Asymptotic properties of MLE for partially observed fractional diffusion system with dependent noises

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Abstract

The paper studies long time asymptotic properties of the Maximum Likelihood Estimator (MLE) for the signal drift parameter in a partially observed fractional diffusion system with dependent noise. Using the method of weak convergence of likelihoods due to I. Ibragimov and R. Khasminskii [7], consistency, asymptotic normality and convergence of the moments are established for MLE. The proof is based on Laplace transform computations which was introduced in [4].

1 Introduction

1.1 Historical survey

The drift parameter estimation problem with partial observations have been given a great deal of interest over the last decades. Unfortunately, for continuous time-models with partial observations, we can refer, for Maximum Likelihood estimation, only to [13, 3] for the classical Kalman-Bucy setting and [5] for finite state Markov signal.

Existence and uniqueness of a solution for stochastic differential equations with fractional Brownian are studied under suitable condition on the coefficients in [15, 23, 19, 22, 18, 16, 6]. For statistical application, fractional analogue of the Ornstein-Uhlenbeck process have been studied in [11, 21, 1] and properties of the MLE in the complete observation case can be found in [4].

Paper [10, 4] gives Kalman-Bucy filters, and for statistical inference, constitency, asymptotical normality and convergence of the moments for the MLE of the signal drift in a linear partially observed fractional diffusion system for the Hurst exponent H in (0,1).

In the present paper, we work with a linear Gaussian system, perturbed by fBm noises which are dependent. It means that the initial observation model is not Markovian.

To analyze the large sample asymptotic properties of the MLE, we use the program proposed by [7]. The main idea of this approach is to deduce strong properties of MLE from the weak convergence of scaled likelihoods in appropriate functional spaces, especially the convergence of moments

The explicit expression of the likelihood can be written using the "transformation of the observation model" method proposed in [10, 4]. Even in our particular situation, this approach is reduced to the analysis of a non homogeneous non ergodic signal. To pass this obstacle, we proposed to use Laplace transform computations based on Cameron-Martin formula.

1.2 The setting and the main result

We consider real-valued processes $X = (X_t,\, t \geq 0)$ and $Y = (Y_t,\, t \geq 0)$, representing the signal and the observation respectively. The signal process is governed by the following homogeneous linear stochastic differential equation interpreted as integral equation and is observed linearly but with perturbations:

$$\left\{ \begin{array}{rcl} dX_t &=& -\vartheta X_t dt + dV_t^H\,, \quad X_0 = 0\,, \\ Y_t &=& \mu X_t + W_t^H\,, \quad Y_0 = 0. \end{array} \right. \tag{1}$$

Here, $V^H = (V_t^H,\, t \geq 0)$ and $W^H = (W_t^H, t \geq 0)$ are independent normalized fBm's with the same Hurst parameter H in $(0,1)$ and the coefficients ϑ and $\mu \neq 0$ are real constants. System (1) has a uniquely defined solution process (X,Y) which is Gaussian but neither Markovian nor a semimartingale for $H \neq \frac{1}{2}$.

Suppose that parameter $\vartheta > 0$ is unknown and is to be estimated given the observed trajectory $Y^T = (Y_t,\, 0 \leq t \leq T)$.

Our goal is to study the long time asymptotic properties of the implicit MLE $\hat{\vartheta}_T$ of ϑ given Y^T. Our main result is:

Theorem 1. *The MLE $\hat{\vartheta}_T$ is uniformly on compacts $\mathbb{K} \subset \Theta$ consistent, i.e. for any $\nu > 0$,*

$$\lim_{T\to\infty} \sup_{\vartheta\in\mathbb{K}} \mathbf{P}_\vartheta^T \left\{ \left| \hat{\vartheta}_T - \vartheta \right| > \nu \right\} = 0\,, \tag{2}$$

uniformly asymptotically normal

$$\sqrt{T}\left(\hat{\vartheta}_T - \vartheta\right) \stackrel{law}{\Longrightarrow} \mathcal{N}\left(0, \mathcal{I}(\vartheta)^{-1}\right) \tag{3}$$

where $\mathcal{I}(\vartheta)$ stands for the Fischer information **which does not depend on** *H:*

$$\mathcal{I}(\vartheta) = \frac{1}{2\vartheta} - \frac{2}{\alpha+\vartheta} + \frac{1}{2\alpha} = \frac{1}{2\vartheta} - \frac{2\vartheta\sqrt{1+\mu^2}}{\alpha(\alpha+\vartheta)} + \frac{\vartheta^2(1+\mu^2)}{2\alpha^3} \tag{4}$$

and $\alpha = \sqrt{(\mu\vartheta)^2 + \vartheta^2}$. We have the uniform on $\vartheta \in \mathbb{K}$ convergence of the moments: for any $p > 0$,

$$\lim_{T\to\infty} \mathbf{E}_\vartheta \left| \sqrt{T}\left(\hat{\vartheta}_T - \vartheta\right)\right|^p = \mathbf{E}\left|\mathcal{I}(\vartheta)^{-\frac{1}{2}}\zeta\right|^p \tag{5}$$

where $\zeta \sim \mathcal{N}(0,1)$.

