On parameter estimation for switching diffusion process

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Abstract

We consider the problem of parameter estimation by the observation of ergodic diffusion process. We suppose that the unknown parameter is two-dimensional and the trend coefficient of the process is discontinuous "sign-type". We describe the asymptotic properties of the maximum likelihood estimator, Bayesian estimator and the estimator of the method of moment in this case.

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Key Words: Maximum likelihood estimator, Bayesian estimator, estimator of the method of moment, limit distribution, limit likelihood ratio.

1 Introduction

We consider the problem of parameter estimation by the observation of diffusion process

$$dX_t = -\vartheta_1 \mathrm{sgn}(X_t - \vartheta_2)dt + dW_t, \quad X_0, \quad 0 \le t \le T, \tag{1}$$

where the unknown parameter $\vartheta = (\vartheta_1, \vartheta_2) \in \Theta = (\alpha_1, \beta_1) \times (\alpha_2, \beta_2)$ with $\alpha_1 > 0$. The problems of parameter estimation for ergodic diffusion processes are well studied. It is shown that under regularity conditions the classical estimators (maximum likelihood, Bayesian and method of moment) are consistent, asymptotically normal (with the rate $\sqrt{T}$). Moreover, the maximum likelihood and Bayesian estimators are efficient (see Kutoyants [7]).

The case of singular estimation, when the Fisher information is infinite is less studied. We can mention here the works by Küchler and Kutoyants [4], Dachian and Kutoyants [1] where such problems of parameter estimation were considered in the situations of discontinuous and cusp-type singular us usual, the rate of convergence of the same estimators (in singular case) is bettre than in regular case.

Statistical estimation for the switching diffusion process first was studied in [6], where the problem of estimations of the one-dimensional parameter $\vartheta = \vartheta_2$ by observation (1) was treated.

In the present work we consider the same model of observation, but we suppose that the both parameters ϑ_1 and ϑ_2 are unknown. We show that the maximum likelihood estimator and bayesian estimator have different limit distributions with the normalizing rates $\sqrt{T}$ for ϑ_1 and T for ϑ_2. We follow the same method as used in [5].

The proof is based on the general theorems by Ibragimov and Khasminskii [3] which allow to describe the asymptotic properties of these estimators through the properties of the properly normalized likelihood ratio process.

2 Main result

We observe a trajectory $X^T = \{X_t, 0 \le t \le T\}$ of the diffusion process given by the stochastic differential equation (1). It is easy to see that the conditions of the existence of the solution and ergodicity (see, e.g., conditions $\mathcal{ES}$ and $\mathcal{A}_0(\Theta)$ in Durrett [2], Kutoyants [5]), are fulfilled and (1) is an ergodic diffusion process with the stationary density $f(\vartheta, x) = \vartheta_1 \, e^{-2\,\vartheta_1 |x-\vartheta_2|}$, for $x \in \mathbb{R}$.

To construct the maximum likelihood and the Bayesian estimators we introduce the likelihood ratio function

$$L\left(\vartheta, X^T\right) = \frac{d\mathbf{P}_\vartheta^{(T)}}{d\mathbf{P}^{(T)}}\left(X^T\right), \qquad \vartheta \in \Theta,$$

(here $\mathbf{P}^{(T)} = \mathbf{P}_{0,0}^{(T)}$) by the formula

$$L\left(\vartheta, X^T\right) = \exp\left\{-\vartheta_1 \int_0^T \mathrm{sgn}(X_t - \vartheta_2)\, dX_t - \frac{\vartheta_1^2}{2}\, T\right\}.$$

The maximum likelihood estimator (MLE) $\hat\vartheta_T$ is defined as solution of the equation

$$L\left(\hat\vartheta_T, X^T\right) = \sup_{\vartheta \in \Theta} L\left(\vartheta, X^T\right).$$

If this equation has more than one solution, then we can take anyone as MLE.

To introduce the Bayesian estimator (BE) we suppose that the unknown parameter ϑ is a random vector with a known prior density $\{p(\theta), \theta \in \Theta\}$, which is continuous and positive. Using the quadratic loss function, the BE $\tilde{\vartheta}_T$ (which minimizes the mean square error) is the conditional mathematical expectation

$$\tilde{\vartheta}_T = \int_\Theta \theta\, p\left(\theta | X^T\right)\, d\theta = \frac{\int_\Theta \theta\, p\,(\theta)\ L\left(\theta, X^T\right)\, d\theta}{\int_\Theta p\,(\theta)\ L\left(\theta, X^T\right)\, d\theta}.$$

To describe the limit behavior of the MLE and BE we need two random vectors $\hat{w} = (\hat{v}, \hat{u})$ and $\tilde{w} = (\tilde{v}, \tilde{u})$ defined with the help of the following stochastic process:

$$Z_\vartheta(w) = Z_{\vartheta_1}(v)\, Z_{\vartheta_2}(u), \quad w = (v, u) \in \mathbb{R}^2$$

where

$$Z_{\vartheta_1}(v) = \exp\left\{v\,\zeta - \frac{v^2}{2}\right\}, \qquad Z_{\vartheta_2}(u) = \exp\left\{2\,\vartheta_1^{3/2}\, W(u) - 2\,\vartheta_1^3\, |u|\right\},$$

as foollows:

$$Z_\vartheta(\hat{w}) = \sup_{w \in \mathbb{R}^2} Z_\vartheta(w), \qquad\qquad \tilde{w} = \frac{\int_{\mathbb{R}^2} w\, Z_\vartheta\,(w)\ dw}{\int_{\mathbb{R}^2} Z_\vartheta\,(w)\ dw}.$$

Here ζ and $W(\cdot)$ are independent, where ζ denote independent and identically distributed $\mathcal{N}(0,1)$ random variable. $W(\cdot)$ is a two-sided Wiener process, i.e., $W(u) = W_+(u)$ for $u \geq 0$ and $W(u) = W_-(-u)$ for $u < 0$, where $W_+(u)$, $W_-(u)$, $u \geq 0$ are two independent standard Wiener processes.

