre inconnu, les seuils $e_{\varepsilon,i}$, pour $i = 1, 2$, sont les $(1 - \varepsilon)$-quantiles de la distribution limite des statistiques. La limite des statistiques dépend généralement du paramètre inconnu. Mais Darling [10] a vérifié que la limite des deux stastistiques ne depend pas des paramètres inconnus pour certains modèles spécifiés (par exemple les modèles à paramètre d'échelle et à paramètre de position), et certains estimateurs comme l'estimateur de maximum de vraisemblance (EMV). Pour ces cas le test ne dépend pas non plus du paramètre inconnu, et le test est dit *asymptotically parameter free* (APF).

Le problème similaire existe pour les processus stochastiques en temps continu, largement utilisés en tant que modèle mathématique dans plusieurs domaines. Le test d'ajustement a été étudié par de nombreux auteurs : par exemple, Kutoyants [28] a discuté des possibilités de la construction de ces tests. En particulier, il a considéré la statistique de K-S et celle de C-vM basées sur l'observation continue. Supposons que l'observation $X^T = \{X_t, 0 \leq t \leq T\}$ est un processus de diffusion en temps continu

$$\mathrm{d}X_t = S\left(X_t\right)\,\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0, \quad 0 \leq t \leq T, \tag{1.2}$$

où $\{W_t,\ t \geq 0\}$ est un processus de Wiener, le coefficient de dérive $S\left(\cdot\right)$ est inconnu et le coefficient de diffusion $\sigma(\cdot)^2$ est connu. Il a considéré les hypothèses suivantes

$$\mathcal{H}_0 : \quad S(\cdot) = S_*(\cdot), \qquad \mathcal{H}_1 : \quad S(\cdot) \neq S_*(\cdot).$$

Il a proposé les tests

$$\psi_T(X^T) = 1\!\!\mathrm{I}_{\{\omega_T > y_\varepsilon\}}, \quad \omega_T = \sup_x \sqrt{T}\left|\widehat{f}_T(x) - f_*(x)\right|,$$

et

$$\Phi_T(X^T) = 1\!\!\mathrm{I}_{\{\Omega_T > Y_\varepsilon\}}, \quad \Omega_T = \sup_x \sqrt{T}\left|\widehat{F}_T(x) - F_*(x)\right|,$$

où $\widehat{f}_T(\cdot)$ est l'estimateur de temps local (ETL) de la densité invariante des observations, $\widehat{F}_T(\cdot)$ est la fonction de répartition empirique (FRE), $f_*(x)$ et $F_*(x)$ sont respectivement la fonction de densité invariante et la fonction de répartition invariante sous l'hypothèse nulle, y_ε et Y_ε sont respectivement le $(1 - \varepsilon)$-quantile de la distribution de limite de ω_T et de celle de Ω_T. La statistique de K-S pour les processus de diffusion ergodiques a été étudiée par Fournie [19] et Fournie & Kutoyants [20]. Toutefois, en raison de la structure de la covariance de la limite de processus, la statistique de K-S définie dans [19] et [20] dépend de la distribution dans les modèles de processus de diffusion. Plus récemment, Kutoyants [29] a proposé une modification de la statistique de C-vM et de K-S pour les modèles de diffusion, qui ne dépendt pas de la distribution. Voir également Dachian & Kutoyants [8] qui ont proposé des tests d'ajustement pour des processus de diffusion et de Poisson non-homogène avec des hypothèses de base simple. Dans le cas des processus d'Ornstein-Uhlenbeck, Kutoyants [30] a montré que le test de C-vM est APF. Un autre test a été étudié par Negri & Nishiyama [40].

1.1.2 Résultats principaux

Dans le chapitre 2, nous considérons le test d'ajustement pour les processus de diffusion dont l'équation est (1.2). La section 2.1 est consacrée aux conditions et aux résultats auxiliaires relatifs aux processus de diffusion. Dans les sections 2.2 et 2.3, nous étudions le modèle défini par (1.2), plus particulièrement dans le cas où la dérive $S(\cdot)$ dépend d'un paramètre inconnu, et le coefficient de diffusion $\sigma(\cdot)^2 = 1$. Nous testons l'hypothèse suivante

$$\mathcal{H}_0 \quad : \qquad S\left(x\right) = S_*\left(x-\vartheta\right), \quad \vartheta \in \Theta = \left(\alpha, \beta\right)$$

où $S_*\left(\cdot\right)$ est une fonction connue et le paramètre de shift ϑ est inconnu. Par conséquent, les coefficients de dérive sous l'hypothèse nulle appartiennent à l'ensemble

$$\mathcal{S}\left(\Theta\right) = \left\{S_*\left(x-\vartheta\right), \quad \vartheta \in \Theta\right\}.$$

L'alternative est définie comme

$$\mathcal{H}_1 \quad : \qquad S\left(\cdot\right) \notin \overline{\mathcal{S}(\Theta)},$$

où $\overline{\mathcal{S}(\Theta)} = \{S_*(x-\vartheta), \vartheta \in [\alpha,\beta]\}$. La section 2.2 est consacrée au test de type C-vM. Nous estimons le paramètre inconnu via l'EMV ou via l'estimateur de distance minimale (EDM), puis nous construisons deux tests de la manière suivante :

$$\psi_T = 1\!\!1_{\{\delta_T > d_\varepsilon\}}, \quad \delta_T = T\int_{-\infty}^{\infty}\left(\widehat{f}_T(x) - f_*(x-\widehat{\vartheta}_T)\right)^2 \mathrm{d}x,$$

et

$$\Psi_T = 1\!\!1_{\{\Delta_T > D_\varepsilon\}}, \quad \Delta_T = T\int_{-\infty}^{\infty}\left(\widehat{F}_T(x) - F_*(x-\widehat{\vartheta}_T)\right)^2 \mathrm{d}x,$$

où $\widehat{\vartheta}_T$ est l'estimateur du paramètre inconnu (l'EMV ou l'EDM). Nous montrons que sous certaines conditions de régularités, les deux statistiques convergent en loi vers deux variables aléatoires δ et Δ respectivement. Ainsi d_ε et D_ε sont définies respectivement comme les $(1-\varepsilon)$-quantiles des distributions de δ et de Δ, c'est à dire les solutions des équations suivantes

$$\mathbf{P}(\delta > d_\varepsilon) = \varepsilon, \quad \mathbf{P}(\Delta > D_\varepsilon) = \varepsilon.$$

Notons que les tests $\psi_T = 1\!\!1_{\{\delta_T > d_\varepsilon\}}$ et $\Psi_T = 1\!\!1_{\{\Delta_T > D_\varepsilon\}}$ sont de taille asymptotique ε, i.e.

$$\mathbf{E}_*\psi_T = \varepsilon + o(1), \quad \mathbf{E}_*\Psi_T = \varepsilon + o(1),$$

où $\mathbf{E}_*$ est l'espérance mathématique sous l'hypothèse nulle. En plus nous démontrons dans les théorèmes 2.2.1 et 2.2.2 que les deux tests sont APF. Dans la proposition 2.2.1, nous montrons qu'ils sont consistants.

Dans la section 2.3, nous étudions les tests de type de K-S pour le même modèle. Les tests sont définis comme suit

$$\phi_T = 1\!\!1_{\{\lambda_T > c_\varepsilon\}}, \quad \lambda_T = \sqrt{T}\sup_{x\in\mathbb{R}}\left|\widehat{f}_T(x) - f_*(x-\widehat{\vartheta}_T)\right|,$$

et

$$\Phi_T = 1\!\!1_{\{\Lambda_T > C_\varepsilon\}}, \quad \Lambda_T = \sqrt{T}\sup_{x\in\mathbb{R}}\left|\widehat{F}_T(x) - F_*(x-\widehat{\vartheta}_T)\right|.$$

Nous démontrons dans les théorèmes 2.3.1 et 2.3.2 que les deux tests possèdent les mêmes propriétés que celle de C-vM.

Notons que les tests de C-vM et de K-S dépendent toujours de la dérive. Par conséquent, nous proposons dans la section 2.4 l'utilisation du test du khi-deux. Supposons que l'observation satisfasse l'équation (1.2), où $S(\cdot)$ est inconnue et $\sigma(\cdot)$ est connue. Nous testons l'hypothèse nulle suivante

$$\mathcal{H}_0 \quad : \qquad S\left(x\right) = S_*\left(x\right),$$

où $S_*(\cdot)$ est une fonction connue. Nous introduisons l'espace $\mathcal{L}^2(f_*)$, l'ensemble des fonctions de carré intégrable avec le poids $f_*(\cdot)$

$$\mathcal{L}^2(f_*) = \left\{ h(\cdot):\ \mathbf{E}_* h(\xi_0)^2 = \int_{-\infty}^{\infty} h(x)^2 f_*(x)\mathrm{d}x < \infty \right\}.$$

Soit $\{\phi_1, \phi_2, ...\}$ une base orthonormée complète de cet espace. Nous introduisons alors l'alternative : pour $N \in \mathbb{N}$ fixé

$$\mathcal{H}_1 : \quad S(\cdot) \in \mathcal{S}_N,$$

où $\mathcal{S}_N$ est le sous-espace des fonctions de carré intégrable suivant

$$\begin{aligned} \mathcal{S}_N \quad = \quad & \left\{ S(\cdot) \in \mathcal{L}^2(f_*) \middle| \sum_{i=1}^N \int_{-\infty}^{\infty} \phi_i(x)^2 f_S(x)\mathrm{d}x < \infty, \right. \\ & \left. \sum_{i=1}^N \left(\int_{-\infty}^{\infty} \left(\frac{S(x) - S_*(x)}{\sigma(x)} \right) \phi_i(x) f_S(x)\mathrm{d}x \right)^2 > 0 \right\}. \end{aligned}$$

Nous définissons le test du khi-deux comme

$$\rho_{T,N} = \mathbb{1}_{\{\mu_{T,N} > z_\varepsilon\}},$$

où

$$\mu_{T,N} = \sum_{i=1}^N \left(\frac{1}{\sqrt{T}} \int_0^T \frac{\phi_i(X_t)}{\sigma(X_t)} [\mathrm{d}X_t - S_*(X_t)\mathrm{d}t] \right)^2,$$

et z_ε est le $(1-\varepsilon)$-quantile de loi du khi-deux $\chi^2(N)$. Nous démontrons dans le théorème 2.4.1 que le test du khi-deux est de taille asymptotique ε, qu'il est consistant, et qu'il ne dépend pas de la distribution. De plus, nous étudions le comportement asymptotique du test pour l'alternative de Pitman. Nous donnons dans le théorème 2.4.2 la puissance de ce test. Nous étudions ensuite le cas, plus intétessant, où $N \to \infty$. Nous démontrons dans la proposition 2.4.1 que la limite de la statistique suit une loi normale standard.

1.2 Approximation des EDS Rétrogrades

Par rapport au chapitre 3, nous étudions le problème statistique des équations différentielles stochastiques rétrogrades (EDSR). Supposons que l'on observe un processus de diffusion $X^T = \{X_t, 0 \leq t \leq T\}$ satisfaisant une équation différentielle stochastique (EDS)

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad 0 \leq t \leq T, \quad X_0 = x_0.$$

Pour deux fonctions $f(t, x, y, z)$ et $\Phi(x)$ données, la question se pose de construire un couple de processus (Y_t, Z_t) qui est la solution de l'équation suivante

$$\mathrm{d}Y_t = -f(t, X_t, Y_t, Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \quad 0 \leq t \leq T, \tag{1.3}$$

avec $Y_T = \Phi(X_T)$ comme valeur finale. La solution de ce problème est très connue, nous citons ici l'article d'El Karoui al. [15]. Dans leur travail, ils ont montré que la solution de cette EDSR est liée à la solution d'une équation différentielle partielle (EDP). En fait, notons $u(t, x)$ la solution de l'équation suivante

$$\begin{cases} \frac{\partial u}{\partial t} + b(x)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma(x)^2\frac{\partial^2 u}{\partial x^2} = -f\left(t, x, u, \sigma(x)\frac{\partial u}{\partial x}\right), \\ u(T, x) = \Phi(x). \end{cases} \tag{1.4}$$

En appliquant la formule d'Itô à $Y_t = u(t, X_t)$, on obtient

$$\begin{aligned} \mathrm{d}Y_t &= \left[\frac{\partial u}{\partial t}(t, X_t) + b(X_t)\frac{\partial u}{\partial x}(t, X_t) + \frac{1}{2}\sigma(X)^2\frac{\partial^2 u}{\partial x^2}(t, X_t)\right]\mathrm{d}t + \frac{\partial u}{\partial x}(t, X_t)\,\sigma(X_t)\,\mathrm{d}W_t, \\ &= -f(t, X_t, Y_t, Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \qquad Y_0 = u(0, x_0), \end{aligned}$$

où $Z_t = \sigma(X_t)\,u'(t, X_t)$. Ainsi, le problème (1.3) est résolu et la solution est

$$Y_t = u(t, X_t), Z_t = \sigma(X_t)\,u'(t, X_t).$$

Le chapitre 3 est consacré au problème suivant

$$\mathrm{d}X_t = S(\vartheta, X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad 0 \leq t \leq T, \quad X_0 = x_0.$$

où S et σ sont des fonctions connues et $\vartheta \in \Theta \subset R^d$ est un paramètre inconnu. Dans ce cas, la solution $u\left(t, x, \vartheta\right)$ de (1.4) dépend également du paramère inconnu. Nous ne pouvons donc plus utiliser $Y_t = u\left(t, X_t, \vartheta\right)$ ni $Z_t = \sigma\left(X_t\right) u'\left(t, X_t, \vartheta\right)$. Par conséquent, nous considérons le problème de construction d'un couple de processus adaptés $(\widehat{Y}_t, \widehat{Z}_t)$, où $\widehat{Y}_t$ et $\widehat{Z}_t$ sont des approximations de (Y_t, Z_t). Cette approximation est réalisée à l'aide de l'EMV $\widehat{\vartheta}$. Nous nous sommes intéressés à une situation où l'erreur de cette approximation est petite. Une des possibilités d'avoir une petite erreur d'approximation est dans un certain sens équivalente à la situation d'avoir une petite erreur d'estimation du paramètre ϑ. Ensuite la continuité de la fonction $u\left(t, x, \vartheta\right)$ par rapport à ϑ nous donne $\widehat{Y}_T \sim Y_T = \Phi\left(X_T\right)$.

Nous pouvons avoir la petite erreur d'estimation dans les situations suivantes : soit lorsque $T \to \infty$, soit lorsque $\sigma\left(\cdot\right)^2 \to 0$ (à voir par exemple, Kutoyants [28] et [27]). Dans le chapitre 3, nous étudions ce modèle avec un *petit bruit*, c'est à dire que le coefficient de diffusion tend vers 0. Cela nous permet de garder le temps final T fixé et cette asymptotique est plus facile à traiter.

La section 3.1 est consacrée aux résultats préliminaires. Dans la section 3.2, nous considérons un cas relativement simple, où la dérive $S\left(\vartheta, x\right)$ est une fonction linéaire de ϑ, le coefficient de diffuson est $\varepsilon^2 \sigma\left(x\right)^2$ et la fonction $f\left(t, x, y, z\right)$ est linéaire par rapport à x. Supposons que l'observation $X^T = \{X_t, 0 \leq t \leq T\}$ satisfait l'EDS suivante

$$\mathrm{d}X_t = \vartheta h(X_t)\,\mathrm{d}t + \varepsilon\sigma(X_t)\,\mathrm{d}W_t, \quad X_0 = x_0,\ 0 \leq t \leq T. \tag{1.5}$$

Notre objetif est de construire un couple de processus $(\widehat{Y}, \widehat{Z})$ qui se rapproche de la solution de l'équation

$$\mathrm{d}Y_t = (k(X_t) + g(X_t)\,Y_t)\,\mathrm{d}t + Z_t \mathrm{d}W_t, \quad 0 \leq t \leq T, \quad Y_T = \Phi(X_T). \tag{1.6}$$

Pour cela, nous estimons tout d'abord ϑ par l'EMV $\widehat{\vartheta}_{t,\varepsilon}$ pour tout $0 \leq t \leq T$. Ensuite, les processus approximés sont définis comme :

$$\widehat{Y}_t = u(t, X_t, \widehat{\vartheta}_{t,\varepsilon}), \quad \widehat{Z}_t = \varepsilon\sigma(X_t) u'(t, X_t, \widehat{\vartheta}_{t,\varepsilon}),$$

où $u(t,x,\vartheta)$ est la solution de l'EDP

$$\frac{\partial u}{\partial t}+\vartheta h\left(x\right)\frac{\partial u}{\partial x}+\frac{\varepsilon^{2}}{2}\sigma\left(x\right)^{2}\frac{\partial^{2}u}{\partial x^{2}}=k\left(x\right)+g\left(x\right)\,u,\quad u\left(T,x\right)=\Phi\left(x\right). \tag{1.7}$$

Nous montrons, sous des conditions de régularité, que $\widehat{Y}_t$ est proche de Y_t pour les petites valeurs de ε. Dans la section 3.3, nous généralisons le résultat au cas non-linéaire. Dans la section 3.4, nous établissons que l'approximation proposée ci-dessus est asymptotiquement efficace. A la fin, nous illustrons nos résultats par la simulation numérique.

Chapter 2

On Goodness-of Fit Tests for Diffusion Process

2.1 Introduction

We consider the problem of goodness of fit (GoF) test for the model of ergodic diffusion process when this process under the null hypothesis belongs to a given family. In Section 2.2 and 2.3, we study the Cramer-von Mises (C-vM) type and the Kolmogorov-Smirnov (K-S) type statistics for parametrical family. To construct the test, we use the local time estimator (LTE) or the empirical distribution function (EDF). We show that the C-vM type and the K-S type statistics converge in both cases to limits which do not depend on the unknown parameter, so the test is called asymptotically parameter free (APF). In Section 2.4, we study the chi-square test for simple basic hypothesis. We show that the limit of the statistic does not depend on the trend coefficient, that is the test is asymptotically distribution free (ADF). In addition, all of these tests are consistent against any fixed alternatives.

Let us remind the similar statement of the problem in the well known case of the observations of independent identically distributed (i.i.d.) random variables (r.v.) $X^n = (X_1, \ldots, X_n)$. Suppose that the distribution of X_j under the basic hypothesis is $F(\vartheta, x) = F_*(x - \vartheta)$, where ϑ is some unknown parameter. This kind of parametrical GoF problem has been studied in Kac al. [23], and then developed by many other works. We mention here for example, Darling [10], Martynov [38] and Lehmann &

Romano [32]. In these works, the C-vM type and the K-S type tests are proposed as follows:

$$\widehat{\psi}_{n,1}\left(X^n\right) = \mathbb{1}_{\left\{\omega_{n,1}^2 > e_{\varepsilon,1}\right\}}, \qquad \omega_{n,1}^2 = n\int_{-\infty}^{\infty}\left[\widehat{F}_n\left(x\right) - F_*\left(x - \widehat{\vartheta}_n\right)\right]^2 \mathrm{d}F_*\left(x - \widehat{\vartheta}_n\right),$$
$$\widehat{\psi}_{n,2}\left(X^n\right) = \mathbb{1}_{\left\{\omega_{n,2} > e_{\varepsilon,2}\right\}}, \qquad \omega_{n,2} = \sup_x \sqrt{n}\left|\widehat{F}_n(x) - F(x - \widehat{\vartheta}_n)\right|,$$

where $\widehat{F}_n(x)$ is the EDF and $\widehat{\theta}_n$ is certain consistent estimator. They proved that under the basic hypothesis, the statistics $\omega_{n,1}^2$ and $\omega_{n,2}$ converge in distribution to some random variables ω_1^2 and ω_2. In addition the limit r.v. ω_1^2 and ω_2 do not depend on ϑ. Thus the threshold $e_{\varepsilon,i}$ can be calculated as solution of the equation

$$\mathbf{P}\left\{\omega_1^2 > e_{\varepsilon,1}\right\} = \varepsilon, \quad \mathbf{P}\left\{\omega_2 > e_{\varepsilon,2}\right\} = \varepsilon.$$

Therefore the tests do not depend on the unknown parameter, that is the C-vM test and the K-S test are all APF. The details concerning this result can be founded in Darling [10] and Kac al. [23]. For more general problems see the works of Durbin [12] or Lehmann & Romano [32].

Otherwise, we are interested in the chi-square test. We mention here the works of Cramer [7], Dahiya & Gurland [9], Watson [49] and Greenwood & Nikulin [21]. For i.i.d. sample $\{X_n, n \in \mathbb{N}\}$, one tests hypothesis H_0 that the data form a sample of n values of a r.v. X with the given probability function $f(x)$. We partition the space of the variable X into r part $I_1, ..., I_r$, and consider the statistic

$$\omega_{n,3} = \sum_{i=0}^{r} \frac{(v_i - np_i)^2}{np_i},$$

where $p_i = P(I_i) > 0$ and $\sum_{i=1}^{r} p_i = 1$, and v_i is the number of sample values which belong to I_i. Thus Cramer [7] showed that as $n \longrightarrow \infty$, $\omega_{n,3}$ is distributed in a χ^2-distribution with $(r-1)$-degrees of freedom $(\chi^2(r-1))$. Thus the test

$$\widehat{\psi}_{n,3}(X^n) = \mathbb{1}_{\{\omega_{n,3} > e_{\varepsilon,3}\}},$$

where $e_{\varepsilon,3}$ is the $(1-\varepsilon)$-quantile of $\chi^2(r-1)$, is ADF, i.e. the test does not depend on the distribution of the sample.

A similar problem exists for the continuous time stochastic processes, which are widely used as mathematic models in many fields. The goodness of fit tests (GoF) are studied by many authors. For example Kutoyants [28] discussed some possibilities of construction of such tests. In particular, he considered the K-S statistics and the C-vM Statistics based on the continuous observation. Note that the K-S statistics for ergodic diffusion process were studied in Fournie [19] and in Fournie and Kutoyants [20]. However, due to the structure of the covariance of the limit process, the K-S statistics are not ADF in diffusion process models. More recently Kutoyants [29] has proposed a modification of the K-S statistics for diffusion models that became ADF. See also Dachian and Kutoyants [8] where they propose some GoF tests for diffusion and inhomogeneous Poisson processes with simple basic hypothesis which are all ADF. In the case of Ornstein-Uhlenbeck process Kutoyants showed that the C-vM type tests are APF in [30]. Another test was studied by Negri and Nishiyama [40].

In this work we are interested in the goodness of fit testing problems for composite and simple case. Suppose that the observation $X^T = \{X_t, 0 \le t \le T\}$ is a continuous-time diffusion process satisfying

$$\mathrm{d}X_t = S\left(X_t\right)\,\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0, \quad 0 \le t \le T, \tag{2.1}$$

where $\{W_t, t \ge 0\}$ is a standard Wiener process, the trend coefficient $S\left(\cdot\right)$ is unknown and the diffusion coefficient $\sigma(\cdot)^2$ is known. We introduce some conditions and auxiliary results in this section. Let us remind the following condition, to ensure that the equation (2.1) has a unique weak solution (See Durett [13]).

$\mathcal{ES}$. *The function $S(\cdot)$ is locally bounded, the function $\sigma(\cdot)^2$ is continuous and for some $C > 0$,*

$$xS(x) + \sigma(x)^2 \le C(1 + x^2).$$

The stochastic process (2.1) has ergodic properties if the functions $S(\cdot)$ and $\sigma(\cdot)$ satisfy the following two conditions:

$\mathcal{RP}$.

$$V(S,x)=\int_0^x \exp\left\{-2\int_0^y \frac{S(z)}{\sigma(z)^2}\mathrm{d}z\right\}\mathrm{d}y \to \pm\infty, \quad \textit{as } x\to\pm\infty$$

and

$$G(S)=\int_{-\infty}^{\infty}\sigma(y)^{-2}\exp\left\{2\int_0^y \frac{S(z)}{\sigma(z)^2}\mathrm{d}z\right\}\mathrm{d}y<\infty.$$

That is under these two conditions, the process is recurrent positive and has the following density of invariant law (See Durett [13])

$$f_S(x)=\frac{1}{G(S)\sigma(x)^2}\exp\left\{2\int_0^x \frac{S(y)}{\sigma(y)^2}\mathrm{d}y\right\}.$$

Denote $\mathcal{P}$ as the class of functions having polynomial majorants i.e.

$$\mathcal{P}=\{h(\cdot):\ |h(x)|\le C(1+|x|^p)\},$$

with some $p>0$. Note that a sufficient condition for $\mathcal{RP}$ is

$\mathcal{A}_0$. *The coefficient functions satisfy:* $\sigma^{\mp 1}\in\mathcal{P}$ *and*

$$\overline{\lim_{|x|\to\infty}}\ \mathrm{sgn}(x)\frac{S(x)}{\sigma(x)^2}<0.$$

We introduce also the condition which provides the equivalence of measures defined by different trend coefficient.

$\mathcal{EM}$. *The function* $S(\cdot)$ *and* $\sigma(\cdot)$ *satisfy condition* $\mathcal{ES}$ *and the densities* $f_S(\cdot), f_0(\cdot)$ *(with respect to the Lebesgue measure) of the corresponding initial values have the same support.*

In this chapter, we study the GoF test for the model (2.1), where some auxiliary results will be required. Therefore, we introduce in the follows some conditions and results about the ergodic diffusion process, including the properties of the maximum likelihood estimator (MLE) and the minimum distance estimator (MDE) for unknown parameter, the LTE for the invariant density function and the EDF.

2.1.1 Auxiliary results

Suppose that we observe an ergodic diffusion process, solution to the following stochastic differential equation (SDE)

$$\mathrm{d}X_t = S(X_t,\vartheta)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0,\ 0 \le t \le T, \tag{2.2}$$

where the functions $S(\cdot)$ and $\sigma(\cdot)$ are known and the parameter ϑ is unknown. In Kutoyants [28], the author introduced some methods to estimate the unknown parameter. Under the condition $\mathcal{A}_0$, the diffusion process is recurrent and its invariant density $f_S(x,\vartheta)$ can be written as:

$$f_S(x,\vartheta) = \frac{1}{G(\vartheta)\sigma(x)^2}\exp\left\{2\int_0^x \frac{S(y,\vartheta)}{\sigma(y)^2}\mathrm{d}y\right\}.$$

Denote by ξ_ϑ a r.v. having this density $f_S(x,\vartheta)$, denote by $\mathbf{E}_\vartheta$ the corresponding mathematic expectation. For any derivable function $h(x,\vartheta)$, we denote $h'(x,\vartheta)$ the derivative w.r.t. x and $\dot{h}(x,\vartheta)$ the derivative w.r.t. ϑ.

Let us introduce the MLE $\widehat{\vartheta}_T$ and some properties. We denote $L(\vartheta,X^T)$ the log-likelihood ratio

$$L(\vartheta,X^T) = \int_0^T \frac{S(X_t,\vartheta)}{\sigma(X_t)^2}\mathrm{d}X_t - \frac{1}{2}\int_0^T \left(\frac{S(X_t,\vartheta)}{\sigma(X_t)}\right)^2 \mathrm{d}t. \tag{2.3}$$

Then the MLE $\widehat{\vartheta}_T$ is defined as the solution of the equation

$$L(\widehat{\vartheta}_T,X^T) = \sup_{\theta\in\Theta} L(\theta,X^T).$$

Let us denote ϑ_0 the true value of the unknown parameter, we introduce the condition $\mathcal{A}$:

$\mathcal{A}_1$. *The function $S(\cdot,\cdot)$ is continuously differentiable w.r.t. ϑ, the derivative $\dot{S}(\cdot,\cdot)\in\mathcal{P}$ and is uniformly continuous in the following sense:*

$$\lim_{\delta\to 0}\sup_{|\vartheta-\vartheta_0|<\delta}\mathbf{E}_{\vartheta_0}\left|\frac{\dot{S}(\xi,\vartheta)-\dot{S}(\xi,\vartheta_0)}{\sigma(\xi)}\right|^2 = 0.$$

$\mathcal{A}_2$. *The Fisher information is positive*

$$I(\vartheta) = \mathbf{E}_\vartheta \left(\frac{\dot S(\xi,\vartheta)}{\sigma(\xi)} \right)^2 > 0, \tag{2.4}$$

and for any $\nu > 0$

$$\inf_{|\vartheta-\vartheta_0|>\delta} \mathbf{E}_{\vartheta_0} \left(\frac{S(\xi,\vartheta)-S(\xi,\vartheta_0)}{\sigma(\xi)} \right)^2 > 0.$$

We have the following result.

Lemma 2.1.1. *(See Kutoyants [28] Theorem 2.8) Let the condition* $\mathcal{A}_0$ *and* $\mathcal{A}$ *be fulfilled, Then the MLE* $\widehat\vartheta_T$ *is consistent, i.e., for any* $\nu > 0$,

$$\lim_{T\to\infty} \mathbf{P}_{\vartheta_0}\big\{ |\widehat\vartheta_T - \vartheta_0| > \nu \big\} = 0,$$

asymptotically normal

$$\mathcal{L}_{\vartheta_0}\big\{ \sqrt{T}(\widehat\vartheta_T - \vartheta_0) \big\} \Rightarrow \mathcal{N}(0, I(\vartheta_0)^{-1}),$$

and the moments converge: for $p > 0$

$$\lim_{T\to\infty} \mathbf{E}_{\vartheta_0} \left| \sqrt{T}(\widehat\vartheta_T - \vartheta_0) \right|^p = \mathbf{E}\, |\widehat u|^p\,,$$

where $\widehat u$ *is a r.v. of normal distribution* $\mathcal{N}(0, I(\vartheta_0)^{-1})$.

Now we introduce the LTE $\widehat f_T(x)$ and the EDF $\widehat F_T(x)$. Suppose that the process observed is a solution to the following SDE

$$\mathrm{d}X_t = S(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0,\ 0 \le t \le T, \tag{2.5}$$

where the trend coefficient $S(\cdot)$ is unknown and the diffusion coefficient $\sigma(\cdot)^2$ is a known continuous positive function. Then the invariant density function is

$$f_S(x) = \frac{1}{G(S)\sigma(x)^2} \exp\left\{ 2\int_0^x \frac{S(y)}{\sigma(y)^2}\mathrm{d}y \right\}.$$

Denote by ξ a r.v. having this density $f_S(x)$, denote by $\mathbf{E}_S$ the corresponding mathematic expectation. Firstly, we introduce the LTE $\widehat f_T(x)$ for this invariant density function. Let us remind the local time for diffusion process (See Corollary 6.1.9 in Revuz & Yor [45]):

$$\Lambda_T(x) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^T \mathbb{1}_{\{|X_t - x| \le \varepsilon\}} \sigma(X_t)^2 \mathrm{d}t.$$

According to Tanaka's formula, it can be written as

$$\Lambda_T(x) = \frac{1}{T}(|X_T - x| - |X_0 - x|) - \frac{1}{T}\int_0^T \mathrm{sgn}(X_t - x)\mathrm{d}X_t.$$

Thus we define the LTE for the invariant density function:

$$\widehat{f}_T(x) = \frac{\Lambda_T(x)}{T\sigma(x)^2}.$$

Let us introduce the condition $\mathcal{O}$:

$\mathcal{O}$. *For some* $p \geq 2$

$$\mathbf{E}_S \left\{ \left| \frac{F_S(\xi) - 1\!\!1_{\{\xi > x\}}}{\sigma(\xi) f_S(\xi)} \right|^p + \left| \int_0^\xi \frac{F_S(v) - 1\!\!1_{\{v > x\}}}{\sigma(v)^2 f_S(v)} \mathrm{d}v \right|^p \right\} < \infty.$$

Note that under the condition $\mathcal{A}_0$, we have the law of large numbers

$$\mathbf{P}_S - \lim_{T \longrightarrow \infty} \frac{4 f_S(x)^2}{T} \int_0^T \left(\frac{F_S(X_t) - 1\!\!1_{\{X_t > x\}}}{\sigma(X_t) f_S(X_t)} \right)^2 \mathrm{d}t = I_f(S, x)$$

where

$$I_f(S, x) = 4 f_S(x)^2 \mathbf{E}_S \left(\frac{F_S(\xi) - 1\!\!1_{\{\xi > x\}}}{\sigma(\xi) f_S(\xi)} \right)^2.$$

We have the following result

Lemma 2.1.2. *(See Kutoyants [28] Theorem 4.11) Let the condition* $\mathcal{O}$ *be fulfilled, then the estimator* $\widehat{f}_T(x)$ *is consistent and asymptotically normal*

$$\mathcal{L}_S \left\{ T^{1/2}(\widehat{f}_T(x) - f_S(x)) \right\} \Longrightarrow \mathcal{N}\left(0,\ I_f(S, x)\right).$$

Concerning the EDF

$$\widehat{F}_T(x) = \frac{1}{T}\int_0^T 1\!\!1_{\{X_t < x\}} \mathrm{d}t,$$

we introduce the condition $\mathcal{N}$:

$\mathcal{N}$. *There exists a number* $p \geq 2$ *such that*

$$\mathbf{E}_S \left\{ \left| \int_x^\xi \frac{F_S(v \wedge x)(F_S(v \vee x) - 1)}{\sigma(v)^2 f_S(v)} \mathrm{d}v \right|^p + \left| \frac{F_S(\xi \wedge x)(F_S(\xi \vee x) - 1)}{\sigma(\xi) f_S(\xi)} \right|^p \right\} < \infty.$$

Let us denote

$$I_F(S, x) = 4 \mathbf{E}_S \left(\frac{F_S(\xi \wedge x)(F_S(\xi \vee x) - 1)}{\sigma(\xi) f_S(\xi)} \right)^2,$$

then we have the following result

Lemma 2.1.3. *(See Kutoyants [28] Theorem 4.6) Let the condition $\mathcal{N}$ be fulfilled. Then the EDF $\widehat{F}_T(x)$ is consistent and asymptotically normal.*

$$\mathcal{L}_S\left\{T^{1/2}(\widehat{F}_T(x)-F_S(x))\right\}\Longrightarrow\mathcal{N}\left(0,\,I_F(S,x)\right).$$

2.1.2 A special case

In Section 2.2 and 2.3, we are interested in the following model. Suppose that the observed ergodic diffusion process satisfies the following SDE

$$\mathrm{d}X_t=S(X_t-\vartheta)\mathrm{d}t+\mathrm{d}W_t,\quad X_0,\ 0\le t\le T, \tag{2.6}$$

where ϑ is the unknown shift parameter.

Under the condition $\mathcal{A}_0$, the density of the invariant law $f_S(\cdot,\cdot)$ can be calculated as follows:

$$\begin{aligned}f_S(x,\vartheta) &= \frac{1}{G(\vartheta)}\exp\left\{2\int_\vartheta^x S(x-\vartheta)\mathrm{d}y\right\}=\frac{\exp\left\{2\int_\vartheta^x S(y-\vartheta)\mathrm{d}y\right\}}{\int_{-\infty}^\infty\exp\left\{2\int_\vartheta^y S(z-\vartheta)\mathrm{d}z\right\}\mathrm{d}y}\\ &= \frac{\exp\left\{2\int_0^{x-\vartheta}S(y)\mathrm{d}y\right\}}{\int_{-\infty}^\infty\exp\left\{2\int_0^v S(u)\mathrm{d}u\right\}\mathrm{d}v}=f(x-\vartheta).\end{aligned} \tag{2.7}$$

where $f(x)$ is

$$f(x)=\left(\int_{-\infty}^\infty\exp\left\{2\int_0^v S(u)\mathrm{d}u\right\}\mathrm{d}v\right)^{-1}\exp\left\{2\int_0^x S(y)\mathrm{d}y\right\}.$$

Let us denote

$$F(x)=\int_{-\infty}^x f(y)\mathrm{d}y,$$

thus the distribution function of this process is

$$F_S(x,\vartheta)=\int_{-\infty}^x f(y-\vartheta)\mathrm{d}y=\int_{-\infty}^{x-\vartheta}f(y)\mathrm{d}y=F(x-\vartheta),$$

Denote by ξ_ϑ a r.v. with density function $f(x-\vartheta)$, denote by $\mathbf{E}_\vartheta$ the corresponding mathematical expectation. Correspondingly, ξ_0 and $\mathbf{E}_0$ are respectively the r.v. and the mathematical expectation for the case $\vartheta=0$.

The MLE $\widehat{\vartheta}_T$ is defined as the solution of the equation

$$L(\widehat{\vartheta}_T, X^T) = \sup_{\theta\in\Theta} L(\theta, X^T),$$

where $L(\vartheta, X^T)$ is the log-likelihood ratio

$$L(\vartheta, X^T) = \int_0^T S(X_t - \vartheta)\mathrm{d}X_t - \frac{1}{2}\int_0^T S(X_t - \vartheta)^2\mathrm{d}t. \tag{2.8}$$

Note that

$$\mathbf{E}_\vartheta h(\xi_\vartheta - \vartheta) = \int_{-\infty}^{\infty} f(x-\vartheta)h(x-\vartheta)\mathrm{d}x = \int_{-\infty}^{\infty} f(x)h(x)\mathrm{d}x = \mathbf{E}_0 h(\xi_0). \tag{2.9}$$

Therefore, the Fisher information in this case does not depend on the unknown parameter ϑ_0, i.e.

$$I = \mathbf{E}_{\vartheta_0} S'(\xi_{\vartheta_0} - \vartheta_0)^2 = \mathbf{E}_0 S'(\xi_0)^2. \tag{2.10}$$

The condition $\mathcal{A}$ in this model can be written as follows:

$\mathcal{A}_1$. *The function $S(\cdot)$ is continuously differentiable, the derivative $S'(\cdot) \in \mathcal{P}$ and it is uniformly continuous in the following sense:*

$$\lim_{\nu\to 0}\sup_{|\tau|<\nu} \mathbf{E}_0 \big|S'(\xi_0) - S'(\xi_0+\tau)\big|^2 = 0.$$

$\mathcal{A}_2$. *The Fisher information*

$$I = \mathbf{E}_0 S'(\xi_0)^2 > 0. \tag{2.11}$$

In addition, for any $\nu > 0$

$$\inf_{|\tau|>\nu} \mathbf{E}_0\big(S(\xi_0) - S(\xi_0+\tau)\big)^2 > 0.$$

As that is shown in Lemma 2.1.1, the MLE $\widehat{\vartheta}_T$ is consistent and asymptotically normal under conditions $\mathcal{A}_0$ and $\mathcal{A}$. Let us denote $\widehat{u}_T = \sqrt{T}(\widehat{\vartheta}_T - \vartheta_0)$ and define

$$\widehat{u} = -\frac{1}{I}\int_{-\infty}^{\infty} S'(y)\sqrt{f(y)}\mathrm{d}W(y),$$

with $W(y) = W_1(y)$ for $y \in \mathbb{R}^+$ and $W(y) = W_2(-y)$ for $y \in \mathbb{R}^-$, where W_1 and W_2 are independent Wiener processes. Then the asymptotical normality can be written as

$$\mathcal{L}_{\vartheta_0}\left\{\widehat{u}_T\right\} \Longrightarrow \mathcal{L}\left\{\widehat{u}\right\}. \tag{2.12}$$

From the condition $\mathcal{A}_0$, it follows that there exist some constants $A > 0$ and $\gamma > 0$ such that for all $|x| > A$,

$$\operatorname{sgn}(x)S(x) < -\gamma. \tag{2.13}$$

It can be shown that for $x > A$,

$$f(x) = \frac{1}{G(S)}\exp\left\{2\left(\int_0^A + \int_A^x\right) S(y)\mathrm{d}y\right\} < C\mathrm{e}^{-2\gamma x}.$$

Similar result can be deduced for $x < -A$, then we have

$$f(x) < C\mathrm{e}^{-2\gamma|x|}, \quad \text{for } |x| > A. \tag{2.14}$$

Moreover, the LTE $\widehat{f}_T(x)$ is

$$\widehat{f}_T(x) = \frac{1}{T}(|X_T - x| - |X_0 - x|) - \frac{1}{T}\int_0^T \operatorname{sgn}(X_t - x)\mathrm{d}X_t,$$

and the EDF is

$$\widehat{F}_T(x) = \frac{1}{T}\int_0^T \mathbb{1}_{\{X_t<x\}}\mathrm{d}t.$$

In fact, these estimators of the invariant density and the invariant distribution function are consistent and asymptotically normal under condition $\mathcal{A}_0$. This will be proved in section 2.2.