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2 The proof

2.1 Preliminaries and transformation of the observation model

In what follows, all random variables and processes are defined on a given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ satisfying the usual conditions and processes are $(\mathcal{F}_t)$-adapted. Moreover the *natural filtration* of a process is understood as the $\mathbf{P}$-completion of the filtration generated by this process.

System (1) can be rewritten, as

$$\begin{cases} dX_t &= -\vartheta X_t dt + dV_t^H\,, \quad X_0 = 0\,, \\ dY_t &= -\mu\vartheta X_t dt + \mu dV_t^H + dW_t^H\,, \quad Y_0 = 0. \end{cases} \tag{6}$$

In this proof, we focus on the case $H > 1/2$. Nerverless, the result is valid for any $H \in (0,1)$ (see Remark 1).

Even if fBm are not martingales, there are simple integral transformations which change the fBm to martingales (see [2, 17, 20]). In particular, defining for $0 < s < t$,

$$k_H(t,s) = \kappa_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}\,, \quad \kappa_H = 2H\Gamma\left(\frac{3}{2}-H\right)\Gamma\left(\frac{1}{2}+H\right)\,,$$

$$\lambda = \frac{H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{2(1-H)\Gamma(\frac{3}{2}-H)}\,, \quad w_H(t) = \frac{1}{2(2-2H)\lambda} t^{2-2H}$$

$$M_t = \int_0^t k_H(t,s) dW_s^H\,, \quad N_t = \int_0^t k_H(t,s) dV_s^H\,,$$

then the process $M = (M_t, t \geq 0)$ is a Gaussian martingale, called in [17] the *fundamental martingale* whose variance function is nothing but the function w_H. Moreover, the natural filtration of the martingale M coincides with the natural filtration of the fBm W^H. Similarly $N = (N_t, t \geq 0)$ stands for the fundamental martingale of V.

Following [10], let us introduce $Z^O = \left(Z_t^O, t \geq 0\right)$ the *fundamental semimartingale* associated to Y, namely

$$Z_t^O = \int_0^t k_H(t,s) dY_s\,. \tag{7}$$

Note that Y can be represented as $Y_t = \int_0^t K_H(t,s) dZ_s^O$ where $K_H(t,s) = H(2H-1)\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}} dr$ for $0 \leq s \leq t$ and therefore that natural filtrations of Y and Z^O coincide.

We can prove that

$$dZ_t^O = -\mu\vartheta\lambda l(t)^*\zeta_t d\langle N\rangle_t + \mu dM_t + dN_t\,,\quad Z_0^O = 0\,, \tag{8}$$

where $\zeta = (\zeta_t, t \geq 0)$ is the solution of the stochastic differential equation

$$d\zeta_t = -\vartheta\lambda\mathbf{A}(t)\zeta_t d\langle M\rangle_t + b(t)dM_t,\quad \zeta_0 = 0\,, \tag{9}$$

with

$$l(t) = \begin{pmatrix} t^{2H-1} \\ 1 \end{pmatrix},\quad \mathbf{A}(t) = \begin{pmatrix} t^{2H-1} & 1 \\ t^{4H-2} & t^{2H-1} \end{pmatrix}\quad \text{and}\quad b(t) = \begin{pmatrix} 1 \\ t^{2H-1} \end{pmatrix}.$$

2.2 Likelihood function and likelihood ratio

In this section, we are interested in the explicit representation of the likelihood function $\mathcal{L}_T\left(\vartheta, Z^{O,T}\right)$ and the likelihood ratio $\mathcal{Z}_T(\vartheta_1,\vartheta_2,Z^{O,T}) = \frac{\mathcal{L}_T(\vartheta_2,Z^{O,T})}{\mathcal{L}_T(\vartheta_1,Z^{O,T})}$. Indeed the classical Girsanov theorem and the general filtering theorem [14, Theorem 7.16] give the following equality

$$\mathcal{L}_T(\vartheta, Z^{O,T}) = \exp\left\{\int_0^T -\frac{\vartheta\mu\lambda}{1+\mu^2} l^*\pi_t(\zeta)dZ_t^0 - \frac{1}{2}\int_0^T \frac{\vartheta^2\mu^2\lambda^2}{(1+\mu^2)}\pi_t(\zeta)^* ll^*\pi_t(\zeta)d\langle N\rangle_t\right\} \tag{10}$$

where $\pi_t(\zeta) = \mathbf{E}_\vartheta\left(\zeta_t|\mathcal{F}_t^Y\right)$ is the conditional expectation.