Introduce the normalizing matrix

$$\varphi_T = \begin{pmatrix} T^{-1/2} & 0 \\ 0 & T^{-1} \end{pmatrix}.$$

The asymptotic properties of the MLE $\hat{\vartheta}_T = (\hat{\vartheta}_T^{(1)}, \hat{\vartheta}_T^{(2)})$ and the BE $\tilde{\vartheta}_T = (\tilde{\vartheta}_T^{(1)}, \tilde{\vartheta}_T^{(2)})$ are described in the following theorems.

Theorem 2.1. *The MLE $\hat{\vartheta}_T$ constructed by the observations X^T of the diffusion process is consistent, i.e., for any $\nu > 0$*

$$\lim_{T \to \infty} \mathbf{P}_\vartheta^{(T)} \left\{|\hat{\vartheta}_T - \vartheta| > \nu\right\} = 0,$$

the distribution of the random vector $\varphi_T^{-1}(\hat\vartheta_T-\vartheta)$ *converge to the distribution of the random vector* $\hat w$ *and for any* $p>0$

$$\lim_{T\to\infty}\mathbf{E}_\vartheta\left|\varphi_T^{-1}(\hat\vartheta_T-\vartheta)\right|^p=\mathbf{E}_\vartheta\left|\hat w\right|^p,$$

Theorem 2.2. *The BE* $\tilde\vartheta_T$ *constructed by the observations* X^T *of the diffusion process is consistent, the normalized difference* $\varphi_T^{-1}(\tilde\vartheta_T-\vartheta)$ *converges in distribution:*

$$\varphi_T^{-1}(\tilde\vartheta_T-\vartheta)\Longrightarrow\tilde w,$$

and for any $p>0$

$$\lim_{T\to\infty}\mathbf{E}_\vartheta\left|\varphi_T^{-1}(\tilde\vartheta_T-\vartheta)\right|^p=\mathbf{E}_\vartheta\left|\tilde w\right|^p,$$

Proof. The proof is based on the general result by Ibragimov and Khasminskii [3], Theorems 1.10.1 and 1.10.2, so we check the conditions of these theorems with the help of the three lemmas presented below. That is, we show the convergence of marginal distributions of the normalized likelihood ratio $Z_T(\cdot)$ and establish two estimates on the increments $Z_T(\cdot)$ and on its decrease.

Introduce the normalized likelihood ratio process

$$Z_T(w)=\frac{d\mathbf{P}^{(T)}_{\vartheta+\varphi_T w}}{d\mathbf{P}^{(T)}_{\vartheta}}(X^T)=\frac{L(\vartheta+\varphi_T\,w,X^T)}{L(\vartheta,X^T)},\qquad w\in\mathbb{W}_T,$$

where ϑ is the true value and the set

$$\mathbb{W}_T=\left(\sqrt{T}\left(\alpha_1-\vartheta_1\right),\sqrt{T}\left(\beta_1-\vartheta_1\right)\right)\times\left(T\left(\alpha_2-\vartheta_2\right),T\left(\beta_2-\vartheta_2\right)\right).$$

Write the BE ($\vartheta_w=\vartheta+\varphi_T\,w$) as

$$\tilde\vartheta_T=\frac{\int_\Theta\theta\,p\left(\theta\right)\,L\left(\theta,X^T\right)\,d\theta}{\int_\Theta p\left(\theta\right)\,L\left(\theta,X^T\right)\,d\theta}=\vartheta+\varphi_T\,\frac{\int_{\mathbb{W}_T}w\,p\left(\vartheta_w\right)\,L\left(\vartheta_w,X^T\right)\,dw}{\int_{\mathbb{W}_T}p\left(\vartheta_w\right)\,L\left(\vartheta_w,X^T\right)\,dw}$$

$$=\vartheta+\varphi_T\,\frac{\int_{\mathbb{W}_T}w\,p\left(\vartheta_w\right)\,\frac{L\left(\vartheta_w,X^T\right)}{L(\vartheta,X^T)}\,dw}{\int_{\mathbb{W}_T}p\left(\vartheta_w\right)\,\frac{L(\vartheta_w,X^T)}{L(\vartheta,X^T)}\,dw}=\vartheta+\varphi_T\,\frac{\int_{\mathbb{W}_T}w\,p(\vartheta_w)\,Z_T(w)dw}{\int_{\mathbb{W}_T}p(\vartheta_w)\,Z_T(w)\,dw}.$$