2.1.3 Main results

Suppose that we observe an ergodic diffusion process

$$\mathrm{d}X_t = S(X_t)\mathrm{d}t + \mathrm{d}W_t, \quad X_0,\ 0 \le t \le T. \tag{2.15}$$

where the trend coefficient $S(\cdot)$ is unknown. We propose three types of GoF test. In section 2.2, we are interested in the following hypotheses test problem. The basic hypothesis is

$$\mathcal{H}_0 \quad : \qquad S\left(x\right) = S_*\left(x-\vartheta\right), \quad \vartheta \in \Theta = (\alpha,\beta)$$

where $S_*\left(\cdot\right)$ is some known function and the shift parameter ϑ is unknown. Therefore, the trend coefficients under hypothesis belong to the family

$$\mathcal{S}\left(\Theta\right) = \left\{S_*\left(x-\vartheta\right), \quad \vartheta\in\Theta\right\}.$$

The alternative is defined as

$$\mathcal{H}_1 \quad : \qquad S\left(\cdot\right) \notin \overline{\mathcal{S}(\Theta)},$$

where $\overline{\mathcal{S}(\Theta)} = \{S_*(x-\vartheta), \vartheta \in [\alpha,\beta]\}$.

Let us fix some $\varepsilon \in (0,1)$, and denote by $\mathcal{K}_\varepsilon$ the class of tests ψ_T of asymptotic size ε, i.e.

$$\mathbf{E}_*\psi_T = \varepsilon + o(1),$$

where $\mathbf{E}_*$ is the mathematical expectation under the basic hypothesis.

We introduce two C-vM type tests. In the first test, we use the LTE $\widehat{f}_T(x)$ and the MLE $\widehat{\vartheta}_T$. The statistic is defined as the following integral

$$\delta_T = T\int_{-\infty}^{\infty}\left(\widehat{f}_T(x) - f_*(x,\widehat{\vartheta}_T)\right)^2 \mathrm{d}x,$$

where $f_*(\cdot,\cdot)$ is the invariant density function under hypothesis $\mathcal{H}_0$. We show that under hypothesis $\mathcal{H}_0$, it converges in distribution to some r.v. δ which does not depend on ϑ. Thus we define the C-vM type test as

$$\psi_T = 1\!\!1_{\{\delta_T > d_\varepsilon\}},$$

with d_ε the $(1-\varepsilon)$-quantile of the distribution of δ, i.e. d_ε is the solution of the following equation

$$\mathbf{P}\left(\delta > d_\varepsilon\right) = \varepsilon.$$

We show in Section 2.2 that the test ψ_T belongs to $\mathcal{K}_\varepsilon$, is consistent and is APF.

The second C-vM type test is based on the EDF $\widehat{F}_T(x)$ and the MLE $\widehat{\vartheta}_T$:

$$\Psi_T = 1\!\!1_{\{\Delta_T > D_\varepsilon\}}, \quad \Delta_T = T\int_{-\infty}^{\infty}\left(\widehat{F}_T(x) - F_*(x,\widehat{\vartheta}_T)\right)^2 \mathrm{d}x,$$

where $F_*(\cdot,\cdot)$ is the invariant distribution function under hypothesis $\mathcal{H}_0$. The statistic Δ_T converges in distribution to some r.v. Δ which does not depend on ϑ, and D_ε is the $(1-\varepsilon)$-quantile of the distribution of Δ. We obtain that the test Ψ_T belongs to $\mathcal{K}_\varepsilon$ and is APF.

In Section 2.3, we study the same hypotheses testing problem, but for the K-S test. We introduce two tests via the LTE $\widehat{f}_T(x)$ and the EDF $\widehat{F}_T(x)$:

$$\phi_T = 1\!\!1_{\{\lambda_T > c_\varepsilon\}}, \quad \Phi_T = 1\!\!1_{\{\Lambda_T > C_\varepsilon\}},$$

where the statistics

$$\lambda_T = \sqrt{T}\sup_{x\in\mathbb{R}}\left|\widehat{f}_T(x) - f_*(x-\widehat{\vartheta}_T)\right|,$$

$$\Lambda_T = \sqrt{T}\sup_{x\in\mathbb{R}}\left|\widehat{F}_T(x) - F_*(x-\widehat{\vartheta}_T)\right|.$$

These statistics converge in distribution to certain r.v. λ and Λ respectively, which do not depend on ϑ. Thus c_ε and C_ε are defined respectively as the $(1-\varepsilon)$-quantile of the distribution of λ and Λ. We show that these tests ϕ_T and Φ_T belong to $\mathcal{K}_\varepsilon$, are consistent and are all APF.

In Section 2.4, we study the chi-square test. Suppose that we observe an ergodic diffusion process

$$\mathrm{d}X_t = S(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0,\ 0 \le t \le T. \tag{2.16}$$

We test the following basic hypothesis:

$$\mathcal{H}_0 \quad : \qquad S\left(x\right) = S_*\left(x\right),$$

where $S_*(\cdot)$ is some known function. We denote always the invariant density function under the basic hypothesis as $f_*(\cdot)$. Let us introduce the space $\mathcal{L}^2(f_*)$ of square integrable functions with weights $f_*(\cdot)$:

$$\mathcal{L}^2(f_*) = \left\{ h(\cdot):\ \mathbf{E}h(\xi_0)^2 = \int_{-\infty}^{\infty} h(x)^2 f_*(x)\mathrm{d}x < \infty \right\}.$$

Denote by $\{\phi_1, \phi_2, ...\}$ a complete orthonormal basis in the space $\mathcal{L}^2(f_*)$. We test the hypothesis $\mathcal{H}_0$ against the alternative

$$\mathcal{H}_{1,N}: \quad S(\cdot) \in \mathcal{S}_N,$$

where $\mathcal{S}_N$ is the subspace of square integrable functions such that for fixed $N \in \mathbb{N}$,

$$\begin{aligned} \mathcal{S}_N \quad = \quad \Bigg\{ S(\cdot) \in \mathcal{L}^2(f_*) \Bigg| & \sum_{i=1}^{N} \int_{-\infty}^{\infty} \phi_i(x)^2 f_S(x)\mathrm{d}x < \infty, \\ & \sum_{i=1}^{N} \left(\int_{-\infty}^{\infty} \left(\frac{S(x) - S_*(x)}{\sigma(x)} \right) \phi_i(x) f_S(x)\mathrm{d}x \right)^2 > 0 \Bigg\}. \end{aligned}$$

The chi-square test will be denoted as

$$\rho_{T,N} = 1\!\!\mathrm{I}_{\{\mu_{T,N} > z_\varepsilon\}}, \quad \mu_{T,N} = \sum_{i=1}^{N} \eta_{T,N}^2$$

where

$$\eta_{T,N} = \frac{1}{\sqrt{T}} \int_0^T \frac{\phi_i(X_t)}{\sigma(X_t)} [\mathrm{d}X_t - S_*(X_t)\mathrm{d}t],$$

and z_ε the $(1-\varepsilon)$-quantile of $\chi^2(N)$.We obtain that the test $\rho_{T,N}$ belongs to $\mathcal{K}_\varepsilon$, is consistent and that it is ADF.

After that, we study the chi-square test for the statistic that $N \to \infty$. We define the statistic

$$\nu_{T,N} = \frac{1}{\sqrt{2N}} \sum_{i=1}^{N} \left(\eta_{T,N}^2 - 1\right),$$

which will be proved to converge to normal distribution $\mathcal{N}(0,1)$ when $T \to \infty$ and $N \to \infty$. Thus the test $\rho_{T,N} = \mathbb{I}_{\{\nu_{T,N} > Z_\varepsilon\}}$, with Z_ε the $(1-\varepsilon)$-quantile of $\mathcal{N}(0,1)$ belongs to $\mathcal{K}_\varepsilon$, is consistent and ADF.

2.2 The Cramer-von Mises Type Tests

This section is based on the work [41]

Suppose that we observe an ergodic diffusion process, solution to the following stochastic differential equation

$$\mathrm{d}X_t = S(X_t)\mathrm{d}t + \mathrm{d}W_t, \quad X_0,\ 0 \le t \le T. \tag{2.17}$$

We want to test the following null hypothesis

$$\mathcal{H}_0 \quad : \qquad S(x) = S_*(x-\vartheta), \quad \vartheta \in \Theta,$$

where $S_*(\cdot)$ is some known function and the shift parameter ϑ is unknown. We suppose that $0 \in \Theta = (\alpha, \beta)$. Let us introduce the family

$$\mathcal{S}(\Theta) = \left\{S_*(x-\vartheta), \quad \vartheta \in \Theta = (\alpha,\beta)\right\}.$$

The alternative is defined as

$$\mathcal{H}_1 \quad : \qquad S(\cdot) \notin \overline{\mathcal{S}(\Theta)},$$

where $\overline{\mathcal{S}(\Theta)} = \left\{S_*(x-\vartheta), \vartheta \in [\alpha,\beta]\right\}$.

We suppose that the trend coefficients $S(\cdot)$ of the observed diffusion process under both hypotheses satisfy the conditions $\mathcal{EM}$, $\mathcal{ES}$ and $\mathcal{A}_0$.

Remind that under these conditions the diffusion process is recurrent and its invariant density $f_{S_*}(x,\vartheta)$ under hypothesis $\mathcal{H}_0$ can be given explicitly as (2.7). Let us denote

$$f_*(x) = \frac{1}{G(S_*)} \exp\left\{2\int_0^x S_*(y)\mathrm{d}y\right\}.$$

then $f_{S_*}(x,\vartheta) = f_*(x-\vartheta)$. Denote by ξ_ϑ a r.v. having this density $f_*(x-\vartheta)$, denote by $\mathbf{E}_\vartheta$ the mathematical expectation. Moreover the unknown parameter is estimated by the MLE $\widehat{\vartheta}_T$

$$L(\widehat{\vartheta}_T, X^T) = \sup_{\theta\in\Theta} L(\theta, X^T),$$

with $L(\vartheta, X^T)$ the log-likelihood ratio (2.8). Remind that we have in Lemma 2.1.1, under the conditions $\mathcal{A}_0$ and $\mathcal{A}$, the MLE $\widehat{\vartheta}_T$ is consistent and asymptotically normal.

2.2.1 The C-vM type test via the LTE

To test the hypothesis $\mathcal{H}_0$, we propose in this subsection the C-vM type test basing on the MLE $\widehat{\vartheta}_T$ and the LTE $\widehat{f}_T(x)$

$$\widehat{f}_T(x) = \frac{1}{T}(|X_T - x| - |X_0 - x|) - \frac{1}{T}\int_0^T \operatorname{sgn}(X_t - x)\mathrm{d}X_t.$$

Let us denote the statistic as follows

$$\delta_T = T\int_{-\infty}^{\infty}\left(\widehat{f}_T(x) - f_*(x-\widehat{\vartheta}_T)\right)^2 \mathrm{d}x.$$

We show that under hypothesis $\mathcal{H}_0$, the statistic δ_T converges in distribution to

$$\delta = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(2f_*(x)\frac{F_*(y) - 1\!\!1_{\{y>x\}}}{\sqrt{f_*(y)}} - \frac{2}{I}S_*(x)f_*(x)S_*'(y)\sqrt{f_*(y)}\right)\mathrm{d}W(y)\right)^2 \mathrm{d}x, \tag{2.18}$$

with $W(y) = W_1(y)$ for $y \in \mathbb{R}^+$ and $W(y) = W_2(-y)$ for $y \in \mathbb{R}^-$, where W_1 and W_2 are independent Wiener processes. The C-vM type test is defined as

$$\psi_T = 1\!\!1_{\{\delta_T > d_\varepsilon\}},$$

where d_ε is the $(1-\varepsilon)$-quantile of the distribution of δ, that is the solution of the following equation

$$\mathbf{P}\Big(\delta \geq d_\varepsilon\Big) = \varepsilon. \tag{2.19}$$

The main result for the C-vM test via the LTE $\widehat{f}_T(x)$ is the following:

Theorem 2.2.1. *Let the conditions $\mathcal{ES}$, $\mathcal{A}_0$ and $\mathcal{A}$ be fulfilled, then the test $\psi_T = 1\!\!1_{\{\delta_T > d_\varepsilon\}}$ belongs to $\mathcal{K}_\varepsilon$.*

Note that neither δ nor d_ε depends on the unknown parameter, this allows us to conclude that the test is APF. To prove this result, we have to introduce three lemmas which will be given later. All these lemmas are given under the assumption that the hypothesis $\mathcal{H}_0$ is true.

Let us define $\eta_T(x) = \sqrt{T}\left(\widehat{f}_T(x) - f_*(x-\vartheta_0)\right)$. According to Kutoyants [28] Theorem 4.11, if the hypothesis $\mathcal{H}_0$ is true, we have the following representation

$$\begin{aligned}\eta_T(x) &= \sqrt{T}(\widehat{f}_T(x) - f_*(x-\vartheta_0)) \\ &= -2\frac{f_*(x-\vartheta_0)}{\sqrt{T}}\int_{X_0}^{X_T}\left(\frac{F_*(y-\vartheta_0)-1\!\!1_{\{y>x\}}}{f_*(y-\vartheta_0)}\right)\mathrm{d}y \\ &\quad +2\frac{f_*(x-\vartheta_0)}{\sqrt{T}}\int_0^T\left(\frac{F_*(X_t-\vartheta_0)-1\!\!1_{\{X_t>x\}}}{f_*(X_t-\vartheta_0)}\right)\mathrm{d}W_t. \qquad (2.20)\end{aligned}$$

Let us put

$$M(y,x) = 2f_*(x)\frac{F_*(y)-1\!\!1_{\{y>x\}}}{f_*(y)}.$$

Then $\eta_T(x)$ can be written as

$$\begin{aligned}\eta_T(x) &= \frac{1}{\sqrt{T}}\int_0^T M(X_t-\vartheta_0, x-\vartheta_0)\mathrm{d}W_t \\ &\quad -\frac{1}{\sqrt{T}}\int_{X_0}^{X_T} M(y-\vartheta_0, x-\vartheta_0)\mathrm{d}y. \qquad (2.21)\end{aligned}$$

We state that

Lemma 2.2.1. *Let the condition $\mathcal{A}_0$ be fulfilled, then*

$$\int_{-\infty}^{\infty}\mathbf{E}_0\left(\int_0^{\xi_0} M(y,x)\mathrm{d}y\right)^2\mathrm{d}x < \infty.$$

Proof. Applying the estimate (2.14), for $x > A$,

$$\begin{aligned}&\mathbf{E}_0\left(\int_0^{\xi_0} M(y,x)\mathrm{d}y\right)^2 \\ &= 4f_*(x)^2\int_{-\infty}^{\infty}\left(\int_0^z\frac{F_*(y)-1\!\!1_{\{y>x\}}}{f_*(y)}\mathrm{d}y\right)^2 f_*(z)\mathrm{d}z \\ &= 4f_*(x)^2\left(\int_{-\infty}^{-A}+\int_{-A}^{A}+\int_A^x\right)\left(\int_0^z\frac{F_*(y)}{f_*(y)}\mathrm{d}y\right)^2 f_*(z)\mathrm{d}z \\ &\quad +4f_*(x)^2\int_x^{\infty}\left(\int_0^x\frac{F_*(y)}{f_*(y)}\mathrm{d}y+\int_x^z\frac{F_*(y)-1}{f_*(y)}\mathrm{d}y\right)^2 f_*(z)\mathrm{d}z\end{aligned}$$

Further,

$$
\begin{aligned}
&f_*(x)^2 \int_{-\infty}^{-A} \left(\int_0^z \frac{F_*(y)}{f_*(y)} \mathrm{d}y \right)^2 f_*(z) \mathrm{d}z \\
&\quad = f_*(x)^2 \int_{-\infty}^{-A} \left(\left(\int_z^{-A} + \int_{-A}^0 \right) \frac{F_*(y)}{f_*(y)} \mathrm{d}y \right)^2 f_*(z) \mathrm{d}z \\
&\quad \le f_*(x)^2 \int_{-\infty}^{-A} \left(\int_z^{-A} \int_{-\infty}^{y} \frac{1}{G(S_*)} \exp\left(-2 \int_u^y S_*(v) \mathrm{d}v \right) \mathrm{d}u \mathrm{d}y + C_1 \right)^2 f_*(z) \mathrm{d}z \\
&\quad \le f_*(x)^2 \int_{-\infty}^{-A} \left(C_2 \int_z^{-A} \int_{-\infty}^{y} \mathrm{e}^{-2\gamma(y-u)} \mathrm{d}u \mathrm{d}y + C_1 \right)^2 f_*(z) \mathrm{d}z \\
&\quad \le C f_*(x)^2 \int_{-\infty}^{-A} (1+z)^2 f_*(z) \mathrm{d}z \le C f_*(x)^2 \le C \mathrm{e}^{-4\gamma x},
\end{aligned}
$$

moreover

$$
\begin{aligned}
&f_*(x)^2 \int_A^x \left(\int_0^z \frac{F_*(y)}{f_*(y)} \mathrm{d}y \right)^2 f_*(z) \mathrm{d}z \\
&\quad \le \int_A^x \left(\left(\int_0^A + \int_A^z \right) \frac{f_*(x)}{f_*(y)} \mathrm{d}y \right)^2 f_*(z) \mathrm{d}z \\
&\quad \le \int_A^x \left(C_1 f(x) + C_2 \int_A^z \mathrm{e}^{-2\gamma(x-y)} \mathrm{d}y \right)^2 f_*(z) \mathrm{d}z \\
&\quad \le \int_A^x \left(C_1 \mathrm{e}^{-2\gamma x} + C_2' \mathrm{e}^{-2\gamma(x-z)} - C_2' \mathrm{e}^{-2\gamma(x-A)} \right)^2 \cdot C \mathrm{e}^{-2\gamma z} \mathrm{d}z \\
&\quad \le \mathrm{e}^{-4\gamma x} \int_A^x \left(C_3 \mathrm{e}^{2\gamma z} + C_4 \mathrm{e}^{-2\gamma z} \right) \mathrm{d}z \le C \mathrm{e}^{-2\gamma x},
\end{aligned}
$$

and finally

$$
\begin{aligned}
&f_*(x)^2 \int_x^\infty \left(\int_x^z \frac{F_*(y) - 1}{f_*(y)} \mathrm{d}y \right)^2 f_*(z) \mathrm{d}z = f_*(x)^2 \int_x^\infty \left(\int_x^z \frac{1 - F_*(y)}{f_*(y)} \mathrm{d}y \right)^2 f_*(z) \mathrm{d}z \\
&\quad \le C f_*(x)^2 \int_x^\infty \left(\int_x^z \int_y^\infty \mathrm{e}^{-2\gamma(u-y)} \mathrm{d}u \mathrm{d}y \right)^2 \mathrm{e}^{-2\gamma z} \mathrm{d}z \\
&\quad \le C f_*(x)^2 \int_x^\infty (z-x)^2 \mathrm{e}^{-2\gamma z} \mathrm{d}z \le C f_*(x)^2 \int_0^\infty s^2 \mathrm{e}^{-2\gamma(s+x)} \mathrm{d}s \le C \mathrm{e}^{-6\gamma x}.
\end{aligned}
$$

Then we have

$$
\mathbf{E}_0 \left(\int_0^{\xi_0} M(y,x) \mathrm{d}y \right)^2 \le C \mathrm{e}^{-2\gamma |x|} \quad \text{for } x > A. \tag{2.22}
$$

Similar estimate can be obtained for $x < -A$, therefore the result holds for $|x| > A$. We obtain finally

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathbf{E}_0 \left(\int_0^{\xi_0} M(y,x)\mathrm{d}y \right)^2 \mathrm{d}x \\
&\quad = \left(\int_{-\infty}^{-A} + \int_{-A}^{A} + \int_{A}^{\infty} \right) \mathbf{E}_0 \left(\int_0^{\xi_0} M(y,x)\mathrm{d}y \right)^2 \mathrm{d}x \\
&\quad \leq C_1 \int_{-\infty}^{-A} \mathrm{e}^{2\gamma x}\mathrm{d}x + C_2 + C_3 \int_A^{\infty} \mathrm{e}^{-2\gamma x}\mathrm{d}x < \infty.
\end{aligned}$$

This result yields directly the conditions $\mathcal{O}$ of Lemma 2.1.2:

$$\mathbf{E}_{\vartheta_0} M(\xi_{\vartheta_0} - \vartheta_0, x - \vartheta_0)^2 = \mathbf{E}_0 M(\xi_0, x - \vartheta_0)^2 < \infty,$$

and

$$\mathbf{E}_{\vartheta_0} \left(\int_0^{\xi_{\vartheta_0}} M(y - \vartheta_0, x - \vartheta_0)\mathrm{d}y \right)^2 < \infty.$$

Thus we deduce the convergence and the asymptotical normality of $\eta_T(x)$. In fact under the condition $\mathcal{A}_0$, the LTE $\widehat{f}_T(x)$ is consistent and asymptotically normal, that is

$$\eta_T(x) = \sqrt{T}\left(\widehat{f}_T(x) - f_*(x - \vartheta_0) \right) \Longrightarrow \eta(x - \vartheta_0),$$

where $\eta(x) \sim \mathcal{N}(0, d(x)^2)$, and

$$d(x)^2 = 4 f_*(x)^2 \mathbf{E}_0 \left(\frac{F_*(\xi_0) - 1\!\!1_{\{\xi_0 > x\}}}{f_*(\xi_0)} \right)^2.$$

Moreover

$$\mathbf{E}_0\left(\eta(x)\eta(y)\right) \quad = 4 f_*(x) f_*(y) \mathbf{E}_0 \left(\frac{\left(F_*(\xi_0) - 1\!\!1_{\{\xi_0 > x\}}\right)\left(F_*(\xi_0) - 1\!\!1_{\{\xi_0 > y\}}\right)}{f_*(\xi_0)^2} \right).$$

Let us define

$$\eta(x) = \int_{-\infty}^{\infty} M(y,x)\sqrt{f_*(y)}\mathrm{d}W(y).$$

The distribution of $\eta(x)$ is $\mathcal{N}(0, \mathbf{E}_0 M(\xi_0, x)^2)$, and we have the following convergence

$$\eta_T(x) \Longrightarrow \eta(x - \vartheta_0). \tag{2.23}$$

Remind that as that is shown in Section 2.1.2 $\widehat{u}_T = \sqrt{T}(\widehat{\vartheta}_T - \vartheta_0)$ converges in distribution to

$$\widehat{u} = -\frac{1}{I}\int_{-\infty}^{\infty} S_*'(y)\sqrt{f_*(y)}\mathrm{d}W(y).$$

We have

Lemma 2.2.2. *Let the conditions $\mathcal{A}_0$ and $\mathcal{A}$ be fulfilled, then $(\eta_T(x_1),...,\eta_T(x_k),\widehat{u}_T)$ is asymptotically normal:*

$$\mathcal{L}\left(\eta_T(x_1),...,\eta_T(x_k),\widehat{u}_T\right) \Longrightarrow \mathcal{L}\left(\eta(x_1-\vartheta_0),...,\eta(x_k-\vartheta_0),\widehat{u}\right),$$

for any $\mathbf{x} = \{x_1, x_2, ..., x_k\} \in \mathbb{R}^k$.

Proof. The second integral in (2.21) converges to zero, so we only need to verify the convergence for the part of Itô's integral. Let us denote for simplicity

$$\eta_T^0(x) = \frac{1}{\sqrt{T}}\int_0^T M(X_t-\vartheta_0, x)\mathrm{d}W_t.$$

It is sufficient to verify that for any $\mathbf{x} = \{x_1, x_2, ..., x_k\}$,

$$\left(\eta_T^0(x_1),...,\eta_T^0(x_k),\widehat{u}_T\right) \Longrightarrow \left(\eta(x_1),...,\eta(x_k),\widehat{u}\right). \tag{2.24}$$

Remind that $\widehat{u}_T$ can be defined as follows,

$$Z_T(\widehat{u}_T) = \sup_{u\in\mathbb{U}_T} Z_T(u), \quad \mathbb{U}_T = \{u : \vartheta + \frac{u}{\sqrt{T}} \in \Theta\}, \tag{2.25}$$

where

$$Z_T(u) = \frac{\mathrm{d}\mathbf{P}^T_{\vartheta+\frac{u}{\sqrt{T}}}}{\mathrm{d}\mathbf{P}^T_{\vartheta}}(X^T) = \exp\left\{u\Lambda_T - \frac{u^2}{2}I + r_T\right\}.$$

Here

$$\Lambda_T = \frac{1}{\sqrt{T}}\int_0^T \dot{S}_*(X_t-\vartheta_0)\mathrm{d}W_t = -\frac{1}{\sqrt{T}}\int_0^T S_*'(X_t-\vartheta_0)\mathrm{d}W_t$$

and $r_T \longrightarrow 0$. It was proved in Kutoyants [28], Theorem 2.8 that $Z_T(\cdot)$ converges in distribution to $Z(\cdot)$, where

$$Z(u) = \exp\left\{u\Lambda - \frac{u^2}{2}I\right\},$$

with Λ is a r.v. with normal distribution $\mathcal{N}(0, I)$, which can be written as

$$\Lambda = -\int_{-\infty}^{\infty} S_*'(y)\sqrt{f(y)}\mathrm{d}W(y).$$

Therefore

$$\widehat{u}_T \Longrightarrow \widehat{u} = \frac{\Lambda}{I}.$$

Take $\mathbf{u} = \{u_1, u_2, ..., u_m\}$. We have to verify that the joint finite-dimensional distribution of Y_T

$$Y_T = \left(\eta_T^0(x_1), \eta_T^0(x_2), ..., \eta_T^0(x_k), Z_T(u_1), Z_T(u_2), ..., Z_T(u_m)\right)$$

converges to the finite-dimensional distribution of Y

$$Y = (\eta(x_1), \eta(x_2), ..., \eta(x_k), Z(u_1), Z(u_2), ..., Z(u_m)).$$

Since that $r_T \longrightarrow 0$, we consider only the stochastic term Λ_T in $Z_T(u)$, so (2.24) is equivalent to

$$\left(\eta_T^0(x_1), \eta_T^0(x_2), ..., \eta_T^0(x_k), \Lambda_T\right) \Longrightarrow (\eta(x_1), \eta(x_2), ..., \eta(x_k), \Lambda). \tag{2.26}$$

Take $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_{k+1}\}$, and put

$$h(y, \mathbf{x}, \lambda) = \sum_{l=1}^{k} \lambda_l M(y, x_l) - \lambda_{k+1} S_*'(y).$$

We have

$$\begin{aligned}
\mathbf{E}_{\vartheta_0} h(\xi_{\vartheta_0} - \vartheta_0, \mathbf{x}, \lambda)^2 &= \mathbf{E}_0 h(\xi_0, \mathbf{x}, \lambda)^2 = \mathbf{E}_0 \left(\sum_{l=1}^{k} \lambda_l M(\xi_0, x_l) - \lambda_{k+1} S_*'(\xi_0) \right)^2 \\
&= \mathbf{E}_0 \left(\sum_{l=1}^{k} 2\lambda_l f_*(x_l) \frac{F_*(y) - 1\!\!1_{\{\xi_0 > x_l\}}}{f_*(\xi_0)} + \lambda_{k+1} S_*'(\xi_0) \right)^2 \\
&= \sum_{l=1}^{k} \sum_{m=1}^{k} 4\lambda_l \lambda_m f_*(x_l) f_*(x_m) \mathbf{E}_0 \left(\frac{(F_*(\xi_0) - 1\!\!1_{\{\xi_0 > x_l\}})(F_*(\xi_0) - 1\!\!1_{\{\xi_0 > x_m\}})}{f_*^2(\xi_0)} \right) \\
&\quad - \sum_{l=1}^{k} 4\lambda_l \lambda_{k+1} f_*(x_l) \mathbf{E}_0 \left(\frac{\left(F_*(\xi_0) - 1\!\!1_{\{\xi_0 > x_l\}}\right)}{f_*(\xi_0)} S_*'(\xi_0) \right) \\
&\quad + \lambda_{k+1}^2 \mathbf{E}_0 \left(S_*'(\xi_0)^2 \right) < \infty.
\end{aligned}$$

The law of large number gives us

$$\frac{1}{T} \int_0^T h(X_t - \vartheta_0, \mathbf{x}, \lambda)^2 \mathrm{d}t \longrightarrow \mathbf{E}_0 h(\xi_0, \mathbf{x}, \lambda)^2.$$

Moreover, the central limit theorem for stochastic integral gives us

$$\frac{1}{\sqrt{T}}\int_0^T h(X_t-\vartheta_0,\mathbf{x},\lambda)\mathrm{d}W_t \Longrightarrow \mathcal{N}\left(0,\mathbf{E}_0 h(\xi_0,\mathbf{x},\lambda)^2\right).$$

In addition $\sum\limits_{l=1}^{k}\lambda_l\eta(x_l)+\lambda_{k+1}\Lambda$ is a zero mean normal r.v. with variance

$$\begin{aligned}&\mathbf{E}_0\left(\sum_{l=1}^{k}\lambda_l\eta(x_l)+\lambda_{k+1}\Lambda\right)^2\\ &\quad=\sum_{l=1}^{k}\sum_{m=1}^{k}\lambda_l\lambda_m\mathbf{E}_0\left(\eta(x_l)\eta(x_m)\right)+2\sum_{l=1}^{k}\lambda_l\lambda_{k+1}\mathbf{E}_0(\eta(x_l)\Lambda)+\lambda_{k+1}^2\mathbf{E}_0(\Lambda)^2.\end{aligned}$$

Furthermore

$$\begin{aligned}&\mathbf{E}_0\left(\eta(x_l)\eta(x_m)\right)\\ &\quad=4f_*(x_l)f_*(x_l)\int_{-\infty}^{\infty}\frac{(F_*(y)-1\!\!1_{\{y>x_l\}})(F_*(y)-1\!\!1_{\{y>x_m\}})}{f_*(y)}\mathrm{d}y\\ &\quad=4f_*(x_l)f_*(x_l)\mathbf{E}_0\left(\frac{(F_*(\xi_0)-1\!\!1_{\{\xi_0>x_l\}})(F_*(\xi_0)-1\!\!1_{\{\xi_0>x_m\}})}{f_*^2(\xi_0)}\right),\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}_0(\eta(x_l)\Lambda)&=-2f_*(x_l)\int_{-\infty}^{\infty}(F_*(y)-1\!\!1_{\{y>x_l\}})S_*'(y)\mathrm{d}y\\ &=-2f_*(x_l)\mathbf{E}_0\left(\frac{F_*(\xi_0)-1\!\!1_{\{\xi_0>x_l\}}}{f_*(\xi_0)}S_*'(\xi_0)\right),\end{aligned}$$

$$\mathbf{E}_0(\Lambda)^2=\int_{-\infty}^{\infty}S_*'(y)^2f_*(y)\mathrm{d}y=\mathbf{E}_0\left(S_*'(\xi_0)^2\right).$$

We find that

$$\mathbf{E}_{\vartheta_0}h(\xi_{\vartheta_0}-\vartheta_0,\mathbf{x},\lambda)^2=\mathbf{E}_0h(\xi_0,\mathbf{x},\lambda)^2=\mathbf{E}_0\left(\sum_{l=1}^{k}\lambda_l\eta(x_l)+\lambda_{k+1}\Lambda\right)^2.$$

This yields that

$$\sum_{l=1}^{k}\lambda_l\eta_T^0(x_l)+\lambda_{k+1}\Lambda_T\Longrightarrow\sum_{l=1}^{k}\lambda_l\eta(x_l)+\lambda_{k+1}\Lambda.$$

Thus we have (2.24) and the lemma is proved.

Lemma 2.2.3. *Let the conditions $\mathcal{A}_0$ and $\mathcal{A}$ be fulfilled, then*

$$\mathcal{L}\left\{\int_{-\infty}^{\infty}\left(\eta_T^0(x)+\widehat{u}_T f_*'(x)\right)^2 \mathrm{d}x\right\} \Longrightarrow \mathcal{L}\left\{\int_{-\infty}^{\infty}\left(\eta(x)+\widehat{u} f_*'(x)\right)^2 \mathrm{d}x\right\}$$

Proof. Denote $\zeta_T(x) = \eta_T^0(x) + \widehat{u}_T f_*'(x)$ and $\zeta(x) = \eta(x) + \widehat{u} f_*'(x)$, we prove the following properties

i) $\forall L > 0$, for $x, y \in [-L, L]$ and $|x - y| \le 1$, there exists a constant C depending on L, such that

$$\mathbf{E}_{\vartheta_0}|\zeta_T(x)^2 - \zeta_T(y)^2|^2 \le C|x-y|. \tag{2.27}$$

ii) $\forall \varepsilon > 0,\ \exists L > 0$, such that

$$\mathbf{E}_{\vartheta_0}\int_{\{|x|>L\}} \zeta_T(x)^2 \mathrm{d}x < \varepsilon, \quad \forall T > 0. \tag{2.28}$$

In fact i) and Lemma 2.2.2 yield the convergence in every bounded set $[-L, L]$:

$$\mathcal{L}\Big\{\int_{-L}^{L}\zeta_T(x)^2\mathrm{d}x\Big\} \Longrightarrow \mathcal{L}\Big\{\int_{-L}^{L}\zeta(x)^2\mathrm{d}x\Big\}.$$

Thus i) and ii) along with and Lemma 2.2.2 give us the result of the lemma.

First we prove i). We have

$$\mathbf{E}_{\vartheta_0}\left(\zeta_T(x)^2\right) \le 2\mathbf{E}_{\vartheta_0}\eta_T^0(x)^2 + 2f(x)^2\mathbf{E}_{\vartheta_0}\widehat{u}_T^2 \le C.$$

$$\begin{aligned}
&\mathbf{E}_{\vartheta_0}\left|\zeta_T(x)^2 - \zeta_T(y)^2\right|^2 \\
&\quad = \mathbf{E}_{\vartheta_0}\left(|\zeta_T(x)+\zeta_T(y)|^2|\zeta_T(x)-\zeta_T(y)|^2\right) \\
&\quad \le C\mathbf{E}_{\vartheta_0}|\zeta_T(x)-\zeta_T(y)|^2 \\
&\quad \le C(f'(x)-f'(y))^2\mathbf{E}_{\vartheta_0}|\widehat{u}_T|^2 + \mathbf{E}_{\vartheta_0}|(\eta_T^0(x)-\eta_T^0(y))|^2.
\end{aligned} \tag{2.29}$$

For the first part, let us recall the following result, given in Kutoyants [28], page 119: for any $p > 0,\ R > 0$, chosen N sufficiently large, we have

$$\mathbf{P}_{\vartheta_0}^T\left\{|\widehat{u}_T|^p > R\right\} \le \frac{C_N}{R^{N/p}}.$$

Let us denote $F_T(u)$ the distribution of $|\widehat{u}_T|$, we have

$$\begin{aligned}
\mathbf{E}_{\vartheta_0}|\widehat{u}_T|^p &= \int_0^{\infty} u^p \mathrm{d}F_T(u) \le 1 - \int_1^{\infty} u^p \mathrm{d}[1-F_T(u)] \\
&\le 1 - [1 - F_T(1)] + p\int_1^{\infty} u^{p-1}\frac{C_N}{u^{N/p}}\mathrm{d}u \le C.
\end{aligned} \tag{2.30}$$

Remind that under the condition $\mathcal{A}_1$, S_* and f_* are sufficiently smooth. Thus for $x, y \in [-L, L]$ we have

$$|f_*(x) - f_*(y)| \;\; = \;\; |f_*'(z)(x-y)| \leq C|x-y|,$$

and

$$|f_*'(x) - f_*'(y)| = |f_*''(z)(x-y)| = \left|4f(z)S_*^2(z) + 2f_*(z)S_*'(z)\right| |x-y| \leq C|x-y|.$$

So we have

$$(f_*'(x) - f_*'(y))^2 \mathbf{E}_{\vartheta_0} |\widehat{u}_T|^2 \leq C|x-y|^2.$$

For the second part in (2.39), note that

$$\begin{aligned}
&\mathbf{E}_{\vartheta_0} |(\eta_T^0(x) - \eta_T^0(y))|^2 \\
&\quad = C_1 \mathbf{E}_{\vartheta_0} \left(\frac{1}{\sqrt{T}} \int_0^T (M(X_t - \vartheta_0, x) - M(X_t - \vartheta_0, y)) \mathrm{d}W_t \right)^2 \\
&\quad \leq \frac{C_1}{T} \int_0^T \mathbf{E}_{\vartheta_0} \left(M(X_t - \vartheta_0, x) - M(X_t - \vartheta_0, y) \right)^2 \mathrm{d}t \\
&\quad = C_1 \mathbf{E}_0 \left(M(\xi_0, x) - M(\xi_0, y) \right)^2.
\end{aligned}$$

Suppose that $x \leq y$,

$$\begin{aligned}
\mathbf{E}_0 \left(M(\xi_0, x) - M(\xi_0, y) \right)^2 &= \int_{-\infty}^x \left(2\frac{F_*(z)}{f_*(z)} (f_*(x) - f_*(y)) \right)^2 f_*(z) \mathrm{d}z \\
&+ \int_x^y \left(2\frac{1}{f_*(z)} \left((1 - F_*(z)) f_*(x) + F_*(z) f_*(y) \right) \right)^2 f_*(z) \mathrm{d}z \\
&+ \int_y^\infty \left(2\frac{1 - F_*(z)}{f_*(z)} (f_*(x) - f_*(y)) \right)^2 f_*(z) \mathrm{d}z \\
&\leq C_1 (x-y)^4 + C_2 (x-y) + C_3 (x-y)^2 \leq C(y-x).
\end{aligned}$$

Similar result holds for $x > y$. Then we obtain

$$\mathbf{E}_{\vartheta_0} \left| \eta_T^0(x)^2 - \eta_T^0(y)^2 \right|^2 \leq C|x-y|, \quad x, y \in \mathbb{R}.$$

Thus we have

$$\mathbf{E}_{\vartheta_0} \left| \zeta_T(x)^2 - \zeta_T(y)^2 \right|^2 \leq C|x-y|.$$

Now we prove ii). As in Lemma 2.2.1, we have deduced that

$$\mathbf{E}_0 M(\xi_0, x)^2 \leq C\mathrm{e}^{-2\gamma x}, \quad \text{for } x > A.$$

Thus for $L > A$,

$$\mathbf{E}_{\vartheta_0}\int_L^\infty \left(\eta_T^0(x)\right)^2 \mathrm{d}x = \mathbf{E}_{\vartheta_0}\int_L^\infty \left(\frac{1}{\sqrt{T}}\int_0^T M(X_t-\vartheta_0,x)\mathrm{d}W_t\right)^2 \mathrm{d}x$$
$$\leq C\int_L^\infty \mathbf{E}_0 M(\xi_0,x)^2\mathrm{d}x \leq C\int_L^\infty \mathrm{e}^{-2\gamma x}\mathrm{d}x \leq C\mathrm{e}^{-2\gamma L}.$$

Note that $f_*'(x) = 2S_*(x)f_*(x)$ and along with (2.30) we have

$$\begin{aligned}\int_L^\infty \mathbf{E}_{\vartheta_0}\left(\eta_T^0(x) - f_*'(x)\widehat{u}_T\right)^2\mathrm{d}x &\leq \int_L^\infty \left(2\mathbf{E}_{\vartheta_0}\eta_T(x)^2 + 2f_*'(x)\mathbf{E}_{\vartheta_0}\widehat{u}_T^2\right)\mathrm{d}x \\ &\leq \int_L^\infty C\mathrm{e}^{-2\gamma x}\mathrm{d}x = C\mathrm{e}^{-2\gamma L}.\end{aligned}$$

For any $\varepsilon > 0$, take $L = -\frac{\ln(\varepsilon/C)}{2\gamma} \vee A$, hence we obtain (2.28).

Proof of Theorem 2.2.1.

