# Design for estimation of drift parameter in fractional diffusion system

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**Abstract**

The paper studies long time asymptotic properties of the Maximum Likelihood Estimator (MLE) of the signal drift parameter in a partially observed controled fractional diffusion system. The optimal estimation input is deduced. The consistency, asymptotic normality and convergence of the moments of the MLE are established.

# 1 Introduction

## 1.1 Historical survey

The paper is devoted to the large sample asymptotic properties of the Maximum Likelihood Estimator (MLE) for the signal drift parameter $\vartheta$ in a partially observed and controlled fractional diffusion system. We discuss the asymptotic (for large observation time) design problem of the input signal which gives an efficient estimator of the drift parameter. This kind of optimization problem has been treated by many authors, see *e.g.* [11, 7, 6] and references therein. Following the paper [11], we can separate the initial problem in two subproblems, when the first is equivalent to maximization of the first eigenvalue of a certain self-adjoint operator and the second one is devoted to the analysis of the asymptotic properties of the MLE. In contrast with the previous works, we propose to use (for the both subproblems) Laplace transform computations, in particular, the Cameron-Martin formula and the link between the Laplace transform and the eigenvalues of a covariance operator. This method has been proposed previously in [1, 2].

## 1.2 The setting and the main result

We consider real-valued functions $x = (x_t,\, t \geq 0)$, $u = (u(t),\, t \geq 0)$ and a process $Y = (Y_t,\, t \geq 0)$, representing the signal and the observation respectively, governed by the following homogeneous linear system of ordinary and stochastic differential equations interpreted as integral equations :

$$\begin{cases} dx_t &= -\vartheta x_t dt + u(t)dt\,, \quad x_0 = 0\,, \\ dY_t &= \quad \mu x_t dt + dV_t^H\,, \quad Y_0 = 0. \end{cases} \tag{1}$$

Here, $V^H = (V_t^H,\, t \geq 0)$ is normalized fBm with Hurst parameter $H \in [\frac{1}{2}, 1)$ and the coefficients $\vartheta$ and $\mu \neq 0$ are real constants. System (1) has a uniquely defined solution process $(x, Y)$ where $Y$ is Gaussian but neither Markovian nor a semimartingale for $H \neq \frac{1}{2}$.

Suppose that parameter $\vartheta > 0$ is unknown and is to be estimated given the observed trajectory $Y^T = (Y_t,\, 0 \leq t \leq T)$ for a control $u$ in the proper class.

For a fixed value of the parameter $\vartheta$, let $\mathbf{P}_\vartheta^T$ denote the probability measure, induced by $Y^T$ on the function space $\mathcal{C}_{[0,T]}$ and let $\mathcal{F}_t^Y$ be the natural filtration of $Y$, $\mathcal{F}_t^Y = \sigma\left(Y_s, 0 \leq s \leq t\right)$.

Let $\mathcal{L}(\vartheta, Y^T)$ be the likelihood, *i.e.* the Radon-Nikodym derivative of $\mathbf{P}_\vartheta^T$, restricted to $\mathcal{F}_T^Y$ with respect to some reference measure on $\mathcal{C}_{[0,T]}$. The explicit representation of the likelihood function can be written thanks to the transformation of observation model (1) proposed in [4]. Therefore, Fisher information stands for :

$$\mathcal{I}_T\left(\vartheta, u\right) = -\mathbf{E}_\vartheta \frac{\partial^2}{\partial \vartheta^2} \ln \mathcal{L}_T(\vartheta, Y^T)\,.$$

Let us denoted $\mathcal{U}_T$ some functional space of controls, that will be defined later. Let us therefore note

$$\mathcal{J}_T(\vartheta) = \sup_{u \in \mathcal{U}_T} \mathcal{I}_T\left(\vartheta, u\right)\,.$$

Our main goal is to find estimator $\overline{\vartheta}_T$ of the parameter $\vartheta$ which are asymptotically efficient in the sense that, for any compact $\mathbb{K} \subset \mathbb{R}^+$,

$$\sup_{\vartheta \in \mathbb{K}} \mathcal{J}_T(\vartheta) \mathbf{E}_\vartheta \left(\overline{\vartheta}_T - \vartheta\right)^2 = 1 + o(1)\,, \tag{2}$$

as $T \to \infty$.

As the optimal input does not depend on $\vartheta$ (see Proposition 1.1), a possible candidate is the Maximum Likelihood Estimator (MLE) $\hat{\vartheta}_T$, defined as the maximizer of the likelihood:

$$\hat{\vartheta}_T = \arg\max_{\vartheta > 0} \mathcal{L}(\vartheta, Y^T). \tag{3}$$

In this paper, we claim that:

**Proposition 1.1.** *The asymptotical optimal input in the class of controls* $\mathcal{U}_T$ *is* $u_{opt}(t) = \frac{\kappa_H}{\sqrt{2\lambda}} t^{H-\frac{1}{2}}$, *where the constants* $\lambda$ *and* $\kappa_H$ *are defined in Section 2.1. As in the classical case* $H = \frac{1}{2}$ *(see [11]), as* $T$ *tends to infinity,*

$$\mathcal{J}_T(\vartheta) \sim \frac{\mu^2}{\vartheta^4} T\,.$$

Moreover, MLE reaches efficiency and we deduce its large samples asymptotic properties:

**Proposition 1.2.** *The MLE is uniformly consistent on compacts* $\mathbb{K} \subset \Theta$, *uniformly on compacts asymptotically normal*

$$\sqrt{T}\left(\hat{\vartheta}_T - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \frac{\vartheta^4}{\mu^2}\right)$$

*as* $T$ *tends to* $+\infty$ *and we have the uniform on* $\vartheta \in \mathbb{K}$ *convergence of the moments: for any* $p > 0$,

$$\lim_{T\to\infty} \mathbf{E}_\vartheta \left|\sqrt{T}\left(\hat{\vartheta}_T - \vartheta\right)\right|^p = \mathbf{E}\left|\frac{\vartheta^2}{\mu}\zeta\right|^p \tag{4}$$

*where* $\zeta \sim \mathcal{N}(0,1)$. *Finally, the MLE is efficient in the sense of* (2).