As in [4, 10], we can show that $\pi_t(\zeta) = \mathbf{E}_\vartheta\left(\zeta_t|\mathcal{F}_t^Y\right)$ satisfies the equation

$$\begin{aligned} d\pi_t(\zeta) &= -\vartheta\lambda\mathbf{A}\pi_t(\zeta)d\langle N\rangle_t + \frac{(-\mu\vartheta\lambda\gamma_{\zeta\zeta}l + \mu b)}{\sqrt{\mu^2+1}}\frac{\left(dZ_t^O + \mu\vartheta\lambda l^*\pi_t(\zeta)d\langle N\rangle_t\right)}{\sqrt{\mu^2+1}},\quad \pi_0(\zeta) = 0, \qquad (11) \\ &= \frac{-\vartheta - (\mu\vartheta)^2\lambda\gamma_{\zeta,\zeta}\mathbf{J}}{\mu^2+1}\lambda\mathbf{A}\pi_t(\zeta) + \frac{\mu}{\mu^2+1}\left(b - \vartheta\lambda\gamma_{\zeta\zeta}l\right)dZ_t^O \quad \pi_0(\zeta) = 0\,. \end{aligned}$$

where

$$\frac{d\gamma_{\zeta,\zeta}(t)}{d\langle N\rangle_t} = -\vartheta\lambda\left(\mathbf{A}\gamma_{\zeta,\zeta} + \gamma_{\zeta,\zeta}\mathbf{A}^*\right) + bb^* - \frac{\left(\mu b - \mu\vartheta\lambda\gamma_{\zeta,\zeta}l\right).\left(\mu b - \mu\vartheta\lambda\gamma_{\zeta,\zeta}l\right)^*}{1+\mu^2},\quad \gamma_{\zeta,\zeta}(0) = 0\,. \tag{12}$$

This can be rewritten as

$$(1+\mu^2)\frac{d\gamma_{\zeta,\zeta}(t)}{d\langle N\rangle_t} = -\vartheta\lambda\left(\mathbf{A}\gamma_{\zeta,\zeta} + \gamma_{\zeta,\zeta}\mathbf{A}^*\right) + bb^* - (\mu\vartheta)^2\lambda^2\gamma_{\zeta,\zeta}ll^*\gamma_{\zeta,\zeta}\,,\quad \gamma_{\zeta,\zeta}(0) = 0\,. \tag{13}$$

Note that equation (11) can be rewritten in the equivalent form

$$d\pi_t(X) = -\vartheta\lambda\mathbf{A}\pi_t(\zeta)d\langle N\rangle_t + \frac{-\mu\vartheta\lambda\gamma_{\zeta\zeta}l + \mu b}{\sqrt{\mu^2+1}}d\nu_t \tag{14}$$

where the innovation process $(\nu_t, t \geq 0)$ is defined by:

$$d\nu_t = \frac{dZ_t^O + \mu\vartheta\lambda l^*\pi_t(\zeta)d\langle N\rangle_t}{\sqrt{\mu^2+1}},\quad \nu_0 = 0\,. \tag{15}$$

For any $\vartheta_1 \in \mathbb{R}$, let us define by $\pi_t^{\vartheta_1}(X)$ the solution of equation (11) and by $\gamma_{\zeta,\zeta}^{\vartheta_1}$ the solution of equation (12), both where $\vartheta = \vartheta_1$. Then, the likelihood ratio $\mathcal{Z}_T(\vartheta_1, \vartheta_2, Y^T)$, which is also the Radon-Nikodym derivative of $\mathbf{P}_{\vartheta_2}^T$ with respect to $\mathbf{P}_{\vartheta_1}^T$, restricted to $\mathcal{F}_T^Y$, *i.e.*

$$\mathcal{Z}_T(\vartheta_1, \vartheta_2, Y^T) = \frac{\mathcal{L}_T(\vartheta_2, Y^T)}{\mathcal{L}_T(\vartheta_1, Y^T)} = \frac{d\mathbf{P}_{\vartheta_2}^T}{d\mathbf{P}_{\vartheta_1}^T} / \mathcal{F}_T^Y \,, \tag{16}$$

can be written in the following form:

$$\mathcal{Z}_T(\vartheta_1, \vartheta_2, Y^T) = \exp\left\{ \int_0^T l^* \delta_{\vartheta_1,\vartheta_2} d\nu_t^{\vartheta_1} - \frac{1}{2}\int_0^T \delta_{\vartheta_1,\vartheta_2}^* l l^* \delta_{\vartheta_1,\vartheta_2} d\langle N\rangle_t \right\} \tag{17}$$