We have

$$\begin{aligned}\delta_T &= T\int_{-\infty}^\infty (\widehat{f}_T(x) - f_*(x-\widehat{\vartheta}_T))^2\mathrm{d}x \\ &= T\int_{-\infty}^\infty \left((\widehat{f}_T(x) - f_*(x-\vartheta_0)) + (f_*(x-\vartheta_0) - f_*(x-\widehat{\vartheta}_T))\right)^2 \mathrm{d}x \\ &= \int_{-\infty}^\infty \left(\sqrt{T}(\widehat{f}_T(x) - f_*(x-\vartheta_0)) + \sqrt{T}(\widehat{\vartheta}_T - \vartheta_0)f_*'(x-\widetilde{\vartheta}_T)\right)^2 \mathrm{d}x \\ &= \int_{-\infty}^\infty \left(\eta_T(x) + \widehat{u}_T f_*'(x-\widetilde{\vartheta}_T)\right)^2 \mathrm{d}x,\end{aligned}$$

where $\widetilde{\vartheta}_T$ is between ϑ_0 and $\widehat{\vartheta}_T$ which comes from the mean value theorem. Note that

$$\mathbf{E}_{\vartheta_0}\int_{-\infty}^\infty \left(\widehat{u}_T^2|f_*'(x-\widetilde{\vartheta}_T) - f_*'(x-\vartheta_0)|^2\right)\mathrm{d}x$$
$$= \mathbf{E}_{\vartheta_0}\int_{-\infty}^\infty \left(\widehat{u}_T^2 f_*''(x-\bar{\vartheta}_T)^2(\widetilde{\vartheta}_T - \vartheta_0)^2\right)\mathrm{d}x.$$

The smoothness of $S_*(\cdot)$ and so that of $f''(\cdot)$ give us the convergence

$$\mathbf{E}_{\vartheta_0}\int_{-\infty}^\infty \left(\widehat{u}_T^2|f_*'(x-\widetilde{\vartheta}_T) - f_*'(x-\vartheta_0)|^2\right)\mathrm{d}x \longrightarrow 0.$$

Applying Lemma 2.2.1 and Lemma 2.2.3 we get

$$\begin{aligned}
\delta_T &= \int_{-\infty}^{\infty} \left(\eta_T^0(x-\vartheta_0) + \widehat{u}_T f_*'(x-\vartheta_0)\right)^2 \mathrm{d}x + o(1) \\
&\Longrightarrow \int_{-\infty}^{\infty} \left(\eta(x-\vartheta_0) + \widehat{u} f_*'(x-\vartheta_0)\right)^2 \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(\eta(y) + \widehat{u} f_*'(y)\right)^2 \mathrm{d}y = \int_{-\infty}^{\infty} \left(\eta(y) + 2\widehat{u} S_*(y) f_*(y)\right)^2 \mathrm{d}y = \delta.
\end{aligned}$$

Note that the limit of the statistic δ does not depend on ϑ_0, and the test $\psi_T = 1\!\!1_{\{\delta_T \geq d_\varepsilon\}}$ with d_ε defined by

$$\mathbf{P}\Big(\delta \geq d_\varepsilon\Big) = \varepsilon$$

belongs to $\mathcal{K}_\varepsilon$.

2.2.2 The C-vM type test via the EDF

We introduce in this subsection the C-vM type test in using the EDF:

$$\widehat{F}_T(x) = \frac{1}{T}\int_0^T 1\!\!1_{\{X_t < x\}} \mathrm{d}t.$$

Let us define the statistic

$$\Delta_T = T\int_{-\infty}^{\infty} \Big(\widehat{F}_T(x) - F_*(x-\widehat{\vartheta}_T)\Big)^2 \mathrm{d}x,$$

and its limit in distribution

$$\Delta = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(2\frac{F_*(y)F_*(x) - F_*(y\wedge x)}{\sqrt{f_*(y)}} - \frac{1}{I} f_*(x) S_*'(y)\sqrt{f_*(y)}\right) \mathrm{d}W(y)\right)^2 \mathrm{d}x. \tag{2.31}$$

This convergence will be proved later. Thus we propose the C-vM type test

$$\Psi_T = 1\!\!1_{\{\Delta_T > D_\varepsilon\}},$$

where D_ε is the solution of the equation

$$\mathbf{P}\Big(\Delta \geq D_\varepsilon\Big) = \varepsilon. \tag{2.32}$$

We have the result

Theorem 2.2.2. *Under the conditions $\mathcal{ES}$, $\mathcal{A}_0$ and $\mathcal{A}$, the test $\Psi_T = 1\!\!I_{\{\Delta_T > D_\varepsilon\}}$ belongs to $\mathcal{K}_\varepsilon$ and is APF.*

Denote $\eta_T^F(x) = \sqrt{T}(\widehat{F}_T(x) - F_*(x-\vartheta_0))$ and

$$H(z,x) = 2\frac{F_*(z)F(x) - F_*(z\wedge x)}{f_*(z)}.$$

In Kutoyants [28] Theorem 4.6, the following equality is presented:

$$\begin{aligned}
\eta_T^F(x) &= \frac{2}{\sqrt{T}}\int_{X_0}^{X_T} \frac{F_*((z\wedge x)-\vartheta_0) - F_*(z-\vartheta_0)F_*(x-\vartheta_0)}{f_*(z-\vartheta_0)}\mathrm{d}z \\
&\quad -\frac{2}{\sqrt{T}}\int_0^T \frac{F_*((X_t\wedge x)-\vartheta_0) - F_*(X_t-\vartheta_0)F_*(x-\vartheta_0)}{f_*(X_t-\vartheta_0)}\mathrm{d}W_t \\
&= -\frac{1}{\sqrt{T}}\left(\int_0^{X_T} H(z-\vartheta_0, x-\vartheta_0)\mathrm{d}z - \int_0^{X_0} H(z-\vartheta_0, x-\vartheta_0)\mathrm{d}z\right) \\
&\quad +\frac{1}{\sqrt{T}}\int_0^T H(X_t-\vartheta_0, x-\vartheta_0)\mathrm{d}W_t. \qquad (2.33)
\end{aligned}$$

We present the following lemma

Lemma 2.2.4. *Let the condition $\mathcal{A}_0$ be fulfilled, then*

$$\int_{-\infty}^{\infty} \mathbf{E}_0\left(\int_0^{\xi_0} H(y,x)\mathrm{d}y\right)^2 \mathrm{d}x < \infty.$$

Proof. In applying (2.13) we have, for $x > A$,

$$1 - F_*(x) = C\int_x^{\infty} \exp\left(2\int_0^y S_*(r)\mathrm{d}r\right)\mathrm{d}y \leq C\mathrm{e}^{-2\gamma x},$$

and

$$\frac{1-F_*(x)}{f_*(x)} \leq C\int_x^{\infty} \mathrm{e}^{-2\gamma(y-x)}\mathrm{d}y \leq C.$$

For $x < -A$ we have $F_*(x) \leq C\mathrm{e}^{-2\gamma|x|}$ and we can write

$$\frac{F_*(x)}{f_*(x)} = C\int_{-\infty}^{x} \exp(2\int_x^y S_*(r)\mathrm{d}r)\mathrm{d}y \leq C.$$

So that for $x > A$,

$$
\begin{aligned}
&\mathbf{E}\left(\int_0^{\xi_0} H(z,x)\mathrm{d}z\right)^2 \\
&= 4\int_{-\infty}^{A} f_*(y)\left(\int_0^y (F_*(x)-1)\frac{F_*(z)}{f_*(z)}\mathrm{d}z\right)^2 \mathrm{d}y \\
&\quad +4\int_A^x f_*(y)\left(\int_0^y (F_*(x)-1)\frac{F_*(z)}{f_*(z)}\mathrm{d}z\right)^2 \mathrm{d}y \\
&\quad +4\int_x^\infty f_*(y)\left(\int_0^x (F_*(x)-1)\frac{F_*(z)}{f_*(z)}\mathrm{d}z + \int_x^y F_*(x)\frac{F_*(z)-1}{f_*(z)}\mathrm{d}z\right)^2 \mathrm{d}y.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_{-\infty}^{A} f_*(y)\left(\int_0^y (F_*(x)-1)\frac{F_*(z)}{f_*(z)}\mathrm{d}z\right)^2 \mathrm{d}y \\
&\quad = \int_{-\infty}^{A} f_*(y)\left(\int_0^y (1-F_*(x))\frac{F_*(z)}{f_*(z)}\mathrm{d}z\right)^2 \mathrm{d}y \\
&\quad \le (1-F_*(x))^2\int_{-\infty}^{A} y^2 f_*(y)\mathrm{d}y \le C(1-F_*(x))^2 \le C\mathrm{e}^{-4\gamma x},
\end{aligned}
$$

Further

$$
\begin{aligned}
&\int_A^x f_*(y)\left(\int_0^y (1-F_*(x))\frac{F_*(z)}{f_*(z)}\mathrm{d}z\right)^2 \mathrm{d}y \le \int_A^x f_*(y)\left(\int_0^y \frac{1-F_*(x)}{f_*(z)}\mathrm{d}z\right)^2 \mathrm{d}y \\
&\quad \le C\int_A^x f_*(y)\left(\int_0^y \int_x^\infty \mathrm{e}^{-2\gamma(u-z)}\mathrm{d}u\mathrm{d}z\right)^2 \mathrm{d}y \\
&\quad \le C\int_A^x f_*(y)\mathrm{e}^{-2\gamma x}\left(1-\mathrm{e}^{2\gamma y}\right)\mathrm{d}y \le C(1+x)\mathrm{e}^{-2\gamma x},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_x^\infty f_*(y)\left(\int_0^x (1-F_*(x))\frac{F_*(z)}{f_*(z)}\mathrm{d}z\right)^2 \mathrm{d}y \\
&\quad = \int_x^\infty f_*(y)\left(\left(\int_0^A + \int_A^x\right)(1-F_*(x))\frac{F_*(z)}{f_*(z)}\mathrm{d}z\right)^2 \mathrm{d}y \\
&\quad \le C\int_x^\infty f_*(y)\left((1-F_*(x)) + \int_A^x \frac{1-F_*(x)}{f_*(z)}\mathrm{d}z\right)^2 \mathrm{d}y \\
&\quad \le C\int_x^\infty f_*(y)(1+\mathrm{e}^{-4\gamma x})\mathrm{d}y \le C\mathrm{e}^{-2\gamma x},
\end{aligned}
$$

and

$$\int_x^\infty f_*(y)\left(\int_x^y F_*(x)\frac{F_*(z)-1}{f_*(z)}\mathrm{d}z\right)^2\mathrm{d}y$$
$$\leq C\int_x^\infty (y-x)f_*(y)\mathrm{d}y \leq C(1+x)\mathrm{e}^{-2\gamma x},$$

thus we have

$$\mathbf{E}_0\left(\int_0^{\xi_0} H(z,x)\mathrm{d}z\right)^2 \leq C\mathrm{e}^{-\gamma x}, \quad x>A. \tag{2.34}$$

Similarly we get

$$\mathbf{E}_0\left(\int_0^{\xi_0} H(z,x)\mathrm{d}z\right)^2 \leq C\mathrm{e}^{-\gamma |x|}, \quad x<-A.$$

and

$$\mathbf{E}_0\left(\int_0^{\xi_0} H(z,x)\mathrm{d}z\right)^2 \leq C, \quad x\in[-A,A].$$

We obtain finally

$$\int_{-\infty}^\infty \mathbf{E}_0\left(\int_0^{\xi_0} H(y,x)\mathrm{d}y\right)^2 \mathrm{d}x<\infty.$$

This inequality allows us to deduce the following bounds

$$\mathbf{E}_{\vartheta_0}H(\xi_{\vartheta_0}-\vartheta_0,x)^2=\mathbf{E}_0H(\xi_0,x)^2<\infty, \tag{2.35}$$

and

$$\mathbf{E}_{\vartheta_0}\left(\int_0^{\xi_{\vartheta_0}-\vartheta_0} H(z,x)\mathrm{d}z\right)^2 = \mathbf{E}_0\left(\int_0^{\xi_0} H(z,x)\mathrm{d}z\right)^2 \leq \infty, \quad |x|>A. \tag{2.36}$$

Hence we get the asymptotic normality of $\eta_T^F(x)$:

$$\eta_T^F(x)\Longrightarrow \eta^F(x-\vartheta_0)\sim\mathcal{N}(0,\mathbf{E}_0\left(H(\xi_0,x-\vartheta_0)\right)^2),$$

where we define

$$\eta^F(x)=\int_{-\infty}^\infty H(y,x)\sqrt{f(y)}\mathrm{d}W(y).$$

As in Lemma 2.2.2 and Lemma 2.2.3, if conditions $\mathcal{A}$ and $\mathcal{A}_0$ hold, we show the convergence of the vector $(\eta_T^F(x_1),...,\eta_T^F(x_k),\widehat{u}_T)$:

Lemma 2.2.5. *Let the conditions $\mathcal{A}_0$ and $\mathcal{A}$ be fulfilled, then*

$$\mathcal{L}_{\vartheta_0}\left(\eta_T^F(x_1),...,\eta_T^F(x_k),\widehat{u}_T\right)\Longrightarrow\mathcal{L}_{\vartheta_0}\left(\eta^F(x_1-\vartheta_0),...,\eta_T^F(x_k-\vartheta_0),\widehat{u}\right)$$

for any $\mathbf{x}=\{x_1,x_2,...,x_k\}\in\mathbb{R}^k$.

Proof. We omit the proof since that it is similar as Lemma 2.2.2.

Let us define

$$\widetilde{\eta}_T^F(x)=\frac{1}{\sqrt{T}}\int_0^T H(X_t-\vartheta_0,x)\mathrm{d}W_t$$

we prove that

Lemma 2.2.6. *Let the conditions $\mathcal{A}$ and $\mathcal{A}_0$ be fulfilled, then*

$$\mathcal{L}_{\vartheta_0}\{\int_{-\infty}^{\infty}\left(\widetilde{\eta}_T^F(x)+\widehat{u}_Tf_*(x)\right)^2\mathrm{d}x\}\quad\Longrightarrow\quad\mathcal{L}\left\{\int_{-\infty}^{\infty}\left(\eta^F(x)+\widehat{u}f_*(x)\right)^2\mathrm{d}x\right\}.$$

Proof. Denote $\zeta_T^F(x)=\widetilde{\eta}_T^F(x)-\widehat{u}_Tf_*(x)$. Similar as Lemma 2.2.3, we need to verify

i) $\forall\ L>0$, for $x,y\in[-L,L]$ and $|x-y|\leq 1$, there exists C depending on L such that

$$\mathbf{E}_{\vartheta_0}|\zeta_T^F(x)^2-\zeta_T^F(y)^2|^2\leq C|x-y|^{1/2}. \tag{2.37}$$

ii)$\forall\ \varepsilon>0,\ \exists L>0$, such that

$$\mathbf{E}_{\vartheta_0}\int_{\{|x|>L\}}\zeta_T^F(x)^2\mathrm{d}x<\varepsilon,\quad\forall T>0. \tag{2.38}$$

Firstly we prove i). Note that

$$\begin{aligned}&\mathbf{E}_{\vartheta_0}\left|\zeta_T^F(x)^2-\zeta_T^F(y)^2\right|^2\\&\quad\leq C\left((f_*(x)-f_*(y))^4\mathbf{E}_{\vartheta_0}|\widehat{u}_T|^4+\mathbf{E}_{\vartheta_0}|(\widetilde{\eta}_T^F(x)-\widetilde{\eta}_T^F(y))|^4\right)^{1/2}.\end{aligned}$$

Moreover,

$$\begin{aligned}&\mathbf{E}_{\vartheta_0}|(\widetilde{\eta}_T^F(x)-\widetilde{\eta}_T^F(y))|^4\\&\quad\leq C_1T^{-2}\mathbf{E}_{\vartheta_0}\left(\frac{1}{\sqrt{T}}\int_0^{\xi_{\vartheta_0}-\vartheta_0}(H(z,x)-H(z,y))\mathrm{d}z\right)^4\\&\qquad+C_2T^{-5/4}\mathbf{E}_{\vartheta_0}\left(\frac{1}{\sqrt{T}}\int_0^T(H(X_t-\vartheta_0,x)-H(X_t-\vartheta_0,y))\mathrm{d}W_t\right)^4\\&\quad\leq C_1T^{-2}\mathbf{E}\left(\frac{1}{\sqrt{T}}\int_0^{\xi_0}(H(z,x)-H(z,y))\mathrm{d}z\right)^4+C_2T^{-1/4}\mathbf{E}\left(H(\xi_0,x)-H(\xi_0,y)\right)^4.\end{aligned}$$

Suppose that $x \leq y$,

$$
\begin{aligned}
&\mathbf{E}_{\vartheta_0}\left(H(X_t,x)-H(X_t,y)\right)^4\\
&\quad=\int_{-\infty}^{x}\frac{F_*(z)}{f_*(z)}(F_*(x)-F_*(y))\mathrm{d}z+\int_{y}^{\infty}\frac{F_*(z)-1}{f_*(z)}(F_*(x)-F_*(y))\mathrm{d}z\\
&\quad\quad+\int_{x}^{y}\frac{1}{f_*(z)}\left(F_*(z)(F_*(x)-F_*(y))+(F_*(z)-F_*(x))\right)\mathrm{d}z\\
&\quad\leq C_1(x-y)^4+C_3(x-y)^4+C_2(x-y)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{E}_{\vartheta_0}\left(\int_0^{\xi}H(z,x)-H(z,y)\mathrm{d}z\right)^4\\
&\quad=2\int_{-\infty}^{x}f_*(s)\left(\int_0^{s}\frac{F_*(z)}{f_*(z)}(F_*(x)-F_*(y))\mathrm{d}z\right)^4\mathrm{d}s\\
&\quad\quad+2\int_{x}^{y}f_*(s)\left(\int_x^{s}\frac{F_*(z)-F_*(x)+F_*(z)(F_*(x)-F_*(y))}{f_*(z)}\mathrm{d}z\right)^4\mathrm{d}s\\
&\quad\quad+8\int_{y}^{\infty}f_*(s)\left(\int_x^{y}\frac{F_*(z)-F_*(x)+F_*(z)(F_*(x)-F_*(y))}{f_*(z)}\mathrm{d}z\right)^4\mathrm{d}s\\
&\quad\quad+8\int_{y}^{\infty}f_*(s)\left(\int_y^{s}\frac{F_*(z)-1}{f_*(z)}(F_*(x)-F_*(y))\mathrm{d}z\right)^4\mathrm{d}s\\
&\quad\leq C_1(y-x)^4+C_2(y-x)+C_3(y-x)^4+C_4(y-x)^4.
\end{aligned}
$$

Similar result for $x \geq y$. We obtain finally

$$
\mathbf{E}_{\vartheta_0}\left|\widetilde{\eta}_T^F(x)-\widetilde{\eta}_T^F(y)\right|^4\leq C|x-y|,
$$

Therefore,

$$
\mathbf{E}_{\vartheta_0}\left|\zeta_T^F(x)^2-\zeta_T^F(y)^2\right|^2\leq|x-y|^{1/2}.
$$

Now we prove ii). Thanks to Lemma 2.2.4, we have

$$
\mathbf{E}_{\vartheta_0}\left|\widetilde{\eta}_T^F(x)\right|^2\leq C\mathrm{e}^{-\gamma|x|},\quad x>A. \tag{2.39}
$$

Hence for $L > A$,

$$
\begin{aligned}
&\int_L^{\infty}\mathbf{E}_{\vartheta_0}\left(\widetilde{\eta}_T^F(x)-f_*(x)\widehat{u}_T\right)^2\mathrm{d}x\\
&\quad\leq\int_L^{\infty}\left(2\mathbf{E}_{\vartheta_0}\eta_T^F(x)^2+2f_*(x)^2\mathbf{E}_{\vartheta_0}\widehat{u}_T^2\right)\mathrm{d}x\\
&\quad\leq\int_L^{\infty}C\mathrm{e}^{-\gamma x}\mathrm{d}x=C\mathrm{e}^{-\gamma L}.
\end{aligned}
$$

For any $\varepsilon > 0$, take $L = -\frac{\ln(\varepsilon/C)}{\gamma} \vee A$, we have (2.38).

Proof of Theorem 2.2.2 We have

$$\begin{aligned}
\Delta_T &= T\int_{-\infty}^{\infty} (\widehat{F}_T(x) - F_*(x, \widehat{\vartheta}_T))^2 \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left[\sqrt{T}(\widehat{F}_T(x) - F_*(x-\vartheta_0)) + \sqrt{T}(\widehat{\vartheta}_T - \vartheta_0)\dot{F}_*(x - \widetilde{\vartheta}_T)\right]^2 \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left[\eta_T^F(x) + \widehat{u}_T f_*(x - \widetilde{\vartheta}_T)\right]^2 \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left[\eta_T^F(x) + \widehat{u}_T f_*(x - \vartheta_0)\right]^2 \mathrm{d}x + o(1) \\
&\Longrightarrow \int_{-\infty}^{\infty} \left[\eta^F(x-\vartheta_0) + \widehat{u} f_*(x - \vartheta_0)\right]^2 \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(\eta^F(y) + \widehat{u} f_*(y)\right)^2 \mathrm{d}y = \Delta.
\end{aligned}$$

Note that the limit of the statistic Δ does not depend on ϑ_0, the test $\Psi_T = 1\!\!\mathrm{I}_{\{\Delta_T \geq D_\varepsilon\}}$ with D_ε the solution of

$$\mathbf{P}\left(\Delta \geq D_\varepsilon\right) = \varepsilon$$

belongs to $\mathcal{K}_\varepsilon$ and it is APF.

2.2.3 Consistency

In this section we discuss the consistency of the proposed tests. We study the tests statistics under the alternative hypothesis that is defined as

$$\mathcal{H}_1: \; S(\cdot) \notin \overline{\mathcal{S}(\Theta)},$$

where $\overline{\mathcal{S}(\Theta)} = \{S_*\left(x - \vartheta\right), \vartheta \in [\alpha, \beta]\}$.

Under this hypothesis we have:

Proposition 2.2.1. *Let all drift coefficients under alternative satisfy the conditions* $\mathcal{ES}$, $\mathcal{A}_0$, *and* $\mathcal{A}$, *then for any* $S(\cdot) \notin \overline{\mathcal{S}(\Theta)}$ *we have*

$$\mathbf{P}_S\left(\delta_T > d_\varepsilon\right) \longrightarrow 1,$$

and

$$\mathbf{P}_S\left(\Delta_T > D_\varepsilon\right) \longrightarrow 1.$$

Proof. Remind that under hypothesis $\mathcal{H}_1$, the MLE $\widehat{\vartheta}_T$ converges to the point which minimizes the distance

$$D(\vartheta) = \mathbf{E}_S \left(S_*(\xi - \vartheta) - S(\xi)\right)^2,$$

where ξ is the random variable of invariant density $f_S(x)$ (See Kutoyants [28], Proposition 2.36):

$$\widehat{\vartheta}_T \longrightarrow \widehat{\vartheta}_0 = \arg \inf_{\vartheta \in \Theta} D(\vartheta).$$

In addition, denoted by $\|\cdot\|$ the norm in L^2, we have

$$\begin{aligned}\mathbf{P}_S\left(\delta_T > d_\varepsilon\right) &= \mathbf{P}_S\left(\left\|\sqrt{T}\left(\widehat{f}_T(\cdot) - f(\cdot, \widehat{\vartheta}_T)\right)\right\|^2 > d_\varepsilon\right)\\ &\geq \mathbf{P}_S\left(\left\|\sqrt{T}\left(f_S(x) - f(x - \widehat{\vartheta}_T)\right)\right\|^2 - \left\|\sqrt{T}\left(\widehat{f}_T(x) - f_S(x)\right)\right\|^2 > d_\varepsilon\right).\end{aligned}$$

Hence

$$\begin{aligned}\left\|\sqrt{T}\left(f_S(x) - f(x - \widehat{\vartheta}_T)\right)\right\|^2 &= T\int_{-\infty}^{\infty}\left(f_S(x) - f(x - \widehat{\vartheta}_T)\right)^2 \mathrm{d}x\\ &= T\int_{-\infty}^{\infty}\left(f_S(x) - f(x - \widehat{\vartheta}_0) + o(1)\right)^2 \mathrm{d}x\\ &= (C + o(1))T \longrightarrow \infty, \quad \text{as } T \longrightarrow \infty.\end{aligned}$$

Moreover

$$\begin{aligned}\mathbf{E}_S\left(\left\|\sqrt{T}\left(\widehat{f}_T(x) - f_S(x)\right)\right\|^2\right) &= \mathbf{E}_S\left(T\int_{-\infty}^{\infty}\left(\widehat{f}_T(x) - f_S(x)\right)^2 \mathrm{d}x\right)\\ &\leq C\int_{-\infty}^{\infty}\mathbf{E}_S(\eta_T(x)^2)\mathrm{d}x \leq C\int_{-\infty}^{\infty}\mathrm{e}^{-2\gamma|x|}\mathrm{d}x < \infty.\end{aligned}$$

Finally we have the result for δ_T:

$$\begin{aligned}&\mathbf{P}_S\left(\delta_T > d_\varepsilon\right)\\ &\quad\geq \mathbf{P}_S\left(\left\|\sqrt{T}\left(f_S(x) - f(x - \widehat{\vartheta}_T)\right)\right\|^2 - \left\|\sqrt{T}\left(\widehat{f}_T(x) - f_S(x)\right)\right\|^2 > d_\varepsilon\right) \longrightarrow 1.\end{aligned}$$

A similar result can be obtained for Δ_T.

2.2.4 C-vM test via the MDE

In this part, we discuss the test where the unknown parameter is estimated by the method of the minimum distance. We consider always the following equation

$$\mathrm{d}X_t = S(X_t)\mathrm{d}t + \mathrm{d}W_t, \quad X_0,\ 0 \le t \le T \tag{2.40}$$

and we have to test the following basic hypothesis

$$\mathcal{H}_0 \quad : \qquad S\left(x\right) = S_*\left(x-\vartheta\right), \quad \vartheta \in \Theta = \left(\alpha,\beta\right),$$

against the alternative

$$\mathcal{H}_1 \quad : \qquad S\left(x\right) \notin \overline{\mathcal{S}\left(\Theta\right)} = \left\{S_*\left(x-\vartheta\right), \vartheta \in \left[\alpha,\beta\right]\right\}.$$

Let us consider the following test. The unknown parameter is estimated by the MDE ϑ_T^* as follows

$$\vartheta_T^* = \arg\inf_{\theta\in\Theta} \|\widehat{F}_T(\cdot) - F_*(\cdot,\theta)\|, \tag{2.41}$$

where $\|\cdot\|$ is the norm in L^2 space:

$$\|h(\cdot)\| = \left(\int_{-\infty}^{\infty} h(x)^2\mathrm{d}x\right)^{1/2}.$$

Thus the test is defined as

$$\varphi_T = 1\!\!\mathrm{I}_{\{\omega_T^2 > e_\varepsilon\}}, \quad \omega_T^2 = T\left\|\widehat{F}_T(\cdot) - F_*(\cdot,\theta_T^*)\right\|^2,$$

where e_ε is the solution of the equation

$$\mathbf{P}\left(\omega^2 > e_\varepsilon\right) = \varepsilon$$

with

$$\omega^2 := \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} 2\frac{F_*(x)F_*(y) - F_*(x\wedge y)}{\sqrt{f_*(y)}} - J^{-1}f_*(x)R(y)\sqrt{f_*(y)}\mathrm{d}W(y)\right)^2\mathrm{d}x$$

and

$$R(y) = 2f_*(y)\int_{-\infty}^{\infty}\left(1-F_*(z)\right)\frac{F_*(z) - 1\!\!\mathrm{I}_{\{z>y\}}}{f_*(z)}\mathrm{d}z, \quad J = \int_{-\infty}^{\infty} f_*(x)^2\mathrm{d}x.$$

We have the following result

Theorem 2.2.3. *Let the conditions $\mathcal{ES}$, $\mathcal{A}_0$ and $\mathcal{A}$ be fulfilled, then the test φ_T belongs to $\mathcal{K}_\varepsilon$ and it is APF.*

To prove this theorem, we introduce firstly two lemmas.

Lemma 2.2.7. *Under the conditions $\mathcal{A}_0$ and $\mathcal{A}$, the MDE is consistent: for any $\nu > 0$*

$$\lim_{T\to\infty} \mathbf{P}_{\vartheta_0}\left(|\vartheta_T^* - \vartheta_0| > \nu\right) = 0,$$

and asymptotically normal

$$\sqrt{T}(\vartheta_T^* - \vartheta_0) \Longrightarrow \mathcal{N}\left(0, J^{-2}\mathbf{E}_0 R(\xi_0)^2\right).$$

Proof. Let us denote

$$g(\nu,\vartheta) = \inf_{|\vartheta-\theta|>\nu} \left\| F_*(x-\vartheta) - F_*(x-\theta)\right\|, \quad g(\nu) = \inf_{\vartheta} g(\nu,\vartheta).$$

Note that under the condition $\mathcal{A}_0$, there exists a constant $\kappa > 0$ such that

$$g(\nu,\vartheta) = \inf_{|\vartheta-\theta|>\nu} \left(\int_{-\infty}^{\infty} \left(f_*(x-\widetilde{\vartheta})(\vartheta-\theta)\right)^2 \mathrm{d}x\right)^{1/2} > \kappa|\nu|, \tag{2.42}$$

thus $g(\nu) > \kappa|\nu|$. For the consistency, we apply Chebyshev's inequality:

$$\begin{aligned}
&\mathbf{P}_{\vartheta_0}\left(|\vartheta_T^* - \vartheta_0| > \nu\right) \\
&\quad = \mathbf{P}_{\vartheta_0}\left(\inf_{|\theta-\vartheta_0|\le\nu} \left\|\widehat{F}_T(x) - F_*(x-\theta)\right\| > \inf_{\sqrt{T}|\theta-\vartheta_0|>\nu} \left\|\widehat{F}_T(x) - F_*(x-\theta)\right\|\right) \\
&\quad \le \mathbf{P}_{\vartheta_0}\Bigg(\inf_{|\theta-\vartheta_0|\le\nu} \left(\left\|\widehat{F}_T(x) - F_*(x-\vartheta_0)\right\| + \left\|F(x-\vartheta_0) - F_*(x-\theta)\right\|\right) \\
&\qquad\qquad > \inf_{|\theta-\vartheta_0|>\nu} \left(\left\|F_*(x-\vartheta_0) - F_*(x-\theta)\right\| - \left\|\widehat{F}_T(x) - F_*(x-\vartheta_0)\right\|\right)\Bigg) \\
&\quad = \mathbf{P}_{\vartheta_0}\left(2\left\|\widehat{F}_T(x) - F_*(x-\vartheta_0)\right\| > \inf_{|\theta-\vartheta_0|>\nu} \left\|F_*(x-\vartheta_0) - F_*(x-\theta)\right\|\right) \\
&\quad = \mathbf{P}_{\vartheta_0}\left(2\left\|\eta_T^F(x)\right\| > \sqrt{T}g(\nu)\right) \le \frac{4\mathbf{E}_{\vartheta_0}\left\|\eta_T^F(x)\right\|^2}{g(\nu)^2 T} \longrightarrow 0, \quad \text{as } T \longrightarrow 0.
\end{aligned}$$

Here we have applied the inequality for norms:

$$\|h\| - \|g\| \le \|h+g\| \le \|h\| + \|g\|,$$

the boundedness of $\mathbf{E}_{\vartheta_0}\left\|\eta_T^F(x)\right\|^2$ can be deduced owing to Lemma 2.2.4.

Now we prove the asymptotical normality. Note that under the regularity conditions, the invariant distribution function is sufficiently smooth. Thus the MDE ϑ_T^* can be written as the solution of the following equation

$$\frac{\partial}{\partial\theta}\left\|\widehat{F}_T(x) - F_*(x-\theta)\right\| = \int_{-\infty}^{\infty} 2\left(\widehat{F}_T(x) - F_*(x-\theta)\right)\dot{F}_*(x-\theta)\mathrm{d}x = 0,$$

which deduces that

$$\int_{-\infty}^{\infty}\left(\left(\widehat{F}_T(x)-F_*(x-\vartheta_0)\right)+\left(F_*(x-\vartheta_0)-F_*(x-\vartheta_T^*)\right)\right)f_*(x-\vartheta_T^*)\mathrm{d}x$$
$$=\int_{-\infty}^{\infty}\left(\left(\widehat{F}_T(x)-F_*(x-\vartheta_0)\right)-(\vartheta_T^*-\vartheta_0)\dot{F}_*(x-\widetilde{\vartheta}_T)\right)f_*(x-\vartheta_T^*)\mathrm{d}x=0.$$

Thus we have

$$u_T^*=\sqrt{T}\,(\vartheta_T^*-\vartheta_0)=-\frac{\sqrt{T}\int_{-\infty}^{\infty}\left(\widehat{F}_T(x)-F_*(x-\vartheta_0)\right)f_*(x-\vartheta_T^*)\mathrm{d}x}{\int_{-\infty}^{\infty}f_*(x-\vartheta_T^*)f_*(x-\widetilde{\vartheta}_T)\mathrm{d}x}. \tag{2.43}$$

Note that owing to the convergence $\vartheta_T^*\to\vartheta_0$ and the continuity of the density function $f_*(\cdot)$, we have

$$\int_{-\infty}^{\infty}f_*(x-\vartheta_T^*)f_*(x-\widetilde{\vartheta}_T)\mathrm{d}x\longrightarrow\int_{-\infty}^{\infty}f_*(x-\vartheta_0)^2\mathrm{d}x=J.$$

In addition,

$$\sqrt{T}\int_{-\infty}^{\infty}\left(\widehat{F}_T(x)-F_*(x-\vartheta_0)\right)f_*(x-\vartheta_T^*)\mathrm{d}x$$
$$=\int_{-\infty}^{\infty}\eta_T^F(x)\left(f_*(x-\vartheta_0)+\dot{f}_*(x-\widetilde{\vartheta}_T)(\vartheta_T^*-\vartheta_0)\right)\mathrm{d}x$$
$$=\int_{-\infty}^{\infty}\eta_T^F(x)f_*(x-\vartheta_0)\mathrm{d}x+r_{1,T}.$$

Remind that under the condition $\mathcal{A}_0$, we have $f_*(x)\leq C\mathrm{e}^{-2\gamma|x|}$ for $|x|>A$. This yields that

$$\mathbf{E}_{\vartheta_0}\left(\int_{-\infty}^{\infty}\eta_T^F(x)\dot{f}_*(x-\widetilde{\vartheta}_T)\mathrm{d}x\right)^4$$
$$\leq\mathbf{E}_{\vartheta_0}\int_{-\infty}^{\infty}\eta_T^F(x)^4\mathrm{d}x\left(\int_{-\infty}^{\infty}\left(2S(x-\widetilde{\vartheta}_T)f_*(x-\widetilde{\vartheta}_T)\right)^{4/3}\mathrm{d}x\right)^3$$
$$\leq\mathbf{E}_{\vartheta_0}\int_{-\infty}^{\infty}\eta_T^F(x)^4\mathrm{d}x\left(\int_{-\infty}^{\infty}\left(2(1+|x-\widetilde{\vartheta}_T|^p)f_*(x-\widetilde{\vartheta}_T)\right)^{4/3}\mathrm{d}x\right)^3$$
$$\leq C\int_{-\infty}^{\infty}\mathbf{E}_{\vartheta_0}\eta_T^F(x)^4\mathrm{d}x\leq C\left(\int_{|x|>A}+\int_{|x|\leq A}\right)\mathbf{E}_{\vartheta_0}\eta_T^F(x)^4\mathrm{d}x\leq C.$$

Thus we have

$$\begin{aligned}\mathbf{E}_{\vartheta_0} r_{1,T}^2 &= J^{-2}\mathbf{E}_{\vartheta_0}\left(\int_{-\infty}^{\infty} \eta_T^F(x)\dot{f}_*(x-\widetilde{\vartheta}_T)(\vartheta_T^*-\vartheta_0)\mathrm{d}x\right)^2\\ &\le J^{-2}\left(\mathbf{E}_{\vartheta_0}(\vartheta_T^*-\vartheta_0)^4\right)^{1/2}\left(\mathbf{E}_{\vartheta_0}\left(\int_{-\infty}^{\infty}\eta_T^F(x)\dot{f}_*(x-\widetilde{\vartheta}_T)\mathrm{d}x\right)^4\right)^{1/2}\\ &\le C\left(\mathbf{E}_{\vartheta_0}(\vartheta_T^*-\vartheta_0)^4\right)^{1/2}\longrightarrow 0.\end{aligned}$$

Note that

$$\begin{aligned}H_y'(z,y) &= 2\frac{\partial}{\partial y}\left(\frac{F_*(y)F_*(z)-F_*(y\wedge z)}{f_*(z)}\right)\\ &= 2f_*(y)\left(\frac{F_*(z)-\mathbb{1}_{\{z>y\}}}{f_*(z)}\right) = M(z,y)\end{aligned}$$

and that

$$\eta_T^F(y)\Longrightarrow \eta^F(y-\vartheta_0)=\int_{-\infty}^{\infty}H(z,y-\vartheta_0)\sqrt{f_*(z)}\mathrm{d}W(z).$$

We have

$$\begin{aligned}&\int_{-\infty}^{\infty}\eta_T^F(x)f_*(x-\vartheta_0)\mathrm{d}x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbb{1}_{\{y<x\}}\mathrm{d}\eta_T^F(y)f_*(x-\vartheta_0)\mathrm{d}x\\ &= J^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbb{1}_{\{x>y\}}f_*(x-\vartheta_0)\mathrm{d}x\mathrm{d}\eta_T^F(y)\\ &= \int_{-\infty}^{\infty}\left(1-F_*(y-\vartheta_0)\right)\mathrm{d}\eta_T^F(y)\\ &\Longrightarrow J^{-1}\int_{-\infty}^{\infty}\left(1-F_*(y-\vartheta_0)\right)\int_{-\infty}^{\infty}H_y'(z,y-\vartheta_0)\sqrt{f_*(z)}\mathrm{d}W(z)\mathrm{d}y\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-F_*(y)\right)M(z,y)\sqrt{f_*(z)}\mathrm{d}y\mathrm{d}W(z).\end{aligned}$$

In replacing these results in (2.43), we obtain the asymptotical normality

$$\begin{aligned}\sqrt{T}\left(\vartheta_T^*-\vartheta_0\right) &\Longrightarrow -J^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-F_*(y)\right)M(z,y)\sqrt{f_*(z)}\mathrm{d}y\mathrm{d}W(z)\\ &\sim\mathcal{N}(0,J^{-2}\mathbf{E}_0R(\xi_0)^2).\end{aligned}$$

We define from now on

$$u^* = -J^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-F_*(y)\right)M(z,y)\sqrt{f_*(z)}\mathrm{d}y\mathrm{d}W(z),$$

then we have the finite-dimensional convergence

Lemma 2.2.8. *Let the conditions $\mathcal{A}_0$ and $\mathcal{A}$ be fulfilled, then*

$$\mathcal{L}_{\vartheta_0}\left(\eta_T^F(x_1), ..., \eta_T^F(x_k), u_T^*\right) \Longrightarrow \mathcal{L}_{\vartheta_0}\left(\eta^F(x_1-\vartheta_0), ..., \eta_T^F(x_k-\vartheta_0), u^*\right)$$

for any $\mathbf{x} = \{x_1, x_2, ..., x_k\} \in \mathbb{R}^k$.

Proof. Remind that in Section 2.2.2, we have defined

$$\widetilde{\eta}_T^F(x) = \frac{1}{\sqrt{T}} \int_0^T H(X_t - \vartheta_0, x) \mathrm{d}W_t.$$

We define in addition

$$\Lambda_T^* = \frac{1}{\sqrt{T}} \int_0^T \int_{-\infty}^{\infty} \left(1 - F_*(z)\right) M(X_t, z) \mathrm{d}z \mathrm{d}W_t,$$

and

$$\Lambda^* = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(1 - F_*(y)\right) M(z, y) \sqrt{f_*(z)} \mathrm{d}y \mathrm{d}W(z),$$

Note the representation (2.43) and (2.33), in omitting the asymptotically null parts, we need to prove the convergence

$$\mathcal{L}_{\vartheta_0}\left(\widetilde{\eta}_T^F(x_1), ..., \widetilde{\eta}_T^F(x_k), \Lambda_T^*\right) \Longrightarrow \mathcal{L}_{\vartheta_0}\left(\eta^F(x_1), ..., \eta_T^F(x_k), \Lambda^*\right)$$

for any $\mathbf{x} = \{x_1, x_2, ..., x_k\}$. Let us take $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_{k+1} \in \mathbb{R}^{k+1}\}$, and denote

$$h(y, \mathbf{x}, \lambda) = \sum_{l=1}^{k} \lambda_l H(y, x_l) + \lambda_{k+1} \int_{-\infty}^{\infty} \left(1 - F_*(z)\right) M(y, z) \mathrm{d}z.$$

We need to verify

$$\frac{1}{\sqrt{T}} \int_0^T h(X_t, \mathbf{x}, \lambda) \mathrm{d}W_t \Longrightarrow \sum_{l=1}^{k} \lambda_l \eta^F(x_l) + \lambda_{k+1} \Lambda^*. \tag{2.44}$$

Note that

$$\frac{1}{\sqrt{T}} \int_0^T h(X_t, \mathbf{x}, \lambda) \mathrm{d}W_t \Rightarrow \mathcal{N}\left(0, \mathbf{E}_0 h(\xi_0, \mathbf{x}, \lambda)^2\right).$$

where

$$\begin{aligned}\mathbf{E}_0 h(\xi_0, \mathbf{x}, \lambda)^2 &= \mathbf{E}_0 \left(\sum_{l=1}^{k} \lambda_l H(\xi_0, x_l) + \lambda_{k+1} \int_{-\infty}^{\infty} \left(1 - F_*(z)\right) M(\xi_0, z) \mathrm{d}z\right)^2 \\ &= \sum_{l=1}^{k} \sum_{m=1}^{k} \lambda_l \lambda_m \mathbf{E}_0 \left(H(\xi_0, x_l) H(\xi_0, x_m)\right) + \lambda_{k+1}^2 \mathbf{E}_0 \left(\int_{-\infty}^{\infty} f_*(z) H(\xi_0, z) \mathrm{d}z\right)^2 \\ &\quad + 2 \sum_{l=1}^{k} \lambda_l \lambda_{k+1} \mathbf{E}_0 \left(H(\xi_0, x_l) \int_{-\infty}^{\infty} \left(1 - F_*(z)\right) M(\xi_0, z) \mathrm{d}z\right).\end{aligned}$$

In addition, $\sum\limits_{l=1}^{k} \lambda_l \eta^F(x_l) + \lambda_{k+1}\Lambda^*$ is a normal variable with null expectation and variation

$$\begin{aligned}
&\mathbf{E}\left(\sum_{l=1}^{k} \lambda_l \eta^F(x_l) + \lambda_{k+1}\Lambda\right)^2 \\
&\quad = \sum_{l=1}^{k}\sum_{m=1}^{k} \lambda_l\lambda_m \mathbf{E}(\eta^F(x_l)\eta^F(x_m)) + \sum_{l=1}^{k} \lambda_l\lambda_{k+1}\mathbf{E}(\eta^F(x_l)\Lambda^*) + \lambda_{k+1}^2\mathbf{E}(\Lambda^*)^2 \\
&\quad = \mathbf{E}_0 h(\xi_0, \mathbf{x}, \lambda)^2
\end{aligned}$$

Thus we obtain (2.44), and so that the result of the lemma.