# 2 The proof

## 2.1 Preliminaries

In what follows, all random variables and processes are defined on a given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ satisfying the usual conditions and processes are $(\mathcal{F}_t)$-adapted. Moreover the *natural filtration* of a process is understood as the $\mathbf{P}$-completion of the filtration generated by this process.

Even if fBm are not martingales, there are simple integral transformations which change the fBm to martingales (see [8, 9]). In particular, defining for $0 < s < t$,

$$k_H(t,s) = \kappa_H^{-1} s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H} \ , \quad \kappa_H = 2H\Gamma\left(\frac{3}{2}-H\right)\Gamma\left(\frac{1}{2}+H\right) \ ,$$

$$\lambda = \frac{H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{2(1-H)\Gamma(\frac{3}{2}-H)} \ , \quad w_H(t) = \frac{1}{2\lambda(2-2H)} t^{2-2H}$$

$$N_t = \int_0^t k_H(t,s) dV_s^H \ ,$$

then the process $N = (N_t, t \geq 0)$ is a Gaussian martingale, called in [8] the *fundamental martingale* whose variance function is nothing but the function $w_H$. Moreover, the natural filtration of the martingale $N$ coincides with the natural filtration of the fBm $V^H$.

Following [4], let us introduce $Z = (Z_t, t \geq 0)$ the *fundamental semi-martingale* associated to $Y$, namely

$$Z_t = \int_0^t k_H(t,s) dY_s \,. \tag{5}$$

Note that $Y$ can be represented as $Y_t = \int_0^t K_H(t,s) dZ_s$ where $K_H(t,s) = H(2H-1)\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}} dr$ for $0 \leq s \leq t$ and therefore that natural filtrations of $Y$ and $Z$ coincide. It can be proved that the following representation holds:

$$dZ_t = \mu Q_t d\langle N\rangle_t + dN_t \,, \quad Z_0 = 0 \,, \tag{6}$$

where

$$Q_t = \frac{d}{d\langle N\rangle_t} \int_0^t k_H(t,s) x_s ds \,. \tag{7}$$

Moreover, the following equation holds (see *e.g.* [4]):

$$dZ_t = \mu\lambda \ell(t)^* \zeta_t d\langle N\rangle_t + dN_t \,, \quad Z_0 = 0 \,, \tag{8}$$

where $\zeta = (\zeta_t, t \geq 0)$ is the solution of the ordinary differential equation

$$\frac{d\zeta_t}{d\langle N\rangle_t} = -\vartheta\lambda \mathbf{A}(t)\zeta_t + b(t)v(t), \quad \zeta_0 = 0 \,, \tag{9}$$

where $v(t) = \frac{d}{d\langle N\rangle_t}\int_0^t k_H(t,s)u(s)ds$ and with

$$\ell(t) = \begin{pmatrix} t^{2H-1} \\ 1 \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} t^{2H-1} & 1 \\ t^{4H-2} & t^{2H-1} \end{pmatrix} \quad \text{and} \quad b(t) = \begin{pmatrix} 1 \\ t^{2H-1} \end{pmatrix}.$$

Let us note, finally, $\mathcal{V}_T$ the class of admissible controls

$$\mathcal{V}_T = \left\{ v \quad | \quad \frac{1}{T}\int_0^T |v(t)|^2 d\langle N\rangle_t \leq 1 \right\}.$$

For the correspondence between controls $u = (u(t),\, 0 \leq t \leq T)$ and $v = (v(t),\, 0 \leq t \leq T)$, see Remark 1.

## 2.2 Likelihood function and the Fischer information

In this section, we are interested in the explicit representation of the likelihood function $\mathcal{L}_T\left(\vartheta, Z^T\right)$. Indeed the classical Girsanov theorem gives the following equality

$$\mathcal{L}_T(\vartheta, Z^T) = \exp\left\{ \mu\lambda \int_0^T \ell(t)^* \zeta_t dZ_t - \frac{\mu^2\lambda^2}{2}\int_0^T \zeta_t^* \ell(t)\ell(t)^* \zeta_t d\langle N\rangle_t \right\} \tag{10}$$

where $\zeta = (\zeta_t, t \geq 0)$ is the solution of the ordinary differential equation (9). the Fischer information stands for

$$\begin{aligned} \mathcal{I}_T\left(\vartheta, v\right) &= -\mathbf{E}_\vartheta \frac{\partial^2}{\partial\vartheta^2} \ln \mathcal{L}_T(\vartheta, Z^T) \\ &= \mathbf{E}_\vartheta \int_0^T \mu^2\lambda^2 \left(\frac{\partial}{\partial\vartheta}\ell(t)^*\zeta_t\right)^2 d\langle N\rangle_t \\ &= \int_0^T \mu^2\lambda^2 \left(\frac{\partial}{\partial\vartheta}\ell(t)^*\zeta_t\right)^2 d\langle N\rangle_t \quad (\zeta \text{ is deterministic}) \qquad (11) \\ &= \int_0^T \left(\frac{\partial\zeta_t}{\partial\vartheta}\right)^* \mu^2\lambda^2 \ell(t)\ell(t)^* \frac{\partial\zeta_t}{\partial\vartheta} d\langle N\rangle_t . \end{aligned}$$