We have the following presentations for the MLE and BE:

$$\hat w_T=\arg\sup_{w\in\mathbb{W}_T}Z_T(w),\qquad\tilde w_T=\frac{\int_{\mathbb{W}_T}w\,p(\vartheta_w)\,Z_T(w)dw}{\int_{\mathbb{W}_T}p(\vartheta_w)\,Z_T(w)\,dw}.$$

where

$$\hat w_T=\left(\sqrt{T}(\hat\vartheta_T^{(1)}-\vartheta_1),\,T(\hat\vartheta_T^{(2)}-\vartheta_2)\right),\quad\tilde w_T=\left(\sqrt{T}(\tilde\vartheta_T^{(1)}-\vartheta_1),\,T(\tilde\vartheta_T^{(2)}-\vartheta_2)\right).$$

Lemma 2.1. *The marginal (finite-dimensional) distributions of the random functions $Z_T(\cdot)$ converge to the marginal distributions of the random functions $Z_\vartheta(\cdot)$.*

Proof. As before, we put $\vartheta_v = \vartheta_1 + \frac{v}{\sqrt{T}}$, $\vartheta_u = \vartheta_2 + \frac{u}{T}$ and denoted

$$\delta(\vartheta_w, \vartheta, x) = -2\,\vartheta_v\, q_\vartheta(u,x) - \frac{v}{\sqrt{T}}\,\mathrm{sgn}(x-\vartheta_2), \quad \vartheta_w = (\vartheta_v, \vartheta_u). \tag{2}$$

where $q_\vartheta(u,x) = \mathrm{sgn}(u)\, 1_{\{\vartheta_2\wedge\vartheta_u \le x \le \vartheta_2 \vee \vartheta_u\}}$. The normalized likelihood ratio $Z_T(\cdot)$ with $\mathbf{P}_\vartheta^{(T)}$ probability 1 admits the representation

$$\begin{aligned} Z_T(v,u) &= \exp\left\{\int_0^T \delta(\vartheta_w,\vartheta,X_t)\,dW_t - \frac{1}{2}\int_0^T \delta(\vartheta_w,\vartheta,X_t)^2 dt\right\} \\ &= \exp\left\{2\,\vartheta_v\, I_T(u,\vartheta) - 2\,\vartheta_1\vartheta_v \int_0^T q_\vartheta(u,X_t)^2 dt + v\,\zeta_T - \frac{v^2}{2}\right\} \quad (3) \end{aligned}$$

where the stochastic integrals:

$$I_T(u,\vartheta) = -\int_0^T q_\vartheta(u,X_t)\,dW_t, \qquad \zeta_T = -\frac{1}{\sqrt{T}}\int_0^T \mathrm{sgn}\,(X_t - \vartheta_2)\,dW_t.$$

We show that for a fixed u

$$\mathbf{P}_\vartheta - \lim_{T\to\infty}\int_0^T q_\vartheta(u,X_t)^2 dt = \mathbf{P}_\vartheta - \lim_{T\to\infty}\int_0^T 1_{\{\vartheta_2\wedge\vartheta_u\le X_t\le\vartheta_2\vee\vartheta_u\}}\,dt = \vartheta_1|u|. \tag{4}$$

Note that the local time $\Lambda_T(x)$ allows us to write the equality (see Revuz and Yor [8]), for any integrable function $h(\cdot)$

$$\frac{1}{T}\int_0^T h(X_t)\,dt = \int_{-\infty}^{\infty} h(x)\,\frac{\Lambda_T(x)}{T}\,dx = \int_{-\infty}^{\infty} h(x)\, f_T^\circ(x)\,dx.$$

Hence we obtain an estimate for the ordinary integral

$$\begin{aligned} &\mathbf{E}_\vartheta\left(\int_0^T q_\vartheta(u,X_t)^2\,dt - \vartheta_1|u|\right)^2 = \mathbf{E}_\vartheta\left(\int_{\vartheta_2\wedge\vartheta_u}^{\vartheta_2\vee\vartheta_u} f_T^\circ(\vartheta,x)\,dx - |u|\,f(\vartheta,\vartheta_2)\right)^2 \\ &\le 2\,T^2\mathbf{E}_\vartheta\left(\int_{\vartheta_2\wedge\vartheta_u}^{\vartheta_2\vee\vartheta_u}\left[f_T^\circ(\vartheta,x) - f(\vartheta,x)\right]\,dx\right)^2 \\ &\quad +2\,T^2\left(\int_{\vartheta_2\wedge\vartheta_u}^{\vartheta_2\vee\vartheta_u}\left[f(\vartheta,x) - f(\vartheta,\vartheta_2)\right]\,dx\right)^2 \\ &\le 2\,\frac{|u|^2}{T}\sup_{x\in(\alpha_2,\beta_2)}\mathbf{E}_\vartheta\left(\sqrt{T}\left[f_T^\circ(\vartheta,x) - f(\vartheta,x)\right]\right)^2 + 2\,\vartheta_1^2\frac{|u|^4}{T^2} \\ &\le 2\left(\frac{16}{T}+4\right)\frac{|u|^2}{T} + 2\,\beta_1^2\frac{|u|^4}{T^2}. \end{aligned} \tag{5}$$

where we used the estimates

$$|f(\vartheta,x)-f(\vartheta,\vartheta_2)|\leq 2\,\vartheta_1|x-\vartheta_2|,$$

and

$$\sup_{x\in(\alpha_2,\beta_2)}\mathbf{E}_\vartheta\left(\sqrt{T}\,[f_T^\circ(\vartheta,x)-f(\vartheta,x)]\right)^2=\sup_{x\in(\alpha_2,\beta_2)}\mathbf{E}_\vartheta\eta_T(\vartheta,x)^2\leq\frac{16}{T}+4.$$