In addition, we have the convergence:

Lemma 2.2.9. *Let the conditions $\mathcal{A}_0$ and $\mathcal{A}$ be fulfilled, then we have*

$$\mathcal{L}_{\vartheta_0}\{\int_{-\infty}^{\infty} \left(\widetilde{\eta}_T^F(x) + u_T^* f_*(x)\right)^2 \mathrm{d}x\} \Longrightarrow \mathcal{L}\left\{\int_{-\infty}^{\infty} \left(\eta^F(x) + u^* f_*(x)\right)^2 \mathrm{d}x\right\}$$

Proof. We introduce firstly an estimate (See for example Lemma 1.1 in Kutoyants [28]): Suppose that

$$\mathbf{E}\int_0^T h(s,\omega)^{2m}\mathrm{d}t < \infty$$

is satisfied, then

$$\mathbf{E}\left(\int_0^T h(s,\omega)\mathrm{d}W_t\right)^{2m} \leq (m(2m-1))^m T^{m-1}\mathbf{E}\int_0^T h(s,\omega)^{2m}\mathrm{d}t.$$

Thus we have

$$\begin{aligned}\mathbf{E}_{\vartheta_0}(u_T^*)^4 &= \mathbf{E}_{\vartheta_0}\left(\frac{1}{\sqrt{T}}\int_0^T J^{-1}\int_{-\infty}^{\infty} f_*(x)H(X_t-\vartheta_0,x)\mathrm{d}x\mathrm{d}W_t + o(1)\right)^4\\
&\le CJ^{-2}\mathbf{E}_{\vartheta_0}\left(\int_{-\infty}^{\infty} f_*(x)H(\xi_{\vartheta_0}-\vartheta_0,x)\mathrm{d}x\right)^4 + o(1)\\
&= CJ^{-2}\mathbf{E}_0\left(\int_{-\infty}^{\infty} f_*(x)H(\xi_0,x)\mathrm{d}x\right)^4 + o(1)\\
&\le CJ^{-2}\left(\int_{-\infty}^{\infty} f_*(x)^{4/3}\mathrm{d}x\right)^3\int_{-\infty}^{\infty}\mathbf{E}_0 H(\xi_0,x)^4\mathrm{d}x + o(1)\\
&\le C\left(\int_{|x|\le A}\mathbf{E}_0 H(\xi_0,x)^4\mathrm{d}x + \int_{|x|>A}\mathbf{E}_0 H(\xi_0,x)^4\mathrm{d}x\right) + o(1)\\
&\le C\left(C + \int_{|x|>A}\mathrm{e}^{-2\gamma|x|}\mathrm{d}x\right) + o(1) \le C\end{aligned}$$

Let us denote $\widetilde{\zeta}_T^D(x) = \widetilde{\eta}_T^F(x) + u_T^* f_*(x)$. Thus following the proof of Lemma 2.2.6, we have
i) $\forall\ L>0$, for $x,y\in[-L,L]$ and $|x-y|\le 1$, there exists C depending on L such that

$$\mathbf{E}_{\vartheta_0}|\widetilde{\zeta}_T^D(x)^2 - \widetilde{\zeta}_T^D(y)^2|^2 \le C|x-y|^{1/2}.$$

ii)$\forall\ \varepsilon>0,\ \exists L>0$, such that

$$\mathbf{E}_{\vartheta_0}\int_{\{|x|>L\}}\widetilde{\zeta}_T^D(x)^2\mathrm{d}x < \varepsilon, \quad \forall T>0.$$

Along with the finite-dimensional convergence in Lemma 2.2.8, we obtain the result of the lemma.

These lemmas above yield the convergence of the test statistic. In fact

$$\begin{aligned}
\omega_T^2 &= T\int_{-\infty}^{\infty}\left(\widehat{F}_T(x)-F_*(x-\theta_T^*)\right)^2\mathrm{d}x\\
&=\int_{-\infty}^{\infty}\left(\sqrt{T}(\widehat{F}_T(x)-F_*(x-\vartheta_0))+\sqrt{T}(F_*(x-\vartheta_0)-F_*(x-\theta_T^*)\right)^2\mathrm{d}x\\
&=\int_{-\infty}^{\infty}\left(\eta_T^F(x)+u_T^*\dot{F}_*(x-\widetilde{\vartheta}_T)\right)^2\mathrm{d}x\\
&=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\mathbb{1}_{\{y<x\}}\mathrm{d}\eta_T^F(x)-f_*(x-\vartheta_0)J^{-1}\int_{-\infty}^{\infty}(1-F_*(y-\vartheta_0))\,\mathrm{d}\eta_T^F(y)\right)^2\mathrm{d}x+o(1)\\
&=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\mathbb{1}_{\{y<x\}}-f_*(x-\vartheta_0)J^{-1}\left(1-F_*(y-\vartheta_0)\right)\right)\mathrm{d}\eta_T^F(y)\right)^2\mathrm{d}x+o(1)\\
&=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\mathbb{1}_{\{y<x\}}-f_*(x-\vartheta_0)J^{-1}\left(1-F_*(y-\vartheta_0)\right)\right)\eta_T(y)\mathrm{d}y\right)^2\mathrm{d}x+o(1)\\
&\Longrightarrow\int_{-\infty}^{\infty}\zeta^D(x)^2\,\mathrm{d}x=\omega^2,
\end{aligned}$$

where

$$\begin{aligned}
\zeta^D(x)&=\eta^F(x)+u^*f_*(x)\\
&=\int_{-\infty}^{\infty}\left(\mathbb{1}_{\{y<x\}}-f_*(x)J^{-1}\left(1-F_*(y)\right)\right)\int_{-\infty}^{\infty}M(z,y)\sqrt{f_*(z)}\mathrm{d}W(z)\mathrm{d}y.
\end{aligned}$$

Thus the test μ_T belongs to $\mathcal{K}_\varepsilon$. Moreover, the limit of the statistic does not depend on ϑ, which means that the test φ_T is APF.

Remark 2.2.1. *Note that the statistic ω_T^2 and its limit ω^2 can be presented as follows:*

$$\begin{aligned}
\omega_T^2 = T\left\|\widehat{F}_T(\cdot)-F(\cdot,\vartheta_T^*)\right\|^2 &= \inf_{\theta\in\Theta}\int_{-\infty}^{\infty}T\left(\widehat{F}_T(x)-F(x-\theta)\right)^2\mathrm{d}x\\
\Longrightarrow \omega^2=\int_{-\infty}^{\infty}\left(\eta(x)+u^*f(x)\right)^2\mathrm{d}x &= \inf_{u\in\mathbb{R}}\int_{-\infty}^{\infty}\left(\eta(x)+uf(x)\right)^2\mathrm{d}x.
\end{aligned}$$

The advantage of this C-vM type test with MDE is that we do not have to calculate the real value of the estimator $\widehat{\vartheta}_T^*$, in fact the minimum value of $\left\|\widehat{F}_T(\cdot)-F(x-\theta)\right\|$ is sufficient to constrcut the test.

Remark 2.2.2. *The same procedure can be applied to the case where the test is constructed by the LTE. In addition, other estimators for the invariant density or the invariant distribution function propose similar result, in providing that the estimators are consistent and asymptotically normal.*

2.2.5 Numerical example

We consider the Ornstein-Uhlenbeck process. Remind that the tests for O-U process were studied in Kutoyants [30] as well. Suppose that the observed process under the null hypothesis is

$$\mathrm{d}X_t = -(X_t - \vartheta_0)\mathrm{d}t + \mathrm{d}W_t, \quad X_0,\ 0 \le t \le T.$$

The invariant density is $f_*(x-\vartheta_0)$, where $f_*(x) = \pi^{-1/2}\mathrm{e}^{-x^2}$.

The log-likelihood ratio is

$$L(X^T,\vartheta) = -\int_0^T (X_t - \vartheta)\mathrm{d}X_t - \frac{1}{2}\int_0^T (X_t-\vartheta)^2\mathrm{d}t,$$

so that the MLE $\widehat{\vartheta}_T$ can be calculated as

$$\widehat{\vartheta}_T = \frac{1}{T}\int_0^T X_t\mathrm{d}t + \frac{X_T - X_0}{T}.$$

The Fisher information in this case equals to 1, and the LTE is

$$\widehat{f}_T(x) = \frac{1}{T}(|X_T - x| - |X_0 - x|) - \frac{1}{T}\int_0^T \mathrm{sgn}(X_t - x)\mathrm{d}X_t.$$

The conditions $\mathcal{A}_0$ and $\mathcal{A}$ are fulfilled, then the statistic is convergent:

$$\delta_T = \int_{-\infty}^{\infty}\left(\widehat{f}_T(x) - f_*(x-\widehat{\vartheta}_T)\right)^2\mathrm{d}x \Longrightarrow \delta = \int_{-\infty}^{\infty}\zeta_1(x)^2\mathrm{d}x,$$

where the limit process $\zeta_1(x) = \eta(x) - \widehat{u}f'(x)$ can be written as

$$\zeta_1(x) = \int_{-\infty}^{\infty}\left(2f_*(x)\frac{F_*(y) - 1\!\!1_{\{y>x\}}}{\sqrt{f_*(y)}} + f_*'(x)\sqrt{f_*(y)}\right)\mathrm{d}W(y).$$

We have a similar result for the test based on the EDF:

$$\Delta_T = \int_{-\infty}^{\infty}\left(\widehat{F}_T(x) - F_*(x-\widehat{\vartheta}_T)\right)^2\mathrm{d}x \Longrightarrow \Delta = \int_{-\infty}^{\infty}\left(\zeta_2(x)\right)^2\mathrm{d}x,$$

where the limit process can be written as

$$\zeta_2(x) = \int_{-\infty}^{\infty}\left(2\frac{F_*(y)F_*(x) - F_*(y\wedge x)}{\sqrt{f_*(y)}} + f_*(x)\sqrt{f_*(y)}\right)\mathrm{d}W(y).$$

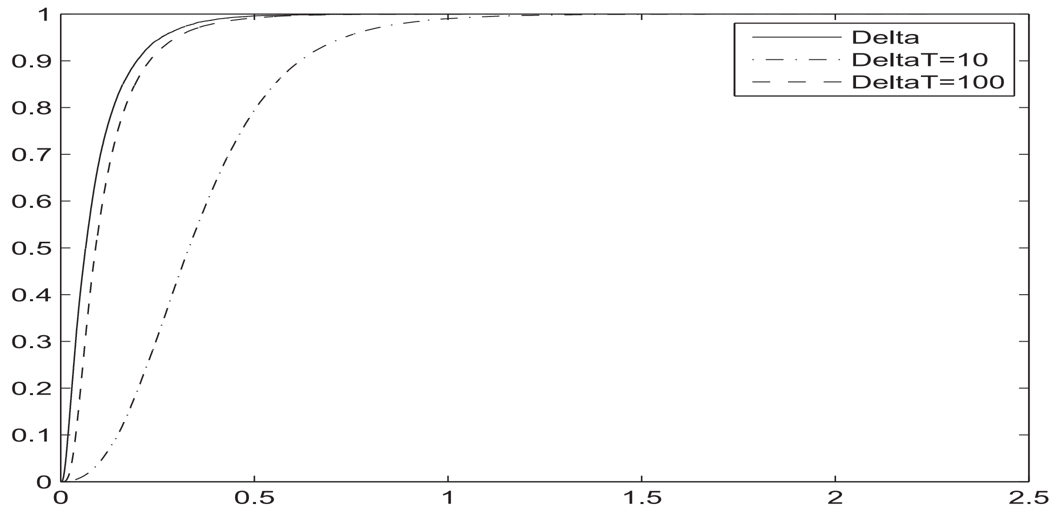

Figure 2.1: Distribution function for test statistics and its limit.

This convergence can be verified by numeric method. We take the test statistic via EDF as an example, Graphic 2.1 is the curves of the distribution function for Δ and Δ_T, $T = 10,\ 100$.

We simulate 10^5 trajectories of δ (resp. Δ) and calculate the empirical $1 - \varepsilon$ quantiles of δ (resp. Δ). We obtain the simulated density for δ and Δ that are showed in Graphic 2.2. The values of the thresholds d_ε for different ε are showed in Graphic 2.3.

2.3 The Kolmogorov-Smirnov Type Tests

This section is based on the work [51]

We consider always the following problem. Suppose that we observe an ergodic diffusion process

$$\mathrm{d}X_t = S(X_t)\mathrm{d}t + \mathrm{d}W_t, \quad X_0,\ 0 \leq t \leq T \tag{2.45}$$

and we have to test the following basic hypothesis

$$\mathcal{H}_0 \quad : \qquad S\left(x\right) = S_*\left(x - \vartheta\right), \quad \vartheta \in \Theta = \left(\alpha, \beta\right)$$

where $S_*\left(\cdot\right)$ is some known function and the shift parameter ϑ is unknown. Therefore,

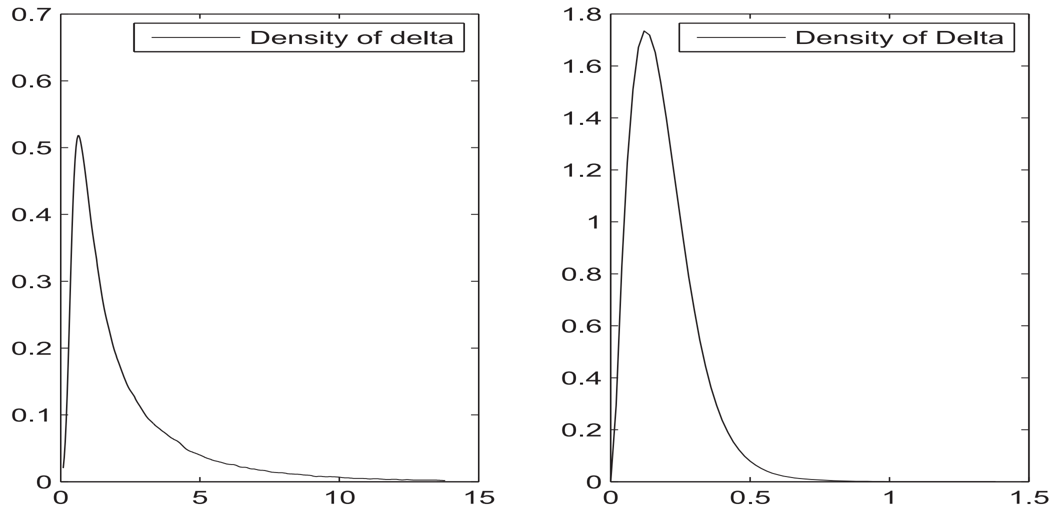

Figure 2.2: Density of δ and Δ.

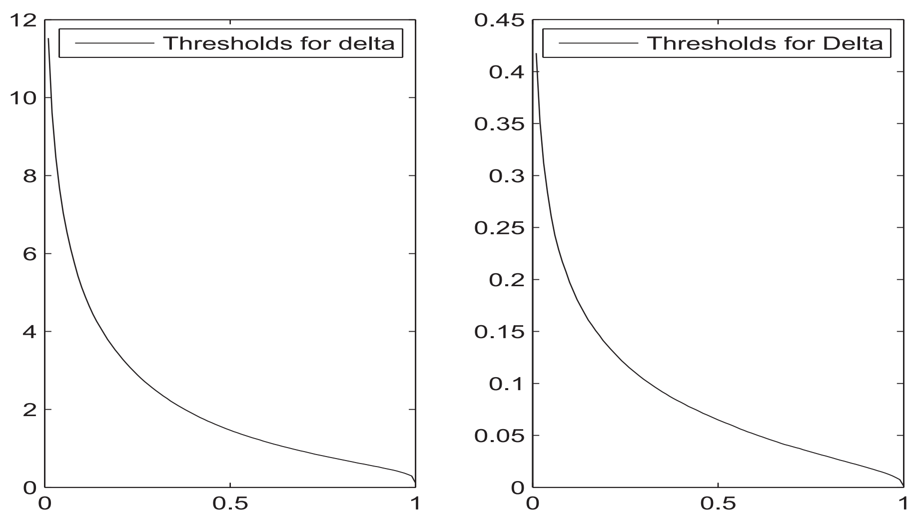

Figure 2.3: Threshold for different ε.

the trend coefficients under hypothesis belong to the parametrical family

$$\mathcal{S}\left(\Theta\right)=\left\{S_*\left(x-\vartheta\right),\quad \vartheta\in\Theta\right\}.$$

The alternative is

$$\mathcal{H}_1 \quad : \qquad S\left(x\right) \notin \overline{\mathcal{S}\left(\Theta\right)},$$

where $\overline{\mathcal{S}\left(\Theta\right)} = \left\{S_*\left(x-\vartheta\right), \vartheta \in [\alpha,\beta]\right\}$.

The invariant density function and the invariant distribution function under $\mathcal{H}_0$ are denoted as $f_*(x-\vartheta)$ and $F_*(x-\vartheta)$.

2.3.1 The K-S test via the LTE

Suppose that the trend coefficients $S\left(\cdot\right)$ of the observed diffusion process under both hypotheses satisfy the conditions $\mathcal{EM}$, $\mathcal{ES}$ and $\mathcal{A}_0$.

The unknown parameter is estimated by the MLE $\widehat{\vartheta}_T$, which is defined as the solution of the equation

$$L(\widehat{\vartheta}_T, X^T) = \sup_{\theta\in\Theta} L(\theta, X^T).$$

The LTE $\widehat{f}_T(x)$ of the invariant density is

$$\widehat{f}_T(x) = \frac{1}{T}(|X_T - x| - |X_0 - x|) - \frac{1}{T}\int_0^T \mathrm{sgn}(X_t - x)\mathrm{d}X_t.$$

Let us propose a statistic which is defined as follows

$$\lambda_T = \sqrt{T}\sup_{x\in\mathbb{R}}\left|\widehat{f}_T(x) - f_*(x-\widehat{\vartheta}_T)\right|,$$

we show that under hypothesis $\mathcal{H}_0$, it converges in distribution to

$$\lambda = \sup_{x\in\mathbb{R}}\left|\int_{-\infty}^{\infty}\left(2f(x)\frac{F_*(y) - 1\!\!1_{\{y>x\}}}{\sqrt{f_*(y)}} - \frac{2}{I}S_*(x)f_*(x)S_*'(y)\sqrt{f_*(y)}\right)\mathrm{d}W(y)\right|. \quad (2.46)$$

The K-S test is defined as

$$\phi_T = 1\!\!1_{\{\lambda_T > c_\varepsilon\}},$$

where c_ε is the $(1-\varepsilon)$-quantile of the distribution of λ, i.e. c_ε is the solution of the following equation

$$\mathbf{P}\Big(\lambda \geq c_\varepsilon\Big) = \varepsilon. \quad (2.47)$$

The main result for the K-S test based on LTE is the following:

Theorem 2.3.1. *Let the conditions $\mathcal{ES}$, $\mathcal{A}_0$ and $\mathcal{A}$ be fulfilled, then the test $\phi_T = 1\!\!1_{\{\lambda_T > c_\varepsilon\}}$ belongs to $\mathcal{K}_\varepsilon$.*

Note that neither λ nor c_ε depends on the unknown parameter. Therefore the test ϕ_T is APF.

Remind that under the condition $\mathcal{A}$ the MLE $\widehat{\vartheta}_T$ is consistent and asymptotically normal. Let us define $\widehat{u}_T = \sqrt{T}(\widehat{\vartheta}_T - \vartheta_0)$, then it converges in distribution to

$$\widehat{u} = -\frac{1}{I}\int_{-\infty}^{\infty} S_*'(y)\sqrt{f_*(y)}\mathrm{d}W(y).$$

Let us define $\eta_T(x) = \sqrt{T}\left(\widehat{f}_T(x) - f_*(x-\vartheta_0)\right)$. As that is shown in (2.21), it admits the following representation

$$\eta_T(x) = -\frac{1}{\sqrt{T}}\int_{X_0}^{X_T} M(y-\vartheta_0, x-\vartheta_0)\mathrm{d}y + \frac{1}{\sqrt{T}}\int_0^T M(X_t-\vartheta_0, x-\vartheta_0)\mathrm{d}W_t,$$

where

$$M(y,x) = 2f(x)\frac{F_*(y) - 1\!\!1_{\{y>x\}}}{f_*(y)}.$$

Remind that $\eta_T(x)$ is convergent and asymptotically normal under the regularity conditions. In addition, in Lemma 2.2.2, we have the convergence of the joint finite-dimensional distribution for $\widehat{u}_T$ and $\eta_T(x)$:

$$\mathcal{L}\left(\eta_T(x_1), ..., \eta_T(x_k), \widehat{u}_T\right) \Longrightarrow \mathcal{L}\left(\eta(x_1-\vartheta_0), ..., \eta(x_k-\vartheta_0), \widehat{u}\right),$$

for any $\mathbf{x} = \{x_1, x_2, ..., x_k\} \in \mathbb{R}^k$ and $k = 1, 2, 3...$, where

$$\eta(x) = 2f_*(x)\int_{-\infty}^{\infty}\frac{F_*(y) - 1\!\!1_{\{y>x\}}}{\sqrt{f_*(y)}}\mathrm{d}W(y).$$

We denote $\zeta_T(x) = \sqrt{T}(\widehat{f}_T(x) - f_*(x-\widehat{\vartheta}_T))$, then

$$\begin{aligned}\zeta_T(x) &= \sqrt{T}(\widehat{f}_T(x) - f_*(x-\widehat{\vartheta}_T)) \\ &= \eta_T(x) + \sqrt{T}\left(f_*(x-\vartheta_0) - f_*(x-\widetilde{\vartheta}_T)\right) \\ &= \eta_T(x) + \widehat{u}_T\dot{f}_*(x-\vartheta_0) + o(\widehat{\vartheta}_T - \vartheta_0).\end{aligned}$$

Denote also

$$\begin{aligned}\zeta(x) &= \eta(x) + \widehat{u}f'(x) \\ &= \int_{-\infty}^{\infty}\left(2f_*(x)\frac{F_*(y) - 1\!\!1_{\{y>x\}}}{\sqrt{f_*(y)}} - \frac{2}{I}S_*(x)f_*(x)S_*'(y)\sqrt{f_*(y)}\right)\mathrm{d}W(y),\end{aligned}$$

We will prove that $\zeta_T(\cdot)$ converges weakly to $\zeta(\cdot)$. For this, we prove firstly two lemmas:

Lemma 2.3.1. *Let conditions $\mathcal{A}_0$ and $\mathcal{A}$ be fulfilled, then*

$$\mathbf{E}_{\vartheta_0} \left|\zeta_T(x)\right|^2 \leq C e^{-\gamma|x|} \quad x \in \mathbb{R}.$$

Proof. We have

$$\begin{aligned}\mathbf{E}_{\vartheta_0} \left|\zeta_T(x)\right|^2 &= \mathbf{E}_{\vartheta_0} \left|\eta_T(x) + \widehat{u}_T f_*'(x - \widetilde{\vartheta}_T)\right|^2 \\ &\leq 2(f_*'(x - \widetilde{\vartheta}_T))^2 \mathbf{E}_{\vartheta_0}|\widehat{u}_T|^2 + 2\mathbf{E}_{\vartheta_0}|\eta_T(x)|^2.\end{aligned}$$

For the first part, let us recall the following result, given in Kutoyants [28], page 119: for any $p > 0$,

$$\mathbf{E}_{\vartheta_0}|\widehat{u}_T|^p \leq C.$$

Beside this, we have

$$f'(x) \leq C\mathrm{e}^{-\gamma|x|}, \quad \forall x \in \mathbb{R},$$

because that for large $|x|$

$$|f_*'(x)| = 2|S_*(x) f_*(x)| \leq C(1 + |x|^p)\mathrm{e}^{-2\gamma|x|} \leq C\mathrm{e}^{-\gamma|x|},$$

and for $|x|$ bounded, both $S_*(\cdot)$ and $f(\cdot)$ are bounded, then we can find some constant C such that $S_*(x) f_*(x) \leq C\mathrm{e}^{-\gamma|x|}$.

In addition according to Lemma 2.2.1

$$\begin{aligned}\mathbf{E}_{\vartheta_0}|\eta_T(x)|^2 &\leq 2\mathbf{E}_{\vartheta_0}\left(\frac{1}{\sqrt{T}}\int_0^T M(X_t - \vartheta_0, x - \vartheta_0)\mathrm{d}W_t\right)^2 \\ &\quad +2\mathbf{E}_{\vartheta_0}\left(\frac{1}{\sqrt{T}}\int_{X_0}^{X_T} M(z - \vartheta_0, x - \vartheta_0)\mathrm{d}z\right)^2 \\ &= 2\mathbf{E}_0 M(\xi_0, x)^2 + \frac{4}{T}\mathbf{E}_0\left(\int_0^{\xi} M(z, x)\mathrm{d}z\right)^2 \leq C\mathrm{e}^{-2\gamma|x|}.\end{aligned}$$

We obtain thus the result of the lemma.

Lemma 2.3.2. *Let conditions $\mathcal{A}_0$ and $\mathcal{A}$ be fulfilled, then*

$$\mathcal{L}_T\left\{\sup_{x\in\mathbb{R}} |\zeta_T(x)|\right\} \Longrightarrow \mathcal{L}\left\{\sup_{x\in\mathbb{R}} |\zeta(x - \vartheta_0)|\right\}.$$

Proof. Remind that $\zeta_T(x) = \eta_T(x) + \sqrt{T}\left(f(x-\vartheta_0) - f(x-\widehat{\vartheta}_T)\right)$. Thus we need to prove the weak convergence for the two parts:

$$\mathcal{L}_T\left\{\sup_{x\in\mathbb{R}} |\eta_T(x)|\right\} \Longrightarrow \mathcal{L}\left\{\sup_{x\in\mathbb{R}} |\eta(x-\vartheta_0)|\right\} \tag{2.48}$$

and

$$\mathcal{L}_T\left\{\sup_{x\in\mathbb{R}} \left|\sqrt{T}\left(f(x-\vartheta_0) - f(x-\widehat{\vartheta}_T)\right)\right|\right\} \Longrightarrow \mathcal{L}\left\{\sup_{x\in\mathbb{R}} |\widehat{u} f'(x-\vartheta_0)|\right\}. \tag{2.49}$$

Along with the convergence of the joint finite-dimensional distribution, we have the result of the lemma.

The convergence (2.48) follows from the Theorem 4.13 in Kutoyants [28]. In applying the Theorem A.20 (Appendix I) in [22], the Lemmae 2.2.2 and 2.3.1 provide us the following result: the distribution Q_T in $C_0(\mathbb{R})$ generated by the process $\eta_T(\cdot)$ converges to the distribution Q generated by the process $\eta(\cdot)$. Therefore we have the weak convergence of $\eta_T(x)$ and further the convergence in distribution of the supremum of $\eta_T(x)$.

For (2.49), note that

$$\sqrt{T}\left(f(x-\vartheta_0) - f(x-\widehat{\vartheta}_T)\right) = \sqrt{T}(\widehat{\vartheta}_T - \vartheta_0) f'(x-\widetilde{\vartheta}_T).$$

Moreover, $f''(x) = 2S_*'(x)f(x) + 4S_*(x)^2 f(x)$ is bounded since that $S_*(\cdot)$ and $S_*'(\cdot)$ belong to $\mathcal{P}$, and $f(x) \leq C\mathrm{e}^{-2\gamma|x|}$ for large $|x|$. Thus we have

$$\sup_{x\in\mathbb{R}} \left|f'(x-\widetilde{\vartheta}_T) - f'(x-\vartheta_0)\right| = \sup_{x\in\mathbb{R}} |f''(\widetilde{x})| \cdot \left|\widetilde{\vartheta}_T - \vartheta_0\right| \longrightarrow 0.$$

Therefore

$$\begin{aligned}\sup_{x\in\mathbb{R}} \left|\sqrt{T}\left(f(x-\vartheta_0) - f(x-\widehat{\vartheta}_T)\right)\right| &= \sup_{x\in\mathbb{R}} \left|\sqrt{T}(\widehat{\vartheta}_T - \vartheta_0) f'(x-\widetilde{\vartheta}_T)\right| \\ &\Longrightarrow |\widehat{u}| \sup_{x\in\mathbb{R}} |f'(x-\vartheta_0)|.\end{aligned}$$

Moreover,

$$\sup_{x\in\mathbb{R}} |\zeta(x-\vartheta_0)| = \sup_{(y+\vartheta_0)\in\mathbb{R}} |\zeta(y)| = \sup_{z\in\mathbb{R}} |\zeta(z)| = \lambda.$$

Thus we have

$$\mathcal{L}_T\{\lambda_T\} \Longrightarrow \mathcal{L}\{\lambda\}.$$

Note that λ does not depend on the unknown parameter ϑ_0, we conclude that the test $\phi_T = \mathbb{1}_{\{\lambda_T > c_\varepsilon\}}$ belongs to $\mathcal{K}_\varepsilon$ and is APF.

2.3.2 The K-S test via the EDF

We introduce in this part the test based on the EDF:

$$\widehat{F}_T(x) = \frac{1}{T}\int_0^T 1\!\!{\rm I}_{\{X_t<x\}}\mathrm{d}t.$$

Let us introduce the statistic

$$\Lambda_T = \sqrt{T}\sup_{x\in\mathbb{R}}\left|\widehat{F}_T(x) - F_*(x-\widehat{\vartheta}_T)\right|,$$

we will prove that it converges in distribution to

$$\Lambda = \sup_{x\in\mathbb{R}}\left|\int_{-\infty}^{\infty}\left(2\frac{F_*(y)F_*(x)-F_*(y\wedge x)}{\sqrt{f_*(y)}} - \frac{1}{I}S'_*(y)\sqrt{f_*(y)}f_*(x)\right)\mathrm{d}W(y)\right|. \quad (2.50)$$

Thus we propose the K-S test

$$\Phi_T = 1\!\!{\rm I}_{\{\Lambda_T>C_\varepsilon\}},$$

where C_ε is the solution of the following equation

$$\mathbf{P}\Big(\Lambda \geq C_\varepsilon\Big) = \varepsilon. \quad (2.51)$$

The main result for the K-S test based on EDF is the following:

Theorem 2.3.2. *Under the conditions $\mathcal{ES}$, $\mathcal{A}_0$ and $\mathcal{A}$, the test $\Phi_T = 1\!\!{\rm I}_{\{\Lambda_T>C_\varepsilon\}}$ belongs to $\mathcal{K}_\varepsilon$.*

Remind that $\eta_T^F(x) = \sqrt{T}\left(\widehat{F}_T(x) - F_*(x-\vartheta_0)\right)$ and

$$\begin{aligned}\eta_T^F(x) &= \sqrt{T}(\widehat{F}_T(x) - F_*(x-\vartheta_0)) \\ &= -\frac{1}{\sqrt{T}}\left(\int_0^{X_T} H(z-\vartheta_0, x-\vartheta_0)\mathrm{d}z - \int_0^{X_0} H(z-\vartheta_0, x-\vartheta_0)\mathrm{d}z\right) \\ &\quad +\frac{1}{\sqrt{T}}\int_0^T H(X_t-\vartheta_0, x-\vartheta_0)\mathrm{d}W_t,\end{aligned}$$

where

$$H(z,x) = 2\frac{F_*(z)F_*(x)-F_*(z\wedge x)}{f_*(z)}.$$

As that is shown in Lemma 2.2.4, under the condition $\mathcal{A}_0$ the EDF $\widehat{F}_T(x)$ is consistent and asymptotically normal, that is

$$\eta_T^F(x) = \sqrt{T}\left(\widehat{F}_T(x) - F_*(x-\vartheta_0)\right) \Longrightarrow \eta^F(x-\vartheta_0),$$

where

$$\eta^F(x) = \int_{-\infty}^{\infty} H(y,x)\sqrt{f_*(y)}\mathrm{d}W(y) \sim \mathcal{N}(0, 4\mathbf{E}_0\left(H(\xi_0, x-\vartheta_0)\right)^2).$$

Moreover we have the convergence of joint finite-dimensional distribution as follows:

$$\mathcal{L}\left(\eta_T^F(x_1), ..., \eta_T^F(x_k), \widehat{u}_T\right) \Longrightarrow \mathcal{L}\left(\eta^F(x_1-\vartheta_0), ..., \eta^F(x_k-\vartheta_0), \widehat{u}\right),$$

for any $\mathbf{x} = \{x_1, x_2, ..., x_k\} \in \mathbb{R}^k$.

Denote $\zeta_T^F(x) = \sqrt{T}(\widehat{F}_T(x) - F_*(x-\widehat{\vartheta}_T))$. As in the section above, we prove that

Lemma 2.3.3. *Under the conditions $\mathcal{A}_0$ and $\mathcal{A}$,*

$$\mathbf{E}_{\vartheta_0}\left|\zeta_T^F(x)\right|^2 \leq C e^{-2\gamma|x|} \quad x \in \mathbb{R}.$$

Proof. We have

$$\begin{aligned}
\mathbf{E}_{\vartheta_0}\left|\zeta_T^F(x)\right|^2 &= \mathbf{E}_{\vartheta_0}\left|\eta_T^F(x) + \widehat{u}_T f_*(x-\widetilde{\vartheta}_T)\right|^2 \\
&\leq 2(f_*(x-\widetilde{\vartheta}_T))^2\mathbf{E}_{\vartheta_0}|\widehat{u}_T|^2 + 2\mathbf{E}_{\vartheta_0}|\eta_T^F(x)|^2.
\end{aligned}$$

In addition

$$\mathbf{E}_{\vartheta_0}|\widehat{u}_T|^2 \leq C, \quad f_*(x) \leq C\mathrm{e}^{-2\gamma|x|},$$

and

$$\begin{aligned}
\mathbf{E}_{\vartheta_0}|\eta_T^F(x)|^2 &\leq 2\mathbf{E}_{\vartheta_0}\left(\frac{1}{\sqrt{T}}\int_0^T H(X_t-\vartheta_0, x-\vartheta_0)\mathrm{d}W_t\right)^2 \\
&\quad +2\mathbf{E}_{\vartheta_0}\left(\frac{1}{\sqrt{T}}\int_{X_0}^{X_T} H(z-\vartheta_0, x-\vartheta_0)\mathrm{d}z\right)^2 \\
&= 2\mathbf{E}_0 H(\xi_0, x)^2 + \frac{4}{T}\mathbf{E}_0\left(\int_0^{\xi} H(z,x)\mathrm{d}z\right)^2 \leq C\mathrm{e}^{-2\gamma|x|}.
\end{aligned}$$

We obtain thus the result of the lemma.

Lemma 2.3.4. *Let conditions $\mathcal{A}_0$ and $\mathcal{A}$ be fulfilled, then*

$$\mathcal{L}_T\left\{\sup_{x\in\mathbb{R}}\left|\zeta_T^F(x)\right|\right\} \Longrightarrow \mathcal{L}\left\{\sup_{x\in\mathbb{R}}\left|\zeta^F(x-\vartheta_0)\right|\right\}.$$

Proof. Remind that $\zeta_T^F(x) = \eta_T^F(x) + \sqrt{T}\left(F(x-\vartheta_0) - F(x-\widehat{\vartheta}_T)\right)$. Thus we need to prove the weak convergence for the two parts:

$$\mathcal{L}_T\left\{\sup_{x\in\mathbb{R}}\left|\eta_T^F(x)\right|\right\} \Longrightarrow \mathcal{L}\left\{\sup_{x\in\mathbb{R}}\left|\eta^F(x-\vartheta_0)\right|\right\} \tag{2.52}$$

and

$$\mathcal{L}_T\left\{\sup_{x\in\mathbb{R}}\left|\sqrt{T}\left(F(x-\vartheta_0) - F(x-\widehat{\vartheta}_T)\right)\right|\right\} \Longrightarrow \mathcal{L}\left\{\sup_{x\in\mathbb{R}}\left|\widehat{u} f(x-\vartheta_0)\right|\right\}. \tag{2.53}$$

Along with the convergence of the joint finite-dimensional distribution, we have the result of the lemma.

The convergence (2.52) follows from the Theorem 4.6 in Kutoyants [28] and the Theorem A.20 in Ibragimov & Hasminskii [22]. For (2.53), note that

$$\sqrt{T}\left(F(x-\vartheta_0) - F(x-\widehat{\vartheta}_T)\right) = \sqrt{T}(\widehat{\vartheta}_T - \vartheta_0) f(x-\widetilde{\vartheta}_T).$$

We have

$$\sup_{x\in\mathbb{R}}\left|f(x-\widetilde{\vartheta}_T) - f(x-\vartheta_0)\right| = \sup_{x\in\mathbb{R}}\left|f'(\widetilde{x})\right| \cdot \left|\widetilde{\vartheta}_T - \vartheta_0\right| \longrightarrow 0.$$

Therefore

$$\begin{aligned}\sup_{x\in\mathbb{R}}\left|\sqrt{T}\left(F(x-\vartheta_0) - F(x-\widehat{\vartheta}_T)\right)\right| &= \sup_{x\in\mathbb{R}}\left|\sqrt{T}(\widehat{\vartheta}_T - \vartheta_0) f(x-\widetilde{\vartheta}_T)\right| \\ &\Longrightarrow \left|\widehat{u}\right| \sup_{x\in\mathbb{R}}\left|f(x-\vartheta_0)\right|.\end{aligned}$$

Moreover,

$$\sup_{x\in\mathbb{R}}\left|\zeta^F(x-\vartheta_0)\right| = \sup_{(y+\vartheta_0)\in\mathbb{R}}\left|\zeta^F(y)\right| = \sup_{z\in\mathbb{R}}\left|\zeta^F(z)\right| = \Lambda.$$

Thus we have

$$\mathcal{L}_T\left\{\Lambda_T\right\} \Longrightarrow \mathcal{L}\left\{\Lambda\right\}.$$

Note that Λ does not depend on the unknown parameter ϑ_0, therefore the test $\Psi_T = 1\!\!1_{\{\Lambda_T > C_\varepsilon\}}$ belongs to $\mathcal{K}_\varepsilon$ and is APF.

2.3.3 Discussions

We presented two tests and there is a question of comparison of these two tests. As usual in nonparametric hypothesis testing, the tests are compared under some parametric alternatives and the result can depend strongly on the choice of these parametric families. In general the tests based on the estimators of the densities can be sensitive to the alternatives with the densities having havy tails. If the goal of the test is to detect such alternatives then the test based on local time estimator can be preferable.

Below we discuss the consistency of the proposed tests and verify the condition $\mathcal{A}_2$. Firstly we study the behavior of the test statistics in the situation when the hypothesis $\mathcal{H}_0$ is not true. We define the alternative hypothesis as

$$\mathcal{H}_1: \ S(\cdot) \notin \overline{\mathcal{S}(\Theta)},$$

where $\overline{\mathcal{S}(\Theta)} = \{S_* (x - \vartheta), \vartheta \in [\alpha, \beta]\}$. Under this hypothesis we have:

Proposition 2.3.1. *Let all drift coefficients under alternative satisfy the conditions $\mathcal{ES}$, $\mathcal{A}_0$, and $\mathcal{A}$, then for any $S(\cdot) \notin \overline{\mathcal{S}(\Theta)}$ we have*

$$\mathbf{P}_S (\lambda_T > c_\varepsilon) \longrightarrow 1,$$

and

$$\mathbf{P}_S (\Lambda_T > c_\varepsilon) \longrightarrow 1.$$

Since that the prove is similar as Proposition 2.2.1, we omit it.