## 2.3 Optimal input and Efficiency

From (9), we get

$$\zeta_t = \varphi(t)\int_0^t \varphi^{-1}(s)b(s)v(s)d\langle N\rangle_s \tag{12}$$

where $\varphi(t)$ is the fundamental matrix, *i.e.*

$$\frac{d\varphi(t)}{d\langle N\rangle_t} = -\vartheta\lambda\mathbf{A}(t)\varphi(t)\,, \quad \varphi(0) = \mathbf{Id}\,,$$

where $\mathbf{Id}$ is the $2\times 2$ identity matrix. Therefore

$$\begin{aligned}\mathcal{I}_T\left(\vartheta, v\right) &= \mu^2\lambda^2\int_0^T \left(\frac{\partial\zeta_t}{\partial\vartheta}\right)^* \ell(t)\ell(t)^* \frac{\partial\zeta_t}{\partial\vartheta} d\langle N\rangle_t \\ &= \int_0^T\int_0^T K_T(s,\sigma)\frac{s^{\frac{1}{2}-H}}{\sqrt{2\lambda}}v(s)\frac{\sigma^{\frac{1}{2}-H}}{\sqrt{2\lambda}}v(\sigma)dsd\sigma,\end{aligned}$$

where

$$K_T(s,\sigma) \;=\; \int_{\max(s,\sigma)}^T G(t,s)G(t,\sigma)dt\,, \tag{13}$$

and

$$G(t,\sigma) = \frac{\partial}{\partial\vartheta}\left(\frac{\mu}{2}t^{\frac{1}{2}-H}\ell(t)^*\varphi(t)\varphi^{-1}(\sigma)b(\sigma)\sigma^{\frac{1}{2}-H}\right)\,.$$

Then

$$\begin{aligned}\mathcal{J}_T(\vartheta) &= \sup_{v\in\mathcal{V}_T}\mathcal{I}_T\left(\vartheta, v\right) \\ &= T\sup_{v\in L^2[0,T]}\int_0^T\int_0^T K_T(s,\sigma)v(s)v(\sigma)dsd\sigma \\ &= T\sup_{v\in L^2[0,T],\,||v||=1}\left(K_T v, v\right)\,.\end{aligned}$$

Because of the stability, we have

$$\lim_{T\to+\infty} K_T(s,\sigma) = K_\infty(s,\sigma)$$

uniformly in any finite interval of $s$ and $\sigma$. Contrary to the classical case $H=\frac{1}{2}$, the limit kernel $K_\infty(s,\sigma)$ is no more of the form

$$C(s-\sigma) = \frac{\mu^2}{4\vartheta^3}e^{-\vartheta|\sigma-s|}\left(\vartheta|\sigma-s|+1\right)$$

(see [11]). But we have the following result:

**Lemma 2.1.** *In this case,*

$$\liminf_{T\to+\infty} \frac{\mathcal{J}_T(\vartheta)}{T} \geq \sup_{w\in L^2(\mathbb{R})} (Cw, w) = \frac{\mu^2}{\vartheta^4}\,.$$

*Proof.* The proof is postponed in Section 3. □

In the classical case, the reminder $K_\infty(s,\sigma) - K_T(s,\sigma)$ corresponds to a positive quadratic form and

$$\sup_{v\in L^2[0,T]} (K_T v, v) \leq \sup_{w\in L^2(\mathbb{R})} (K_\infty w, w) = \sup_{w\in L^2(\mathbb{R})} (Cw, w)\,.$$

In the fractional case, we only have that:

$$\sup_{v\in L^2[0,T]} (K_T v, v) \leq \sup_{w\in L^2(\mathbb{R})} (K_\infty w, w)\,.$$

Nevertheless, we claim the following result:

**Proposition 2.1.**

$$\lim_{T\to+\infty} \sup_{v\in L^2[0,T],\,||v||=1} (K_T v, v) = \sup_{w\in L^2(\mathbb{R}),\,||w||=1} (Cw, w) = \frac{\mu^2}{\vartheta^4}\,.$$

*Proof.* Lemma 2.1 gives the lower bound. Therefore, we have to show the upper bound:

$$\lim_{T\to+\infty} \sup_{v\in L^2[0,T],\,||v||=1} (K_T v, v) \leq \sup_{w\in L^2(\mathbb{R}),\,||w||=1} (Cw, w) = \frac{\mu^2}{\vartheta^4}\,.$$

Let us introduce the pair process $\xi = \left((\xi_t^1, \xi_t^2), 0 \leq t \leq T\right)$ with

$$\xi_t^1 = \left(\int_t^T \sigma^{\frac{1}{2}-H} \ell(\sigma)^* \varphi(\sigma) * dW_\sigma\right) \varphi^{-1}(t) \quad \text{and} \quad \xi_t^2 = \frac{\partial}{\partial\vartheta}\xi_t^1\,, \tag{14}$$

where $*dW_\sigma$ denotes the Itô backward integral (see *e.g* [12]). It is worth emphasizing that

$$K_T(s,\sigma) = \frac{\mu^2}{4}\mathbf{E}\left(\xi_s^2 b(s) s^{\frac{1}{2}-H} \xi_\sigma^2 b(\sigma)\sigma^{\frac{1}{2}-H}\right)\,.$$

This process also satisfies the following dynamic

$$-d\xi_t = \xi_t \mathcal{A}(t) d\langle N\rangle_t + \mathcal{L}(t)\frac{t^{\frac{1}{2}-H}}{\sqrt{2\lambda}} * dW_t,$$

or

$$-d\xi_t = \xi_t \mathcal{A}(t) d\langle M\rangle_t + \mathcal{L}(t) * dM_t,\ \xi_T = 0,$$

with $M_t$ a martingale of the same variance function as $N_t$,

$$\mathcal{A}(t) = \begin{pmatrix} -\vartheta & 0 \\ -1 & -\vartheta \end{pmatrix} \otimes \lambda \mathbf{A}(t) \quad \text{and} \quad \mathcal{L}(t) = \sqrt{2\lambda} \begin{pmatrix} \ell(t)^* & 0 \end{pmatrix}.$$