The first one is trivial and the second follows from the representation (see Kutoyants [5] p. 29)

$$\begin{aligned}\eta_T(\vartheta,x)&=\frac{2\,f(\vartheta,x)}{\sqrt{T}}\int_{X_0}^{X_T}\frac{1_{\{v>x\}}-F(\vartheta,v)}{f(\vartheta,v)}\,dv\\&\quad-\frac{2\,f(\vartheta,x)}{\sqrt{T}}\int_0^T\frac{1_{\{X_t>x\}}-F(\vartheta,X_t)}{f(\vartheta,X_t)}\,dW_t.\end{aligned}$$

To simplify the exposition we suppose that the process X^T is stationary. So the direct calculation gives us the estimate, for any $p\geq 1$

$$\begin{aligned}\mathbf{E}_\vartheta\,|\eta_T(\vartheta,x)|^{2p}&\leq 2^{\,4p-1}\,T^{-p}\,f(\vartheta,x)^{2p}\,\mathbf{E}_\vartheta\left|\int_{X_0}^{X_T}\frac{1_{\{v>x\}}-F(\vartheta,v)}{f(\vartheta,v)}\,dv\right|^{2p}\\&\quad+2^{\,4p-1}\,f(\vartheta,x)^{2p}\,\mathbf{E}_\vartheta\left|\frac{1_{\{\xi>x\}}-F(\vartheta,\xi)}{f(\vartheta,\xi)}\right|^{2p}\\&\leq\frac{2^{\,6p-3}\,\Gamma(2p+1)}{(2p-1)^{2p+1}\,\vartheta_1}\,T^{-p}\,f(\vartheta,x)+\frac{2^{\,4p-2}}{(2p-1)\,\vartheta_1}\,f(\vartheta,x).\end{aligned}$$

Therefore, for any $\delta>0$

$$\begin{aligned}&\lim_{T\to\infty}\mathbf{P}_\vartheta\left\{\left|\int_0^T 1_{\{\vartheta_2\wedge\vartheta_u\leq X_t\leq\vartheta_2\vee\vartheta_u\}}\,dt-\vartheta_1|u|\right|>\delta\right\}\\&\qquad\leq\lim_{T\to\infty}\frac{1}{\delta^2}\mathbf{E}_\vartheta\left(\int_0^T 1_{\{\vartheta_2\wedge\vartheta_u\leq X_t\leq\vartheta_2\vee\vartheta_u\}}\,dt-\vartheta_1|u|\right)^2=0.\end{aligned}$$

The central limit theorem for stochastic integrals (see Kutoyants [5], Theorem 1.19, p.43) and convergence (3) provide the asymptotic normality

$$\mathcal{L}_\vartheta\left\{I_T(u,\vartheta)\right\}\Longrightarrow\mathcal{N}\left(0,\vartheta_1|u|\right),\qquad\mathcal{L}_\vartheta\left\{\zeta_T\right\}\Longrightarrow\mathcal{L}_\vartheta\left\{\zeta\right\}=\mathcal{N}(0,1).$$

Note that the integrals I_T and ζ_T are independent because

$$\lim_{T\to\infty}\mathbf{E}_\vartheta\left(I_T\zeta_T\right)=\lim_{T\to\infty}\sqrt{T}\int_{\vartheta_2\wedge\vartheta_u}^{\vartheta_2\vee\vartheta_u}f(\vartheta,x)\,dx\leq\lim_{T\to\infty}\beta_1\frac{|u|}{\sqrt{T}}=0.$$

These provide the convergence of one-dimensional distributions of $Z_T(v,u)$ to those of $Z_\vartheta(v,u)$ only.

It can be shown by a similar argument to the convergence (4) that for any $u_l, u_m \in \mathbb{R}$

$$\mathbf{P}_\vartheta - \lim_{T\to\infty} \int_0^T q_\vartheta(u_l, X_t)\, q_\vartheta(u_m, X_t)\, dt = \begin{cases} 0 & \text{if} \quad u_l\, u_m \leq 0, \\ |u_l| \wedge |u_m| & \text{if} \quad u_l\, u_m > 0 \end{cases} \tag{6}$$

Further, the same central limit theorem and the convergence (6) provides the joint asymptotic normality

$$\mathcal{L}_\vartheta \left\{ I_T(u_1, \vartheta), \ldots, I_T(u_k, , \vartheta) \right\} \Longrightarrow \mathcal{L}_\vartheta \left\{ \vartheta_1^{1/2} W(u_1), \ldots, \vartheta_1^{1/2} W(u_k) \right\},$$

for any $k = 1, 2, \ldots$. Finally, we obtain the convergence of the finite-dimensional distributions

$$\mathcal{L}_\vartheta \left\{ Z_T(v_1, u_1), \ldots, Z_T(v_k, u_k) \right\} \Longrightarrow \mathcal{L}_\vartheta \left\{ Z_\vartheta(v_1, u_1), \ldots, Z_\vartheta(v_k, u_k) \right\}.$$

Lemma 2.2. *There exist constants C_1 and C_2 such that*

$$\mathbf{E}_\vartheta \left| Z_T^{1/8}(v_1, u_1) - Z_T^{1/8}(v_2, u_2) \right|^4 \leq C_1 \left(|v_1 - v_2|^4 + |u_1 - u_2|^2 \right), \tag{7}$$

$$\mathbf{E}_\vartheta \left| Z_T^{1/2}(v_1, u_1) - Z_T^{1/2}(v_2, u_2) \right|^2 \leq C_2 \left(|v_1 - v_2|^2 + |u_1 - u_2| \right). \tag{8}$$