Remind that our results are obtained under the assumptions $\mathcal{A}_0$ and $\mathcal{A}$. For the properties of $\widehat{u}_T$, we have applied the condition $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$. In the case of shift parameter these assumptions can be reduced to $\mathcal{A}_0$ and $\mathcal{A}_1$. This is to say that the condition $\mathcal{A}_2$ can be deduced from $\mathcal{A}_0$ and $\mathcal{A}_1$.

Proposition 2.3.2. *Let the conditions $\mathcal{A}_0$ and $\mathcal{A}_1$ be fulfilled, then we have:*

$$0 < \mathbf{E}_0 S'(\xi_0)^2 < \infty.$$

Proof. Remind that under $\mathcal{A}_0$ we have (2.13), which means that

$$S(x) < -\gamma \text{ for } x > A, \quad S(x) > \gamma \text{ for } x < -A.$$

Thus there exists at least one point x_0 such that $S'(x_0) \neq 0$. Owing to the continuity of S', there exists $\rho > 0$ such that for $x \in (x_0 - \rho, \ x_0 + \rho)$, $S'(x) \neq 0$, then

$$\mathbf{E}_0 S'(\xi_0)^2 = \int_{-\infty}^{\infty} S'(x)^2 f_*(x) \mathrm{d}x \geq \int_{x_0-\rho}^{x_0+\rho} S'(x)^2 f_*(x) \mathrm{d}x > 0.$$

On the other hand, $S'(\cdot) \in \mathcal{P}$ is of p-polynomial majorants, thus

$$\begin{aligned} \mathbf{E}_0 S'(\xi_0)^2 &= \int_{-\infty}^{\infty} S'(x)^2 f_*(x)\mathrm{d}x \\ &\leq C\int_{-\infty}^{\infty} \left(1+|x|^p\right)^2 \mathrm{e}^{-2\gamma|x|}\mathrm{d}x < \infty. \end{aligned}$$

Proposition 2.3.3. *Let the conditions $\mathcal{A}_0$ and $\mathcal{A}_1$ be fulfilled, then we have: for any $\nu > 0$*

$$\inf_{|\tau|>\nu} \mathbf{E}_0\big(S(\xi_0) - S(\xi_0+\tau)\big)^2 > 0.$$

Proof. In Proposition 2.3.2, we have shown that there exists $\rho > 0$, such that $S'(x) \neq 0$ for $x \in (x_0-\rho,\ x_0+\rho)$. Thus for $\tau < \rho$,

$$\begin{aligned} \mathbf{E}_0\left(S(\xi_0) - S(\xi_0+\tau)\right)^2 &= \int_{-\infty}^{\infty} \left(S(x) - S(x+\tau)\right)^2 f_*(x)\mathrm{d}x \\ &\geq \int_{x_0-\rho+\tau}^{x_0+\rho-\tau} \left(S(x) - S(x+\tau)\right)^2 f_*(x)\mathrm{d}x \\ &= \tau^2 \int_{x_0-\rho+\tau}^{x_0+\rho-\tau} S'(\widetilde{x})^2 f_*(x)\mathrm{d}x \geq\ C\tau^2. \end{aligned}$$

On other hand for any $\tau \geq \rho$, according to (2.13), $S(x+n\tau) \neq S(x-n\tau)$ for n sufficiently large. Thus S can not be a τ-periodic function, then

$$\mathbf{E}_0\big(S(\xi_0) - S(\xi_0+\tau)\big)^2 \neq 0.$$

We obtain thus the result of the proposition.

2.3.4 Numerical example

We consider always the Ornstein-Uhlenbeck process. Suppose that the observed process under the null hypothesis is

$$\mathrm{d}X_t = -(X_t - \vartheta_0)\mathrm{d}t + \mathrm{d}W_t, \quad X_0,\ 0 \leq t \leq T.$$

Remind that the invariant density under $\mathcal{H}_0$ is $f_*(x-\vartheta_0)$, where $f_*(x) = \pi^{-1/2}\mathrm{e}^{-x^2}$. The MLE $\widehat{\vartheta}_T$ can be calculated as

$$\widehat{\vartheta}_T = \frac{1}{T}\int_0^T X_t \mathrm{d}t + \frac{X_T - X_0}{T}.$$

The Fisher information in this case equals to 1, and the LTE is

$$\widehat{f}_T(x) = \frac{1}{T}(|X_T - x| - |X_0 - x|) - \frac{1}{T}\int_0^T \mathrm{sgn}(X_t - x)\mathrm{d}X_t.$$

The conditions $\mathcal{A}_0$ and $\mathcal{A}$ are fulfilled, then the statistic is convergent:

$$\begin{aligned}\lambda_T &= \sqrt{T}\sup_{x\in\mathbb{R}}\left|\widehat{f}_T(x) - f_*(x - \widehat{\vartheta}_T)\right| \\ &\Longrightarrow \sup_{x\in\mathbb{R}}\left|\int_{-\infty}^{\infty}\left(2f_*(x)\frac{F_*(y) - 1\!\!\mathrm{I}_{\{y>x\}}}{\sqrt{f_*(y)}} + f_*'(x)\sqrt{f_*(y)}\right)\mathrm{d}W(y)\right| = \lambda,\end{aligned}$$

Similar result for the test based on the EDF:

$$\Lambda_T = \int_{-\infty}^{\infty}\left(\widehat{F}_T(x) - F_*(x - \widehat{\vartheta}_T)\right)^2\mathrm{d}x \Longrightarrow \Lambda = \int_{-\infty}^{\infty}\left(\zeta_2(x)\right)^2\mathrm{d}x,$$

$$\begin{aligned}\Lambda_T &= \sqrt{T}\sup_{x\in\mathbb{R}}\left|\widehat{F}_T(x) - F_*(x - \widehat{\vartheta}_T)\right| \\ &\Longrightarrow \sup_{x\in\mathbb{R}}\left|\int_{-\infty}^{\infty}\left(2\frac{F_*(y\wedge x) - F_*(y)F_*(x)}{\sqrt{f_*(y)}} + f_*(x)\sqrt{f_*(y)}\right)\mathrm{d}W(y)\right| = \Lambda,\end{aligned}$$

We simulate 10^5 trajectories of λ (resp. Λ) and calculate the empirical $(1-\varepsilon)$-quantiles of λ (resp. Λ).

We obtain the simulated density for λ and Λ that are showed in Graphic 2.4. The values of the thresholds c_ε for different ε are showed in Graphic 2.5.

2.4 The Chi-Square Tests

This chapter is based on the work [50]

2.4.1 Problem statement

We consider the following problem. Suppose that we observe an ergodic diffusion process

$$\mathrm{d}X_t = S(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0,\ 0 \le t \le T. \tag{2.54}$$

and we have to test the following basic hypothesis

$$\mathcal{H}_0 \quad : \qquad S\left(x\right) = S_*\left(x\right),$$

where $S_*(\cdot)$ is some known function.

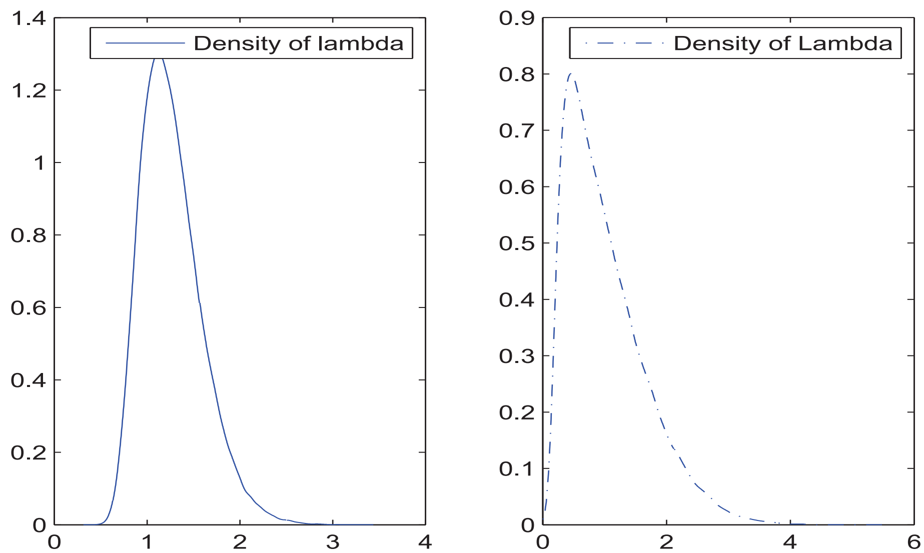

Figure 2.4: Densities of the statistics. On the left the density of λ, on the right the density of Λ.

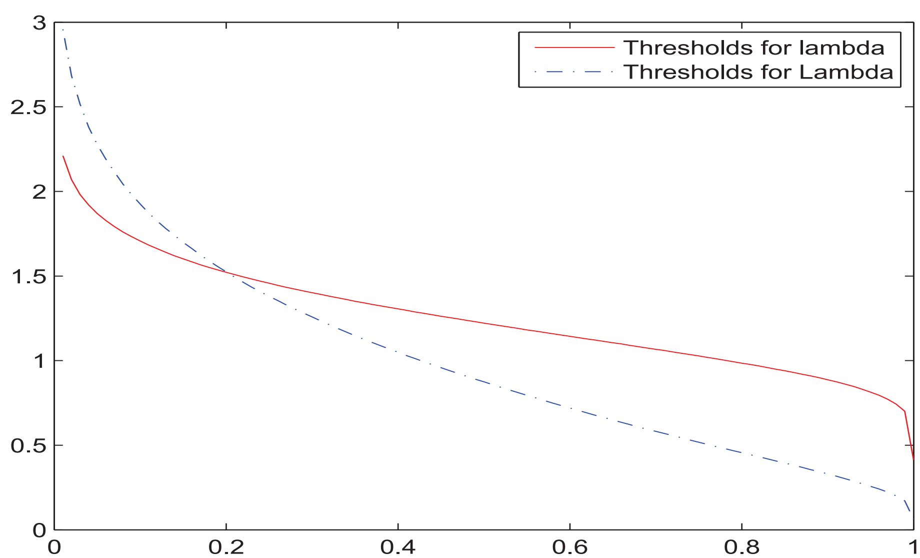

Figure 2.5: Thresholds for different ε. The solid line represents the values for λ, the dotted line represents the values for Λ.

Suppose that the trend coefficient $S(\cdot)$ of the observed diffusion process satisfies the conditions $\mathcal{ES}$ and $\mathcal{A}_0$. Remind that under these conditions, the equation (2.54) has a unique weak solution, the diffusion process is recurrent and its invariant density $f_S(x)$ is

$$f_S(x) = \frac{1}{G(S)\sigma(x)^2} \exp\left\{2\int_0^x \frac{S(y)}{\sigma(y)^2}\mathrm{d}y\right\}.$$

Thus the distribution function is

$$F_S(x) = \int_{-\infty}^x f_S(y)\mathrm{d}y, \quad x \in \mathbb{R}.$$

Denote by ξ_S a r.v. with invariant density $f_S(x)$ and by $\mathbf{E}_S$ the corresponding mathematic expectation. To simplify the notations, the invariant density under hypothesis $\mathcal{H}_0$ is denoted as $f_*(x)$ and the mathematical expectation is $\mathbf{E}_*$.

Let us introduce the space $\mathcal{L}^2(f)$ of square integrable functions with weight $f(\cdot)$:

$$\mathcal{L}^2(f) = \left\{h(\cdot):\ \mathbf{E}h(\xi)^2 = \int_{-\infty}^{\infty} h(x)^2 f(x)\mathrm{d}x < \infty\right\}.$$

Correspondingly, we have $\mathcal{L}^2(f_*)$ the square integrable function space with weight $f_*(\cdot)$. Remind that according to (2.14), under the condition $\mathcal{A}_0$ the density function f_* is of negative exponential majorant. Thus $\mathbf{E}_*\left(S_*(\xi_{S_*})\right)^2 < \infty$ and so that $S_* \in \mathcal{L}^2(f_*)$.

Denote by $\{\phi_1, \phi_2, ...\}$ a complete orthonormal basis in the space $\mathcal{L}^2(f_*)$. The alternative is as follows: for some $N \in \mathbb{N}$ fixed

$$\mathcal{H}_{1,N}: \quad S(\cdot) \in \mathcal{S}_N,$$

where $\mathcal{S}_N$ is the subspace of square integrable function such that

$$\begin{aligned}\mathcal{S}_N \;=\; \Bigg\{S(\cdot) \in \mathcal{L}^2(f_*) \Bigg| & \sum_{i=1}^N \int_{-\infty}^{\infty} \phi_i(x)^2 f_S(x)\mathrm{d}x < \infty, \\ & \sum_{i=1}^N \left(\int_{-\infty}^{\infty}\left(\frac{S(x) - S_*(x)}{\sigma(x)}\right)\phi_i(x) f_S(x)\mathrm{d}x\right)^2 > 0\Bigg\}.\end{aligned}$$

2.4.2 The properties of a chi-square test

We construct the chi-square test. Let us denote

$$\eta_{i,T} = \frac{1}{\sqrt{T}}\int_0^T \frac{\phi_i(X_t)}{\sigma(X_t)}[\mathrm{d}X_t - S_*(X_t)\mathrm{d}t].$$

For the N fixed, we denote

$$\mu_{T,N} = \sum_{i=1}^{N} \eta_{i,T}^2.$$

Then we have

Theorem 2.4.1. *The test* $\rho_{T,N} = 1\!\!1_{\{\mu_{T,N} > z_\varepsilon\}}$, *with* z_ε *the* $(1-\varepsilon)$*-quantile of* $\chi^2(N)$ *law, is ADF, belongs to* $\mathcal{K}_\varepsilon$ *and is consistent against the alternative* $\mathcal{H}_{1,N}$.

Proof. Since that $(\phi_1, \phi_2, ...)$ is an orthonormal basis in $\mathcal{L}^2(f_*)$, we have $\mathbf{E}_*(\phi_i(\xi)\phi_j(\xi)) = \delta_{ij}$. Thus according to the central limit theorem in Kutoyants [28], we have under the hypothesis $\mathcal{H}_0$:

$$\begin{aligned}
\eta_{i,T} &= \frac{1}{\sqrt{T}} \int_0^T \frac{\phi_i(X_t)}{\sigma(X_t)} [\mathrm{d}X_t - S_*(X_t)\mathrm{d}t] \\
&= \frac{1}{\sqrt{T}} \int_0^T \phi_i(X_t)\mathrm{d}W_t \Longrightarrow \eta_i \sim \mathcal{N}(0,1), \quad \text{as } T \longrightarrow \infty.
\end{aligned}$$

Moreover, for any $\mathbf{k} = (k_1, ..., k_N) \in \mathbb{R}^N$,

$$\sum_{i=1}^{N} k_i \eta_{i,T} = \frac{1}{\sqrt{T}} \int_0^T \sum_{i=1}^{N} k_i \phi_i(X_t)\mathrm{d}W_t,$$

and

$$\begin{aligned}
\frac{1}{T} \int_0^T \left(\sum_{i=1}^{N} k_i \phi_{i,T}(X_t) \right)^2 \mathrm{d}t &= \sum_{i,j=1}^{N} k_i k_j \left(\frac{1}{T} \int_0^T \phi_{i,T}(X_t)\phi_{j,T}(X_t)\mathrm{d}t \right) \\
&\longrightarrow \sum_{i,j=1}^{N} k_i k_j \mathbf{E}_* \left(\phi_i(\xi)\phi_j(\xi)\right) = \sum_{i=1}^{N} k_i^2.
\end{aligned}$$

We have the convergence in distribution:

$$\frac{1}{\sqrt{T}} \int_0^T \sum_{i=1}^{N} k_i \phi_i(X_t)\mathrm{d}W_t \Longrightarrow \mathcal{N}\left(0, \sum_{i=1}^{N} k_i^2\right).$$

Thus

$$(\eta_{i,T},\ i = 1, ..., k) \Longrightarrow (\eta_i,\ i = 1, ..., k)$$

where $(\eta_1, ..., \eta_N)$ are N independent gaussian variables: $\eta_i \sim \mathcal{N}(0,1)$. Thus we have $\mu_{T,N} \Longrightarrow \chi^2(N)$. We conclude that the test $\rho_{T,N} = 1\!\!1_{\{\mu_{T,N} > z_\varepsilon\}}$ belongs to $\mathcal{K}_\varepsilon$, with z_ε the solution of

$$\mathbf{P}\left(\chi^2(N) \geq z_\varepsilon\right) = \varepsilon.$$

Now we verify the consistency. To simplify the notations, we denote

$$\zeta_{i,T} = \frac{1}{\sqrt{T}} \int_0^T \phi_i(X_t) \mathrm{d}W_t,$$

and

$$\theta_{i,T} = \frac{1}{T} \int_0^T \frac{\phi_i(X_t)}{\sigma(X_t)} \left(S(X_t) - S_*(X_t)\right) \mathrm{d}t,$$
$$\theta_i = \int_{-\infty}^{\infty} \frac{\phi_i(x)}{\sigma(x)} \left(S(x) - S_*(x)\right) f_S(x) \mathrm{d}x.$$

We denote the vectors $\theta_{\mathbf{T}} = (\theta_{i,T},\ i = 1, ..., N)$ and $\zeta_{\mathbf{T}} = (\zeta_{i,T},\ i = 1, ..., N)$, $\|\cdot\|$ is the Euclidean norm: for vector $x = (x_1, x_2, ..., x_n)$, $\|x\| = \sqrt{x_1^2 + ... + x_n^2}$.

Under the hypothesis $\mathcal{H}_{1,N}$, we have

$$\eta_{i,T} \quad = \quad \frac{1}{\sqrt{T}} \int_0^T \frac{\phi_i(X_t)}{\sigma(X_t)} [\mathrm{d}X_t - S_*(X_t)\mathrm{d}t] = \sqrt{T}\theta_{i,T} + \zeta_{i,T},$$

Note that $\theta_{i,T} \longrightarrow \theta_i$ according the law of large numbers, and that $\sum\limits_{i=1}^{N} \theta_i^2 > 0$. Thus we have

$$\sqrt{T} \|\theta_T\| = \sqrt{T} \left(\sum_{i=1}^{N} \theta_{i,T}^2 \right)^{1/2} \longrightarrow \infty.$$

In addition

$$\mathbf{E}_S \left\|\zeta_T\right\|^2 = \mathbf{E}_S \left(\sum_{i=1}^{N} \zeta_{i,T}^2 \right) = \sum_{i=1}^{N} \mathbf{E}_S \left(\phi_i(\xi_S)^2 \right) < \infty,$$

according to the definition of the alternative. We obtain that

$$\begin{aligned} \mathbf{P}_S \left(\mu_{T,N} > z_\varepsilon \right) \quad &= \quad \mathbf{P}_S \left(\sum_{i=1}^{N} \left(\sqrt{T}\theta_{i,T} + \zeta_{i,T} \right)^2 > z_\varepsilon \right) \\ &= \quad \mathbf{P}_S \left(\left\| \sqrt{T}\theta_{\mathbf{T}} + \zeta_{\mathbf{T}} \right\| > \sqrt{z_\varepsilon} \right) \\ &\geq \quad \mathbf{P}_S \left(\sqrt{T} \left\|\theta_{\mathbf{T}}\right\| - \left\|\zeta_{\mathbf{T}}\right\| > \sqrt{z_\varepsilon} \right) \longrightarrow 1. \end{aligned}$$

2.4.3 Pitman alternative

Let us consider the asymptotic behavior under the Pitman alternative:

$$\mathcal{H}_1 : \ S(x) = S_*(x) + \frac{1}{\sqrt{T}} h(x),$$

where $h \in \mathcal{L}^2(f_*)$. Remind that the likelihood ratio in this case is asymptotically non-degenerate. We construct the test as in the above subsection

$$\eta_{i,T} = \frac{1}{\sqrt{T}} \int_0^T \frac{\phi_i(X_t)}{\sigma(X_t)} [\mathrm{d}X_t - S_*(X_t)\mathrm{d}t].$$

For the N fixed, let us denote

$$\mu_{T,N} = \sum_{i=1}^{N} \eta_{i,T}^2.$$

The chi-square test is $\rho_{T,N} = 1\!\!1_{\{\mu_{T,N} > z_\varepsilon\}}$, with z_ε the $(1-\varepsilon)$-quantile of $\chi^2(N)$ law.

Let us denote $(\eta_1, ..., \eta_N)$ a N dimensional independent standard Gaussian random vector and

$$\theta_i = \int_{-\infty}^{\infty} \frac{\phi_i(x)}{\sigma(x)} h(x) f_*(x) \mathrm{d}x.$$

We have the following result

Theorem 2.4.2. *Let the conditions $\mathcal{ES}$ and $\mathcal{A}_0$ be fulfilled, then the power function of the test $\rho_{T,N}$ is*

$$\beta(h, \rho_{T,N}) = \mathbf{P}\left(\sum_{i=1}^{N} \eta_{i,T}^2 > z_\varepsilon \right) \longrightarrow \mathbf{P}\left(\sum_{i=1}^{N} (\eta_i + \theta_i)^2 > z_\varepsilon \right).$$

Proof. The invariant density function under the alternative is

$$f_S(x) = \frac{1}{G(S)} \exp\left\{ 2 \int_0^x \frac{S(v)}{\sigma(v)^2} \mathrm{d}v \right\} = f_*(x) \frac{G(S_*)}{G(S)} \exp\left\{ \frac{2}{\sqrt{T}} \int_0^x \frac{h(v)}{\sigma(v)^2} \mathrm{d}v \right\},$$

where

$$\exp\left\{ \frac{2}{\sqrt{T}} \int_0^x \frac{h(v)}{\sigma(v)^2} \mathrm{d}v \right\} = 1 + \frac{2}{\sqrt{T}} \int_0^x \frac{h(v)}{\sigma(v)^2} \mathrm{d}v + o\left(\frac{1}{\sqrt{T}} \right),$$

and

$$
\begin{aligned}
G(S) &= \int_{-\infty}^{\infty} \exp\left\{2\int_0^x \frac{S_*(v)+\frac{1}{\sqrt{T}}h(v)}{\sigma(v)^2}\mathrm{d}v\right\}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \exp\left\{2\int_0^x \frac{S_*(v)}{\sigma(v)^2}\mathrm{d}v\right\}\exp\left\{\frac{2}{\sqrt{T}}\int_0^x \frac{h(v)}{\sigma(v)^2}\mathrm{d}v\right\}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \exp\left\{2\int_0^x \frac{S_*(v)}{\sigma(v)^2}\mathrm{d}v\right\}\left(1+\frac{2}{\sqrt{T}}\int_0^x \frac{h(v)}{\sigma(v)^2}\mathrm{d}v + o\left(\frac{1}{\sqrt{T}}\right)\right)\mathrm{d}x \\
&= G(S_*) + \frac{2}{\sqrt{T}}\int_{-\infty}^{\infty} \exp\left\{2\int_0^x \frac{S_*(v)}{\sigma(v)^2}\mathrm{d}v\right\}\int_0^x \frac{h(v)}{\sigma(v)^2}\mathrm{d}v\mathrm{d}x + o\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}
$$

Thus we have for $T \longrightarrow \infty$

$$
\begin{aligned}
f_S(x) = f_*(x)\frac{G(S_*)}{G(S)}\exp\left\{\frac{2}{\sqrt{T}}\int_0^x \frac{h(v)}{\sigma(v)^2}\mathrm{d}v\right\} &= f_*(x)\exp\left\{\frac{2}{\sqrt{T}}\int_0^x \frac{h(v)}{\sigma(v)^2}\mathrm{d}v\right\} \\
-\frac{2}{\sqrt{T}}\left(\int_{-\infty}^{\infty} \mathrm{e}^{2\int_0^x \frac{S_*(v)}{\sigma(v)^2}\mathrm{d}v}\int_0^x \frac{h(v)}{\sigma(v)^2}\mathrm{d}v\mathrm{d}x\right)&\frac{f_*(x)}{G(S)}\mathrm{e}^{\frac{2}{\sqrt{T}}\int_0^x \frac{h(v)}{\sigma(v)^2}\mathrm{d}v} + o\left(\frac{1}{\sqrt{T}}\right) \\
\longrightarrow f_*(x),&
\end{aligned}
$$

Therefore,

$$
\int_{-\infty}^{\infty} \frac{\phi_i(x)}{\sigma(x)}h(x)f_S(x)\mathrm{d}x - \theta_i = \int_{-\infty}^{\infty} \frac{\phi_i(x)}{\sigma(x)}h(x)\left(f_S(x)-f_*(x)\right)\mathrm{d}x \longrightarrow 0.
$$

Furthermore, according to the law of large numbers

$$
\theta_{i,T} - \int_{-\infty}^{\infty} \frac{\phi_i(x)}{\sigma(x)}h(x)f_S(x)\mathrm{d}x \longrightarrow 0.
$$

We obtain thus $\theta_{i,T} \longrightarrow \theta_i$ and then

$$
\sum_{i=1}^{N} \eta_{i,T}^2 \Longrightarrow \sum_{i=1}^{N} (\eta_i + \theta_i)^2.
$$

Therefore the power

$$
\beta(h, \rho_{T,N}) = \mathbf{P}\left(\sum_{i=1}^{N} \eta_{i,T}^2 > z_\varepsilon\right) \longrightarrow \mathbf{P}\left(\sum_{i=1}^{N} (\eta_i + \theta_i)^2 > z_\varepsilon\right).
$$

2.4.4 Example

Let us propose an example. Suppose that the observed process satisfies the following equation under the hypothesis $\mathcal{H}_0$:

$$\mathrm{d}X_t = -aX_t\mathrm{d}t + \sigma\mathrm{d}W_t,$$

where a and σ are known parameters. We have the invariant density under this hypothesis

$$f_*(x) = \sqrt{\frac{a}{\pi\sigma^2}}\mathrm{e}^{-\frac{a}{\sigma^2}x^2}.$$

Let us define $(\phi_1(x), \phi_2(x), ...)$ the basis in the space $\mathcal{L}^2(f_*)$ as follows

$$\begin{aligned}
&\phi_1(x) = 1, \quad \phi_2(x) = \sqrt{\frac{2a}{\sigma^2}}\,x, \quad \phi_3(x) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}\,a}{\sigma^2}\,x^2,\\
&\phi_4(x) = -\sqrt{\frac{3a}{\sigma^2}}\,x + \sqrt{\frac{4a^3}{3\sigma^6}}\,x^3, \quad ...
\end{aligned}$$

In taking $N = 4$, we have the statistic for chi-square test as follows

$$\mu_{T,4} = \sum_{i=1}^{4}\left(\frac{1}{\sqrt{T}}\int_0^T \frac{\phi_i(X_t)}{\sigma(X_t)}[\mathrm{d}X_t - S_*(X_t)\mathrm{d}t]\right)^2 \Longrightarrow \mu_4 \sim \chi^2(4).$$

Then the chi-square test $\rho_{T,4} = \mathbb{1}_{\{\mu_{T,4} > z_\varepsilon\}}$ with z_ε the $(1-\varepsilon)$-quantile of $\chi^2(4)$ law is ADF.

We show the convergence of the statistic in graphic 2.6. Note that as T increases, the curve is more and more approach to the density curve of $\chi^2(4)$.

2.4.5 Discussions

We consider this kind of test for the advantage that it is ADF, that is the limit of the statistic does not depend on the coefficient function. But in considering the consistency, it is not a good choice to fix the number of basis N. In fact, more basis we take, better test we obtain. Thus it is natural to consider the case where $N \longrightarrow \infty$. For this purpose, we remind that

Lemma 2.4.1. *If $X \sim \chi^2(N)$, then as N tends to infinity, the distribution of $\frac{(X-N)}{\sqrt{2N}} \sim \mathcal{N}(0,1)$.*

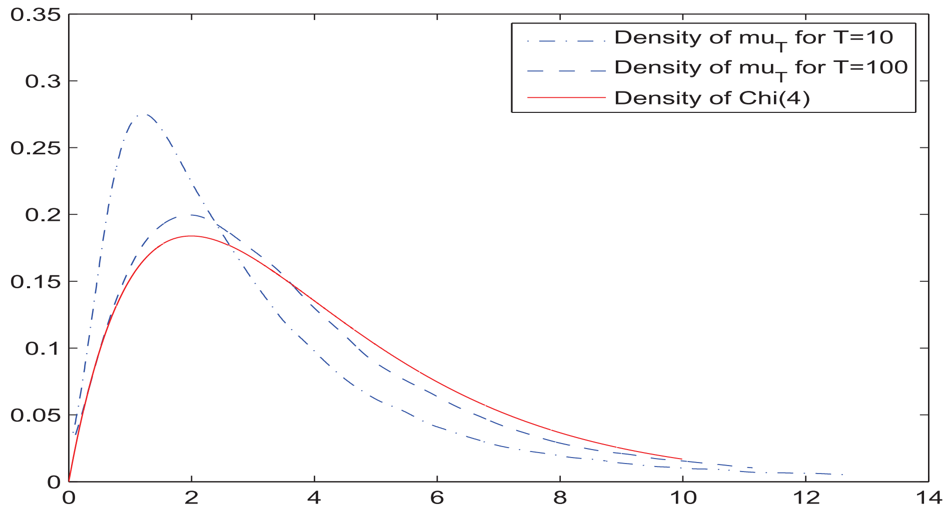

Figure 2.6: The density of $\mu_{T,4}$ and $\chi^2(4)$.

Thus we test the hypothesis $\mathcal{H}_0$ under the alternative $\mathcal{H}_{1,\infty}$. Let us consider the following statistic

$$\nu_{T,N} = \frac{1}{\sqrt{2N}} \sum_{i=1}^{N} \left(\eta_{i,T}^2 - 1\right),$$

where

$$\eta_{i,T} = \frac{1}{\sqrt{T}} \int_0^T \frac{\phi_i(X_t)}{\sigma(X_t)} [\mathrm{d}X_t - S_*(X_t)\mathrm{d}t].$$

Then we have

Proposition 2.4.1. *The test $\rho_{T,N} = 1\!\!1_{\{\nu_{T,N} > Z_\varepsilon\}}$, with Z_ε the $(1-\varepsilon)$-quantile of $\mathcal{N}(0,1)$ law, belongs to $\mathcal{K}_\varepsilon$ for $T \to \infty$ and then $N \to \infty$. Moreover, the test is ADF, and consistent against the alternative $\mathcal{H}_{1,\infty}$.*

Remark 2.4.1. *A test more interesting is the case where N depend on T, noting N_T, such that T and N_T converge to infinity simultaneously. That is, we consider the following statistic*

$$\mu_T = \frac{1}{\sqrt{2N_T}} \sum_{i=1}^{N_T} \left(\eta_{i,T}^2 - 1\right).$$

We look for N_T such that when T converges to infinity, the statistic is convergent. This was our first purpose to consider the chi-square test. But it has not yet been resolved account for the dependance between η_{i,t_1} and η_{i,t_2} even for long time distance, that is for $|t_1 - t_2| \longrightarrow \infty$. We will try to resolve this problem in the future.

Chapter 3

Approximation of BSDE

This chapter is based on the work [31]

3.1 Introduction

We consider the following problem.

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + a(X_t)\,\mathrm{d}W_t, \quad X_0 = x_0,\ 0 \le t \le T \tag{3.1}$$

and we are given functions $f(t, x, y, z)$ and $\Phi(x)$. We have to construct a couple of processes (Y_t, Z_t) such that the solution of the equation

$$\mathrm{d}Y_t = -f(t, X_t, Y_t, Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t,\ 0 \le t \le T, \tag{3.2}$$

has the final value $Y_T = \Phi(X_T)$.

The existence and uniqueness of the solution for backward stochastic differential equations (BSDE) is well-known owing to Pardoux and Peng [43]. The problem that we considered above was introduced as forward-backward stochastic differential equations (FBSDE) in El Karoui & al. [15], the solution for this FBSDEs is presented as a triple of process $(X_t,\, Y_t,\, Z_t)_{t\ge 0}$. They proved in [15] that the solution $(X_t,\, Y_t,\, Z_t)_{t\ge 0}$ exists and is unique under the condition that the coefficient functions are all Lipschitzian and that they are of linear growth. In addition, they introduced the relation between a FBSDE and a partial differential equation (PDE). In fact, suppose that $u(t, x)$ is the solution of the following equation

$$\frac{\partial u}{\partial t} + b(x)\frac{\partial u}{\partial x} + \frac{1}{2}a(x)^2\frac{\partial^2 u}{\partial x^2} = -f\left(t, y, u, a(x)\frac{\partial u}{\partial x}\right), \quad u(T, x) = \Phi(x). \tag{3.3}$$

Then applying the Itô formula to process $Y_t = u(t, X_t)$, we have the stochastic differential

$$\mathrm{d}Y_t = \left[\frac{\partial u}{\partial t}(t, X_t) + b(X_t)\frac{\partial u}{\partial x}(t, X_t) + \frac{1}{2}a(x)^2\frac{\partial^2 u}{\partial x^2}(t, X_t)\right]\mathrm{d}t + a(X_t)\frac{\partial u}{\partial x}(t, X_t)\,\mathrm{d}W_t,$$
$$= -f(t, X_t, Y_t, Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t \qquad Y_0 = u(0, X_0),$$

where $Z_t = a(X_t)\,u'(t, X_t)$. Therefore the problem is solved and the couple (Y_t, Z_t) provides the desired solution. More details and explication can be founded in El Karoui & Mazliak [14] and Ma & Yong [36].

In the present work we consider the similar statement but in the situation when the trend coefficient $b(x)$ of the diffusion process (3.1) depends on the unknown parameter $\vartheta \in \Theta \subset R^d$, i.e., $b(x) = S(\vartheta, x)$. In this case the function $u(t, x)$ satisfying the equation (3.3) depends on unknown parameter ϑ and we can not put $Y_t = u(t, X_t, \vartheta)$. Therefore, we consider the problem of adaptive construction of the couple $(\widehat{Y}_t, \widehat{Z}_t)$, where $\widehat{Y}_t$ and $\widehat{Z}_t$ are some approximations of (Y_t, Z_t). This approximation is done with the help of the maximum likelihood estimator $\widehat{\vartheta}$. We are interested by a situation when the error of this approximation is small. One of the possibilities to have a small error of approximations is in some sense equivalent to the situation with the small error of estimation of the parameter ϑ, then from the continuity of the function $u(t, x, \vartheta)$ w.r.t. ϑ, we obtain $\widehat{Y}_T \sim Y_T = \Phi(X_T)$. The small error of estimation we can have, besides others, in the situations when $T \to \infty$ or when $a(\cdot) \to 0$ (see, e.g., Kutoyants [28] and [27]). In our statement we propose to study this model in the asymptotics of *small noise*, i.e. the diffusion coefficient tends to 0. This allows us to keep the final time T fixed and, what is as well important, this asymptotics is easier to treat. At the beginning, we consider a relatively simple case when the trend coefficient $S(\vartheta, x)$ is a linear function of ϑ, diffusion coefficient of (3.1) is $a(x)^2 = \varepsilon^2\sigma(x)^2$ and the function $f(t, x, y, z)$ is linear w.r.t. x. We show (under regularity conditions) that the proposed $\widehat{Y}_t$ is close to Y_t for the small values of ε.

We believe that the presented results can be valid (generalized) for essentially more general, say, nonlinear models and the conditions of regularity can be weakened.

3.1.1 Preliminaries

We introduce in this section some regularity results for solutions of PDEs. To be more clear, we introduce firstly the linear case. Suppose that the observed process $X^T = (X_t, 0 \le t \le T)$ satisfies the stochastic differential equation

$$\mathrm{d}X_t = \vartheta h(X_t)\,\mathrm{d}t + \varepsilon\sigma(X_t)\,\mathrm{d}W_t, \quad X_0 = x_0,\ 0 \le t \le T, \tag{3.4}$$

where $h(\cdot)$ and $\sigma(\cdot)$ are some given functions and $\vartheta \in \Theta = (\alpha, \beta)$ is unknown parameter. We are given as well the functions $k(\cdot)$, $g(\cdot)$ and $\Phi(\cdot)$ and our goal is to construct the couple of process (Y, Z) such that the process Y_t satisfies the equation

$$\mathrm{d}Y_t = (k(X_t) + g(X_t)\,Y_t)\,\mathrm{d}t + Z_t \mathrm{d}W_t, \quad 0 \le t \le T, \tag{3.5}$$

with the final value $Y_T = \Phi(X_T)$. The corresponding PDE is

$$\frac{\partial u}{\partial t} + \vartheta h\left(x\right) \frac{\partial u}{\partial x} + \frac{\varepsilon^2}{2} \sigma\left(x\right)^2 \frac{\partial^2 u}{\partial x^2} = k\left(x\right) + g\left(x\right)\,u, \qquad u\left(T, x\right) = \Phi\left(x\right) \tag{3.6}$$

The solution of this problem with unknown ϑ is probably impossible to express and we seek the approximate $(\widehat{Y}_t, \widehat{Z}_t)$ which are close to (Y_t, Z_t) for the small values of ε.

In section 3.2, we consider the system (3.4)-(3.5) with the coefficient functions satisfying the following conditions:

Condition $\mathcal{A}$.

$\mathcal{A}1$. *The functions $\sigma(x)$, $h(x)$,$k(x)$ and $g(x)$ are bounded and have continuous bounded derivatives $\sigma'\left(x\right), h'\left(x\right), k'\left(x\right)$ and $g'\left(x\right)$.*
$\mathcal{A}2$. *The function $\Phi(x)$ is bounded and continuous.*
$\mathcal{A}3$. *There exists $\kappa_0 > 0$ such that $h(x_0)^2 > \kappa_0$ and $\sigma(x)^2 > \kappa_0,\ \forall x \in \mathbb{R}$.*

Below we remind some preliminary results.

Deterministic case.

Suppose that $\varepsilon = 0$. Then the system (3.4)-(3.5) becomes a system of ordinary differential equations

$$\begin{cases} \dfrac{\partial x_t}{\partial t} = \vartheta\, h\left(x_t\right), \qquad x_0, \quad 0 \le t \le T, & \text{(3.7a)} \\ \dfrac{\partial y_t}{\partial t} = k\left(x_t\right) + g\left(x_t\right)\,y_t, \qquad y_T = \Phi(x_T), \quad 0 \le t \le T. & \text{(3.7b)} \end{cases}$$

Note that in this case the parameter ϑ can be calculated without error. For example, we have the equality

$$\vartheta = h\left(x_t\right)^{-1} \frac{\partial x_t}{\partial t}$$

which is valid for all $t \in (0, T]$. To have the final value $y_T = \Phi\left(x_T\right)$ we can first solve the equation

$$\frac{\partial u^0}{\partial t} + \vartheta h\left(x\right) \frac{\partial u^0}{\partial x} = k\left(x\right) + g\left(x\right)\,u^0 \tag{3.8}$$

with the final value $u^0\left(T, x\right) = \Phi\left(x\right)$ and then to put $y_t = u^0\left(t, x_t\right)$. Then we obtain the soluton y_t which satisfies the equation (3.7b) and has the final value $\Phi\left(x_T\right)$.

Therefore, the only thing we need is the initial value $y_0 = u^0(0, x_0)$ for the equation (3.7b).

Note that the solution of (3.7b) can be written explicitly

$$y_t = y_0 \exp\left\{\int_0^t g(x_s)\,\mathrm{d}s\right\} + \int_0^t \exp\left\{\int_s^t g(x_v)\,\mathrm{d}v\right\} k(x_s)\ \mathrm{d}s$$

and the initial value y_0 can be found from the following equality

$$\Phi(x_T) = y_0 \exp\left\{\int_0^T g(x_s)\,\mathrm{d}s\right\} + \int_0^T \exp\left\{\int_s^T g(x_v)\,\mathrm{d}v\right\} k(x_s)\ \mathrm{d}s.$$

Let us change the variables

$$\int_0^T g(x_s)\,\mathrm{d}s = \int_0^T \frac{g(x_s)}{\vartheta h(x_s)}\mathrm{d}x_s = \int_{x_0}^{x_T} \frac{g(z)}{\vartheta h(z)}\mathrm{d}z \equiv \ln \Psi(x_T).$$

Hence

$$y_T = u^0(T, x_T) = \Psi(x_T)\left[y_0 + \int_{x_0}^{x_T} \Psi(z)^{-1}\frac{k(z)}{\vartheta h(z)}\mathrm{d}z\right]$$

but this solution is not satisfactory because to calculate y_t at the instant $t = 0$ we have to use the value x_T from the *future*.

Non-deterministic case.

The approximation $\widehat{Y}_t$ we construct with the help of the solution $u(t, x)$ of the equation (3.6) and as we are interested by the asymptotics $\varepsilon \to 0$, we need the convergence of the solution of (3.6) to the solution $u^0(t, x)$ of the equation (3.9).