Obviously, we should estimate the spectral gap (the first eigenvalue $\nu_1(T)$) of the operator associated to the kernel $K_T$. The estimation of the spectral gap is based on the Laplace transform computation.
Let us compute, for sufficiently small negative $a < 0$ the Laplace transform:

$$\begin{aligned} L_T(a) &= \mathbf{E}_\vartheta \exp\left\{-a \int_0^T \left[\frac{\mu}{2}\left(\frac{\partial}{\partial\vartheta}\xi_t^1\right) b(t) t^{\frac{1}{2}-H}\right]^2 dt\right\} \\ &= \mathbf{E}_\vartheta \exp\left\{-a\frac{\mu^2\lambda}{2}\int_0^T \xi_t \mathcal{M}(t)\xi_t^* d\langle N\rangle_t\right\} \end{aligned}$$

where

$$\mathcal{M}(t) = \begin{pmatrix} 0 & 0 \\ 0 & b(t)b(t)^* \end{pmatrix}.$$

On the one hand, for $a > -\frac{1}{\nu_1(T)}$, $L_T(a)$ can be represented as:

$$L_T(a) = \prod_{i\geq 1}(1 + 2a\nu_i(T))^{-\frac{1}{2}}, \tag{15}$$

where $\nu_i(T)$, $i \geq 1$ is the sequence of positive eigenvalues of the covariance operator. On the other hand,

$$L_T(a) = \exp\left\{\frac{1}{2}\int_0^T \mathrm{trace}(\mathcal{H}(t)\mathcal{L}(t)^*\mathcal{L}(t)) d\langle N\rangle_t\right\},$$

where $\mathcal{H}(t)$ is the solution of Ricatti differential equation:

$$\frac{d\mathcal{H}(t)}{d\langle N\rangle_t} = \mathcal{H}(t)\mathcal{A}(t)^* + \mathcal{A}(t)\mathcal{H}(t) + \mathcal{H}(t)\mathcal{L}(t)^*\mathcal{L}(t)\mathcal{H}(t) - a\mu^2\lambda\mathcal{M}(t), \tag{16}$$

with initial condition $\mathcal{H}(0) = 0$, provided that the solution of equation (16) exists for any $0 \le t \le T$.

It is well known that if $\det\Psi_1(t) > 0$, for any $t \in [0,T]$, then the solution $\mathcal{H}(t)$ of equation (16) can be written as $\mathcal{H}(t) = \Psi_1^{-1}(t)\Psi_2(t)$, where the pair of $4 \times 4$ matrices $(\Psi_1, \Psi_2)$ satisfies the system of linear differential equations:

$$\frac{d\Psi_1(t)}{d\langle N\rangle_t} = -\Psi_1(t)\mathcal{A}(t) - \Psi_2(t)\mathcal{L}(t)^*\mathcal{L}(t)\,, \quad \Psi_1(0) = \mathcal{I}d\,, \tag{17}$$

$$\frac{d\Psi_2(t)}{d\langle N\rangle_t} = -a\mu^2\lambda\Psi_1(t)\mathcal{M}(t) + \Psi_2(t)\mathcal{A}(t)^*\,, \quad \Psi_2(0) = 0\,, \tag{18}$$

and $\mathcal{I}d$ is the $4 \times 4$ identity matrix.

Moreover, under the condition $\det\Psi_1(t) > 0$, for any $t \in [0,T]$, the following equality holds:

$$\begin{aligned} L_T(a) &= \exp\left\{-\frac{1}{2}\int_0^T \mathrm{trace}\,\mathcal{A}(t)d\langle N\rangle_t\right\}(\det\Psi_1(T))^{-\frac{1}{2}} \\ &= \exp\{\vartheta T\}(\det\Psi_1(T))^{-\frac{1}{2}}\,, \end{aligned} \tag{19}$$

or, equivalently,

$$\prod_{i\ge1}(1 + 2a\nu_i(T)) = \exp\{-2\vartheta T\}(\det\Psi_1(T))\,. \tag{20}$$

Let us note here that the solution of linear system (17) exists for any $t > 0$ and for any $a \in \mathbb{C}$. For $a = 0$, $\det\Psi_1(t) = \exp\{2\vartheta t\} > 0$. Due to the continuity property of the solutions of linear differential equations with respect to a parameter, for all $T > 0$, there exists $a(T) < 0$ such that

$$\inf_{t\in[0,T]} \det\Psi_1(t) > 0\,.$$

Therefore, equality (20) holds in an open set in $\mathbb{C}$, containing 0. Compactness of the covariance operator, namely, $\int_0^T K_T(s,s)ds < \infty$, implies, due to the Weierstrass theorem, the analytic property of $\prod_{i\ge1}(1 + 2a\nu_i(T))$ with respect to $a$. Hence, equality (20) holds for any $a \in \mathbb{C}$.

Now let us show that for $T$ sufficiently large and for $-\frac{\vartheta^4}{2\mu^2} < a < 0$ $\det\Psi_1(T) > 0$. Indeed, linear system (17) can be rewritten as

$$\frac{d\left(\Psi_1(t), \Psi_2(t) \otimes \mathbf{J}\right)}{d\langle N\rangle_t} = \left(\Psi_1(t), \Psi_2(t) \otimes \mathbf{J}\right) \cdot \left(\beth \otimes \lambda\mathbf{A}(t)\right) \tag{21}$$

where

$$\daleth = \begin{pmatrix} \vartheta & 1 & 0 & 0 \\ 0 & \vartheta & 0 & -a\mu^2 \\ -2 & 0 & -\vartheta & 0 \\ 0 & 0 & -1 & -\vartheta \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly, system (21) has an explicit solution:

$$(\Psi_1(t), \Psi_2(t) \otimes \mathbf{J}) = (\mathcal{I}d, 0) \cdot (\mathcal{P} \otimes \mathbf{Id})\, \mathcal{G} \left(\mathcal{P}^{-1} \otimes \mathbf{Id}\right) \tag{22}$$

where $\mathcal{G} = \mathrm{diag}\,(\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3, \mathbf{G}_4)$ and

$$\frac{d\mathbf{G}_i(t)}{d\langle N\rangle_t} = \lambda x_i \mathbf{G}_i \mathbf{A} \quad \mathbf{G}_i(0) = \mathbf{Id}\,, \quad i = 1 \ldots 4\,, \tag{23}$$

with $(x_i)_{i=1\ldots4}$ the eigenvalues of matrix $\daleth$ and $\mathcal{P}$ the matrix of its eigenvectors. For

$$-\frac{\vartheta^4}{2\mu^2} < a < 0,$$

eigenvalues of matrix $\daleth$ are of the form $x_i = \pm\sqrt{\vartheta^2 \pm \mu\sqrt{-2a}}$.