Proof. Let us denote

$$V_T = \left(\frac{d\mathbf{P}_{\vartheta_{w_1}}^{(T)}}{d\mathbf{P}_{\vartheta_{w_2}}^{(T)}} (X^T) \right)^{1/8} = \left(\frac{Z_T(v_1, u_1)}{Z_T(v_2, u_2)} \right)^{1/8}$$

(here $\vartheta_{w_i} = (\vartheta_{v_i}, \vartheta_{u_i}) = \left(\vartheta_1 + v_i/\sqrt{T}, \vartheta_2 + u_i/T \right), i = 1, 2$). The stochastic process $V^T = \{V_t, 0 \leq t \leq T\}$ by the Itô formula admits (with $\mathbf{P}_{\vartheta_{w_1}}^{(T)}$ probability 1) the differential equation (see Kutoyants [5], Lemma 1.13)

$$V_T = 1 - \frac{7}{128} \int_0^T V_t\, \delta(\vartheta_{w_1}, \vartheta_{w_2}, X_t)^2\, dt + \frac{1}{8} \int_0^T V_t\, \delta(\vartheta_{w_1}, \vartheta_{w_2}, X_t)\, dW_t,$$

where

$$\begin{aligned} \delta(\vartheta_{w_1}, \vartheta_{w_2}, X_t) = &-2\, \mathrm{sgn}(u_2 - u_1)\, \vartheta_{v_1} 1_{\left\{ \vartheta_{u_1} \wedge \vartheta_{u_2} \leq X_t \leq \vartheta_{u_1} \vee \vartheta_{u_2} \right\}} \\ &- \frac{v_1 - v_2}{\sqrt{T}}\, \mathrm{sgn}\left(X_t - \vartheta_{u_2} \right). \end{aligned}$$

To prove the inequality (7), we can write

$$\mathbf{E}_\vartheta \left|Z_T^{1/8}(v_1,u_1)-Z_T^{1/8}(v_2,u_2)\right|^4=\mathbf{E}_\vartheta Z_T^{1/2}(v_2,u_2)\left|V_T-1\right|^4$$
$$\leq\left(\frac{1}{8}\right)^4\mathbf{E}_\vartheta Z_T^{1/2}(v_2,u_2)\left(\int_0^T V_t\,\delta(\vartheta_{w_1},\vartheta_{w_2},X_t)\,dW_t\right)^4$$
$$\leq\left(\frac{1}{8}\right)^4\left(\mathbf{E}_{\vartheta_{w_2}}\left(\sup_{0\leq t\leq T}V_t^8\right)\right)^{1/2}\left(\mathbf{E}_\vartheta\left(\int_0^T\delta(\vartheta_{w_1},\vartheta_{w_2},X_t)^2\,dt\right)^4\right)^{1/2}$$

The first expectation is bounded

$$\mathbf{E}_{\omega_2}\left(\sup_{0\leq t\leq T}V_t^8\right)\leq 1.$$

Indeed,

$$0<V_t^8\leq 1+\int_0^t V_s^8\,\delta(\vartheta_{w_1},\vartheta_{w_2},X_s)\,dW_s.$$

For the second expectation we have

$$\mathbf{E}_\vartheta\left(\int_0^T\delta(\vartheta_{w_1},\vartheta_{w_2},X_t)^2\,dt\right)^4$$
$$\leq 8\,(4\,\vartheta_{v_1}\vartheta_{v_2})^2\,\mathbf{E}_\vartheta\left(\int_0^T 1_{\left\{\vartheta_{u_1}\wedge\vartheta_{u_2}\leq X_t\leq\vartheta_{u_1}\vee\vartheta_{u_2}\right\}}\,dt\right)^4+8\left|v_1-v_2\right|^8,$$

and we have the same estimate as above, this gives us

$$\mathbf{E}_\vartheta\left(\int_0^T 1_{\left\{\vartheta_{u_1}\wedge\vartheta_{u_2}\leq X_t\leq\vartheta_{u_1}\vee\vartheta_{u_2}\right\}}\,dt\right)^4$$
$$\leq 8\,T^2\,\mathbf{E}_\vartheta\left(\int_{\vartheta_{u_1}\wedge\vartheta_{u_2}}^{\vartheta_{u_1}\vee\vartheta_{u_2}}\eta_T(\vartheta,x)\,dx\right)^4+8\,T^4\,\mathbf{E}_\vartheta\left(\int_{\vartheta_{u_1}\wedge\vartheta_{u_2}}^{\vartheta_{u_1}\vee\vartheta_{u_2}}f(\vartheta,x)\,dx\right)^4$$
$$\leq\left(\frac{8^4\,\Gamma(5)}{3^5\,T^4}+\frac{8^3}{3\,T^2}\right)\left|u_1-u_2\right|^4+8\,\beta_1^4\left|u_1-u_2\right|^4$$