As the solution of (3.6), u depends also on ϑ, we write from now on $u(t, x, \vartheta)$. We are interested in the regularities of $u(t, x, \vartheta)$ w.r.t. x and ϑ, which was studied by Friedman [18]. Here they studied a similar kind of PDE where the initial value but not the terminal was given. This does not change a lot because that a change of variable $v(t, x) = u(T - t, x)$ makes the results coincide with our case. First of all we give the existence of the solution

Lemma 3.1.1. *Let the condition $\mathcal{A}$ be fulfilled, then the solution of* (3.6) $u(t, x, \vartheta)$ *exists for all* $(t, x) \in [0, T] \otimes \mathbb{R}$, *and*

$$|u(t, x, \vartheta)| \leq Ce^{\nu|x^2|}, \quad \textit{for } (t, x, \vartheta) \in [0, T] \times \mathbb{R} \times \Theta.$$

See Theorem 1.12 in Friedman [18].

Lemma 3.1.2. *Let the condition $\mathcal{A}$ be fulfilled, then the solution of* (3.6) $u(t,x,\vartheta)$

- is 2 times differentiable w.r.t. x in bounded domain $D \in \mathbb{R}$.

- is infinitely differentiable w.r.t. ϑ, and these derivatives have derivatives of any order w.r.t. x.

The first result was given in Theorem 1.16 in Friedman[18]. For the second result, suppose that $\Gamma(t,x;\tau,\lambda)$ is the fundamental solution of (3.6) (solution for the case $k=0$). According to Lemma 9.3 in Friedman[18], $\Gamma(t,x;\tau,\lambda)$ is infinitely differentiable w.r.t. ϑ, and these derivatives have derivatives of any order w.r.t. x. Note that the solution of (3.6) can be presented as (See Theorem 1.12 in Friedman [18])

$$u(t,x,\vartheta) = \int_{\mathbb{R}} \Gamma(t,x;0,\lambda;\vartheta)\Phi(\lambda)\mathrm{d}\lambda - \int_0^t \int_{\mathbb{R}} \Gamma(t,x;\tau,\lambda,\vartheta)k(\lambda)\mathrm{d}\lambda\mathrm{d}\tau.$$

We have this differentiable result w.r.t. ϑ for $u(t,x,\vartheta)$.

The convergence of u to u^0 was studied by Freidlin and Wentzell [17]. We present in the following lemma

Lemma 3.1.3. *Suppose that the conditions $\mathcal{A}_1$ and $\mathcal{A}_3$ are fulfilled, then the solution of* (3.6) *converges to the solution of* (3.9)*:*

$$\lim_{\varepsilon\to 0} u(t,x,\vartheta) = u^0(t,x,\vartheta).$$

See Theorem 1.3.1 in Freidlin and Wentzell [17].

General case.

Let us consider a more general case. We deal with the diffusion process

$$\mathrm{d}X_t = S(\vartheta,X_t)\mathrm{d}t + \varepsilon\sigma(X_t)\mathrm{d}W_t, \quad X_0 = x_0, \quad 0\le t\le T. \tag{3.9}$$

where $\vartheta \in \Theta = (\alpha,\,\beta)$ is an unknown parameter. The parameter $\varepsilon \in (0,1]$, and the limit corresponds to $\varepsilon \to 0$. We have to construct the process $\left(\widehat{Y}_t, \widehat{Z}_t\right)$ which is close to the solution of the equation (Y_t, Z_t)

$$\mathrm{d}Y_t = \left[k\left(X_t\right) + g\left(X_t\right)\,Y_t\right]\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \qquad Y_T = \Phi\left(X_T\right), \quad 0\le t\le T. \tag{3.10}$$

The PDE corresponding to this problem is

$$\frac{\partial u}{\partial t}(t,x,\vartheta) + S(\vartheta,x)\frac{\partial u}{\partial x}(t,x,\vartheta) + \frac{1}{2}\varepsilon^2\sigma(x)^2\frac{\partial^2 u}{\partial x^2}(t,x,\vartheta) = k(x) + g(x)u(t,x,\vartheta), \tag{3.11}$$

with terminal condition $u(t,x,\vartheta)=\Phi(x)$. For $\varepsilon=0$, we have the deterministic PDE

$$\frac{\partial u^0}{\partial t}(t,x,\vartheta)+S(\vartheta,x)\frac{\partial u^0}{\partial x}(t,x,\vartheta)=k(x)+g(x)u^0(t,x,\vartheta),\quad u^0(T,x,\vartheta)=\Phi(x). \tag{3.12}$$

For any function $h(t,x,\vartheta)$, $h'(t,x,\vartheta)$ is defined as the derivative w.r.t. x. We define in addition $\dot{h}(t,x,\vartheta)$ and $\ddot{h}(t,x,\vartheta)$ the derivatives w.r.t. ϑ, and $\dot{h}'(t,x,\vartheta)$ is the second order derivative w.r.t. x and ϑ, etc. Let us introduce the regularity conditions $\mathcal{B}$:

$\mathcal{B}_1$. *The functions $\sigma(x)$ and $S(\vartheta,x)$ are differentiable w.r.t. x, the function $S(\vartheta,x)\in\mathcal{C}_\vartheta^{(5)}$, and all these derivatives are continuous and bounded. In addition, there exists $\kappa_1>0$ such that $\sigma(x)^2>\kappa_1,\ x\in\mathbb{R}$.*

$\mathcal{B}_2$. *The function $\Phi(x)$ is bounded and continuous. The function $k(x)$ is bounded and has continuous bounded derivative $k'(x)$.*

$\mathcal{B}_3$. *For a fixed time δ, the Fisher information is positive:*

$$I(x^\delta,\vartheta)=\int_0^\delta\frac{\dot{S}(\vartheta,x_s)^2}{\sigma(x_s)^2}\mathrm{d}s>0,$$

and for any $\nu>0$,

$$\inf_{|\theta-\vartheta|>\nu}\left\|\frac{S(\theta,x)-S(\vartheta,x)}{\sigma(x)}\right\|_\delta>0.$$

Here $\|\cdot\|_t$ is the norm in the space of square integrable functions:

$$\|f\|_t=\left(\int_0^t f(s,\omega)^2\mathrm{d}s\right)^{1/2}.$$

The following result by Friedman [18] and Freidlin & Wentzell [17] will be used in the sequel.

Lemma 3.1.4. *Suppose that the conditions $\mathcal{B}_1$ and $\mathcal{B}_2$ are fulfilled, then the solution of PDE* (3.11) *$u(t,x,\vartheta)$ and that of* (3.12) *$u^0(t,x,\vartheta)$ exist and*

- *$u(t,x,\vartheta)\in\mathcal{C}_x^{(2)}$ in any bounded domain $D\in\mathbb{R}$.*
- *$u(t,x,\vartheta)\in\mathcal{C}_\vartheta^{(5)}$ and these derivatives have derivatives of any order w.r.t. x.*
- *the solution of* (3.11) *converges to the solution of* (3.12)*:*

$$\lim_{\varepsilon\to0}u(t,x,\vartheta)=u^0(t,x,\vartheta).$$

We introduce in addition the following condition $\mathcal{C}$:

$\mathcal{C}_1$. Suppose that $\dot{u}^0(t,x,\vartheta)$ and $\dot{u}^{0\prime}(t,x,\vartheta_0)$ exist and that they are continuous.

$\mathcal{C}_{\mathbf{2}}$. Suppose that $u(t,x,\vartheta),\ \dot{u}(t,x,\vartheta),\ u'(t,x,\vartheta),\ \dot{u}'(t,x,\vartheta)\in\mathcal{P}$, i.e. they are all of polynomial majorants w.r.t. x.

We show in Section 3.4 that this condition $\mathcal{C}$ gives us the asymptotical efficiency of the approximations.

Remark 3.1.1. *We remark that in the following, we will introduce properties of the approximations in the following sense: $\widehat{X}=X+o(\varepsilon)$ means that for any $\nu>0$,*

$$\mathbf{P}(\varepsilon^{-1}\left|\widehat{X}-X\right|>\nu)\longrightarrow 0,$$

and $\widehat{X}=X+O(\varepsilon)$ means that for any $\nu>0$,

$$\lim_{C\to\infty}\mathbf{P}(\varepsilon^{-1}\left|\widehat{X}-X\right|>C)\longrightarrow 0.$$

3.1.2 Main results

We study the following problem: Suppose that our observation $X^T=(X_t,\ 0\leq t\leq T)$ satisfies the SDE (3.9), we have to construct a couple of process $(\widehat{Y},\ \widehat{Z})$, such that it approximates the solution of the BSDE (3.10). For this, we denote

$$\widehat{Y}_t=u(t,X_t,\widehat{\vartheta}_{t,\varepsilon}),\quad \widehat{Z}_t=\varepsilon\sigma(X_t)u'(t,X_t,\widehat{\vartheta}_{t,\varepsilon})$$

where u is the solution of the PDE (3.11) and $\widehat{\vartheta}^T_\varepsilon=\left(\widehat{\vartheta}_{t,\varepsilon},\ 0\leq t\leq T\right)$ is the maximum likelihood estimator-process (MLE-process).

Remind that we have introduced the MLT $\widehat{\vartheta}_T$ in Section 2.1. In fact, this estimator can be defined as a function of time t by introducing the observations $X^t=\{X_s,0\leq s\leq t\}$. Let us introduce the likelihood ratio

$$L(X^t,\vartheta)=\exp\left\{\frac{1}{\varepsilon^2}\int_0^t\frac{S(\vartheta,X_s)}{\sigma(X_s)^2}\mathrm{d}X_s-\frac{1}{2\varepsilon^2}\int_0^t\frac{S(\vartheta,X_s)^2}{\sigma(X_s)^2}\mathrm{d}s\right\}.$$

Then the MLE-process $\widehat{\vartheta}_{t,\varepsilon}$ is defined as

$$\widehat{\vartheta}_{t,\varepsilon}=\arg\max_{\theta\in\Theta}L(X^t,\theta).$$

Particularly, for the linear case (3.7a)–(3.7b), the likelihood ratio is

$$L\left(X^t,\vartheta\right)=\exp\left\{\int_0^t\frac{\vartheta h\left(X_s\right)}{\varepsilon^2\sigma\left(X_s\right)^2}\,\mathrm{d}X_s-\int_0^t\frac{\vartheta^2 h\left(X_s\right)^2}{2\varepsilon^2\sigma\left(X_s\right)^2}\,\mathrm{d}s\right\},\qquad \vartheta\in\Theta.$$

and the MLE-process can be written explicitly

$$\widehat{\vartheta}_{t,\varepsilon}=\left(\int_0^t \frac{h\left(X_s\right)^2}{\sigma\left(X_s\right)^2}\,\mathrm{d}s\right)^{-1}\int_0^t \frac{h\left(X_s\right)}{\sigma\left(X_s\right)^2}\,\mathrm{d}X_s.$$

In Section 3.2, we study this problem for linear case. The following result is presented:

Theorem 3.2.1. Under the regularity condition $\mathcal{A}$, the couple $\left(\widehat{Y}_t,\widehat{Z}_t\right)$ admits the representation

$$\widehat{Y}_t=Y_t+\varepsilon\,\xi_{t,1}(x^t)\,\dot{u}\left(t,X_t,\vartheta\right)+O\left(\varepsilon^2\right),\qquad \widehat{Z}_t=Z_t+\varepsilon^2\,\xi_{t,1}(x^t)\,\sigma(X_t)\,\dot{u}'\left(t,X_t,\vartheta\right)+O\left(\varepsilon^3\right),$$

where

$$\xi_{t,1}(x^t)=\left(\int_0^t \frac{h\left(x_s\right)^2}{\sigma\left(x_s\right)^2}\,\mathrm{d}s\right)^{-1}\int_0^t \frac{h\left(x_s\right)}{\sigma\left(x_s\right)}\,\mathrm{d}W_s,\qquad \delta\leq t\leq T.$$

We study the general case in Section 3.3, where we obtain the following result

Theorem 3.3.1. Under the regularity condition $\mathcal{B}$, the couple $\left(\widehat{Y}_t,\widehat{Z}_t\right)$ admits the representation:

$$\begin{aligned}\widehat{Y}_t&=Y_t+\varepsilon\,\xi_{t,1}(x^t,\vartheta)\,\dot{u}\left(t,X_t,\vartheta\right)\\&\quad+\varepsilon^2\left(\xi_{t,2}(x^t,\vartheta)^2\,\dot{u}\left(t,X_t,\vartheta\right)+\frac{1}{2}\xi_{t,1}(x^t,\vartheta)\,\ddot{u}\left(t,X_t,\vartheta\right)\right)+O\left(\varepsilon^3\right)\\ \widehat{Z}_t&=Z_t+\varepsilon^2\,\xi_{t,2}(x^t,\vartheta)\,\sigma(X_t)\,\dot{u}'\left(t,X_t,\vartheta\right)+O\left(\varepsilon^3\right),\end{aligned}$$

where $\xi_{t,1}(x^t,\vartheta)$ and $\xi_{t,2}(x^t,\vartheta)$ are defined in (3.23) and (3.24).

At last, we show in Theorem 3.4.2 that our approximation is efficient.

3.2 Linear Forward Equation

We consider problem for linear system (3.4)-(3.5). Remind that the corresponding PDE is (3.6).

3.2.1 Maximum Likelihood Estimator.

Our objectif is to use the solution $u\left(t,x,\vartheta\right)$ of the equation (3.6) to define $\widehat{Y}_t=u(t,X_t,\widehat{\vartheta}_\varepsilon)$, where $\widehat{\vartheta}_\varepsilon$ is the MLE of the parameter ϑ. Remind that the likelihood ratio in our problem is the random function

$$L\left(X^T,\vartheta\right)=\exp\left\{\int_0^T \frac{\vartheta h\left(X_s\right)}{\varepsilon^2\sigma\left(X_s\right)^2}\,\mathrm{d}X_s-\int_0^T \frac{\vartheta^2 h\left(X_s\right)^2}{2\varepsilon^2\sigma\left(X_s\right)^2}\,\mathrm{d}s\right\},\qquad \vartheta\in\Theta$$

and the MLE $\widehat{\vartheta}_\varepsilon$ can be written as

$$\widehat{\vartheta}_\varepsilon = \left(\int_0^T \frac{h\left(X_s\right)^2}{\sigma\left(X_s\right)^2}\,{\rm d}s\right)^{-1}\int_0^T \frac{h\left(X_s\right)}{\sigma\left(X_s\right)^2}\,{\rm d}X_s.$$

Unfortunately we can not use this estimator for $\widehat{Y}_t$ because it depends on the whole trajectory X^T. That is why we introduce the MLE-process $\widehat{\vartheta}_{t,\varepsilon}$ defined by the observations up to time t. The likelihood ratio function is

$$L\left(X^t,\vartheta\right) = \exp\left\{\int_0^t \frac{\vartheta h\left(X_s\right)}{\varepsilon^2\sigma\left(X_s\right)^2}\,{\rm d}X_s - \int_0^t \frac{\vartheta^2 h\left(X_s\right)^2}{2\varepsilon^2\sigma\left(X_s\right)^2}\,{\rm d}s\right\}, \qquad \vartheta\in\Theta$$

and the MLE-process is

$$\widehat{\vartheta}_{t,\varepsilon} = \left(\int_0^t \frac{h\left(X_s\right)^2}{\sigma\left(X_s\right)^2}\,{\rm d}s\right)^{-1}\int_0^t \frac{h\left(X_s\right)}{\sigma\left(X_s\right)^2}\,{\rm d}X_s.$$

Now we can put $\widehat{Y}_t = u(t,X_t,\widehat{\vartheta}_{t,\varepsilon})$ but we need this estimator to be consistent as $\varepsilon\to 0$.

We consider two different strategies. The first one uses the MLE-process on the time interval $[\delta_\varepsilon,T]$, where $\delta_\varepsilon\to 0$ and the rate of convergence is such that the estimator $\widehat{\vartheta}_{\delta_\varepsilon,\varepsilon}$ is consistent. The second strategy is based on the estimator $\widehat{\vartheta}_{t,\varepsilon}$, where $t\in[\delta,T]$ with fixed δ. In this case we have an opportunity to improve the approximation of the process (Y_t,Z_t).

To simplify the notations, let us denote

$$J\left(X^t\right) = \int_0^t \frac{h\left(X_s\right)}{\sigma\left(X_s\right)}\,{\rm d}W_s,\quad I\left(X^t\right) = \int_0^t \left(\frac{h\left(X_s\right)}{\sigma\left(X_s\right)}\right)^2{\rm d}s,$$
$$J\left(x^t\right) = \int_0^t \frac{h\left(x_s\right)}{\sigma\left(x_s\right)}\,{\rm d}W_s,\quad I\left(x^t\right) = \int_0^t \left(\frac{h\left(x_s\right)}{\sigma\left(x_s\right)}\right)^2{\rm d}s.$$

Note that in this linear case, the Fisher information for time t is $I(x^t)$ which does not depend on the unknown parameter.

Case $\delta_\varepsilon\to 0$. Let us put $\delta_\varepsilon = \varepsilon^2\ln\frac{1}{\varepsilon}$.

Lemma 3.2.1. *For any $\nu>0$ we have*

$$\mathbf{P}_\vartheta\left\{\left|\widehat{\vartheta}_{t,\varepsilon}-\vartheta\right|\,1\!\!1_{\{\delta_\varepsilon\leq t\leq T\}}>\nu\right\}\longrightarrow 0, \tag{3.13}$$

as $\varepsilon\to 0$.

Proof. We have for the estimator $\widehat{\vartheta}_{t,\varepsilon}$ the representation

$$\widehat{\vartheta}_{t,\varepsilon} = \vartheta + \varepsilon \left(\int_0^t \frac{h\left(X_s\right)^2}{\sigma\left(X_s\right)^2} \,\mathrm{d}s \right)^{-1} \int_0^t \frac{h\left(X_s\right)}{\sigma\left(X_s\right)} \,\mathrm{d}W_s.$$

By conditions $\mathcal{A}1, \mathcal{A}3$ there exists a constant $\kappa_* > 0$ such that

$$\left| \frac{h\left(x\right)}{\sigma\left(x\right)} \right| > \kappa_*.$$

Therefore for $t \in [\delta_\varepsilon, T]$ we can write

$$\begin{aligned}
\mathbf{P}_\vartheta \left\{ \left| \widehat{\vartheta}_{t,\varepsilon} - \vartheta \right| > \nu \right\} &\leq \nu^{-2} \mathbf{E}_\vartheta \left| \widehat{\vartheta}_{t,\varepsilon} - \vartheta \right|^2 \leq \left(\nu \kappa_*^2 t\right)^{-2} \varepsilon^2 \mathbf{E}_\vartheta \left(\int_0^t \frac{h\left(X_s\right)}{\sigma\left(X_s\right)} \,\mathrm{d}W_s \right)^2 \\
&= \left(\nu \kappa_*^2 t\right)^{-2} \varepsilon^2 \int_0^t \mathbf{E}_\vartheta \left(\frac{h\left(X_s\right)}{\sigma\left(X_s\right)} \right)^2 \,\mathrm{d}s \leq C \,\frac{\varepsilon^2}{\nu^2 \delta_\varepsilon} = \frac{C}{\nu^2 \ln \frac{1}{\varepsilon}} \longrightarrow 0.
\end{aligned}$$

Case $\delta > 0$ fixe. Let us consider the MLE-process $\widehat{\vartheta}_{t,\varepsilon}, \delta \leq t \leq T$ and introduce the Gaussian process

$$\xi_{t,1}(x^t) = \frac{J(x^t)}{I(x^t)}, \qquad \delta \leq t \leq T.$$

We have the following result.

Lemma 3.2.2. *The MLE-process $\widehat{\vartheta}_{t,\varepsilon}$ is uniformly asymptotically normal in probability: for any $\nu > 0$*

$$\mathbf{P}_\vartheta \left\{ \sup_{\delta \leq t \leq T} \left| \frac{\widehat{\vartheta}_{t,\varepsilon} - \vartheta}{\varepsilon} - \xi_{t,1}(x^t) \right| > \nu \right\} \to 0. \tag{3.14}$$

Proof. For the process $\eta_{t,\varepsilon} = \varepsilon^{-1} \left(\widehat{\vartheta}_{t,\varepsilon} - \vartheta \right) - \xi_{t,1}$ we can write

$$\begin{aligned}
\mathbf{P}_\vartheta \left\{ \left| \eta_{t,\varepsilon} \right| > \nu \right\} &= \mathbf{P}_\vartheta \left\{ \left| \frac{J\left(X^t\right)}{I\left(X^t\right)} - \frac{J\left(x^t\right)}{I\left(x^t\right)} \right| > \nu \right\} \\
&= \mathbf{P}_\vartheta \left\{ \left| \frac{J\left(X^t\right) - J\left(x^t\right)}{I\left(X^t\right)} + \frac{J\left(x^t\right)\left(I\left(x^t\right) - I\left(X^t\right)\right)}{I\left(x^t\right) I\left(X^t\right)} \right| > \nu \right\} \\
&\leq \mathbf{P}_\vartheta \left\{ \left| \frac{J\left(X^t\right) - J\left(x^t\right)}{I\left(X^t\right)} \right| > \frac{\nu}{2} \right\} + \mathbf{P}_\vartheta \left\{ \left| \frac{J\left(x^t\right)\left(I\left(x^t\right) - I\left(X^t\right)\right)}{I\left(x^t\right) I\left(X^t\right)} \right| > \frac{\nu}{2} \right\}.
\end{aligned}$$

Using the estimate $I\left(X^t\right) \geq \kappa_*^2 t$, we obtain that: for any $\mu > 0$ (see Lemma 4.6 in Lipster and Shiryaev [35])

$$\begin{aligned}
\mathbf{P}_\vartheta &\left\{\sup_{\delta\leq t\leq T}\left|\frac{J\left(X^t\right)-J\left(x^t\right)}{I\left(X^t\right)}\right| > \frac{\nu}{2}\right\} \leq \mathbf{P}_\vartheta\left\{\sup_{\delta\leq t\leq T}\left|J\left(X^t\right)-J\left(x^t\right)\right| > \frac{\delta\kappa_*^2\nu}{2}\right\}\\
&\leq \mathbf{P}_\vartheta\left\{\sup_{\delta\leq t\leq T}\left|\int_0^t\left[\frac{h\left(X_s\right)}{\sigma\left(X_s\right)}-\frac{h\left(x_s\right)}{\sigma\left(x_s\right)}\right]\,\mathrm{d}W_s\right| > \frac{\delta\kappa_*^2\nu}{2}\right\}\\
&\leq \frac{4\mu}{\delta^2\kappa_*^4\nu^2}+\mathbf{P}_\vartheta\left\{\int_0^T\left[\frac{h\left(X_s\right)}{\sigma\left(X_s\right)}-\frac{h\left(x_s\right)}{\sigma\left(x_s\right)}\right]^2\,\mathrm{d}s\geq\mu\right\}\\
&\leq \frac{4\mu}{\delta^2\kappa_*^4\nu^2}+\mu^{-1}\mathbf{E}_\vartheta\int_0^T\left[\frac{h\left(X_s\right)}{\sigma\left(X_s\right)}-\frac{h\left(x_s\right)}{\sigma\left(x_s\right)}\right]^2\,\mathrm{d}s.
\end{aligned}$$

The condition $\mathcal{A}$ allows us to write (see, e.g., Lemma 1.19, [27])

$$\left|\frac{h\left(X_s\right)}{\sigma\left(X_s\right)}-\frac{h\left(x_s\right)}{\sigma\left(x_s\right)}\right|\leq L\left|X_s-x_s\right|,\qquad \mathbf{E}_\vartheta\left|X_s-x_s\right|^2\leq C\,\varepsilon^2.$$

Hence

$$\begin{aligned}
\mathbf{P}_\vartheta\left\{\sup_{\delta\leq t\leq T}\left|\frac{J\left(X^t\right)-J\left(x^t\right)}{I\left(X^t\right)}\right| > \frac{\nu}{2}\right\} &\leq \frac{4\mu}{\delta^2\kappa_*^4\nu^2}+\frac{TL^2C\varepsilon^2}{\mu}\\
&\leq\left(\frac{4}{\delta^2\kappa_*^4\nu^2}+TL^2C\right)\varepsilon\longrightarrow 0,
\end{aligned}$$

where we put $\mu=\varepsilon$.

By a similar way we prove the convergence

$$\mathbf{P}_\vartheta\left\{\sup_{\delta\leq t\leq T}\left|\frac{J\left(x^t\right)\left(I\left(x^t\right)-I\left(X^t\right)\right)}{I\left(x^t\right)I\left(X^t\right)}\right| > \frac{\nu}{2}\right\}\longrightarrow 0.$$

Remark 3.2.1. *In fact, if we suppose that the coefficient functions h and σ are infinitely derivable and these derivatives are bounded, then applying the Itô formula, we have the following representation for $\widehat{\vartheta}_{t,\varepsilon}-\vartheta$:*

$$\widehat{\vartheta}_{t,\varepsilon}-\vartheta=\varepsilon\xi_{t,1}(x^t)+\varepsilon^2\xi_{t,2}(x^t)+...$$

for example

$$\xi_{t,2}(x^t)=\left(\frac{M(x^t)}{I(x^t)}-J(x^t)\frac{N(x^t)}{I(x^t)^2}\right)$$

where

$$M(x^t) = \int_0^t \frac{h'(x_s)\sigma(x_s) - h(x_s)\sigma'(x_s)}{\sigma(x_s)^2} x_s^{(1)} \mathrm{d}W_s,$$
$$N(x^t) = \int_0^t \frac{2h(x_s)h'(x_s)\sigma(x_s) - 2h(x_s)^2\sigma'(x_s)}{\sigma(x_s)^3} x_s^{(1)} \mathrm{d}s$$

and $x^{(1)}$ *comes from the decomposition of* $X_t = x_t + \varepsilon x_t^{(1)} + ...$, *which is the solution of the following equation (see Chapter 3 in Kutoyants [27])*

$$\mathrm{d}x_t^{(1)} = \vartheta h'(x_t) x_t^{(1)} \mathrm{d}t + \sigma(x_t)\mathrm{d}W_t, \quad x_0^{(1)} = 0.$$

In fact we can prove in a similar way as in Lemma 3.2.2:

$$\begin{aligned}
&\mathbf{P}_\vartheta \left\{ \varepsilon^{-2} \left| \widehat{\vartheta}_{t,\varepsilon} - \vartheta - \varepsilon \xi_{t,1}(x^t) - \varepsilon^2 \xi_{t,2}(x^t) \right| > \nu \right\} \\
&\leq \mathbf{P}_\vartheta \left\{ \varepsilon^{-1} \left| \frac{J(X^t) - J(x^t)}{I(X^t)} - \varepsilon \frac{M(x^t)}{I(x^t)} \right| > \frac{\nu}{2} \right\} \\
&\quad + \mathbf{P}_\vartheta \left\{ \varepsilon^{-1} \left| \frac{J(x^t)(I(x^t) - I(X^t))}{I(x^t) I(X^t)} + \varepsilon J(x^t) \frac{N(x^t)}{I(x^t)^2} \right| > \frac{\nu}{2} \right\} \\
&\leq \mathbf{P}_\vartheta \left\{ \varepsilon^{-1} \left| \frac{1}{I(X^t)} \left(J\left(X^t\right) - J\left(x^t\right) - \varepsilon M(x^t) \right) \right| > \frac{\nu}{4} \right\} \\
&\quad + \mathbf{P}_\vartheta \left\{ \left| \frac{M(x^t)}{I(X^t) I(x^t)} \left(I(X^t) - I(x^t) \right) \right| > \frac{\nu}{4} \right\} \\
&\quad + \mathbf{P}_\vartheta \left\{ \varepsilon^{-1} \left| \frac{J(x^t)(I(x^t) - I(X^t)) + \varepsilon N(x^t)}{I(x^t) I(X^t)} \right| > \frac{\nu}{4} \right\} \\
&\quad + \mathbf{P}_\vartheta \left\{ \left| \frac{N(x^t) J(x^t)}{I(x^t)^2 I(X^t)} \left(I\left(x^t\right) - I\left(X^t\right) \right) \right| > \frac{\nu}{4} \right\}.
\end{aligned}$$

Each term on the right side converges to zero, thus we have

$$\mathbf{P}_\vartheta \left\{ \varepsilon^{-2} \left| \widehat{\vartheta}_{t,\varepsilon} - \vartheta - \varepsilon \xi_{t,1}(x^t) - \varepsilon^2 \xi_{t,2}(x^t) \right| > \nu \right\} \longrightarrow 0.$$

3.2.2 Approximation process

We observe the stochastic process

$$\mathrm{d}X_t = \vartheta h(X_t)\,\mathrm{d}t + \varepsilon\sigma(X_t)\,\mathrm{d}W_t, \qquad x_0, \qquad 0 \leq t \leq T,$$

and have to construct a couple of process $\left(\widehat{Y}_t, \widehat{Z}_t\right)$ which is close to the true solution (Y_t, Z_t). This process is given by the equalities $Y_t = u(t, X_t, \vartheta)$ and $Z_t = \varepsilon\sigma(X_t)\,u'(t, X_t, \vartheta)$ and satisfies the equation

$$\mathrm{d}Y_t = \left[k(X_t) + g(X_t)\,Y_t\right]\mathrm{d}t + Z_t\,\mathrm{d}W_t, \qquad Y_0, \quad 0 \leq t \leq T. \qquad (3.15)$$

The initial and final values are $Y_0 = u\left(0, X_0, \vartheta\right)$ and $Y_T = \Phi\left(X_T\right)$ respectively.

Let us define the processes $\widehat{Y}_t = u\left(t, X_t, \widehat{\vartheta}_{t,\varepsilon}\right)$ and $\widehat{Z}_t = \varepsilon\sigma\left(X_t\right) u'\left(t, X_t, \widehat{\vartheta}_{t,\varepsilon}\right)$. Of course, these processes do not start at $t = 0$ because that we have no estimator for ϑ.

If we start at the moment $t = \delta_\varepsilon$, then due to the continuity w.r.t. ϑ of the function $u\left(t, x, \vartheta\right)$ and boundness of $u'\left(t, y, \vartheta\right)$ it follows that $\left(\widehat{Y}_t, \delta_\varepsilon \leq t \leq T\right)$ converges to $\left(y_t, 0 \leq t \leq T\right)$, the process $\widehat{Z}_t \to 0$ and therefore $\widehat{Y}_T \to \Phi\left(x_T\right)$. This (non random) limit is probably not satisfactory.

Let us start at $t = \delta$ and consider the approximation of $\left(Y_t, Z_t,\ \delta \leq t \leq T\right)$ satisfying the equation

$$\mathrm{d}Y_t = \left[k\left(X_t\right) + g\left(X_t\right) Y_t\right] \mathrm{d}t + Z_t\, \mathrm{d}W_t, \quad Y_\delta = u\left(\delta, X_\delta, \vartheta\right) \tag{3.16}$$

by $\left(\widehat{Y}_t, \widehat{Z}_t,\ \delta \leq t \leq T\right)$.

Theorem 3.2.1. *Let the regularity condition $\mathcal{A}$ be fulfilled, then the couple $\left(\widehat{Y}_t, \widehat{Z}_t\right)$ admits the representation*

$$\begin{aligned}\widehat{Y}_t &= Y_t + \varepsilon\, \xi_{t,1}(x^t)\, \dot{u}\left(t, X_t, \vartheta\right) + O\left(\varepsilon^2\right), \\ \widehat{Z}_t &= Z_t + \varepsilon^2\, \xi_{t,1}(x^t)\, \sigma(X_t)\, \dot{u}'\left(t, X_t, \vartheta\right) + O\left(\varepsilon^3\right),\end{aligned} \tag{3.17}$$

where $Y_t = u\left(t, X_t, \vartheta\right)$ and $Z_t = \varepsilon\sigma\left(X_t\right) u'\left(t, X_t, \vartheta\right)$.

Proof. The proof follows directly from the Lemma 3.2.2 and Taylor formula. Remind that the functions $u\left(t, x, \vartheta\right)$ and $u'\left(t, x, \vartheta\right)$ have continuous derivatives w.r.t. ϑ.

Remark 3.2.2. *Applying the Taylor formula, we develop the representation for a higher order. For example*

$$\widehat{Y}_t = Y_t + \varepsilon\, \xi_{t,1}(x^t)\, \dot{u}\left(t, X_t, \vartheta\right) + \varepsilon^2 \left(\xi_{t,2}(x^t)^2\, \dot{u}\left(t, X_t, \vartheta\right) + \frac{1}{2}\xi_{t,1}(x^t)\, \ddot{u}\left(t, X_t, \vartheta\right)\right) + O\left(\varepsilon^3\right). \tag{3.18}$$

Note that the process $\widehat{Y}_t$ does not satisfy the equation (3.16) but has the following stochastic differential form (by Itô's formula)

$$\begin{aligned}
\mathrm{d}\widehat{Y}_t = &\left[\frac{\partial u}{\partial t} - \frac{\varepsilon h\left(X_t\right)^2 J\left(X^t\right)}{\sigma\left(X_t\right)^2 I\left(X^t\right)^2}\frac{\partial u}{\partial \vartheta} + \vartheta h\left(X_t\right)\frac{\partial u}{\partial y}\right]\mathrm{d}t \\
&+ \left[\frac{1}{2}\varepsilon^2\sigma\left(X_t\right)^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\varepsilon^2 h\left(X_t\right)^2}{\sigma\left(X_t\right)^2 I\left(X^t\right)^2}\frac{\partial^2 u}{\partial \vartheta^2} + \frac{\varepsilon^2 h\left(X_t\right)}{\sigma(X_t) I\left(X^t\right)}\frac{\partial^2 u}{\partial\vartheta\partial x}\right]\mathrm{d}t \\
&+ \left[\varepsilon\sigma\left(X_t\right)\frac{\partial u}{\partial y} + \frac{\varepsilon h\left(X_t\right)}{\sigma\left(X_t\right) I\left(X^t\right)}\frac{\partial u}{\partial \vartheta}\right]\mathrm{d}W_t \\
= &\left[k\left(X_t\right) + g\left(X_t\right)\widehat{Y}_t\right]\mathrm{d}t + \widehat{Z}_t\mathrm{d}W_t \\
&+ \left[\left(\vartheta - \widehat{\vartheta}_{t,\varepsilon}\right)h(X_t)\frac{\partial u}{\partial x} - \frac{\varepsilon h\left(X_t\right)^2 J\left(X^t\right)}{\sigma\left(X_t\right)^2 I\left(X^t\right)^2}\frac{\partial u}{\partial \vartheta}\right]\mathrm{d}t \\
&+ \left[\frac{1}{2}\frac{\varepsilon^2 h\left(X_t\right)^2}{\sigma\left(X_t\right)^2 I\left(X^t\right)^2}\frac{\partial^2 u}{\partial \vartheta^2} + \frac{\varepsilon^2 h\left(X_t\right)}{\sigma\left(X_t\right)^2 I\left(X^t\right)^2}\frac{\partial^2 u}{\partial\vartheta\partial x}\right]\mathrm{d}t + \frac{\varepsilon h\left(X_t\right)}{\sigma\left(X_t\right) I\left(X^t\right)}\frac{\partial u}{\partial \vartheta}\mathrm{d}W_t
\end{aligned}$$

where we has used the stochastic differential form of $\widehat{\vartheta}_{t,\varepsilon}$:

$$\mathrm{d}\widehat{\vartheta}_{t,\varepsilon} = \mathrm{d}\left(\vartheta + \varepsilon\frac{J_t}{I_t}\right) = -\frac{\varepsilon h(X_t)^2 J_t}{\sigma(X_t)^2 I_t^2}\mathrm{d}t + \frac{\varepsilon h(X_t)}{\sigma(X_t) I_t}\mathrm{d}W_t.$$

Remark 3.2.3. *Note that we can simplify the equation for $\widehat{Y}_t$ if we take the estimator $\widehat{\vartheta}_{\delta,\varepsilon}$ and put $\check{Y}_t = u(t, X_t, \widehat{\vartheta}_{\delta,\varepsilon})$. Then the SDE for $\check{Y}_t$ becomes*

$$\begin{aligned}
\mathrm{d}\check{Y}_t &= \left[\frac{\partial u}{\partial t} + \vartheta h\left(X_t\right)\frac{\partial u}{\partial x} + \frac{1}{2}\varepsilon^2\sigma\left(X_t\right)^2\frac{\partial^2 u}{\partial x^2}\right]\mathrm{d}t + \varepsilon\sigma\left(X_t\right)\frac{\partial u}{\partial x}\mathrm{d}W_t \\
&= \left[k\left(X_t\right) + g\left(X_t\right)\widehat{Y}_t\right]\mathrm{d}t + \widehat{Z}_t\mathrm{d}W_t + \left(\vartheta - \widehat{\vartheta}_{\delta,\varepsilon}\right)h(X_t)\frac{\partial u}{\partial x}\mathrm{d}t.
\end{aligned}$$

3.3 Nonlinear Forward Equation

In this section, we deal with the diffusion process

$$\mathrm{d}X_t = S(\vartheta, X_t)\mathrm{d}t + \varepsilon\sigma(X_t)\mathrm{d}W_t, \quad X_0 = x_0, \quad 0 \leq t \leq T. \tag{3.19}$$

where $\vartheta \in \Theta$ is the unknown parameter, Θ is an open, bounded, convex set. Parameter $\varepsilon \in (0, 1]$, and the limits correspond to $\varepsilon \to 0$. We have to construct the process $\left(\widehat{Y}_t, \widehat{Z}_t\right)$ which is close to the exact solution (Y_t, Z_t) which satisfies the equation

$$\mathrm{d}Y_t = \left[k\left(X_t\right) + g\left(X_t\right) Y_t\right]\mathrm{d}t + Z_t\,\mathrm{d}W_t, \qquad Y_T = \Phi\left(X_T\right), \quad 0 \leq t \leq T. \tag{3.20}$$

For this purpose, we first estimate ϑ by observations $X^t = \{X_s, 0 \leq s \leq t\}$.

Seeing that the case that the beginning time converges to 0, that is $t \geq \delta_\varepsilon$ with $\delta_\varepsilon \to 0$ does not help so much in the construction of the approximate process, we discuss in this section the case where δ is fixed, and the approximate process $(\widehat{X}, \widehat{Z})$ is defined for $\delta \leq t \leq T$.

Denote by $x^T = \{x_t, 0 \leq t \leq T\}$ the solution of the equation where $\varepsilon = 0$:

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = S(\vartheta, x_t), \quad x_0, \quad 0 \leq t \leq T.$$

As that is shown in Kutoyants [27], there exists an expansion for X_t at the point x_t:

$$X_t = x_t + \varepsilon x_t^{(1)} + \varepsilon^2 x_t^{(2)} + ... \tag{3.21}$$

where $x_t^{(1)}$ is the solution of the following equation

$$\mathrm{d}x_t^{(1)} = S'(\vartheta, x_t)x_t^{(1)}\mathrm{d}t + \sigma(x_t)\mathrm{d}W_t, \quad x_0^{(1)} = 0.$$

There exist also equations for higher orders in (3.21). We do not present the details here, the interested reader can find in Chapter 3 in Kutoyants [27].