It can be checked that there exists a constant $C > 0$ such that

$$\det\Psi_1(T) \;=\; \exp\left((x_1 + x_3)T\right) \left(C + O\left(\frac{1}{T}\right)\right),$$

where $x_1 = \sqrt{\vartheta^2 + \mu\sqrt{-2a}} > x_3 = \sqrt{\vartheta^2 - \mu\sqrt{-2a}}$.
Therefore, due to equality (20), we have that $\prod_{i\geq 1}(1 + 2a\nu_i(T)) > 0$ for any $a > -\frac{\vartheta^4}{2\mu^2}$. It means that $-\frac{1}{\nu_1(T)} \leq -\frac{\vartheta^4}{\mu^2}$, or equivalently, that

$$\nu_1(T) \leq \frac{\mu^2}{\vartheta^4} \tag{24}$$

which achieves the proof. □

Combining the proof of Lemma 2.1 and the upper bound (24), we obtain that

$$v_{opt}(t) = \sqrt{2\lambda}\, t^{H-\frac{1}{2}}, \; 0 \leq t \leq T$$

is optimal in the class $\mathcal{V}_T$. As in [11],

$$\frac{1}{T}\int_0^T |v_{opt}(t)|^2 d\langle N\rangle_t = 1$$

Using Remark 1, we have

$$u_{opt}(t) = \frac{d}{dt}\int_0^t K_H(t,s) v_{opt}(s) d\langle N\rangle_s = \frac{\kappa_H}{\sqrt{2\lambda}} t^{H-\frac{1}{2}}.$$

## 2.4 MLE large sample asymptotic properties

As the optimal input does not depend on $\vartheta$, it is possible to compute directly the MLE on the following system:

$$\left\{\begin{array}{rcl} dx_t &=& -\vartheta x_t dt + u_{opt}(t)dt\,, \quad x_0 = 0\,, \\ dY_t &=& \mu x_t dt + dV_t^H\,, \quad Y_0 = 0. \end{array}\right. \tag{25}$$

where $u_{opt}(t)$ is the optimal input found in previous Section 2.3. After transformation,

$$dZ_t = \mu\lambda \ell(t)^* \zeta_t d\langle N\rangle_t + dN_t\,, \quad Z_0 = 0\,, \tag{26}$$

where

$$\frac{d\zeta_t}{d\langle N\rangle_t} = -\vartheta\lambda \mathbf{A}(t)\zeta_t + b(t) v_{opt}(t)\,.$$

Actually, to compute large sample asymptotic properties of the implicit MLE, we need the explicit representation of the likelihood ratio

$$\mathcal{Z}_T(\vartheta_1, \vartheta_2, Z^{O,T}) = \frac{\mathcal{L}_T(\vartheta_2, Z^{O,T})}{\mathcal{L}_T(\vartheta_1, Z^{O,T})}\,, \tag{27}$$

which is also the Radon-Nikodym derivative of $\mathbf{P}_{\vartheta_2}^T$ with respect to $\mathbf{P}_{\vartheta_1}^T$, restricted to $\mathcal{F}_T^Y$, *i.e.*

$$\mathcal{Z}_T(\vartheta_1, \vartheta_2, Z^{O,T}) = \frac{\mathcal{L}_T(\vartheta_2, \zeta^{O,T})}{\mathcal{L}_T(\vartheta_1, \zeta^{O,T})} = \mathbf{E}_\vartheta\left(\frac{d\mathbf{P}_{\vartheta_2}^T}{d\mathbf{P}_{\vartheta_1}^T}\,|\mathcal{F}_T^Y\right)\,.$$

From Equation (9), this ratio can be written in the following form:

$$\mathcal{Z}_T(\vartheta_1, \vartheta_2, Z^{O,T}) = \exp\left\{\mu\lambda\int_0^T l^*\delta_{\vartheta_1,\vartheta_2} d\nu_t^{\vartheta_1} - \frac{\mu^2\lambda^2}{2}\int_0^T \delta_{\vartheta_1,\vartheta_2}^* \ell\ell^* \delta_{\vartheta_1,\vartheta_2} d\langle N\rangle_t\right\}$$

where $\delta_{\vartheta_1,\vartheta_2}(t)$ is the difference $\zeta_t^{\vartheta_2} - \zeta_t^{\vartheta_1}$ and $\left(\nu_t^{\vartheta_1}, t \geq 0\right)$ is defined by:

$$d\nu_t^{\vartheta_1} = dZ_t^O - \mu\lambda\ell(t)^*\zeta_t^{\vartheta_1} d\langle N\rangle_t\,, \quad \nu_0^{\vartheta_1} = 0\,.$$

We will denote by $\mathcal{Z}_T(x, Z^{O,T})$ the perturbation of $\mathcal{Z}_T(\vartheta, \vartheta_2, Z^{O,T})$, when $\vartheta_2 = \vartheta + \frac{x}{\sqrt{T}}$. Namely, $\mathcal{Z}_T(x, Z^{O,T}) = \mathcal{Z}_T(\vartheta, \vartheta + \frac{x}{\sqrt{T}}, Z^{O,T})$. For this case, we will denote $\delta_{\vartheta,x,T} = \delta_{\vartheta,\vartheta+\frac{x}{\sqrt{T}}}$.