Hence choosing $C_2=\max\left(4\,\beta_1^4,\sqrt{2}/4\right)/8^3$ we obtain the estimate (7). For the inequality (8) we have

$$\mathbf{E}_\vartheta\left|Z_T^{1/2}(v_1,u_1)-Z_T^{1/2}(v_2,u_2)\right|^2=\mathbf{E}_\vartheta Z_T(v_2,u_2)\left|V_T^4-1\right|^2=\mathbf{E}_{\vartheta_{w_2}}\left|V_T^4-1\right|^2$$
$$\leq\frac{1}{4}\,\mathbf{E}_{\vartheta_{w_2}}\left(\int_0^T V_t^4\,\delta(\vartheta_{w_1},\vartheta_{w_2},X_t)\,dW_t\right)^2=\frac{1}{4}\,T\,\mathbf{E}_{\vartheta_{w_1}}\delta(\vartheta_{w_1},\vartheta_{w_2},\xi)^2$$
$$\leq\beta_1^3\left|u_1-u_2\right|+\frac{1}{4}\left|v_1-v_2\right|^2,$$

where we used the representation

$$V_T^4 = 1 - \frac{1}{8}\int_0^T V_t^4\,\delta(\vartheta_{w_1},\vartheta_{w_2},X_t)^2\,dt + \frac{1}{2}\int_0^T V_t^4\,\delta(\vartheta_{w_1},\vartheta_{w_2},X_t)\,dW_t,$$

Therefore, for $C_1 = \max\left(\beta_1^3, 1/4\right)$ the inequality (8) is proved.

The last estimate is given in the following lemma.

Lemma 2.3. *For any $N>0$ there exist constants $C = C\,(N) > 0$ and $k>0$ such that*

$$\mathbf{P}_\vartheta^{(T)}\left\{Z_T(v,u) > e^{-k(v^2+|u|)}\right\} \leq \frac{C}{(v^2+|u|)^N}. \tag{9}$$

Proof. In this proof we use the same arguments as in the proof of Theorem 3.17 and 3.18 in [5]. Let us introduce the set

$$\mathbb{A} = \left\{\omega : \int_0^T \delta(\vartheta_w,\vartheta,X_t)^2\,dt \geq 8\,k\,(v^2+|u|)\right\}$$

where $\delta(\vartheta_w,\vartheta,X_t)$ is defined above in (2) and the number $k>0$ will be chosen later. Then we can write

$$\begin{aligned}
&\mathbf{P}_\vartheta^{(T)}\left\{Z_T(v,u) > e^{-k(v^2+|u|)}\right\}\\
&\quad= \mathbf{P}_\vartheta^{(T)}\left\{\int_0^T \delta(\vartheta_w,\vartheta,X_t)\,dW_t - \frac{1}{2}\int_0^T \delta(\vartheta_w,\vartheta,X_t)^2\,dt \geq -k(v^2+|u|)\right\}\\
&\quad\leq \mathbf{P}_\vartheta^{(T)}\left\{\int_0^T \frac{\delta(\vartheta_w,\vartheta,X_t)}{2}\,dW_t - \int_0^T \frac{\delta(\vartheta_w,\vartheta,X_t)^2}{8}\,dt \geq \frac{k}{2}(v^2+|u|),\mathbb{A}\right\}\\
&\qquad+\mathbf{P}_\vartheta^{(T)}\left\{\mathbb{A}^c\right\}\\
&\quad\leq e^{-\frac{k}{2}(v^2+|u|)} + \mathbf{P}_\vartheta^{(T)}\left\{\mathbb{A}^c\right\},
\end{aligned}$$

where we used the Markov inequality and the equality

$$\mathbf{E}_\vartheta \exp\left\{\int_0^T \frac{\delta(\vartheta_w,\vartheta,X_t)}{2}\,dW_t - \int_0^T \frac{\delta(\vartheta_w,\vartheta,X_t)^2}{8}\,dt\right\} = 1.$$

Further, put $Y_t = \mathbf{E}_\vartheta\delta(\vartheta_w,\vartheta,X_t)^2 - \delta(\vartheta_w,\vartheta,X_t)^2$. Then we have

$$\begin{aligned}
\mathbf{P}_\vartheta^{(T)}\left\{\mathbb{A}^c\right\} &= \mathbf{P}_\vartheta^{(T)}\left\{\int_0^T \delta(\vartheta_w,\vartheta,X_t)^2\,dt < 8\,k\,(v^2+|u|)\right\} \qquad (10)\\
&= \mathbf{P}_\vartheta^{(T)}\left\{\int_0^T Y_t\,dt > \int_0^T \mathbf{E}_\vartheta\delta(\vartheta_w,\vartheta,X_t)^2\,dt - 8\,k\,(v^2+|u|)\right\}
\end{aligned}$$

For the last mathematical expectation we can write

$$\mathbf{E}_\vartheta \int_0^T \delta(\vartheta_w, \vartheta, X_t)^2 \, dt = 4\, \vartheta_1\, \vartheta_v\, T \int_{\vartheta_2 \wedge \vartheta_u}^{\vartheta_2 \vee \vartheta_u} f(\vartheta, x)\, dx + v^2 \geq k_* \,(v^2 + |u|),$$

where $k_* = min\left(4\, \alpha_1^3 e^{-2\,\beta_1 |\beta_2 - \alpha_2|}, 1\right)$. Now we choose $k = k_*/16$ and (10) becomes

$$\mathbf{P}_\vartheta^{(T)}\left\{\mathbb{A}^c\right\} \leq \mathbf{P}_\vartheta^{(T)}\left\{\int_0^T Y_t\, dt > \frac{k_*}{2}\,(v^2 + |u|)\right\}.$$