First of all, we estimate the unknown parameter ϑ by the MLE-process $\widehat{\vartheta}_{t,\varepsilon}$ which is defined as follows:

$$L(X^t, \widehat{\vartheta}_{t,\varepsilon}) = \sup_{\vartheta \in \Theta} L(X^t, \vartheta),$$

where $L(X^t, \vartheta)$ is the likelihood ratio:

$$L(X^t, \vartheta) = \exp\left\{\frac{1}{\varepsilon^2}\int_0^t \frac{S(\vartheta, X_s)}{\sigma(X_s)^2}\mathrm{d}X_s - \frac{1}{2\varepsilon^2}\int_0^t \frac{S(\vartheta, X_s)^2}{\sigma(X_s)^2}\mathrm{d}s\right\}. \tag{3.22}$$

To simplify the notations, let us denote $K(\vartheta, y) = \frac{S(\vartheta,x)}{\sigma(y)}$, then

$$\dot{K}(\vartheta, y) = \frac{\dot{S}(\vartheta, x)}{\sigma(y)}, \quad K'(\vartheta, y) = \frac{S'(\vartheta, y)\sigma(y) - S(\vartheta, x)\sigma'(y)}{\sigma(y)^2},$$
$$\ddot{K}(\vartheta, y) = \frac{\ddot{S}(\vartheta, x)}{\sigma(y)}, \quad \dot{K}'(\vartheta, y) = \frac{\dot{S}'(\vartheta, y)\sigma(y) - \dot{S}(\vartheta, x)\sigma'(y)}{\sigma(y)^2}.$$

Moreover, we denote

$$J\left(X^t, \vartheta\right) = \int_0^t \dot{K}(\vartheta, X_s)\,\mathrm{d}W_s, \quad I\left(X^t, \vartheta\right) = \int_0^t \left(\dot{K}(\vartheta, X_s)\right)^2 \mathrm{d}s,$$
$$J\left(x^t, \vartheta\right) = \int_0^t \dot{K}(\vartheta, x_s)\,\mathrm{d}W_s, \quad I\left(x^t, \vartheta\right) = \int_0^t \left(\dot{K}(\vartheta, x_s)\right)^2 \mathrm{d}s.$$

We introduce in addition the Gaussian process for $\delta \le t \le T$:

$$\xi_{t,1}(x^t,\vartheta) = \frac{J\left(x^t,\vartheta\right)}{I\left(x^t,\vartheta\right)} \tag{3.23}$$

and

$$\begin{aligned}\xi_{t,2}(x^t,\vartheta) &= \frac{J\left(x^t,\vartheta\right)}{I\left(x^t,\vartheta\right)^2}\int_0^t \ddot{K}(\vartheta,x_s)\mathrm{d}W_s - \frac{3J\left(x^t,\vartheta\right)^2}{2I\left(x^t,\vartheta\right)^3}\int_0^t \ddot{K}(\vartheta,x_s)\dot{K}(\vartheta,x_s)\mathrm{d}s\\ &+ I\left(x^t,\vartheta\right)^{-1}\int_0^t x_s^{(1)}\dot{K}'(\vartheta,x_s)\mathrm{d}W_s - \frac{2J\left(x^t,\vartheta\right)}{I\left(x^t,\vartheta\right)}\int_0^t x_s^{(1)}\dot{K}(\vartheta,x_s)\dot{K}'(\vartheta,x_s)\mathrm{d}s.\end{aligned} \tag{3.24}$$

Note that under the condition $\mathcal{B}_2$, the positive Fisher information and the identifiability are obtained for all $t \ge \delta$:

$$\begin{aligned}I(x^t,\vartheta) = \int_0^t \frac{\dot{S}(\vartheta,x_s)^2}{\sigma(x_s)^2}\mathrm{d}s \ge \int_0^\delta \frac{\dot{S}(\vartheta,x_s)^2}{\sigma(x_s)^2}\mathrm{d}s > 0,\\ \inf_{|\theta-\vartheta|>\nu}\left\|\frac{S(\theta,x)-S(\vartheta,x)}{\sigma(x)}\right\|_t \ge \inf_{|\theta-\vartheta|>\nu}\left\|\frac{S(\theta,x)-S(\vartheta,x)}{\sigma(x)}\right\|_\delta > 0.\end{aligned}$$

We have the following result.

Lemma 3.3.1. *The MLE-process $\widehat{\vartheta}_{t,\varepsilon}$ admits the following representation: for any $\nu > 0$*

$$\sup_{\delta\le t\le T}\mathbf{P}_\vartheta\left\{\left|\frac{\widehat{\vartheta}_{t,\varepsilon}-\vartheta}{\varepsilon^2} - \frac{\xi_{t,1}(x^t,\vartheta)}{\varepsilon} - \xi_{t,2}(x^t,\vartheta)\right| > \nu\right\} \to 0. \tag{3.25}$$

Proof. As that is shown in Theorem 3.1 in Kutoyants [27], under the regularity conditions, there exist random variables $X_{T,i},\ i=1,2,3,\ \zeta_T$ and a set $\mathcal{M}_T$ such that for sufficiently small ε, the MLE $\widehat{\vartheta}_{T,\varepsilon}$ can be presented as follows:

$$\widehat{\vartheta}_{T,\varepsilon} = \vartheta + \left\{X_{T,1}\varepsilon + X_{T,2}\varepsilon^2 + X_{T,3}\varepsilon^{\frac{5}{2}}\right\}\mathbb{1}_{\{\mathcal{M}_T\}} + \zeta_T\mathbb{1}_{\{\mathcal{M}_T^c\}},$$

where $|X_{T,3}| < 1$, $|\zeta_T|$ and $\mathbf{P}\left(\mathcal{M}_T^c\right)$ are small. Applying this result for all $\widehat{\vartheta}_{t,\varepsilon},\ \delta \le t \le T$ we have: there exist random variables $X_{t,i},\ i=1,2,3,\ \zeta_t$ and set $\mathcal{M}_t$ such that for sufficiently small ε,

$$\widehat{\vartheta}_{t,\varepsilon} = \vartheta + \left\{X_{t,1}\varepsilon + X_{t,2}\varepsilon^2 + X_{t,3}\varepsilon^{\frac{5}{2}}\right\}\mathbb{1}_{\{\mathcal{M}_t\}} + \zeta_t\mathbb{1}_{\{\mathcal{M}_t^c\}}$$

where $|X_{t,3}| < 1$ and for $\delta \in (1,\frac{1}{2})$,

$$\sup_{\theta\in\mathbb{K}}\mathbf{P}_\theta^{(\varepsilon)}(\mathcal{M}_t^c) \le C_{t,1}\exp\left\{-c_{t,1}\varepsilon^{-\gamma_{t,1}}\right\},\quad \sup_{\theta\in\mathbb{K}}\mathbf{P}_\theta^{(\varepsilon)}(|\zeta_t|>\varepsilon^\delta) \le C_{t,2}\exp\left\{-c_{t,2}\varepsilon^{-\gamma_{t,2}}\right\}, \tag{3.26}$$

with positive constants $C_{t,i},\ c_{t,i},\ \gamma_{t,i},\ i=1,2$. Following the proof of Theorem 3.1 in Kutoyants [27], we have fixed $C,\ c,\ \gamma$ such that (3.26) holds for all $\delta \le t \le T$. Thus:

$$\sup_{\delta\le t\le T}\sup_{\theta\in\mathbb{K}}\mathbf{P}_\theta^{(\varepsilon)}(\mathcal{M}_t^c)\le C\exp\left\{-c\varepsilon^{-\gamma}\right\}.$$

Then we have

$$\begin{aligned}\mathbf{P}_\vartheta\left\{\left|\frac{\widehat{\vartheta}_{t,\varepsilon}-\vartheta}{\varepsilon^2}-\frac{X_{t,1}}{\varepsilon}-X_{t,2}\right|>\nu\right\}&=\mathbf{P}_\vartheta\left\{\left|\frac{\widehat{\vartheta}_{t,\varepsilon}-\vartheta}{\varepsilon^2}-\frac{X_{t,1}}{\varepsilon}-X_{t,2}\right|>\nu,\,\mathcal{M}_t\right\}\\&+\mathbf{P}_\vartheta\left\{\left|\frac{\widehat{\vartheta}_{t,\varepsilon}-\vartheta}{\varepsilon^2}-\frac{X_{t,1}}{\varepsilon}-X_{t,2}\right|>\nu,\,\mathcal{M}_t^c\right\}\le O(\varepsilon^{\frac{1}{2}})+C\exp\left\{-c\varepsilon^{-\gamma}\right\}\longrightarrow 0.\end{aligned}\tag{3.27}$$

Now we verify that $X_{t,1}=\xi_{t,1}(x^t,\vartheta)$ and $X_{t,2}=\xi_{t,2}(x^t,\vartheta)$. Denote $\widehat{\tau}_{t,\varepsilon}=\widehat{\vartheta}_{t,\varepsilon}-\vartheta$, then in the set $\mathcal{M}-t$, $\widehat{\tau}_{t,\varepsilon}$ is the unique solution for the maximum likelihood equation

$$\varepsilon\int_0^t\frac{\dot{S}(\vartheta+\tau,X_s)}{\sigma(X_s)}\mathrm{d}W_s-\int_0^t\frac{\dot{S}(\vartheta+\tau,X_s)}{\sigma(X_s)^2}\left[S(\vartheta+\tau,X_s)-S(\vartheta,X_s)\right]\mathrm{d}s=0.$$

which is equal to

$$\varepsilon\int_0^t\dot{K}(\vartheta+\tau,X_s)\mathrm{d}W_s-\int_0^t\dot{K}(\vartheta+\tau,X_s)\left[K(\vartheta+\tau,X_s)-K(\vartheta,X_s)\right]\mathrm{d}s=0.\tag{3.28}$$

We denote the left part as $F_t(\varepsilon,\tau)$, the equation becomes $F_t(\varepsilon,\tau)=0$. Under the regularity conditions, this equation has a unique solution which depends on ε, denoting as $\widehat{\tau}_t(\varepsilon)$. Moreover, $\widehat{\tau}_t=0$ is the solution for the case where $\varepsilon=0$, then we can apply the Taylor formula to $\widehat{\tau}_t(\varepsilon)$:

$$\widehat{\tau}_t(\varepsilon)=\widehat{\vartheta}_{t,\varepsilon}-\vartheta=\varepsilon\widehat{\tau}_t'(0)+\frac{1}{2}\varepsilon^2\widehat{\tau}_t''(0)+...\tag{3.29}$$

where

$$\begin{aligned}\widehat{\tau}_t'(\varepsilon)&=-\frac{\partial F_t(\varepsilon,\tau)}{\partial\varepsilon}\left(\frac{\partial F_t(\varepsilon,\tau)}{\partial\tau}\right)^{-1},\\\widehat{\tau}_t''(\varepsilon)&=-\frac{1}{2}\left(\frac{\partial F_t(\varepsilon,\tau)}{\partial\tau}\right)^{-3}\left[\frac{\partial^2F_t(\varepsilon,\tau)}{\partial\varepsilon^2}\left(\frac{\partial F_t(\varepsilon,\tau)}{\partial\tau}\right)^2\right.\\&\qquad\left.-2\frac{\partial^2F_t(\varepsilon,\tau)}{\partial\varepsilon\partial\tau}\frac{\partial F_t(\varepsilon,\tau)}{\partial\varepsilon}\frac{\partial F_t(\varepsilon,\tau)}{\partial\tau}+\frac{\partial^2F_t(\varepsilon,\tau)}{\partial\tau^2}\left(\frac{\partial F_t(\varepsilon,\tau)}{\partial\varepsilon}\right)^2\right].\end{aligned}$$

Note that X is a process depending on ε, and under the regularity conditions, it is derivable w.r.t. ε. Let us denote $X_t^{(1)} = \frac{\partial X_t}{\partial \varepsilon}$ and $X_t^{(2)} = \frac{\partial^2 X_t}{\partial \varepsilon^2}$, then we have (see Chapter 3 in Kutoyants [27])

$$X_t^{(1)}\Big|_{\varepsilon=0} = x_t^{(1)}, \quad X_t^{(2)}\Big|_{\varepsilon=0} = x_t^{(2)}.$$

Thus we have

$$\begin{aligned}\frac{\partial F_t(\varepsilon,\tau)}{\partial \tau}\bigg|_{\varepsilon=0} &= \bigg(\varepsilon \int_0^t \ddot{K}(\vartheta+\tau,X_s)\mathrm{d}W_s - \int_0^t \dot{K}(\vartheta+\tau,X_s)^2 \mathrm{d}s \\ &\quad - \int_0^t \ddot{K}(\vartheta+\tau,X_s)\left[K(\vartheta+\tau,X_s)-K(\vartheta,X_s)\right]\mathrm{d}s\bigg)\bigg|_{\varepsilon=0} = -I(x^t,\vartheta),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial F_t(\varepsilon,\tau)}{\partial \varepsilon}\bigg|_{\varepsilon=0} &= \bigg(\int_0^t \dot{K}(\vartheta+\tau,X_s)\mathrm{d}W_s + \varepsilon\int_0^t \dot{K}'(\vartheta+\tau,X_s)X_s^{(1)}\mathrm{d}W_s \\ &\quad - \int_0^t X_s^{(1)}\dot{K}'(\vartheta+\tau,X_s)\left[K(\vartheta+\tau,X_s)-K(\vartheta,X_s)\right]\mathrm{d}s \\ &\quad - \int_0^t X_s^{(1)}\dot{K}(\vartheta+\tau,X_s)\left[K'(\vartheta+\tau,X_s)-K'(\vartheta,X_s)\right]\mathrm{d}s\bigg)\bigg|_{\varepsilon=0} = J(x^t,\vartheta).\end{aligned}$$

Hence

$$X_{t,1} = \widehat{\tau}_t'(\varepsilon) = -\frac{\partial F_t(\varepsilon,\tau)}{\partial \varepsilon}\left(\frac{\partial F_t(\varepsilon,\tau)}{\partial \tau}\right)^{-1}\bigg|_{\varepsilon=0} = \frac{J\left(x^t,\vartheta\right)}{I\left(x^t,\vartheta\right)} = \xi_{t,1}(x^t,\vartheta). \tag{3.30}$$

Similarly, we have

$$\begin{aligned}\frac{\partial^2 F_t(\varepsilon,\tau)}{\partial \varepsilon \partial \tau}\bigg|_{\varepsilon=0} &= \bigg(\int_0^t \ddot{K}(\vartheta+\tau,X_s)\mathrm{d}W_s + \varepsilon\int_0^t X_s^{(1)}\ddot{K}'(\vartheta+\tau,X_s)\mathrm{d}W_s \\ &\quad - 2\int_0^t X_s^{(1)}\dot{K}(\vartheta+\tau,X_s)\dot{K}'(\vartheta+\tau,X_s)\mathrm{d}s \\ &\quad - \int_0^t X_s^{(1)}\ddot{K}'(\vartheta+\tau,X_s)\left[K(\vartheta+\tau,X_s)-K(\vartheta,X_s)\right]\mathrm{d}s \\ &\quad - \int_0^t X_s^{(1)}\ddot{K}(\vartheta+\tau,X_s)\left[K'(\vartheta+\tau,X_s)-K'(\vartheta,X_s)\right]\mathrm{d}s\bigg)\bigg|_{\varepsilon=0} \\ &= \int_0^t \ddot{K}(\vartheta,x_s)\mathrm{d}W_s - 2\int_0^t x_s^{(1)}\dot{K}(\vartheta,x_s)\dot{K}'(\vartheta,x_s)\mathrm{d}s,\end{aligned}$$

$$\left.\frac{\partial^2 F_t(\varepsilon,\tau)}{\partial \tau^2}\right|_{\varepsilon=0} = \left(\varepsilon\int_0^t \dddot{K}(\vartheta+\tau,X_s)\mathrm{d}W_s - 3\int_0^t \ddot{K}(\vartheta+\tau,X_s)\dot{K}(\vartheta+\tau,X_s)\mathrm{d}s\right.$$
$$\left.\left. - \int_0^t \dddot{K}(\vartheta+\tau,X_s)\left[K(\vartheta+\tau,X_s)-K(\vartheta,X_s)\right]\mathrm{d}s\right)\right|_{\varepsilon=0}$$
$$= -3\int_0^t \ddot{K}(\vartheta,x_s)\dot{K}(\vartheta,x_s)\mathrm{d}s,$$

and

$$\left.\frac{\partial^2 F_t(\varepsilon,\tau)}{\partial \varepsilon^2}\right|_{\varepsilon=0} = \left(\int_0^t X_s^{(1)}\dot{K}'(\vartheta+\tau,X_s)\mathrm{d}W_s + \int_0^t X_s^{(1)}\dot{K}'(\vartheta+\tau,X_s)\mathrm{d}W_s\right.$$
$$+\varepsilon\int_0^t \dot{K}''(\vartheta+\tau,X_s)\left(X_s^{(1)}\right)^2\mathrm{d}W_s + \varepsilon\int_0^t X_s^{(2)}\dot{K}'(\vartheta+\tau,X_s)\mathrm{d}W_s$$
$$-\int_0^t X_s^{(2)}\dot{K}'(\vartheta+\tau,X_s)\left[K(\vartheta+\tau,X_s)-K(\vartheta,X_s)\right]\mathrm{d}s$$
$$-\int_0^t \left(X_s^{(1)}\right)^2\dot{K}''(\vartheta+\tau,X_s)\left[K(\vartheta+\tau,X_s)-K(\vartheta,X_s)\right]\mathrm{d}s$$
$$-\int_0^t \left(X_s^{(1)}\right)^2\dot{K}'(\vartheta+\tau,X_s)\left[K'(\vartheta+\tau,X_s)-K'(\vartheta,X_s)\right]\mathrm{d}s$$
$$-\int_0^t X_s^{(2)}X_s^{(1)}\dot{K}(\vartheta+\tau,X_s)\left[K'(\vartheta+\tau,X_s)-K'(\vartheta,X_s)\right]\mathrm{d}s$$
$$-\int_0^t \left(X_s^{(1)}\right)^2\dot{K}'(\vartheta+\tau,X_s)\left[K'(\vartheta+\tau,X_s)-K'(\vartheta,X_s)\right]\mathrm{d}s$$
$$\left.\left.-\int_0^t \left(X_s^{(1)}\right)^2\dot{K}(\vartheta+\tau,X_s)\left[K''(\vartheta+\tau,X_s)-K''(\vartheta,X_s)\right]\mathrm{d}s\right)\right|_{\varepsilon=0}$$
$$= 2\int_0^t x_s^{(1)}\dot{K}'(\vartheta,x_s)\mathrm{d}W_s,$$

so that

$$X_{t,2} = \frac{1}{2}\widehat{\tau}_t''(0) = -\frac{1}{2}\left(\frac{\partial F_t(\varepsilon,\tau)}{\partial\tau}\right)^{-3}\left[\frac{\partial^2 F_t(\varepsilon,\tau)}{\partial\varepsilon^2}\left(\frac{\partial F_t(\varepsilon,\tau)}{\partial\tau}\right)^2\right.$$
$$\left.\left. -2\frac{\partial^2 F_t(\varepsilon,\tau)}{\partial\varepsilon\partial\tau}\frac{\partial F_t(\varepsilon,\tau)}{\partial\varepsilon}\frac{\partial F_t(\varepsilon,\tau)}{\partial\tau} + \frac{\partial^2 F_t(\varepsilon,\tau)}{\partial\tau^2}\left(\frac{\partial F_t(\varepsilon,\tau)}{\partial\varepsilon}\right)^2\right]\right|_{\varepsilon=0} = \xi_{t,2}(x^t,\vartheta).$$

Remark 3.3.1. *In fact, as in Section 3.2, we have a better convergence than that is proved in the theorem: for any* $\nu > 0$

$$\mathbf{P}_\vartheta \left\{ \sup_{\delta \le t \le T} \left| \frac{\widehat{\vartheta}_{t,\varepsilon} - \vartheta}{\varepsilon^2} - \frac{\xi_{t,1}(x^t, \vartheta)}{\varepsilon} - \xi_{t,2}(x^t, \vartheta) \right| > \nu \right\} \longrightarrow 0.$$

This result needs a lot of further work, we present in appendix.

Now we construct a couple of processes which approximates to (Y_t, Z_t) for $\delta \le t \le T$. Denote $u(t, x, \vartheta)$ the solution of PDE

$$\frac{\partial u}{\partial t}(t,x,\vartheta) + S(\vartheta, x)\frac{\partial u}{\partial x}(t,x,\vartheta) + \frac{1}{2}\varepsilon^2\sigma(x)^2\frac{\partial^2 u}{\partial t^2}(t,x,\vartheta) = k(x) + g(x)u(t,x,\vartheta), \quad (3.31)$$

with terminal condition $u(t, x, \vartheta) = \Phi(x)$. Denote $u^0(t, x, \vartheta)$ the solution for the case $\varepsilon = 0$

$$\frac{\partial u^0}{\partial t}(t,x,\vartheta) + S(\vartheta, x)\frac{\partial u^0}{\partial x}(t,x,\vartheta) = k(x) + g(x)u(t,x,\vartheta), \quad u^0(T,x,\vartheta) = \Phi(x). \quad (3.32)$$

Define the process $((\widehat{Y}_t, \widehat{Z}_t), \delta \le t \le T)$ as follows:

$$\widehat{Y}_t = u(t, X_t, \widehat{\vartheta}_{t,\varepsilon}), \;\; \widehat{Z}_t = \varepsilon\sigma(X_t)u'(t, X_t, \widehat{\vartheta}_{t,\varepsilon}),$$

where $u(t, x, \vartheta)$ is the solution of (3.31). We have

Theorem 3.3.1. *Let the regularity condition* $\mathcal{B}$ *be fulfilled, then the couple* $\left(\widehat{Y}_t, \widehat{Z}_t\right)$ *admits the representation:*

$$\begin{aligned}
\widehat{Y}_t &= Y_t + \varepsilon\, \xi_{t,1}(x^t, \vartheta)\, \dot{u}\,(t, X_t, \vartheta) \\
&\quad + \varepsilon^2 \left(\xi_{t,2}(x^t, \vartheta)^2\, \dot{u}\,(t, X_t, \vartheta) + \frac{1}{2}\xi_{t,1}(x^t, \vartheta)\, \ddot{u}\,(t, X_t, \vartheta) \right) + O\left(\varepsilon^3\right) \\
\widehat{Z}_t &= Z_t + \varepsilon^2\, \xi_{t,1}(x^t, \vartheta)\, \sigma(X_t)\, \dot{u}'\,(t, X_t, \vartheta) + O\left(\varepsilon^3\right),
\end{aligned}$$

where $Y_t = u\,(t, X_t, \vartheta)$ *and* $Z_t = \varepsilon\sigma\,(X_t)\, u'\,(t, X_t, \vartheta)$.

The proof follows directly from the Lemma 5.1 and the Taylor formula.

Remark 3.3.2. *All these results can be applied to other consistent estimators. For example we can take the minimum distance estimator (MDE)* $\vartheta^*_{t,\varepsilon}$*:*

$$\vartheta^*_{t,\varepsilon} = \arg\inf_{\theta \in \Theta} \int_0^t |X_t - x_t|^2 \mathrm{d}t.$$

3.4 On Asymptotic Efficiency of the Approximation

Remind that as $\varepsilon \longrightarrow 0$, the solution of the PDE (3.31) converges to the solution of the PDE (3.32). We introduce in this section the asymptotical efficiency of the approximation $\widehat{Y}_t$ and $\widehat{Z}_t$ under the condition $\mathcal{C}$ that has been introduced in Section 3.1.1.

Under the condition $\mathcal{C}1$, the representations obtained in Theorem 3.3.1 in the above section for the stochastic process $(Y_t,\, Z_t)$ allow us to verify the consistency and the asymptotical normality of these estimators, i.e. we have the convergence:

$$\varepsilon^{-1}\left(\widehat{Y}_t - Y_t\right) \Longrightarrow \xi_{t,1}(x^t,\vartheta)\dot{u}^0(t,x_t,\vartheta_0) \sim \mathcal{N}\left(0,\, d_1^2(t,\vartheta)^2\right) \tag{3.33}$$

and

$$\varepsilon^{-2}\left(\widehat{Z}_t - Z_t\right) \Longrightarrow \xi_{t,1}(x^t,\vartheta)\sigma(x_t)\dot{u}^{0\prime}(t,x_t,\vartheta) \sim \mathcal{N}\left(0,\, d_2^2(t,\vartheta)^2\right), \tag{3.34}$$

where

$$d_1^2 = \frac{\dot{u}^0(t,x_t,\vartheta_0)^2}{I(x^t,\vartheta_0)}, \quad d_2^2 = \sigma(x_t)^2\frac{\dot{u}^{0\prime}(t,x_t,\vartheta_0)^2}{I(x^t,\vartheta_0)}.$$

Let us consider the following problem: Is it possible to construct the other estimators of the process (Y_t, Z_t) with the limit variance smaller than d_1^2 and d_2^2? In fact, we have the following result:

Theorem 3.4.1. *For any approximation $(\bar{Y}_t,\, \bar{Z}_t)$ of $(Y_t,\, Z_t)$ for $\delta \le t \le T$,*

$$\lim_{\nu\to 0}\lim_{\varepsilon\to 0}\sup_{|\vartheta-\vartheta_0|<\nu} \varepsilon^{-2}\mathbf{E}_\vartheta\left(\bar{Y}_t - Y_t\right)^2 \ge \frac{\dot{u}^0(t,x_t,\vartheta_0)^2}{I(x^t,\vartheta_0)},$$

and

$$\lim_{\nu\to 0}\lim_{\varepsilon\to 0}\sup_{|\vartheta-\vartheta_0|<\nu} \varepsilon^{-4}\mathbf{E}_\vartheta\left(\bar{Z}_t - Z_t\right)^2 \ge \sigma(x_t)^2\frac{\dot{u}^{0\prime}(t,x_t,\vartheta_0)^2}{I(x^t,\vartheta_0)}.$$

Proof. Suppose that the unknown parameter ϑ is a random variable belonging to an interval $[\vartheta_0-\nu,\, \vartheta_0+\nu]$ for $\nu>0$. Let us introduce a probability density $p(\vartheta)$, $\vartheta\in[\vartheta_0-\nu,\, \vartheta_0+\nu]$ and $p(\vartheta_0-\nu)=p(\vartheta_0+\nu)=0$. Then we can write

$$\sup_{|\vartheta-\vartheta_0|<\nu} \varepsilon^{-2}\mathbf{E}_\vartheta\left(\bar{Y}_t - u(t,X_t,\vartheta)\right)^2 \ge \int_{\vartheta_0-\nu}^{\vartheta_0+\nu}\mathbf{E}_\vartheta\left(\bar{Y}_t - u(t,X_t,\vartheta)\right)^2 p(\vartheta)\mathrm{d}\vartheta. \tag{3.35}$$

In addition, we have

$$\begin{aligned}
&\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \dot{u}(t,X_t,\vartheta)L(X^t,\vartheta)p(\vartheta)\mathrm{d}\vartheta \\
&\quad = u(t,X_t,\vartheta)L(X^t,\vartheta)p(\vartheta)\Big|_{\vartheta_0-\nu}^{\vartheta_0+\nu} - \int_{\vartheta_0-\nu}^{\vartheta_0+\nu} u(t,X_t,\vartheta)\frac{\partial}{\partial\vartheta}\left(L(X^t,\vartheta)p(\vartheta)\right)\mathrm{d}\vartheta \\
&\quad = -\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} u(t,X_t,\vartheta)\frac{\partial}{\partial\vartheta}\ln\left(L(X^t,\vartheta)p(\vartheta)\right)L(X^t,\vartheta)p(\vartheta)\mathrm{d}\vartheta.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \bar{Y}_t\frac{\partial}{\partial\vartheta}\ln\left(L(X^t,\vartheta)p(\vartheta)\right)L(X^t,\vartheta)p(\vartheta)\mathrm{d}\vartheta \\
&\quad = \bar{Y}_t\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \frac{\partial}{\partial\vartheta}\left(L(X^t,\vartheta)p(\vartheta)\right)\mathrm{d}\vartheta = \bar{Y}_t\left(L(X^t,\vartheta)p(\vartheta)\right)\Big|_{\vartheta_0-\nu}^{\vartheta_0+\nu} = 0.
\end{aligned}$$

This gives us

$$\begin{aligned}
&\mathbf{E}_0\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \dot{u}(t,X_t,\vartheta)L(X^t,\vartheta)p(\vartheta)\mathrm{d}\vartheta \\
&\quad = \mathbf{E}_0\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \left(\bar{Y}_t - u(t,X_t,\vartheta)\right)\frac{\partial}{\partial\vartheta}\ln\left(L(X^t,\vartheta)p(\vartheta)\right)L(X^t,\vartheta)p(\vartheta)\mathrm{d}\vartheta.
\end{aligned}$$

The Cauchy-Schwarz inequality yields that

$$\begin{aligned}
&\left(\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \mathbf{E}_\vartheta\dot{u}(t,X_t,\vartheta)p(\vartheta)\mathrm{d}\vartheta\right)^2 \\
&\quad \leq \mathbf{E}_0\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \left(\bar{Y}_t - u(t,X_t,\vartheta)\right)^2 L(X^t,\vartheta)p(\vartheta)\mathrm{d}\vartheta \\
&\qquad \cdot\mathbf{E}_0\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \left(\frac{\partial}{\partial\vartheta}\ln\left(L(X^t,\vartheta)p(\vartheta)\right)\right)^2 L(X^t,\vartheta)p(\vartheta)\mathrm{d}\vartheta \\
&\quad = \int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \mathbf{E}_\vartheta\left(\bar{Y}_t - u(t,X_t,\vartheta)\right)^2 p(\vartheta)\mathrm{d}\vartheta\cdot\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \mathbf{E}_\vartheta\left(\frac{\partial}{\partial\vartheta}\ln\left(L(X^t,\vartheta)p(\vartheta)\right)\right)^2 p(\vartheta)\mathrm{d}\vartheta.
\end{aligned}$$

We obtain thus

$$\sup_{|\vartheta-\vartheta_0|<\nu} \varepsilon^{-2}\mathbf{E}_\vartheta \left(\bar{Y}_t - u(t,X_t,\vartheta)\right)^2 \geq \int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \mathbf{E}_\vartheta \left(\bar{Y}_t - u(t,X_t,\vartheta)\right)^2 p(\vartheta)\mathrm{d}\vartheta$$

$$\geq \frac{\left(\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \mathbf{E}_\vartheta \dot{u}(t,X_t,\vartheta)p(\vartheta)\mathrm{d}\vartheta\right)^2}{\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \mathbf{E}_\vartheta \left(\frac{\partial}{\partial\vartheta}\ln\left(L(X^t,\vartheta)p(\vartheta)\right)\right)^2 p(\vartheta)\mathrm{d}\vartheta}. \tag{3.36}$$

Let us put $\varepsilon \longrightarrow 0$. We have In fact

$$\begin{aligned}&\left|\mathbf{E}_\vartheta \dot{u}(t,X_t,\vartheta) - \dot{u}^0(t,x_t,\vartheta)\right| \\ &\quad \leq \left|\mathbf{E}_\vartheta \dot{u}(t,X_t,\vartheta) - \dot{u}(t,x_t,\vartheta)\right| + \left|\dot{u}(t,x_t,\vartheta) - \dot{u}^0(t,x_t,\vartheta)\right|\end{aligned}$$

The regularities of u and u^0 along with Lemma 3.1.4 yield that the second term converges to zero. For the first term, remind that in Lemma 1.13 in Kutoyants [27], there is

$$\mathbf{E}_\vartheta \left|X_t - x_t\right|^2 \leq C\varepsilon^2.$$

Therefore

$$\begin{aligned}&\left|\mathbf{E}_\vartheta \dot{u}(t,X_t,\vartheta) - \dot{u}(t,x_t,\vartheta)\right| \leq \mathbf{E}_\vartheta \left|\dot{u}(t,X_t,\vartheta) - \dot{u}(t,x_t,\vartheta)\right| \\ &\quad \leq \left(\mathbf{E}_\vartheta \left|\dot{u}'(t,\widetilde{X}_t,\vartheta)\right|^2 \mathbf{E}_\vartheta \left|X_t - x_t\right|^2\right)^{1/2} \leq C\varepsilon \left(\mathbf{E}_\vartheta \left(1 + |\widetilde{X}_t|^p\right)^2\right)^{1/2} \leq C'\varepsilon \longrightarrow 0.\end{aligned}$$

Thus we have as $\varepsilon \longrightarrow 0$,

$$\mathbf{E}_\vartheta \dot{u}(t,X_t,\vartheta) \longrightarrow \dot{u}^0(t,x_t,\vartheta). \tag{3.37}$$

In addition, note that

$$\mathbf{E}_\vartheta \left(\frac{\partial}{\partial\vartheta}\ln L(X^t,\vartheta)\right) = 0,$$

and

$$\mathbf{E}_\vartheta \left(\frac{\partial}{\partial\vartheta}\ln L(X^t,\vartheta)\right)^2 = \mathbf{E}_\vartheta \left(\frac{1}{\varepsilon}\int_0^t \frac{\dot{S}(\vartheta,X_s)}{\sigma(X_s)}\mathrm{d}W_s\right)^2 = \varepsilon^{-2} I(X^t,\vartheta).$$

Then as $\varepsilon \longrightarrow 0$,

$$\begin{aligned}
&\varepsilon^2 \int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \mathbf{E}_\vartheta \left(\frac{\partial}{\partial \vartheta} \ln \left(L(X^t,\vartheta) p(\vartheta) \right) \right)^2 p(\vartheta) \mathrm{d}\vartheta \\
&\quad = \varepsilon^2 \int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \mathbf{E}_\vartheta \left(\frac{\partial}{\partial \vartheta} \ln L(X^t,\vartheta) + \frac{\partial}{\partial \vartheta} \ln p(\vartheta) \right)^2 p(\vartheta) \mathrm{d}\vartheta \\
&\quad = \varepsilon^2 \int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \left(\mathbf{E}_\vartheta \left(\frac{\partial}{\partial \vartheta} \ln L(X^t,\vartheta) \right)^2 + \left(\frac{\dot p(\vartheta)}{p(\vartheta)} \right)^2 \right) p(\vartheta) \mathrm{d}\vartheta \\
&\quad = \int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \mathbf{E}_{\vartheta_0} I(X^t,\vartheta) p(\vartheta) \mathrm{d}\vartheta + \varepsilon^2 \int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \frac{\dot p(\vartheta)^2}{p(\vartheta)} \mathrm{d}\vartheta \\
&\quad \longrightarrow \int_{\vartheta_0-\nu}^{\vartheta_0+\nu} I(x^t,\vartheta) p(\vartheta) \mathrm{d}\vartheta.
\end{aligned}$$

These convergences and (3.36) give us

$$\lim_{\varepsilon \to 0} \sup_{|\vartheta-\vartheta_0|<\nu} \varepsilon^{-2} \mathbf{E}_\vartheta \left(\bar Y_t - u(t,X_t,\vartheta) \right)^2 \geq \frac{\left(\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \dot u^0(t,x_t,\vartheta) p(\vartheta) \mathrm{d}\vartheta \right)^2}{\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} I(x^t,\vartheta) p(\vartheta) \mathrm{d}\vartheta}.$$

Now we put $\nu \longrightarrow 0$. Note that for any continuous function f

$$\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} f(\vartheta) p(\vartheta) \mathrm{d}\vartheta = f(\widetilde\vartheta) \int_{\vartheta_0-\nu}^{\vartheta_0+\nu} p(\vartheta) \mathrm{d}\vartheta = f(\widetilde\vartheta) \longrightarrow f(\vartheta_0),$$

here $\widetilde\vartheta \in [\vartheta_0-\nu,\, \vartheta_0+\nu]$. Then we have

$$\left(\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} \dot u^0(t,x_t,\vartheta) p(\vartheta) \mathrm{d}\vartheta \right)^2 \longrightarrow \dot u^0(t,x_t,\vartheta_0)^2,$$

and

$$\int_{\vartheta_0-\nu}^{\vartheta_0+\nu} I(x^t,\vartheta) p(\vartheta) \mathrm{d}\vartheta \longrightarrow I(x^t,\vartheta_0)$$

Therefore we have

$$\lim_{\nu \to 0} \lim_{\varepsilon \to 0} \sup_{|\vartheta-\vartheta_0|<\nu} \varepsilon^{-2} \mathbf{E}_\vartheta \left(\bar Y_t - Y_t \right)^2 \geq \frac{\dot u^0(t,x_t,\vartheta_0)^2}{I(x^t,\vartheta_0)}.$$

Similarly we have this Cramér-Rao bound for the estimators of Z_t.

We define the asymptotically efficient approximation as follows:

Definition 3.4.1. *We say that an approximation $\bar{Y}$ or $\bar{Z}$ is asymptotically efficient, if for all $\vartheta_0 \in (\alpha, \beta)$ and $t \in [\delta, T]$, we have the equalities:*

$$\lim_{\nu\to 0}\lim_{\varepsilon\to 0}\sup_{|\vartheta-\vartheta_0|<\nu} \varepsilon^{-2}\mathbf{E}_\vartheta\left(\bar{Y}_t - Y_t\right)^2 = \frac{\dot{u}^0(t, x_t, \vartheta_0)^2}{I(x^t, \vartheta_0)},$$

and

$$\lim_{\nu\to 0}\lim_{\varepsilon\to 0}\sup_{|\vartheta-\vartheta_0|<\nu} \varepsilon^{-4}\mathbf{E}_\vartheta\left(\bar{Z}_t - Z_t\right)^2 = \sigma(x_t)^2\frac{\dot{u}^{0\prime}(t, x_t, \vartheta_0)^2}{I(x^t, \vartheta_0)}.$$

Therefore, the approximate process that we proposed above $(\widehat{Y}_t,\ \widehat{Z}_t)$ is asymptotically efficient.

Theorem 3.4.2. *Let the conditions $\mathcal{B}$ and $\mathcal{C}$ be fulfilled, then*

$$\widehat{Y}_t = u(t, X_t, \widehat{\vartheta}_{t,\varepsilon}), \quad \widehat{Z}_t = \varepsilon\sigma(X_t)u'(t, X_t, \widehat{\vartheta}_{t,\varepsilon})$$

are asymptotically efficient:

Proof. For $\widehat{Y}_t$ we have

$$\begin{aligned}
\varepsilon^{-2}\mathbf{E}_\vartheta\left(\widehat{Y}_t - Y_t\right)^2 &= \mathbf{E}_\vartheta\left(\xi_{t,1}(x^t,\vartheta)\dot{u}(t, X_t, \vartheta)\right)^2 + O(\varepsilon)\\
&= \mathbf{E}_\vartheta\left(\frac{\int_0^t \dot{K}(\vartheta, x_s)\mathrm{d}W_s}{I(x^s,\vartheta)}\left(\dot{u}(t, x_t, \vartheta) - \varepsilon\dot{u}'(t, x_t, \vartheta)x_t^{(1)}\right)\right)^2 + O(\varepsilon)\\
&= \frac{\dot{u}(t, x_t, \vartheta)^2}{I(x^t, \vartheta)} + O(\varepsilon) \longrightarrow \frac{\dot{u}^0(t, x_t, \vartheta_0)^2}{I(x^t, \vartheta_0)}.
\end{aligned}$$

It can be show that this convergence is uniform w.r.t. $\vartheta \in [\vartheta_0 - \nu,\ \vartheta_0 + \nu]$. Therefore we obtain that for any $t \in [\delta, T]$, there is the equality

$$\lim_{\nu\to 0}\lim_{\varepsilon\to 0}\sup_{|\vartheta-\vartheta_0|<\nu} \varepsilon^{-2}\mathbf{E}_\vartheta\left(\widehat{Y}_t - Y_t\right)^2 = \frac{\dot{u}^0(t, x_t, \vartheta_0)^2}{I(x^t, \vartheta_0)}.$$

Similarly, we have the same result for $\widehat{Z}_t$.

3.5 Example

We consider the linear FBSDE

$$\begin{cases} \mathrm{d}X_t = \vartheta\mathrm{d}t + \varepsilon\sigma\mathrm{d}W_t,\ 0 \le t \le T,\ X_0 = x_0, \\ \mathrm{d}Y_t = -(\beta Y_t + \gamma Z_t)\mathrm{d}t + Z_t\mathrm{d}W_t,\ 0 \le t \le T,\ Y_T = \Phi(X_T). \end{cases} \tag{3.38}$$

where $\vartheta,\ \sigma,\ \beta$ are constants and ϑ is a unknown parameter. Here the trend coefficient function of the backward depends also on Z, we see later that this does not influent the convergence of u to the deterministic case u^0. The corresponding PDE of (3.38) is

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{1}{2}\varepsilon^2\sigma^2\dfrac{\partial^2 u}{\partial x^2}+(\vartheta+\varepsilon\sigma\gamma)\dfrac{\partial u}{\partial x}+\beta u=0,\ 0\le t\le T, y\in\mathbb{R}, \\ u(T,x)=\Phi(x),\ y\in\mathbb{R}. \end{cases} \tag{3.39}$$

with the solution

$$\begin{aligned} &u(t,x,\vartheta)\\ &\quad=\frac{1}{\sqrt{2\pi\varepsilon^2\sigma^2(T-t)}}\int_{-\infty}^{\infty}\exp\{\beta(T-t)-\frac{(y+(\vartheta+\varepsilon\sigma\gamma)(T-t)-z)^2}{2\varepsilon^2\sigma^2(T-t)}\}\Phi(z)\mathrm{d}z\\ &\quad=\frac{\mathrm{e}^{\beta(T-t)}}{\sqrt{2\pi\varepsilon^2\sigma^2(T-t)}}\int_{-\infty}^{\infty}\exp\{-\frac{z^2}{2\varepsilon^2\sigma^2(T-t)}\}\Phi(y+(\vartheta+\varepsilon\sigma\gamma)(T-t)-z)\mathrm{d}z. \end{aligned}$$

Then the solution of the BSDE satisfies

$$\begin{aligned} Y_t&=u(t,X_t,\vartheta)=e^{\beta(T-t)}G(t,X_t,\vartheta),\\ Z_t&=\varepsilon\sigma u'(t,X_t,\vartheta)=\varepsilon\sigma e^{\beta(T-t)}G'(t,X_t,\vartheta), \end{aligned}$$

where

$$G(t,x,\vartheta)=\frac{1}{\sqrt{2\pi\varepsilon^2\sigma^2(T-t)}}\int_{-\infty}^{\infty}\exp\{-\frac{z^2}{2\sigma^2(T-t)}\}\Phi(y+(\vartheta+\varepsilon\sigma\gamma)(T-t)-z)\mathrm{d}z.$$

Note that

$$\begin{aligned} &G'(t,x,\vartheta)\\ &\quad=\frac{1}{\sqrt{2\pi\varepsilon^2\sigma^2(T-t)}}\int_{-\infty}^{\infty}\exp\{-\frac{z^2}{2\sigma^2(T-t)}\}\Phi'(y+(\vartheta+\varepsilon\sigma\gamma)(T-t)-z)\mathrm{d}z. \end{aligned}$$

We have

$$\begin{aligned} &\dot{u}_\vartheta(t,x,\vartheta)\\ &\quad=\frac{(T-t)e^{\beta(T-t)}}{\sqrt{2\pi\varepsilon^2\sigma^2(T-t)}}\int_{-\infty}^{\infty}\exp\{-\frac{z^2}{2\sigma^2(T-t)}\}\Phi'(y+(\vartheta+\varepsilon\sigma\gamma)(T-t)-z)\mathrm{d}z\\ &\quad=(T-t)e^{\beta(T-t)}G'(t,x,\vartheta), \end{aligned}$$

and

$$\begin{aligned} \dot{u}'(t,x,\vartheta)&=(T-t)e^{\beta(T-t)}G''(t,x,\vartheta),\\ \ddot{u}(t,x,\vartheta)&=(T-t)^2e^{\beta(T-t)}G''(t,x,\vartheta) \end{aligned}$$

Suppose that $\varepsilon = 0$. Then the differential equation becomes

$$\begin{cases} \mathrm{d}x_t = \vartheta \mathrm{d}t,\ 0 \le t \le T,\ X_0 = x_0, \\ \mathrm{d}y_t = -\beta y_t \mathrm{d}t,\ 0 \le t \le T,\ y_T = \Phi(x_T). \end{cases} \tag{3.40}$$

We solve the PDE

$$\frac{\partial u^0}{\partial t} + \vartheta \frac{\partial u^0}{\partial x} + \beta u^0 = 0, \quad u^0(T,x) = \Phi(x),$$

which can be written explicitly as

$$u^0(t,x) = e^{\beta(T-t)} \Phi(x + \vartheta(T-t)).$$

Note that the convergence of u to u^0 is obvious if $\Phi(\cdot)$ is derivable.