where $\delta_{\vartheta_1,\vartheta_2}(t)$ is the difference $-\frac{\mu\vartheta_2\lambda}{\sqrt{\mu^2+1}}\pi_t^{\vartheta_2}(X) + \frac{\mu\vartheta_1\lambda}{\sqrt{\mu^2+1}}\pi_t^{\vartheta_1}(X)$ and $\left(\nu_t^{\vartheta_1}, t \geq 0\right)$ is defined by:

$$d\nu_t^{\vartheta_1} = \frac{dZ_t^O + \mu\vartheta_1\lambda l^* \pi_t^{\vartheta_1}(\zeta) d\langle N\rangle_t}{\sqrt{\mu^2+1}} \,, \quad \nu_0^{\vartheta_1} = 0 \,. \tag{18}$$

It is worth emphasizing that in the case of $\vartheta = \vartheta_1$, then $\left(\nu_t^{\vartheta_1}, t \geq 0\right)$ is the innovation process again. We will denote by $\mathcal{Z}_T(u, Y^T)$ the perturbation of $\mathcal{Z}_T(\vartheta, \vartheta_2, Y^T)$, when $\vartheta_2 = \vartheta + \frac{u}{\sqrt{T}}$. Namely, $\mathcal{Z}_T(u, Y^T) = \mathcal{Z}_T(\vartheta, \vartheta + \frac{u}{\sqrt{T}}, Y^T)$. For this case, we will denote $\delta_{\vartheta,u,T} = \delta_{\vartheta,\vartheta+\frac{u}{\sqrt{T}}}$.

2.3 Ibragimov–Khasminskii program

The proof of Theorem 1 is based on [7]. It follows from [7, Theorem I.10.1] that in order to prove Theorem 1, it is sufficient to check the three following conditions:

(A.1)

$$\mathcal{Z}_T(u, Y^T) \overset{law}{\Longrightarrow} \exp\left\{ u.\eta - \frac{u^2}{2}\mathcal{I}(\vartheta) \right\} \text{ with } \quad \eta \sim \mathcal{N}\left(0, \mathcal{I}(\vartheta)\right), za \tag{19}$$

(A.2) for some $\chi > 0$:

$$\mathbf{E}_\vartheta \sqrt{\mathcal{Z}_T(u, Y^T)} \leq \exp\left(-\chi u^2\right) \tag{20}$$

(A.3) there exists $C > 0$ such that

$$\mathbf{E}_\vartheta \left(\sqrt{\mathcal{Z}_T(u_1, Y^T)} - \sqrt{\mathcal{Z}_T(u_2, Y^T)} \right)^2 \leq C|u_1 - u_2|^2 \,. \tag{21}$$

2.4 Laplace Transform proof

As it was mentioned in Introduction, we propose to use Laplace transform method to check conditions (A1–A3).

Let $L_T(a,\vartheta_1,\vartheta_2)$ be the Laplace transform of the integral of the quadratic form of the difference $\delta_{\vartheta_1,\vartheta_2}(t) = -\frac{\mu\vartheta_2\lambda}{\sqrt{\mu^2+1}}\pi_t^{\vartheta_2}(X) + \frac{\mu\vartheta_1\lambda}{\sqrt{\mu^2+1}}\pi_t^{\vartheta_1}(X)$:

$$L_T(a,\vartheta_1,\vartheta_2) = \mathbf{E}_{\vartheta_1}\exp\left\{-\frac{a}{2}\int_0^T \delta^*_{\vartheta_1,\vartheta_2} ll^* \delta_{\vartheta_1,\vartheta_2} d\langle N\rangle_t\right\}. \tag{22}$$

Let us introduce the following condition (L):

(L) There exists $a_0 < 0$ such that for all $a > a_0$, $\forall u_1, u_2 \in \mathbb{R}$,

$$\lim_{T\to\infty} L_T(a,\vartheta+\frac{u_1}{\sqrt{T}},\vartheta+\frac{u_2}{\sqrt{T}}) = \exp\left(-a\frac{(u_2-u_1)^2}{2}\mathcal{I}(\vartheta)\right),$$

where $\mathcal{I}(\vartheta)$ is defined by (4).

Now, we state the Proposition which is the key point of our proof.