For the last expression we consider separately two sets. The first one is

$$\left\{v, u \,:\, v^2 + |u| \leq T^{3/4}\right\}.$$

and we at first note that

$$\int_{\vartheta_2 \wedge \vartheta_u}^{\vartheta_2 \vee \vartheta_u} f(\vartheta, x)\, dx = \vartheta_1 \frac{|u|}{T} + o\left(T^{-1/4}\right)$$

Therefore we consider the main term $\frac{\vartheta_1 |u|}{T}$ only. We can write

$$\begin{aligned}
&\mathbf{P}_\vartheta^{(T)}\left\{\int_0^T Y_t\, dt > \frac{k_*}{2}\,(v^2 + |u|)\right\} \\
&\quad \leq \mathbf{P}_\vartheta^{(T)}\left\{\left|\int_0^T 1_{\{\vartheta_2 \wedge \vartheta_u \leq X_t \leq \vartheta_2 \vee \vartheta_u\}}\, dt - \vartheta_1\, |u|\right| > c\,(v^2 + |u|)\right\} \\
&\quad \leq \frac{C}{(v^2 + |u|)^{2k}}\, \mathbf{E}_\vartheta \left|\int_0^T 1_{\{\vartheta_2 \wedge \vartheta_u \leq X_t \leq \vartheta_2 \vee \vartheta_u\}}\, dt - \vartheta_1\, |u|\right|^{2k} \\
&\quad \leq \frac{C}{(v^2 + |u|)^{2k}} \left(\frac{u^{2k}}{T^k} + \frac{u^{4k}}{T^{2k}}\right) \leq \frac{C}{(v^2 + |u|)^k} + \frac{C\, u^{2k}}{(v^2 + |u|)^{2k}}\, T^{-k/2} \\
&\quad \leq \frac{C}{(v^2 + |u|)^{k/2}}.
\end{aligned}$$

The last estimates were obtained using the same arguments as that for obtaining (5) and we used the inequality $(v^2 + |u|)^k \leq T^{3k/4}$. On the set

$$\left\{v, u \,:\, T^{3/4} \leq v^2 + |u| < \left[(\beta_1 - \alpha_1)^2 + \beta_2 - \alpha_2\right] T\right\}$$

we have the estimate

$$
\begin{aligned}
&\mathbf{P}_\vartheta^{(T)}\left\{\int_0^T Y_t\,dt > \frac{k_*}{2}\,(v^2+|u|)\right\}\\
&\quad\le \mathbf{P}_\vartheta^{(T)}\left\{\left|\int_{\vartheta_2\wedge\vartheta_u}^{\vartheta_2\vee\vartheta_u}\left[\Lambda_T(\vartheta,x)-T\,f(\vartheta,x)\right]dx\right| > c\,(v^2+|u|)\right\}\\
&\quad\le \frac{C\,T^k}{(v^2+|u|)^{2k}}\,\mathbf{E}_\vartheta\left|\int_{\vartheta_2\wedge\vartheta_u}^{\vartheta_2\vee\vartheta_u}\sqrt{T}\left[f_T^\circ(\vartheta,x)-f(\vartheta,x)\right]dx\right|^{2k}\\
&\quad\le \frac{C}{(v^2+|u|)^{2k}}\,\frac{u^{2k}}{T^k}\le\frac{C}{T^k}\le\frac{C}{(v^2+|u|)^k}.
\end{aligned}
$$

Therefore we obtain (9) with $N=k/2$.

The properties of the likelihood ratio $Z_T(\cdot)$ established in Lemmas 2.1-2.3 allow us to apply Theorems 2.1 and 2.2 and so to obtain the desired properties of the MLE and BE.

The asymptotic properties of the estimator of the method of moment (EMM) $\bar\vartheta_T=(\bar\vartheta_T^{(1)},\bar\vartheta_T^{(2)})$ is described in the following theorem.

Theorem 2.3. *The EMM $\bar\vartheta_T$ is consistent: for any $\nu>0$*

$$
\lim_{T\to\infty}\mathbf{P}_\vartheta^{(T)}\left\{\left|\bar\vartheta_T-\vartheta\right|>\nu\right\}=0
$$

is asymptotically normal

$$
\mathcal{L}_\vartheta\left\{\sqrt{T}\left(\bar\vartheta_T-\vartheta\right)\right\}\Longrightarrow\mathcal{N}\left(0,d(\vartheta)^2\right),\quad d(\vartheta)^2=\begin{pmatrix}\frac{7}{2\,\vartheta_1^2} & 0\\ 0 & \frac{5}{4\,\vartheta_1^4}\end{pmatrix},
$$

and for any $p>0$ the moments converge

$$
\lim_{T\to\infty}\mathbf{E}_\vartheta\left|\sqrt{T}\left(\bar\vartheta_T-\vartheta\right)\right|^p=\mathbf{E}_\vartheta\left|\zeta\right|^p,\quad \mathcal{L}(\zeta)=\mathcal{N}\left(0,d(\vartheta)^2\right).
$$