### 2.4.1 Ibragimov–Khasminskii program

It follows from [3, Theorem I.10.1] that in order to prove Theorem 1.2, it is sufficient to check the three following conditions:

(A.1)

$$\mathcal{Z}_T(x, Z^{O,T}) \overset{law}{\Longrightarrow} \exp\left\{x.\eta - \frac{x^2}{2}\mathcal{I}^{opt}(\vartheta)\right\} \text{ with } \quad \eta \sim \mathcal{N}\left(0, \mathcal{I}^{opt}(\vartheta)\right)\,,$$

(A.2) for some $\chi > 0$:

$$\mathbf{E}_\vartheta \sqrt{\mathcal{Z}_T(x, Z^{O,T})} \leq \exp\left(-\chi x^2\right)$$

(A.3) there exists $C > 0$ such that

$$\mathbf{E}_\vartheta \left(\sqrt{\mathcal{Z}_T(x_1, Z^{O,T})} - \sqrt{\mathcal{Z}_T(x_2, Z^{O,T})}\right)^2 \leq C|x_1 - x_2|^2\,.$$

We present here the proof of Theorem 1.2 by checking the three conditions.

*Proof.* As the deterministic quantity (see Lemma 2.1 and Proposition 2.1)

$$\frac{\mu^2\lambda^2}{2}\int_0^T \delta_{\vartheta,x,T}^*\ell\ell^*\delta_{\vartheta,x,T} d\langle N\rangle_t \overset{T\to\infty}{\longrightarrow} \frac{x^2}{2}\lim_{T\to\infty}\frac{\mathcal{I}(\vartheta, v_{opt})}{T} = \frac{x^2}{2}\mathcal{I}^{opt}(\vartheta)\,,$$

with $\mathcal{I}^{opt}(\vartheta) = \frac{\mu^2}{\vartheta^4}$ and then

$$\mu\lambda\int_0^T \ell^*\delta_{\vartheta_1,\vartheta_2} d\nu_t^{\vartheta_1} \overset{law}{\Longrightarrow} \mathcal{N}(0, \mathcal{I}^{opt}(\vartheta))$$

and (A.1) is checked. The condition (A.2) holds thanks to the following chain of inequalities:

$$\begin{aligned}
\mathbf{E}_\vartheta \sqrt{\mathcal{Z}_T(x)} &= \mathbf{E}_\vartheta \exp\left(\frac{\mu\lambda}{2}\int_0^T \ell^* \delta_{\vartheta,x,T} d\nu_t^\vartheta - \frac{\mu^2\lambda^2}{4}\int_0^T \delta_{\vartheta,x,T}^* \ell\ell^* \delta_{\vartheta,x,T} d\langle N\rangle_t\right) \\
&= \mathbf{E}_\vartheta \exp\left(\frac{\mu\lambda}{2}\int_0^T \ell^* \delta_{\vartheta,x,T} d\nu_t^\vartheta - \frac{\mu^2\lambda^2}{8}\int_0^T \delta_{\vartheta,x,T}^* \ell\ell^* \delta_{\vartheta,x,T} d\langle N\rangle_t\right) \\
&\quad \times \exp\left(-\frac{\mu^2\lambda^2}{8}\int_0^T \delta_{\vartheta,x,T}^* \ell\ell^* \delta_{\vartheta,x,T} d\langle N\rangle_t\right) \\
&\overset{(a)}{\leq} \exp\left(-\frac{\mu^2\lambda^2}{8}\int_0^T \delta_{\vartheta,x,T}^* \ell\ell^* \delta_{\vartheta,x,T} d\langle N\rangle_t\right) \\
&\overset{(b)}{\leq} \exp\left(-\chi x^2\right) ,
\end{aligned}$$

where $(a)$ is Girsanov Theorem since

$$\mathbf{E}_\vartheta \exp\left(\frac{\mu\lambda}{2}\int_0^T \ell^* \delta_{\vartheta,x,T} d\nu_t^\vartheta - \frac{\mu^2\lambda^2}{8}\int_0^T \delta_{\vartheta,x,T}^* \ell\ell^* \delta_{\vartheta,x,T} d\langle N\rangle_t\right) \leq 1 ,$$

and $(b)$ comes from the proof of (A.1). To prove (A.3), let us note that

$$\begin{aligned}
\mathbf{E}_\vartheta \left(\sqrt{\mathcal{Z}_T(x_1)} - \sqrt{\mathcal{Z}_T(x_2)}\right)^2 &= 2\left(1 - \mathbf{E}_\vartheta \mathcal{Z}_T(x_1)\sqrt{\frac{\mathcal{Z}_T(x_2)}{\mathcal{Z}_T(x_1)}}\right) \\
&= 2\left(1 - \mathbf{E}_{\vartheta_1}\sqrt{\mathcal{Z}_T(\vartheta_1,\vartheta_2)}\right) .
\end{aligned}$$

The same chain of inequalities (with Reverse Hölder Inequality and Girsanov Theorem) gives:

$$\begin{aligned}
\mathbf{E}_\vartheta \left(\sqrt{\mathcal{Z}_T(x_1)} - \sqrt{\mathcal{Z}_T(x_2)}\right)^2 &\leq 2\left(1 - \exp\left(-\chi_{_1}\left(x_2 - x_1\right)^2\right)\right) \\
&\leq C|x_1 - x_2|^2 .
\end{aligned}$$

□

**Remark 1.** *We have denoted by*

$$\mathcal{V}_T = \left\{ v \quad | \quad \frac{1}{T}\int_0^T |v(t)|^2 d\langle N\rangle_t \leq 1 \right\}$$