Proposition 2.1. *Suppose that the condition (L) is satisfied. Then properties (A.1.–A.3) hold.*

Proof. Actually, (A.1) is a direct consequence of (L). Indeed, for $u_1 = 0$ and $u_2 = u$, we have:

$$\lim_{T\to\infty} L_T(a,\vartheta,\vartheta+\frac{u}{\sqrt{T}}) = \exp\left(-a\frac{u^2}{2}\mathcal{I}(\vartheta)\right).$$

The last equation gives the convergence of the following integrals:

$$\frac{1}{2}\int_0^T \delta^*_{\vartheta_1,\vartheta_2} ll^* \delta_{\vartheta_1,\vartheta_2} d\langle N\rangle_t \longrightarrow \frac{u^2}{2}\mathcal{I}(\vartheta) \quad \text{a.s.}$$

and

$$\int_0^T l^*\delta_{\vartheta,u,T} d\nu_t \overset{law}{\Longrightarrow} \mathcal{N}\left(0, u^2\mathcal{I}(\vartheta)\right),$$

which achieves the proof of (A.1). The condition (A.2) holds thanks to the following chain of inequalities:

$$
\begin{aligned}
\mathbf{E}_\vartheta \sqrt{\mathcal{Z}_T(u)} &= \mathbf{E}_\vartheta \exp\left(\frac{1}{2}\int_0^T l^*\delta_{\vartheta,u,T}d\nu_t^\vartheta - \frac{1}{4}\int_0^T \delta^*_{\vartheta_1,\vartheta_2}ll^*\delta_{\vartheta_1,\vartheta_2}d\langle N\rangle_t\right)\\
&= \mathbf{E}_\vartheta \exp\left(\frac{1}{2}\int_0^T l^*\delta_{\vartheta,u,T}d\nu_t^\vartheta - \frac{p}{8}\int_0^T \delta^*_{\vartheta_1,\vartheta_2}ll^*\delta_{\vartheta_1,\vartheta_2}d\langle N\rangle_t\right)\times\\
&\quad\times \exp\left(\frac{1}{4}\left(\frac{p}{2}-1\right)\int_0^T \delta^*_{\vartheta_1,\vartheta_2}ll^*\delta_{\vartheta_1,\vartheta_2}d\langle N\rangle_t\right)\\
&\overset{(a)}{\leq} \left(\mathbf{E}_\vartheta \exp\left(\frac{p}{2}\int_0^T l^*\delta_{\vartheta,u,T}d\nu_t^\vartheta - \frac{p^2}{8}\int_0^T \delta^*_{\vartheta_1,\vartheta_2}ll^*\delta_{\vartheta_1,\vartheta_2}d\langle N\rangle_t\right)\right)^{\frac{1}{p}}\times\\
&\quad\times\left(\mathbf{E}_\vartheta \exp\left(\frac{q}{4}\left(\frac{p}{2}-1\right)\int_0^T \delta^*_{\vartheta_1,\vartheta_2}ll^*\delta_{\vartheta_1,\vartheta_2}d\langle N\rangle_t\right)\right)^{\frac{1}{q}}\\
&\overset{(b)}{\leq} L_T\left(\frac{q}{2}\left(\frac{p}{2}-1\right),\vartheta,\vartheta+\frac{u}{\sqrt{T}}\right)^{\frac{1}{q}} \leq \exp\left(-\chi u^2\right)\,,
\end{aligned}
$$

where (a) is Hölder inequality and (b) is Girsanov theorem. To prove (A.3), let us note that

$$
\begin{aligned}
\mathbf{E}_\vartheta\left(\sqrt{\mathcal{Z}_T(u_1)}-\sqrt{\mathcal{Z}_T(u_2)}\right)^2 &= 2\left(1-\mathbf{E}_\vartheta \mathcal{Z}_T(u_1)\sqrt{\frac{\mathcal{Z}_T(u_2)}{\mathcal{Z}_T(u_1)}}\right)\\
&= 2\left(1-\mathbf{E}_{\vartheta_1}\sqrt{\mathcal{Z}_T(\vartheta_1,\vartheta_2)}\right)\,.
\end{aligned}
$$

The same chain of inequalities gives:

$$
\begin{aligned}
\mathbf{E}_\vartheta\left(\sqrt{\mathcal{Z}_T(u_1)}-\sqrt{\mathcal{Z}_T(u_2)}\right)^2 &\leq 2\left(1-\exp\left(-\chi\left(u_2-u_1\right)^2\right)\right)\\
&\leq C|u_1-u_2|^2\,.
\end{aligned}
$$

□

3 Laplace Transform computation

3.1 Ricatti equation

In this section, we want to check the condition (L). Actually, the computation of the Laplace transform is based on the Cameron-Martin formula, developed in [9]. In order to apply the approach proposed in [9], let us recall that, for $\vartheta = \vartheta_1$, the optimal filter $\pi_t^{\vartheta_1}(X)$ and the difference $\delta_{\vartheta_1,\vartheta_2}(t) = -\frac{\mu\vartheta_2\lambda}{\sqrt{\mu^2+1}}\pi_t^{\vartheta_2}(X) + \frac{\mu\vartheta_1\lambda}{\sqrt{\mu^2+1}}\pi_t^{\vartheta_1}(X)$ are governed by the stochastic differential equation:

$$
d\tilde{\pi}_t = \mathcal{A}(t)\tilde{\pi}_t d\langle N\rangle_t + \mathcal{B}(t)d\nu_t^{\vartheta_1}\,, \tag{23}
$$

where $\tilde{\pi}_t = \begin{pmatrix} -\frac{\mu\vartheta_1\lambda}{\sqrt{\mu^2+1}}\pi_t^{\vartheta_1}(\zeta) \\ \delta_{\vartheta_1,\vartheta_2} \end{pmatrix}$, $a^{\vartheta_2}(t) = \frac{-\vartheta_2\mathbf{Id}-(\mu\vartheta_2)^2\lambda\gamma_{\zeta,\zeta}^{\vartheta_2}\mathbf{J}}{\mu^2+1}$, $\mathbf{D}_\gamma^{\vartheta_1,\vartheta_2} = \vartheta_2\left(b-\vartheta_2\lambda\gamma_{\zeta,\zeta}^{\vartheta_2}(t)l\right) - \vartheta_1\left(b-\vartheta_1\lambda\gamma_{\zeta,\zeta}^{\vartheta_1}(t)l\right)$, $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{Id}$ is the 2×2 identity matrix,

$$\mathcal{A}(t) = \begin{pmatrix} -\vartheta_1 & 0 \\ -(\vartheta_2-\vartheta_1) & a^{\vartheta_2} \end{pmatrix}\otimes\lambda\mathbf{A} \quad \text{and} \quad \mathcal{B}(t) = \frac{-\mu^2\lambda}{\mu^2+1}\begin{pmatrix} \vartheta_1\left(b-\vartheta_1\lambda\gamma_{\zeta,\zeta}^{\vartheta_1}(t)l\right) \\ \mathbf{D}_\gamma^{\vartheta_1,\vartheta_2} \end{pmatrix}.$$

It follows from [9], that

$$\begin{aligned} L_T(a,\vartheta_1,\vartheta_2) &= \mathbf{E}_{\vartheta_1}\exp\left\{-\frac{a}{2}\int_0^T \delta_{\vartheta_1,\vartheta_2}^* ll^*\delta_{\vartheta_1,\vartheta_2}d\langle N\rangle_t\right\} \\ &= \mathbf{E}_{\vartheta_1}\exp\left\{-\frac{a}{2}\int_0^T \begin{pmatrix} \frac{\mu\vartheta_2\lambda}{\sqrt{\mu^2+1}}\pi_t^{\vartheta_1}(\zeta) \\ \delta_{\vartheta_1,\vartheta_2} \end{pmatrix}^* \mathcal{M}(t)\begin{pmatrix} \frac{\mu\vartheta_2\lambda}{\sqrt{\mu^2+1}}\pi_t^{\vartheta_1}(\zeta) \\ \delta_{\vartheta_1,\vartheta_2} \end{pmatrix} d\langle N\rangle_t\right\} \\ &= \exp\left\{-\frac{a}{2}\int_0^T \operatorname{trace}(\mathcal{H}(t)\mathcal{M}(t))d\langle N\rangle_t\right\}, \end{aligned}$$

where $\mathcal{M}(t) = \begin{pmatrix} 0 & 0 \\ 0 & ll^* \end{pmatrix}$ and $\mathcal{H}(t)$ is the solution of Ricatti differential equation:

$$\frac{d\mathcal{H}(t)}{d\langle N\rangle_t} = \mathcal{A}(t)\mathcal{H}(t)+\mathcal{H}(t)\mathcal{A}(t)^*+\mathcal{B}(t)\mathcal{B}(t)^*-a\mathcal{H}(t)\mathcal{M}(t)\mathcal{H}(t), \quad \mathcal{H}(0)=0. \tag{24}$$

It is known that solution $\mathcal{H}(t)$ can be written as $\mathcal{H}(t) = \Psi_1^{-1}(t)\Psi_2(t)$, where the pair of 4×4 matrices (Ψ_1,Ψ_2) satisfies the system of linear differential equations:

$$\frac{d\Psi_1(t)}{d\langle N\rangle_t} = -\Psi_1(t)\mathcal{A}(t)+a\Psi_2(t)\mathcal{M}(t) \quad \Psi_1(0)=\mathcal{I}d \tag{25}$$

$$\frac{d\Psi_2(t)}{d\langle N\rangle_t} = \Psi_1(t)\mathcal{B}(t)\mathcal{B}(t)^*+\Psi_2(t)\mathcal{A}^*(t) \quad \Psi_2(t)=0, \tag{26}$$

and $\mathcal{I}d$ is the 4×4 identity matrix. Now,

$$L_T(a,\vartheta_1,\vartheta_2) = \exp\left\{-\frac{1}{2}\int_0^T \operatorname{trace}\mathcal{A}(t)d\langle N\rangle_t\right\}(\det\Psi_1(T))^{-\frac{1}{2}}. \tag{27}$$