Proof. We have $\mathbf{E}_\vartheta\xi=\vartheta_2$ and $\mathbf{E}_\vartheta\xi^2=\vartheta_2^2+1/2\vartheta_1^2$. Hence the EMM $\bar\vartheta_T=(\bar\vartheta_T^{(1)},\bar\vartheta_T^{(2)})$ is

$$
\bar\vartheta_T^{(1)}=\frac{1}{\sqrt{2|Y_2-Y_1^2|}},\qquad\qquad \bar\vartheta_T^{(2)}=Y_1,
$$

where

$$
Y_1=\frac{1}{T}\int_0^T X_t\,dt\to\vartheta_2,\qquad\qquad Y_2=\frac{1}{T}\int_0^T X_t^2\,dt\to\vartheta_2^2+\frac{1}{2\vartheta_1^2}.
$$

Hence this estimator is consistent. Let us put $\delta_T = T^{-1/2}$ and

$$\zeta_T = \frac{1}{\sqrt{T}} \int_0^T \left[X_t - \vartheta_2\right] dt, \qquad \eta_T = \frac{1}{\sqrt{T}} \int_0^T \left[X_t^2 - \vartheta_2^2 - \frac{1}{2\vartheta_1^2}\right] dt.$$

Then we can write

$$\begin{aligned}\bar\vartheta_T^{(1)} &= \frac{1}{\sqrt{2\left|\frac{1}{2\vartheta_1^2} + \delta_T\,\eta_T - 2\,\vartheta_2\,\delta_T\,\zeta_T\right|}} \left(1 + o(1)\right) \\ &= \vartheta_1 \left[1 - \delta_T \vartheta_1^2 \left(\eta_T - 2\,\vartheta_2 \zeta_T\right)\right] \left(1 + o(1)\right).\end{aligned}$$

Therefore

$$\sqrt{T}\left(\bar\vartheta_T^{(1)} - \vartheta_1\right) = -\vartheta_1^2 \frac{1}{\sqrt{T}} \int_0^T \left[X_t^2 - 2\,\vartheta_2 X_t + \vartheta_2^2 - \frac{1}{2\vartheta_1^2}\right] dt \left(1 + o(1)\right).$$

Note that for $h(\vartheta, x) = x^2 - 2\,\vartheta_2 x + \vartheta_2^2 - 1/2\vartheta_1^2$ we have $\mathbf{E}_\vartheta h(\vartheta, \xi) = 0$. Hence by the Itô formula we have

$$\begin{aligned}\sqrt{T}\left(\bar\vartheta_T^{(1)} - \vartheta_1\right) &= -\vartheta_1^2 \frac{1}{\sqrt{T}} \int_0^T h(\vartheta, X_t) dt \left(1 + o(1)\right) \\ &= \vartheta_1^2 \frac{2}{\sqrt{T}} \int_0^T \frac{1}{f(\vartheta, X_t)} \int_{-\infty}^{X_t} h(\vartheta, u) f(\vartheta, u)\, du\, dW_t \left(1 + o(1)\right).\end{aligned}$$

Finally

$$\mathcal{L}_\vartheta \left\{\sqrt{T}\left(\bar\vartheta_T^{(1)} - \vartheta_1\right)\right\} \Longrightarrow \mathcal{N}\left(0, d_{\vartheta_1}(\vartheta)^2\right),$$

with

$$d_{\vartheta_1}(\vartheta)^2 = 4\,\vartheta_1^4 \mathbf{E}_\vartheta \left(\int_{-\infty}^{\xi} \frac{h(\vartheta, u) f(\vartheta, u)}{f(\vartheta, \xi)}\, du\right)^2 = \frac{7}{2\,\vartheta_1^2}.$$

The EMM $\bar\vartheta_T^{(2)}$ admit the representation

$$\begin{aligned}\sqrt{T}\left(\bar\vartheta_T^{(2)} - \vartheta_2\right) &= \frac{1}{\sqrt{T}} \int_0^T \left[X_t - \vartheta_2\right] dt = \frac{1}{\sqrt{T}} \int_0^T g(\vartheta, X_t) dt \\ &= -\frac{2}{\sqrt{T}} \int_0^T \frac{1}{f(\vartheta, X_t)} \int_{-\infty}^{X_t} g(\vartheta, u) f(\vartheta, u)\, du\, dW_t.\end{aligned}$$

and the limit variance is

$$d_{\vartheta_2}(\vartheta)^2 = 4\,\mathbf{E}_\vartheta \left(\int_{-\infty}^{\xi} \frac{g(\vartheta, u) f(\vartheta, u)}{f(\vartheta, \xi)}\, du\right)^2 = \frac{5}{4\,\vartheta_1^4}.$$

These representations of the estimators give us the joint asymptotic normality

$$\mathcal{L}_\vartheta\left\{\sqrt{T}\left(\bar\vartheta_T-\vartheta\right)\right\}\Longrightarrow\mathcal{N}\left(0,d(\vartheta)^2\right),$$

where the matrix

$$d(\vartheta)^2=\begin{pmatrix} d_{\vartheta_1}(\vartheta)^2 & d_{\vartheta_1\vartheta_2}(\vartheta)\\ d_{\vartheta_1\vartheta_2}(\vartheta) & d_{\vartheta_2}(\vartheta)^2\end{pmatrix},$$

and

$$d_{\vartheta_1\vartheta_2}(\vartheta)=4\,\vartheta_1^2\mathbf{E}_\vartheta\left(\int_{-\infty}^{\xi}\frac{h(\vartheta,v)f(\vartheta,v)}{f(\vartheta,\xi)}\,dv\int_{-\infty}^{\xi}\frac{g(\vartheta,u)f(\vartheta,u)}{f(\vartheta,\xi)}\,du\right)=0.$$

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