*the class of admissible transformed controls. Remark that the following relation between $u(t)$ and transformation $v(t) = \frac{d}{d\langle N\rangle_t}\int_0^t k_H(t,s)u(s)ds$ holds:*

$$u(t) = \frac{d}{dt}\int_0^t K_H(t,s)v(s)d\langle N\rangle_s.$$

*At the first glance, we can set the admissible controls as:*

$$\mathcal{U}_T = \{u \quad | \quad v \in \mathcal{V}_T\}.$$

*Note that these sets are non empty.*

# 3 Proof of Lemma 2.1

From (12), we have

$$\begin{aligned}
\ell(t)^*\zeta_t &= \ell(t)^*\varphi(t)\int_0^t \varphi^{-1}(s)b(s)v(s)d\langle N\rangle_s \\
&= t^{H-\frac{1}{2}}\int_0^t \left(t^{\frac{1}{2}-H}\ell(t)^*\varphi(t)\varphi^{-1}(s)b(s)s^{\frac{1}{2}-H}\right)\frac{s^{\frac{1}{2}-H}}{2\lambda}v_s ds \\
&= t^{H-\frac{1}{2}}\int_0^t g(t,s)\frac{s^{\frac{1}{2}-H}}{2\lambda}v_s ds\,,
\end{aligned}$$

where

$$g(t,s) = t^{\frac{1}{2}-H}\ell(t)^*\varphi(t)\varphi^{-1}(s)b(s)s^{\frac{1}{2}-H}\,.$$

From [5], explicit expression can be deduced, namely

$$\varphi(t) = e^{-\frac{\vartheta t}{2}}\begin{pmatrix} f_{2,2}(t) & -f_{1,2}(t) \\ -f_{2,1}(t) & f_{1,1}(t)\end{pmatrix} \quad , \quad \det\varphi(t) = e^{-\vartheta t}$$

and functions defined by

$$\begin{aligned}
f_{1,1}(t) &= \left(\frac{\vartheta}{4}\right)^H \Gamma(1-H)\,t^H I_{-H}\left(\frac{\vartheta t}{2}\right) \\
f_{1,2}(t) &= \left(\frac{\vartheta}{4}\right)^H \Gamma(1-H)\,t^{1-H} I_{1-H}\left(\frac{\vartheta t}{2}\right) \\
f_{2,1}(t) &= \left(\frac{\vartheta}{4}\right)^{1-H} \Gamma(H)\,t^H I_{H}\left(\frac{\vartheta t}{2}\right) \\
f_{2,2}(t) &= \left(\frac{\vartheta}{4}\right)^{1-H} \Gamma(H)\,t^{1-H} I_{H-1}\left(\frac{\vartheta t}{2}\right),
\end{aligned}$$

where $I_\nu$ is the modified Bessel function of the first kind and order $\nu$. Direct computation leads to

$$t^{\frac{1}{2}-H}\ell(t)^*\varphi(t) = e^{-\frac{\vartheta t}{2}}\sqrt{t}\left(C_H\left(I_{H-1}-I_H\right)\left(\frac{\vartheta t}{2}\right), C_{1-H}\left(I_{-H}-I_{1-H}\right)\left(\frac{\vartheta t}{2}\right)\right)$$

where $C_H = \left(\frac{\vartheta}{4}\right)^{1-H}\Gamma(H)$.
Let us remark that,

$$\mathbf{Id} = \frac{1}{C_H C_{1-H}}\begin{pmatrix} C_{1-H} & 0 \\ 0 & C_H \end{pmatrix}\begin{pmatrix} C_H & 0 \\ 0 & C_{1-H} \end{pmatrix}$$

and

$$\mathbf{Id} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Therefore, we can compute

$$\begin{aligned}(a) &= t^{\frac{1}{2}-H}\ell(t)^*\varphi(t)\begin{pmatrix} C_{1-H} & 0 \\ 0 & C_H \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= e^{-\frac{\vartheta t}{2}}\left(g_1(t), g_2(t)\right)\end{aligned}$$

with

$$\begin{aligned} g_1(t) &= C_H C_{1-H}\sqrt{t}\left(I_{H-1}-I_H+I_{-H}-I_{1-H}\right)\left(\frac{\vartheta t}{2}\right) \\ g_2(t) &= C_H C_{1-H}\sqrt{t}\left(I_{H-1}-I_H-I_{-H}+I_{1-H}\right)\left(\frac{\vartheta t}{2}\right).\end{aligned}$$

With same computation,

$$\begin{aligned}(b) &= \frac{1}{2C_H C_{1-H}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} C_H & 0 \\ 0 & C_{1-H} \end{pmatrix}\varphi^{-1}(s)b(s)s^{\frac{1}{2}-H} \\ &= e^{\frac{\vartheta s}{2}}\begin{pmatrix} \tilde{g}_1(s) \\ \tilde{g}_2(s) \end{pmatrix}\end{aligned}$$

with

$$\begin{aligned} \tilde{g}_1(s) &= \frac{\sqrt{s}}{2}\left(I_{H-1}+I_H+I_{-H}+I_{1-H}\right)\left(\frac{\vartheta s}{2}\right) \\ \tilde{g}_2(s) &= \frac{\sqrt{s}}{2}\left(-I_{H-1}-I_H+I_{-H}+I_{1-H}\right)\left(\frac{\vartheta s}{2}\right).\end{aligned}$$

Hence

$$
\begin{aligned}
g(t,s) &= (a)(b) = e^{-\frac{\vartheta t}{2}} e^{\frac{\vartheta s}{2}} \left(g_1(t)\tilde{g}_1(s) + g_2(t)\tilde{g}_2(s)\right) \\
&= e^{-\vartheta t} e^{\vartheta s} \left(e^{\frac{\vartheta t}{2}} g_1(t) e^{-\frac{\vartheta s}{2}} \tilde{g}_1(s)\right) + e^{-\frac{\vartheta t}{2}} g_2(t) e^{\frac{\vartheta s}{2}} \tilde{g}_2(s).
\end{aligned}
$$