3.2 Asymptotical results for $L_T(a,\vartheta_1,\vartheta_2)$

In order to check that the condition (L) is fulfilled, we propose to use the asymptotical behavior of the matrix $\gamma_{\zeta,\zeta}^{\vartheta}$ which is the unique solution of (12). We can remark that the algebraic equation of ricatti equation (13) is the same as [4] where μ is replace by $\mu\vartheta$. Therefore, it was shown in that the following limit exists:

$$\lim_{t\to\infty} \Delta\gamma^{\vartheta_2}_{\zeta,\zeta}\Delta = \frac{\alpha_2 \mathbf{Id} + (-\vartheta_2 + (\alpha_2+\vartheta_2)\sin(\pi H))\mathbf{J}}{\lambda(\alpha_2+\vartheta_2)(\alpha_2-\vartheta_2+(\alpha_2+\vartheta_2)\sin(\pi H))} = \gamma_\infty \,, \tag{28}$$

where $\Delta = \begin{pmatrix} t^{H-\frac12} & 0 \\ 0 & t^{\frac12 - H}\end{pmatrix}$, $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{Id}$ is the 2×2 identity matrix and $\alpha = \sqrt{(\mu\vartheta)^2+\vartheta_2^2}$.

The standard arguments (see, *e.g.*, [12]) and the explicit representation of $\gamma^{\vartheta}_{\zeta,\zeta}$ imply that we can replace $\gamma^{\vartheta}_{\zeta,\zeta}$ by $\Delta^{-1}\gamma_\infty\Delta^{-1}$ in the coefficients of Equation (24) and therefore

$$\lim_{T\to\infty} L_T(a,\vartheta_1,\vartheta_2) = \exp\left\{-\frac12\int_0^T \mathrm{trace}\,\mathcal{A}_\infty(t)d\langle N\rangle_t\right\}\left(\det\Psi_{1,\infty}(T)\right)^{-\frac12} \,,$$

where

$$\begin{aligned} \frac{d\Psi_{1,\infty}(t)}{dt} &= -\Psi_{1,\infty}(t)\mathcal{A}_\infty(t) + a\Psi_{2,\infty}(t)\mathcal{M}(t) \quad \Psi_{1,\infty}(0) = \mathcal{I}d \\ \frac{d\Psi_{2,\infty}(t)}{dt} &= \Psi_1(t)\mathcal{B}_\infty(t)\mathcal{B}_\infty(t)^* + \Psi_{2,\infty}(t)\mathcal{A}^*_\infty(t) \quad \Psi_{2,\infty}(t) = 0\,, \end{aligned} \tag{29}$$

with

$$\mathcal{A}_\infty(t) = \begin{pmatrix} -\vartheta_1 & 0 \\ -(\vartheta_2-\vartheta_1) & -\frac{\alpha_2}{1+\mu^2}\end{pmatrix}\otimes\lambda\mathbf{A}\,, \quad \mathcal{B}_\infty\mathcal{B}^*_\infty = \begin{pmatrix} g_1^2 & g_1g_2 \\ g_1g_2 & g_2^2\end{pmatrix}\otimes\lambda\mathbf{AJ}\,,$$

$\alpha_2 = \sqrt{\left(\vartheta_2\mu\right)^2+\vartheta_2^2}$, $g_1 = \frac{(\alpha_1-\vartheta_1)\sqrt{\lambda}}{\sqrt{\mu^2+1}}$ and $g_2 = \frac{(\alpha_2-\vartheta_2)\sqrt{\lambda}}{\sqrt{\mu^2+1}} - \frac{(\alpha_1-\vartheta_1)\sqrt{\lambda}}{\sqrt{\mu^2+1}}$. Here simple relations $l(t)l(t)^* = \mathbf{JA}(t)$, $\Delta^{-1}\Delta^{-1}\mathbf{JA} = \mathbf{A}$, $\mathbf{A}^* = \mathbf{JAJ}$ and $\left(\Delta^{-1}\mathbf{J}\right)^2 = \mathbf{Id}$ have been used.