The MLE estimator for ϑ is $\widehat{\vartheta}_{t,\varepsilon} = \frac{X_t - x_0}{t}$. We have

$$\frac{\widehat{\vartheta}_{t,\varepsilon} - \vartheta}{\varepsilon} = \frac{\sigma}{t} W_t, \quad \text{for } \delta < t \le T.$$

We construct the process $(\widehat{Y}_t, \widehat{Z}_t)$ as follows:

$$\widehat{Y}_t = u(t, X_t, \widehat{\vartheta}_{t,\varepsilon}),\ \widehat{Z}_t = \varepsilon \sigma u'(t, X_t, \widehat{\vartheta}_{t,\varepsilon}).$$

Note that $\dot{u}(t,x,\vartheta) = 0$, and $|\dot{u}(t,x,\vartheta)| \le C$ for $\Phi'(\cdot)$ bounded. Thus we have

$$\widehat{Y}_T = Y_T = \Phi(X_T),$$

and

$$\widehat{Y}_t = Y_t + \frac{\varepsilon \sigma (T-t)}{t} e^{\beta(T-t)} G'(t,x,\vartheta) W_t + O\left(\varepsilon^2\right),$$
$$\widehat{Z}_t = Z_t + \frac{\varepsilon^2 \sigma (T-t)}{t} e^{\beta(T-t)} G''(t,x,\vartheta) W_t + O\left(\varepsilon^3\right).$$

Moreover, applying the Itô's formula to $\widehat{\vartheta}_{t,\varepsilon}$ and $\widehat{Y}_t$, we have

$$\mathrm{d}\widehat{\vartheta}_{t,\varepsilon} = -\frac{\widehat{\vartheta}_{t,\varepsilon}}{t} \mathrm{d}t + \frac{1}{t} \mathrm{d}X_t = -\frac{1}{t^2} \varepsilon \sigma W_t \mathrm{d}t + \frac{1}{t} \varepsilon \sigma \mathrm{d}W_t$$

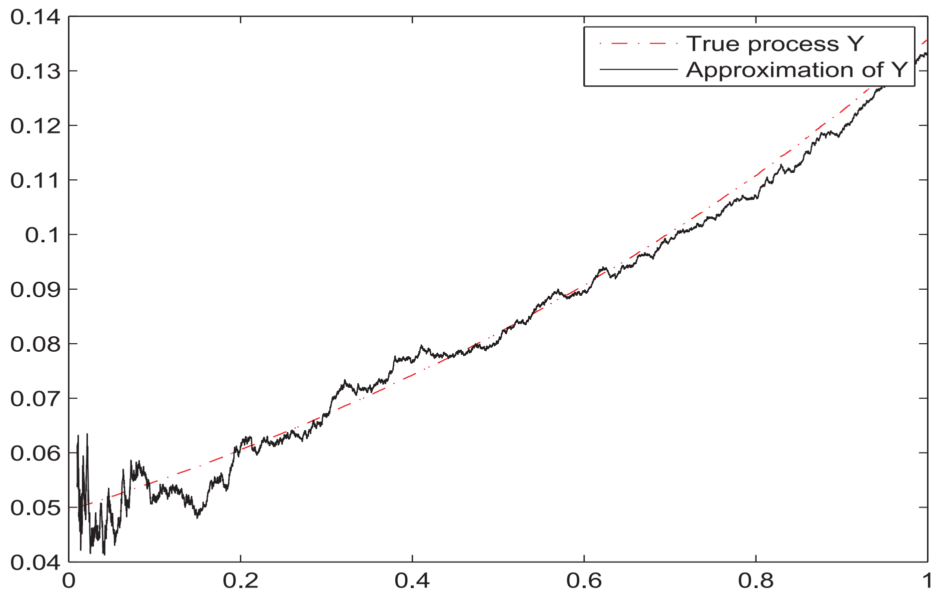

Figure 3.1: Approximation of process $\widehat{Y}$ to Y.

and

$$\begin{aligned}
\mathrm{d}\widehat{Y}_t &= \frac{\partial u}{\partial t}\mathrm{d}t + \frac{\partial u}{\partial x}\mathrm{d}X_t + \frac{\partial u}{\partial \vartheta}\mathrm{d}\widehat{\vartheta}_{t,\varepsilon} + \left(\frac{1}{2}\varepsilon^2\sigma^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{2t^2}\varepsilon^2\sigma^2\frac{\partial^2 u}{\partial \vartheta^2} + \frac{1}{t}\varepsilon^2\sigma^2\frac{\partial^2 u}{\partial x\partial\vartheta}\right)\mathrm{d}t \\
&= -\left(\beta\widehat{Y}_t + \gamma\widehat{Z}_t\right)\mathrm{d}t + \widehat{Z}_t\mathrm{d}W_t + \frac{1}{t}\varepsilon\sigma\dot{u}_\vartheta\mathrm{d}W_t \\
&\quad + \left(\left(\vartheta - \widehat{\vartheta}_{t,\varepsilon}\right)u' - \frac{1}{t^2}\varepsilon\sigma W_t\,\dot{u} + \frac{1}{2t^2}\varepsilon^2\sigma^2\ddot{u} + \frac{1}{t}\varepsilon^2\sigma^2\dot{u}'\right)\mathrm{d}t \\
&= -\left(\beta\widehat{Y}_t + \gamma\widehat{Z}_t\right)\mathrm{d}t + \widehat{Z}_t\mathrm{d}W_t + \frac{T-t}{t}\varepsilon\sigma e^{\beta(T-t)}G'(t,X_t,\vartheta)\mathrm{d}W_t \\
&\quad + \left(-\frac{T}{t^2}\varepsilon\sigma W_t e^{\beta(T-t)}G'(t,x,\vartheta) + \frac{T^2-t^2}{2t^2}\varepsilon^2\sigma^2 e^{\beta(T-t)}G''(t,x,\vartheta)\right)\mathrm{d}t
\end{aligned}$$

Let us present the numeric result. We fix the value for parameters: $\sigma = 5$ and the true value for unknown parameter $\vartheta = -3$ to simulate the process X, choosing $\beta = -1$, $\gamma = 5$, $\varepsilon = 0.1$ we plot the true process of Y and then the approximate process $\widehat{Y}$. See that the approximate process is close to the solution of the BSDE.

3.6 Appendix

In this section, we prove the result in the Remark 3.3.1:

Theorem 3.6.1. *For any* $\nu > 0$

$$\mathbf{P}_\vartheta \left\{ \sup_{\delta \le t \le T} \left| \frac{\widehat{\vartheta}_{t,\varepsilon} - \vartheta}{\varepsilon^2} - \frac{\xi_{t,1}(x^t, \vartheta)}{\varepsilon} - \xi_{t,2}(x^t, \vartheta) \right| > \nu \right\} \longrightarrow 0.$$

First of all, we improve the result of Lemma 1.4 and Lemma 1.5 in kutoyants [27].

Lemma 3.6.1. *Let* $\{f_t(w),\ 0 \le t \le T\}$ *be an adaptive process which is square integrable and*

$$M = \mathbf{E} \exp \left\{ \int_0^T f_t^2 \mathrm{d}t \right\} < \infty,$$

then for $N > 0$

$$\mathbf{P} \left(\sup_{\delta \le t \le T} \int_0^t f_s \mathrm{d}W_s > N \right) \le (2 + M) e^{-N}.$$

Proof. Put $p = 1$ in Lemma 1.4 in Kutoyants [27], we have

$$\mathbf{P} \left(\int_0^T f_t \mathrm{d}W_t > N \right) \le (1 + M) \mathrm{e}^{-N}.$$

Thus

$$\begin{aligned}
&\mathbf{P} \left(\sup_{\delta \le t \le T} \int_0^t f_s \mathrm{d}W_s > N \right) \\
&\le \mathbf{P} \left(\sup_{\delta \le t \le T} \left(\int_0^t f_s \mathrm{d}W_s - \frac{1}{2} \int_0^t f_s^2 \mathrm{d}s \right) > \frac{N}{2} \right) + \mathbf{P} \left(\sup_{\delta \le t \le T} \left(\frac{1}{2} \int_0^t f_s^2 \mathrm{d}s \right) > \frac{N}{2} \right) \\
&\le \mathbf{P} \left(\sup_{\delta \le t \le T} \exp \left(\int_0^t f_s \mathrm{d}W_s - \frac{1}{2} \int_0^t f_s^2 \mathrm{d}s \right) > \mathrm{e}^{\frac{N}{2}} \right) + \mathbf{P} \left(\frac{1}{2} \int_0^T f_s^2 \mathrm{d}s > \frac{N}{2} \right) \\
&\le \mathrm{e}^{-\frac{N}{2}} \mathbf{E} \left(\exp \left(\int_0^T f_s \mathrm{d}W_s - \frac{1}{2} \int_0^T f_s^2 \mathrm{d}s \right) \right) + \mathbf{P} \left(\frac{1}{2} \int_0^T f_s^2 \mathrm{d}s > \frac{N}{2} \right) \\
&\le (2 + M) \mathrm{e}^{-\frac{N}{2}},
\end{aligned}$$

where we applied the Doob's inequality

$$\begin{aligned}
&\mathbf{P} \left(\sup_{0 \le t \le T} \exp \left(\int_0^t f_s \mathrm{d}W_s - \frac{1}{2} \int_0^t f_s^2 \mathrm{d}s \right) > K \right) \\
&\qquad \le K^{-1} \mathbf{E} \exp \left(\int_0^T f_s \mathrm{d}W_s - \frac{1}{2} \int_0^T f_s^2 \mathrm{d}s \right) \le K^{-1}.
\end{aligned}$$

Lemma 3.6.2. *Let $\{f_t(\vartheta),\ 0\le t\le T\}$ be an adaptive square integrable process for all $\vartheta\in[\alpha,\,\beta]$ and for some $p\ge 1$*

$$M=\sup_{\alpha\le\vartheta\le\beta}\mathbf{E}\left(\int_0^T \dot f_t(\vartheta)^2\mathrm{d}t\right)^p<\infty,$$

then for $N>0$, there exists a constant $C>0$ such that

$$\mathbf{P}\left(\sup_{\delta\le t\le T}\sup_{\alpha\le\vartheta\le\beta}\left(\int_0^t f_s(\vartheta)\mathrm{d}W_s-\int_0^t f_s(\alpha)\mathrm{d}W_s\right)>N\right)\le CN^{-2p}.$$

Proof. Below we use the Burkholder-Davis-Gundy inequality: For any $p\ge 1$ there exist positive constants c_p and C_p such that, for all local martingales X with $X_0=0$, the following inequality holds.

$$c_p\mathbf{E}\left([X]_T^p\right)\le\mathbf{E}\left(\sup_{0\le t\le T}|X_t|\right)^{2p}\le C_p\mathbf{E}\left([X]_T^p\right).$$

Thus we have

$$\begin{aligned}
&\mathbf{P}\left(\sup_{\delta\le t\le T}\sup_{\alpha\le\vartheta\le\beta}\left(\int_0^t f_s(\vartheta)\mathrm{d}W_s-\int_0^t f_s(\alpha)\mathrm{d}W_s\right)>N\right)\\
&\quad=\mathbf{P}\left(\sup_{\delta\le t\le T}\sup_{\alpha\le\vartheta\le\beta}\left(\int_0^t\int_\alpha^\vartheta \dot f_s(v)\mathrm{d}v\mathrm{d}W_s\right)>N\right)\\
&\quad\le\mathbf{P}\left(\sup_{\delta\le t\le T}\sup_{\alpha\le\vartheta\le\beta}\left(\int_\alpha^\vartheta\left|\int_0^t \dot f_s(v)\mathrm{d}W_s\right|\mathrm{d}v\right)>N\right)\\
&\quad\le\mathbf{P}\left(\sup_{\delta\le t\le T}\left(\int_\alpha^\beta\left|\int_0^t \dot f_s(v)\mathrm{d}W_s\right|\mathrm{d}v\right)>N\right)\le N^{-2p}\mathbf{E}\left(\int_\alpha^\beta\sup_{\delta\le t\le T}\left|\int_0^t \dot f_s(v)\mathrm{d}W_s\right|\mathrm{d}v\right)^{2p}\\
&\quad\le N^{-2p}(\beta-\alpha)^{2p-1}\int_\alpha^\beta\mathbf{E}\left(\sup_{\delta\le t\le T}\left|\int_0^t \dot f_s(v)\mathrm{d}W_s\right|\right)^{2p}\mathrm{d}v\\
&\quad\le C_pN^{-2p}(\beta-\alpha)^{2p-1}\int_\alpha^\beta\mathbf{E}\left(\int_0^T \dot f_s(v)^2\mathrm{d}s\right)^p\mathrm{d}v\le MC_p(\beta-\alpha)^{2p}N^{-2p}=CN^{-2p}.
\end{aligned}$$

Proof of the Theorem. According to Chapter 3 in Kutoyants [27], $\mathcal{M}_t$ is constructed by three part $\mathcal{M}_{1,t},\ \mathcal{M}_{2,t}, \mathcal{M}_{3,t}$ which can be presented in our case as

$$\mathcal{M}_{1,t} = \{\omega : \sup_{|h|>v\varepsilon} \ln L(X^t, \vartheta + h) < 0\}$$

$$\mathcal{M}_{2,t} = \begin{cases} \sup\limits_{0\leq s\leq t} |W_s| < \varepsilon^{-1+\delta}, \\ \omega : \sup\limits_{|h|<v_\varepsilon} \displaystyle\int_0^t \ddot{K}(\vartheta + h, X_s)\mathrm{d}W_s < \varepsilon^{-1+\delta}, \\ \displaystyle\int_0^t \dot{K}(\vartheta, X_s)\mathrm{d}W_s < \frac{1}{2}\varepsilon^{-1+\delta} I(X^t, \vartheta) \end{cases}$$

$$\mathcal{M}_{3,t} = \{\omega : \sup_{|h|<v_\varepsilon} \left|h^{(3)}(\varepsilon_0, h)\right| < 6\varepsilon^{-\frac{1}{2}}\}$$

Let us define $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ where

$$\mathcal{M}_1 = \{\omega : \sup_{\delta\leq t\leq T} \sup_{|h|>v\varepsilon} \ln L(X^t, \vartheta + h) < 0\}$$

$$\mathcal{M}_2 = \begin{cases} \sup\limits_{0\leq s\leq T} |W_s| < \varepsilon^{-1+\delta}, \\ \omega : \sup\limits_{\delta\leq t\leq T} \sup\limits_{|h|<v_\varepsilon} \displaystyle\int_0^t \ddot{K}(\vartheta + h, X_s)\mathrm{d}W_s < \varepsilon^{-1+\delta}, \\ \sup\limits_{\delta\leq t\leq T} \displaystyle\int_0^t \dot{K}(\vartheta, X_s)\mathrm{d}W_s < \frac{1}{2}\varepsilon^{-1+\delta} I_\delta(\vartheta) \end{cases}$$

$$\mathcal{M}_3 = \{\omega : \sup_{\delta\leq t\leq T} \sup_{|h|<v_\varepsilon} \left|h^{(3)}(\varepsilon_0, h)\right| < 6\varepsilon^{-\frac{1}{2}}\}$$

We prove that

$$\mathbf{P}(\mathcal{M}_i^c) \leq C_i \mathrm{e}^{-c_i\varepsilon^{-\gamma_i}}.$$

Denote $\Delta K_\varepsilon(h) = K(\vartheta + h, X_s) - K(\vartheta, X_s)$ and $\Delta K_0(h) = K(\vartheta + h, x_s) - K(\vartheta, x_s)$.

For the first set $\mathcal{A}_1$, we have

$$\begin{aligned}
\mathbf{P}_\vartheta\left(\mathcal{M}_1^c\right) &= \mathbf{P}_\vartheta\left(\sup_{\delta\le t\le T}\sup_{|h|>v_\varepsilon}\left(\int_0^t \Delta K_\varepsilon(h)\mathrm{d}W_s - \frac{1}{2\varepsilon}\left\|\Delta K_\varepsilon(h)\right\|^2\right)\ge 0\right)\\
&\le \mathbf{P}_\vartheta\left(\sup_{\delta\le t\le T}\sup_{|h|>v_\varepsilon}\left(\int_0^t \Delta K_\varepsilon(h)\mathrm{d}W_s - \frac{1}{4\varepsilon}\left\|\Delta K_0(h)\right\|_t^2\right)\ge 0\right)\\
&\quad + \mathbf{P}_\vartheta\left(\sup_{\delta\le t\le T}\sup_{|h|>v_\varepsilon}\left(\frac{1}{2\varepsilon}\int_0^t \left|\Delta K_0(h)^2-\Delta K_\varepsilon(h)^2\right|\mathrm{d}s - \frac{1}{4\varepsilon}\left\|\Delta S_0(h)\right\|_t^2\right)\ge 0\right)\\
&\le \mathbf{P}_\vartheta\left(\sup_{\delta\le t\le T}\sup_{|h|>v_\varepsilon}\int_0^t \Delta K_\varepsilon(h)\mathrm{d}W_s \ge \inf_{|h|>v_\varepsilon}\frac{1}{4\varepsilon}\left\|\Delta K_0(h)\right\|_\delta^2\right)\\
&\quad + \mathbf{P}_\vartheta\Bigg(\sup_{\delta\le t\le T}\sup_{|h|>v_\varepsilon}\frac{1}{2\varepsilon}\int_0^t \left|\Delta K_0(h)-\Delta K_\varepsilon(h)\right|\left|\Delta K_0(h)+\Delta K_\varepsilon(h)\right|\mathrm{d}s\\
&\qquad \ge \inf_{|h|>v_\varepsilon}\frac{1}{4\varepsilon}\left\|\Delta K_0(h)\right\|_\delta^2\Bigg)\\
&\le \mathbf{P}_\vartheta\left(\sup_{\delta\le t\le T}\sup_{|h|>v_\varepsilon}\int_0^t \Delta K_\varepsilon(h)\mathrm{d}W_s \ge \frac{\kappa}{4\varepsilon}v_\varepsilon^2\right)\\
&\quad + \mathbf{P}_\vartheta\left(\sup_{\delta\le t\le T}\sup_{|h|>v_\varepsilon}\frac{1}{\varepsilon}\int_0^t \left|\Delta K_0(h)-\Delta K_\varepsilon(h)\right|\mathrm{d}s \ge \frac{\kappa v_\varepsilon^2}{2C_0\varepsilon}\right).
\end{aligned}$$

We consider separately on $h\in(v_\varepsilon,\beta-\vartheta)$ and $h\in(\alpha-\vartheta,-v_\varepsilon)$

$$\begin{aligned}
&\mathbf{P}_\vartheta\left(\sup_{\delta\le t\le T}\sup_{v_\varepsilon<h<\beta-\vartheta}\int_0^t \Delta K_\varepsilon(h)\mathrm{d}W_s \ge \frac{\kappa}{4}\varepsilon^{-1+2\delta}\right)\\
&\quad\le \mathbf{P}_\vartheta\left(\sup_{\delta\le t\le T}\sup_{v_\varepsilon<h<\beta-\vartheta}\int_0^t \left(\Delta K_\varepsilon(h)-\Delta K_\varepsilon(v_\varepsilon)\right)\mathrm{d}W_s \ge \frac{\kappa}{8}\varepsilon^{-1+2\delta}\right)\\
&\qquad + \mathbf{P}_\vartheta\left(\sup_{\delta\le t\le T}\int_0^t \Delta K_\varepsilon(v_\varepsilon)\mathrm{d}W_s \ge \frac{\kappa}{8}\varepsilon^{-1+2\delta}\right)\\
&\quad\le C_1\varepsilon^{2p(1-2\delta)} + C_2\mathrm{e}^{-\kappa_2\varepsilon^{-1+2\delta}} \le C\varepsilon^m,
\end{aligned}$$

for any $m\ge 3$. Here we have applied Lemma 3.6.1 and the Lemma 3.6.2 in choosing $p=\frac{m}{2-4\delta}$.
Similarly we have

$$\mathbf{P}_\vartheta\left(\sup_{\delta\le t\le T}\sup_{\alpha-\vartheta<h<v_\varepsilon}\int_0^t \Delta K_\varepsilon(h)\mathrm{d}W_s \ge \frac{\kappa}{4}\varepsilon^{-1+2\delta}\right)\le C\varepsilon^m.$$

Further

$$\frac{1}{\varepsilon}\int_0^t |\Delta K_0(h) - \Delta K_\varepsilon(h)| \mathrm{d}s \leq C \sup_{0\leq s\leq t} |W_s|.$$

In Chapter 1 in Kutoyants [27], there is the following inequality

$$\mathbf{P}\left\{\sup_{0\leq t\leq T} |W_t| > N\right\} \leq 4\mathbf{P}\{W_T > N\} \leq \min\left(2, \frac{4}{N}\sqrt{\frac{T}{2\pi}}\right) \mathrm{e}^{-\frac{N^2}{2T}}.$$

Thus we have

$$\begin{aligned}
&\mathbf{P}_\vartheta\left(\sup_{\delta\leq t\leq T}\sup_{|h|>v_\varepsilon} \frac{1}{\varepsilon}\int_0^t |\Delta K_0(h) - \Delta K_\varepsilon(h)|\, \mathrm{d}s \geq \frac{\kappa v_\varepsilon^2}{2C_0\varepsilon}\right) \\
&\quad\leq \mathbf{P}_\vartheta\left(\sup_{0\leq s\leq T} \frac{1}{\varepsilon}|W_s| \geq \frac{\kappa}{2C_0CT}\varepsilon^{-1+2\delta}\right) \\
&\quad\leq 4\mathbf{P}\left\{W_T > \frac{\kappa}{2C_0CT}\varepsilon^{-1+2\delta}\right\} \leq 2\exp\left\{\frac{\kappa^2}{8C_0^2C^2T^3}\varepsilon^{-2+4\delta}\right\}.
\end{aligned}$$

All these estimates propose us

$$\sup_{\vartheta\in\mathbb{K}} \mathbf{P}_\vartheta\left(\mathcal{M}_1^c\right) \leq C\varepsilon^m. \tag{3.41}$$

For the complement of $\mathcal{M}_2$, we have

$$\begin{aligned}
\mathbf{P}_\vartheta(\mathcal{A}_2^c) &\leq \mathbf{P}\left(\sup_{0\leq s\leq T}|W_s| \geq \varepsilon^{-1+\delta}\right) + \mathbf{P}\left(\sup_{\delta\leq t\leq T}\sup_{|h|<v_\varepsilon}\int_0^t \ddot{K}(\vartheta+h, X_s)\mathrm{d}W_s \geq \varepsilon^{-1+\delta}\right) \\
&\quad + \mathbf{P}\left(\sup_{\delta\leq t\leq T}\int_0^t \dot{K}(\vartheta, X_s)\mathrm{d}W_s \geq \frac{1}{2}\varepsilon^{-1+\delta} I_\delta(\vartheta)\right) \\
&\leq 2\mathrm{e}^{-\frac{1}{2T}\varepsilon^{-2+2\delta}} + \mathbf{P}\left(\sup_{\delta\leq t\leq T}\int_0^t \ddot{K}(\vartheta - v_\varepsilon, X_s)\mathrm{d}W_s \geq \frac{1}{2}\varepsilon^{-1+\delta}\right) \\
&\quad + \mathbf{P}\left(\sup_{\delta\leq t\leq T}\sup_{|h|<v_\varepsilon}\int_0^t \left(\ddot{K}(\vartheta+h, X_s) - \ddot{K}(\vartheta - v_\varepsilon, X_s)\right)\mathrm{d}W_s \geq \frac{1}{2}\varepsilon^{-1+\delta}\right) \\
&\quad + C_1\mathrm{e}^{-\lambda\varepsilon^{-1+\delta}} \leq 2\mathrm{e}^{-\frac{1}{2T}\varepsilon^{-2+2\delta}} + C_2\varepsilon^m + C_3\mathrm{e}^{-\lambda\varepsilon^{-1+\delta}} \leq C\varepsilon^3,
\end{aligned}$$

where we have applied the Lemma 3.6.1 and the Lemma 3.6.2 in choosing $p = \frac{m}{2-2\delta}$. Thus we have

$$\sup_{\vartheta\in\mathbb{K}} \mathbf{P}_\vartheta\left(\mathcal{M}_2^c\right) \leq C\varepsilon^m. \tag{3.42}$$

For the complement of $\mathcal{M}_3$, note that

$$h^{(3)}(\varepsilon) = -\left(F_h'\right)^{-5}\Big[\left(3F_{hh}''F_\varepsilon'F_h' - 2F_{h\varepsilon}''\right)\left(F_{\varepsilon\varepsilon}''(F_h')^2 - 2F_{h\varepsilon}''F_\varepsilon'F_h' + F_{hh}''(F_\varepsilon')^2\right)$$
$$- F_{hhh}'''F_h'(F_\varepsilon')^3 + 2F_{hh\varepsilon}'''(F_\varepsilon')^2(F_h')^2 - 2F_{h\varepsilon\varepsilon}'''F_\varepsilon'(F_h')^3 + F_{\varepsilon\varepsilon\varepsilon}'''(F_h')^4\Big],$$

where $\frac{\partial^{i+j}F(h,\varepsilon)}{\partial h^i\partial\varepsilon^j}$ are all functions similar as in Lemma 3.3.1. Applying the result in Lemma 3.5 in Kutoyants [27]:

$$\sup_{\vartheta\in\mathbb{K}}\sup_{0\leq t\leq T}\left|X_t^{(j)}\right| \leq M_j\left(\sup_{0\leq t\leq T}|W_t|\right)^j,\ j=1,2,3...,k,$$

with M_j are some positive constants, we obtain

$$\sup_{\vartheta\in\mathbb{K}}\mathbf{P}_\vartheta\left(\mathcal{M}_3^c\right) \leq C\varepsilon^m. \tag{3.43}$$

Moreover,

$$\mathbf{P}_\vartheta\left(\sup_{\delta\leq t\leq T}|\zeta_t| > \varepsilon^\delta\right) = \mathbf{P}_\vartheta\left(\sup_{\delta\leq t\leq T}\sup_{|h|<v_\varepsilon}L_t(\vartheta+h,Y) < \sup_{\delta\leq t\leq T}\sup_{|h|\geq v_\varepsilon}L_t(\vartheta+h,Y)\right)$$
$$\leq \mathbf{P}_\vartheta\left(\sup_{\delta\leq t\leq T}\sup_{|h|\geq v_\varepsilon}L_t(\vartheta+h,Y) > 0\right) = \mathbf{P}_\vartheta\left(\mathcal{M}_1^c\right).$$

We obtain finally

$$\mathbf{P}_\vartheta\left\{\sup_{\delta\leq t\leq T}\left|\frac{\widehat{\vartheta}_{t,\varepsilon}-\vartheta}{\varepsilon^2} - \frac{\xi_{t,1}(x^t,\vartheta)}{\varepsilon} - \xi_{t,2}(x^t,\vartheta)\right| > \nu\right\}$$
$$= \mathbf{P}_\vartheta\left\{\sup_{\delta\leq t\leq T}\left|X_{t,3}\varepsilon^{\frac{3}{2}}1\!\!1_{\{\mathcal{M}_t\}} + \varepsilon^{-2}\zeta_t 1\!\!1_{\{\mathcal{M}_t^c\}}\right| > \nu\right\}$$
$$\leq \mathbf{P}_\vartheta\left\{\sup_{\delta\leq t\leq T}\left|X_{t,3}\varepsilon^{\frac{3}{2}}1\!\!1_{\{\mathcal{M}\}}\right| > \frac{\nu}{2}\right\} + \mathbf{P}_\vartheta\left\{\sup_{\delta\leq t\leq T}\left|\varepsilon^{-2}\zeta_t 1\!\!1_{\{\mathcal{M}^c\}}\right| > \frac{\nu}{2}\right\}$$
$$\leq \mathbf{P}_\vartheta\left\{\varepsilon^{\frac{3}{2}}1\!\!1_{\{\mathcal{M}\}} > \frac{\nu}{2}\right\} + \mathbf{P}_\vartheta\left\{\sup_{\delta\leq t\leq T}\left|\zeta_t 1\!\!1_{\{\mathcal{M}^c\}}\right| > \frac{\nu}{2}\varepsilon^2,\ \sup_{\delta\leq t\leq T}|\zeta_t| > \varepsilon^\delta\right\}$$
$$+ \mathbf{P}_\vartheta\left\{\sup_{\delta\leq t\leq T}\left|\zeta_t 1\!\!1_{\{\mathcal{M}^c\}}\right| > \frac{\nu}{2}\varepsilon^2,\ \sup_{\delta\leq t\leq T}|\zeta_t| \leq \varepsilon^\delta\right\}$$
$$\leq \left(\frac{\nu}{2}\varepsilon^{-\frac{3}{2}}\right)^{-2}\mathbf{P}_\vartheta(\mathcal{M}) + \mathbf{P}_\vartheta\left(\sup_{\delta\leq t\leq T}|\zeta_t| > \varepsilon^\delta\right) + \mathbf{P}_\vartheta\left(\varepsilon^\delta 1\!\!1_{\{\mathcal{M}^c\}} > \frac{\nu}{2}\varepsilon^2\right)$$
$$\leq C_1\varepsilon^3 + C_2\varepsilon^{m-2} + C_3\varepsilon^{m-4+2\delta} \longrightarrow 0.$$

Conclusions

We have shown in Chapter 2 our works on GoF and in Chapter 3 the works on approximation of FBSDE.

In Chapter 2, we have shown three types of test for diffusion process: the Cramer-von Mises type test, the Kolrogomov-Smirnov type test and the chi-square test. The C-vM and K-S test for diffusion process with shift parameter are shown to be consistent and APF in Section 2.2 and 2.3. Note that in these two sections, we consider only the SDEs with constant diffusion coefficient: $\sigma^2 = 1$. This is a technical assumption to obtain the APF property for the test. Then it is natural to consider the generalization of the model. In fact, Kutoyants [30] has considered another possibility of construction of models which are also APF. In [30], he consider the diffusion process with scale and location parameters in the drift coefficient S, and with a diffusion coefficient σ^2 as a function of x. But the limitation of the model is that the drift coefficient functions are of form fixed: $S(x) = \beta \mathrm{sgn}(x - \alpha)|x - \alpha|^{\gamma}$. Thus we have not yet resolved the problem for a more general case. In Section 2.4, we introduced the chi-square test for simple case, where the test is shown to be ADF. As that we remarked at last of the section, our goal is to obtain an ADF test for the whole space of $\mathcal{L}^2(f_*)$. That is we consider the case where N converges to infinity. This is an interesting problem but not yet been treated.

In Chapter 3, we have considered the approximation problem for solution of FBSDE. This is a starting work to explore how to put statistical problems for FBSDE. Remind that the FBSDE model could wildly be applied in many fields. However, as that is shown in this chapter, our result is limited to the linear case for the backward equation. Moreover, the conditions on coefficients are very strong. These situations limit the application of the model. So the further work will be concentrated on weakening the conditions and on generalizing the models. In addition, we will consider other statements of statistical problems for BSDEs.

Bibliography

[1] Babu G.J. and Rao C.R. (2004). Goodness-of-fit test when parameters are estimated. *The Indian Journal of Statistics*, Vol. 66, 63-74.

[2] Balakrishnan N., Nikulin M. S., Mesbah M. and Huber-Carol C. (Eds) (2002). Goodness-of-Fit Tests and Model Validity. *Birkhäuser*, Boston.

[3] Bosq D. (2012). Statistique Mathématique et Statistique des Processus. *Hermès & Lavoisier*, Paris.

[4] Billingsley P. (2009) Convergence of Probability Measure. *J. Wiley*, New York.

[5] Cochran W.G.. (1952). The χ^2 test of goodness of fit. *Ann. Math. Stat.*, Vol 23, 315-345.

[6] Cramer H. (1928). On the composition of elementary errors. *II Skand. Aktuarietids*, Vol 11, 141-180.

[7] Cramer H. (1999). Mathematical Methods of Statistics. *Princeton University Press*.

[8] Dachian S. and Kutoyants Y.A. (2007). On the goodness-of-fit tests for some continuous time processes. Statistical Models and Methods for Biomedical and Technical Systems, F.Vonta et al. (Eds), *Birkhauser*, Boston, 395-413.

[9] Dahiya R.C. and Gurland J. (1972). Pearson chi-squared test of fit with random intervals. *Biometrika Trust*, 59(1), 147-153.

[10] Darling D.A. (1955). The Cramer-Smirnov test in the parametric case. *Ann. Math. Statist.*, 26, 1-20.

[11] Dehay D. and Kutoyants Y.A. (2004). On confidence intervals for distribution function and density of ergodic diffusion process. *Journal of Statistical Planning and Inference*, 124, 63-73.

[12] Durbin J. (1973). Distribution Theory for Tests Based on the Sample Distribution Function. *Society for Industrial and Applied Mathematics*, Philadelphia.

[13] Durett R. (1996). Statistic Calculus: A Practical Introduction. *Boca Raton: CRC Press.*

[14] El Karoui N., Mazliak L. (Eds). (1997). Backward Stochastic Differential Equations. Pitman Research Notes in Mathematics Series 364. *Longman*, Harlow.

[15] El Karoui N., Peng S. and Quenez M. (1997). Backward stochastic differential equations in finance, *Math. Finance* 7 1-71.

[16] Fasano G. and Franceschini A. (1987). A multidimensional version of the Kolmogorov-Smirnov test. *Monthly Notices of the Royal Astronomical Society*, 225: 155-170.

[17] Freidlin M. and Wentzell A. (1998). Random Perturbations of Dynamical Systems. *Springer*, New York.

[18] Friedman A. (2008). Partial Differential Equations of Parabolic Type. *Prentice-Hall*, New York.

[19] Fournie E. (1992). Un test de type Kolmogorov-Smirnov pour processus de diffusion ergodiques. Rapports de Recherche, No 1696, INRIA, Sophia-Antipolis.

[20] Fournie E. and Kutoyants Y.A. (1993). Estimateur de la distance minimale pour des processus de diffusion ergodiques. Rapports de Recherche, No 1952, INRIA, Sophia-Antipolis.

[21] Greenwood P.E. and Nikulin M.S. (1996). A Guide to Chi-squared Testing. *Wiley*, New York.

[22] Ibragimov I.A. and Khasminskii R. (1981). Statistical Estimation - Asymptotic Theorey. *Springer-Verlag*, New York.

[23] Kac M., Kiefer J. and Wolfowitz J. (1955). On tests of normality and other tests of goodness-of-fit based on distance methods. *Ann. Math. Statist.*, 26:189-211.

[24] Karatzas I. and Shreve S.E. (1991). Brownian Motion and Stochastic Calculus. *Springer-Verlag*, New York.

[25] Khasminskii R. (2012). Stochastic Stability of Differential Equations. *Springer-Verlag*, Berlin Heidelberg.

[26] Kolmogorov A (1933). Sulla determinazione empirica di una legge di distribuzione. *Giornale dell'Istituto Italiano degli Attuari*, 4: 83-91.

[27] Kutoyants Y.A. (1994). Identification of Dynamical Systems with Small Noise, *Kluwer Academic Publisher*, Dordrecht.

[28] Kutoyants Y.A. (2004). Statiatical Inference for Ergodic Diffusion Processes. *Springer*, London.

[29] Kutoyants Y.A. (2010). On the goodness-of-fit testing for ergodic diffusion process. *Journal of Nonparametric Statistic*, 22, 4, 529-543.

[30] Kutoyants Y.A. (2012). On asymptotically distribution and parameter free goodness-of-fit tests for ergodic diffusion processes. arxiv.org/abs/1302.1026, submitted.

[31] Kutoyants Y.A. and Zhou L. (2013). On approximation of forward-backward stochastic differential equation, submitted.

[32] Lehmann E.L. and Romano J.P (1986). Testing Statistical Hypotheses. *Springer-Verlag*, New York.

[33] Levanony D., Shwartz A. and Zeitouni O. (1994). Recursive identification in continuous-time stochastic process. *Stochastic Process Appl.*, 49, 245-275.

[34] Levanony D., Shwartz A. and Zeitouni O. (1990). Continuous-time recursive identification. SIn E. Arikan (ed) Communication, *Control and Signal Processisng*. Amsterdam:Elsevier, 1725-1732.

[35] Lipster R.S. and Shiryaev A.N. (1978). A.N. Statistics of Random Process I II. *Springer* New York.

[36] Ma J., Yong J.(1999). Forward-Backward Stochastic Differential Equations and their Applications. Lecture Notes in Mathematics. *Springer-Verlag*, Berlin.

[37] Martynov G.V. (1979). The Omega Square Tests, *Nauka*, Moscow.

[38] Martynov G.V. (1992). Statistical tests based on EDF empirical process and related question. *J. Soviet. Math.* 61, 2195-2271.

[39] Negri, I. (1998). Stationary distribution function estimation for ergodic diffusion process, *Statistical Inference for Stochastic Processes*, 1, 61–84.

[40] Negri, I. and Nishiyama, Y. (2009). Goodness of fit test for ergodic diffusion processes, Ann. Inst. Statist. Math., 61, 919-928.

[41] Negri, I. and Zhou L. (2012). On goodness-of-fit testing for ergodic diffusion process with shift parameter, submitted.

[42] Nikulin M.S. (1973). Chi-squared test for normality. In: Proceedings of the International Vilnius Conference on Probability Theory and Mathematical Statistics, v.2, 119-122.

[43] Pardoux E. and Peng S. (1990), Adapted solution of a backward stochastic differential equation. *System Control Letter*, 14 55-61.

[44] Pardoux E. and Peng S. (1992). Backward stochastic differential equations and quasilinear parabolic partial differential equations. *Stochastic Partial Differential Equations and their Applications* (Lect. Notes Control Inf. Sci. 176), 200-217, *Springer*, Berlin.

[45] Revuz D. and Yor M. (1991). Continuous Martingles and Brownian Motion. *Spring-Verlag*, New-York.

[46] Smirnov N. (1936). Sur la distribution de ω^2. *C. R. Acad. Sci. Paris*, Vol 202, 449-452.

[47] Smirnov N. (1948). Tables for estimating the goodness of fit of empirical distributions. *Annals of Mathematical Statistics* 19: 279.

[48] Skorokhod A.V. (1989). Asymptotic Methods in the Theory of Stochastic Differential Equations. *American Mathematical Society*, 78, Providence-Rhode Island.

[49] Watson G.S. (1957). The χ^2 goodness-of -fit test for normal distributions. *Biometrika Trust*, 44(4), 336–348.

[50] Zhou L. (2012). On asymptotically parameter free test for ergodic diffusion process, submitted.

[51] Zhou L. (2013). On chi-square test for ergodic diffusion process, submitted.