Moreover $C_H C_{1-H} = \frac{\vartheta}{4}\frac{\pi}{\sin \pi H}$, $(I_H - I_{-H})\left(\frac{\vartheta t}{2}\right) \simeq \frac{2\sin \pi H}{\sqrt{\vartheta \pi t}} e^{-\frac{\vartheta t}{2}}$ that gives

$$
e^{\frac{\vartheta t}{2}} g_1(t) \underset{t\to\infty}{\longrightarrow} \sqrt{\pi\vartheta}\,.
$$

With property (see [10])

$$
I_\nu\left(\frac{\vartheta t}{2}\right) \underset{t\to\infty}{\simeq} \frac{e^{\frac{\vartheta t}{2}}}{\sqrt{\pi \vartheta t}} \left(1 - \frac{4\nu^2 - 1}{4\vartheta t} + O\left(\frac{1}{t^2}\right)\right),
$$

we have that

$$
\begin{aligned}
e^{-\frac{\vartheta s}{2}} \tilde{g}_1(s) &\underset{s\to\infty}{\longrightarrow} \frac{2}{\sqrt{\pi\vartheta}}, \\
e^{-\frac{\vartheta t}{2}} g_2(t) &\underset{t\to\infty}{\simeq} \frac{2H-1}{2t\sin(\pi H)}\sqrt{\frac{\pi}{\vartheta}}, \\
e^{\frac{\vartheta s}{2}} \tilde{g}_2(s) &\underset{s\to\infty}{\simeq} \frac{(2H-1)\sin(\pi H)}{s\vartheta\sqrt{\pi\vartheta}}.
\end{aligned}
$$

From (11)

$$
\begin{aligned}
\mathcal{I}_T(\vartheta, v) &= \int_0^T \mu^2\lambda^2 \left(\frac{\partial}{\partial\vartheta}\ell(t)^*\zeta_t\right)^2 d\langle N\rangle_t \\
&= \int_0^T \mu^2\lambda^2 \left(\frac{\partial}{\partial\vartheta} t^{H-\frac{1}{2}} \int_0^t g(t,s)\frac{s^{\frac{1}{2}-H}}{2\lambda} v_s ds\right)^2 \frac{t^{1-2H}}{2\lambda} dt \\
&= \frac{T\mu^2}{4}\left[\frac{1}{T}\int_0^T \left(\int_0^t \frac{\partial}{\partial\vartheta} g(t,s)\frac{s^{\frac{1}{2}-H} v_s}{\sqrt{2\lambda}} ds\right)^2 dt\right].
\end{aligned}
$$

Now if we take $v_{opt}(s) = \sqrt{2\lambda}s^{H-\frac{1}{2}}$ then

$$
\frac{1}{T}\int_0^T (v_{opt}(s))^2 d\langle N\rangle_s = \frac{1}{T}\int_0^T (v_{opt}(s))^2 \frac{s^{1-2H}}{2\lambda} ds = 1
$$

and

$$\mathcal{I}_T(\vartheta, v_{opt}) = \frac{T\mu^2}{4}\left[\frac{1}{T}\int_0^T \left(\int_0^t \frac{\partial}{\partial\vartheta} g(t,s)ds\right)^2 dt\right] = \frac{T\mu^2}{4}\left[\frac{1}{T}\int_0^T (\Psi(t))^2\, dt\right].$$

It follows from the previous asymptotic estimates that for $s \geq 0$ and $M \geq 0$ we have

$$\lim_{t\to+\infty} \frac{\partial}{\partial\vartheta} g(t,s) = 0 \Longrightarrow \int_0^M \frac{\partial}{\partial\vartheta} g(t,s)\mathbf{1}_{(0,t)}(s)ds \underset{t\to\infty}{\longrightarrow} 0\,.$$

Moreover for $s$ and $t$ large enough we obtain

$$g(t,s) \simeq 2e^{-\vartheta(t-s)} + \frac{(2H-1)^4}{2\vartheta^2 ts}\,.$$

The recurrence relation for the derivatives of Bessel functions $I'_\nu = I_{\nu+1} + \frac{\nu}{x} I_\nu$ implies that

$$\frac{\partial}{\partial\vartheta} g(t,s) \simeq -2(t-s)e^{-\vartheta(t-s)} - \frac{2(2H-1)^4}{2\vartheta^3 ts}\,.$$

Therefore

$$\int_M^t \frac{\partial}{\partial\vartheta} g(t,s)ds \simeq -2\int_M^t (t-s)e^{-\vartheta(t-s)}ds - \frac{2(2H-1)^4}{2\vartheta^3}\frac{\ln t - \ln M}{t}$$
$$= \frac{-2}{\vartheta^2}\left[1-(1+\vartheta(t-M))e^{-\vartheta(t-M)}\right] - \frac{2(2H-1)^4}{2\vartheta^3}\frac{\ln t - \ln M}{t}\,.$$

Finally we obtain

$$\Psi_t \underset{t\to\infty}{\longrightarrow} \frac{-2}{\vartheta^2} \Longrightarrow \mathcal{I}_T(\vartheta, v_{opt}) \underset{T\to\infty}{\simeq} \frac{T\mu^2}{\vartheta^4}\,.$$

To conclude recall that

$$\mathcal{J}_T(\vartheta) = \sup_{v\in\mathcal{V}_T} \mathcal{I}_T(\vartheta, v) \geq \mathcal{I}_T(\vartheta, v_{opt})\,.$$

which implies

$$\liminf_{T\to+\infty} \frac{\mathcal{J}_T(\vartheta)}{T} \geq \frac{\mu^2}{\vartheta^4} = \sup_{w\in L^2(\mathbb{R})} (Cw, w).$$

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