Linear system (29) can be rewritten as

$$\frac{d\left(\Psi_1(t),\Psi_2(t)\otimes\mathbf{J}\right)}{d\langle N\rangle_t} = \left(\Psi_1(t),\Psi_2(t)\otimes\mathbf{J}\right)\cdot\left(\beth\otimes\lambda\mathbf{A}(t)\right) \tag{30}$$

where

$$\beth = \begin{pmatrix} \vartheta_1 & 0 & g_1^2 & g_1g_2 \\ (\vartheta_2-\vartheta_1) & \frac{\alpha_2}{\mu^2+1} & g_1g_2 & g_2^2 \\ 0 & 0 & -\vartheta_1 & -(\vartheta_2-\vartheta_1) \\ 0 & \frac{a}{\lambda} & 0 & -\frac{\alpha_2}{\mu^2+1}\end{pmatrix}.$$

Clearly, system (30) has an explicit solution:

$$\left(\Psi_1(t),\Psi_2(t)\otimes\mathbf{J}\right) = (\mathcal{I}d,0)\cdot\left(\mathcal{P}\otimes\mathbf{Id}\right)\mathcal{G}\left(\mathcal{P}^{-1}\otimes\mathbf{Id}\right) \tag{31}$$

where $\mathcal{G} = \mathrm{diag}\left(\mathbf{G}_1,\mathbf{G}_2,\mathbf{G}_3,\mathbf{G}_4\right)$ and

$$\frac{d\mathbf{G}_i(t)}{d\langle N\rangle_t} = \lambda x_i\mathbf{G}_i\mathbf{A} \quad \mathbf{G}_i(0) = \mathbf{Id}\,, \quad i = 1\ldots 4\,, \tag{32}$$

with $(x_i)_{i=1\dots4}$ the eigenvalues of matrix $\beth$ and $\mathcal{P}$ the matrix of its eigenvectors.

For $\vartheta_1 = \vartheta + \frac{u_1}{\sqrt{T}}$ and $\vartheta_2 = \vartheta + \frac{u_2}{\sqrt{T}}$, we can compute the eigenvalues and we obtain:

$$\begin{aligned}
x_1 &= \vartheta_1 + C_1 \frac{(u_2-u_1)^2}{T} + o\left(\frac{1}{T}\right), \\
x_2 &= -\vartheta_1 + C_2 \frac{(u_2-u_1)^2}{T} + o\left(\frac{1}{T}\right), \\
x_3 &= \alpha_2 + C_3 \frac{(u_2-u_1)^2}{T} + o\left(\frac{1}{T}\right), \\
x_4 &= -\alpha_2 + C_4 \frac{(u_2-u_1)^2}{T} + o\left(\frac{1}{T}\right),
\end{aligned}$$

where $C_1 = \frac{a(\alpha-\vartheta)}{2\vartheta(\alpha+\vartheta)}$ and $C_3 = -\frac{a(\alpha-\vartheta)}{2\alpha(\alpha+\vartheta)}$. It can be easily checked that

$$\begin{aligned}
\det \Psi_{1,\infty}(T) &= \det\left(G_1.G_3\right)\left(1+o\left(\frac{1}{T}\right)\right) \\
&= \exp\left((x_1+x_3)T\right)\left(1+o\left(\frac{1}{T}\right)\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
\lim_{T\to\infty} L_T(a,\vartheta+\frac{u_1}{\sqrt{T}},\vartheta+\frac{u_2}{\sqrt{T}}) &= \lim_{T\to\infty} \exp\left\{-\frac{1}{2}(x_1+x_3-\alpha_2-\vartheta_1)T\right\} \\
&= \lim_{T\to\infty} \exp\left\{-\frac{1}{2}\left(C_1+C_3\right)\left(u_2-u_1\right)^2\right\} \\
&= \exp\left(-a\frac{(u_2-u_1)^2}{2}\mathcal{I}\left(\vartheta\right)\right).
\end{aligned}$$

Remark 1. *Thanks to [8, Corollary 5.2], for $H < 1/2$, we have the relation between fBm processes of indexes H and $1-H$:*

$$W_t^H = \left(\frac{2H}{\Gamma(2H}\Gamma(3-2H)\right)^{\frac{1}{2}} \int_0^t (t-s)^{2H-1}\, dW_s^{1-H}. \tag{33}$$

Using this relation, we can transform the observation model **??** *to the following observation model:*

$$\left\{\begin{array}{lcl}
d\tilde{X}_t &=& -\vartheta\tilde{X}_t dt + dV_t^{1-H}, \quad \tilde{X}_0 = 0, \\
d\tilde{Y}_t &=& -\mu\vartheta\tilde{X}_t dt + \mu dV_t^{1-H} + dW_t^{1-H}, \quad \tilde{Y}_0 = 0,
\end{array}\right. \tag{34}$$

with

$$\tilde{X}_t = \left(\frac{2H}{\Gamma(2H}\Gamma(3-2H)\right)^{\frac{1}{2}} \int_0^t (t-s)^{1-2H}\, dX_s$$

and

$$\tilde{Y}_t = \left(\frac{2H}{\Gamma(2H}\Gamma(3-2H)\right)^{\frac{1}{2}} \int_0^t (t-s)^{1-2H}\, dY_s\,.$$

Then, $1 - H > \frac{1}{2}$ *and the result of Theorem 1 is valid for any* $H \in (0,1)$*.*

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