Estimating discontinuous periodic signals in a time inhomogeneous diffusion

Reinhard Höpfner and Yury Kutoyants

Johannes-Gutenberg-Universität, Mainz, and Université du Maine, Le Mans

Abstract: We consider a diffusion $(\xi_t)_{t\geq 0}$ with T-periodic time dependence in its drift; under an unknown parameter $\vartheta \in \Theta$, some discontinuity – an additional periodic signal – occurs at times $kT+\vartheta$, $k \in \mathbb{N}$. Assuming positive Harris recurrence of $(\xi_{kT})_{k\in\mathbb{N}_0}$ and exploiting the periodicity structure, we prove limit theorems for certain martingales and functionals of the process $(\xi_t)_{t\geq 0}$. They allow to consider the statistical model parametrized by $\vartheta \in \Theta$ locally in small neighbourhoods of some fixed ϑ, with radius $\frac{1}{n}$ as $n \to \infty$. We prove convergence of local models to a limit experiment studied by Ibragimov and Khasminskii [IH 81] which is not quadratic in its parameter. We discuss the behaviour of estimators under contiguous alternatives, and prove a local asymptotic minimax bound under quadratic loss which is attained by the corresponding Bayes estimator sequence.

Key words: diffusions, inhomogeneity in time, discontinuous signal, periodicity, limit theorems; likelihood ratio processes, convergence of experiments, contiguity, maximum likelihood estimators, Bayes estimators, local asymptotic minimax theorem.

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We consider a problem of parameter estimation in a Markov process $(\xi_t)_{t\geq 0}$ whose drift is T–periodic in the time variable. A parameter ϑ comes in through a periodic signal with periodicity T, and represents a time of discontinuity. Our main assumption on the process is positive Harris recurrence of $(\xi_{kT})_{k\in\mathbb{N}_0}$ from which – exploiting the periodicity structure of the semigroup – we prove positive Harris recurrence of the chain of T–segments $\left((\xi_{(k-1)T+s})_{0\leq s\leq T}\right)_{k\in\mathbb{N}_0}$. This allows to deduce limit theorems for certain martingales and strong laws of large numbers for certain functionals of the continuous-time process $(\xi_t)_{t\geq 0}$. Based on these, we deal with convergence of local models at ϑ – corresponding to observation of the process $(\xi_t)_{t\geq 0}$ up to time nT, local scale at ϑ turns out to be $\frac{1}{n}$ as $n \to \infty$ – to a limit model whose likelihoods are of type $u \to e^{W_u - \frac{1}{2}|u|}$ with double-sided Brownian motion W. This limit model, investigated by Ibragimov and Khasminskii [IH 81, sections VII.2–3], has $\frac{1}{2}$–Hölder continuous parametrization in the metric defined by Hellinger distance. So we are far from the framework of local asymptotic normality (LAN, well-studied since LeCam [L 68] and Hájek [H 70]; for smoothly parametrized ergodic diffusions, cf. Kutoyants [K 04, section II]), or more generally from local asymptotic mixed normality (LAMN) or local asymptotic quadraticity (LAQ, cf. [D 85], [LY 90]). Ibragimov and Khasminskii considered this in a 'signal in white noise' setting, proved convergence of maximum likelihood and Bayes estimators at ϑ, calculated the limit variance of the maximum likelihood estimator, and pointed out that in a limit model with likelihood ratios of type $u \to e^{W_u - \frac{1}{2}|u|}$ a Bayes estimator is better than the maximum likelihood estimator.

Convergence to the same limit model has been explored in their spirit in several settings since then. Küchler and Kutoyants ([KK 00]) obtained it in a framework of delay equations, and Kutoyants [K 04, section 3.4] in time homogeneous ergodic diffusions where the drift has a discontinuity in the space variable. In the context of a change point in iid observations the limit model appears in [DP 84, section 3]. The approach of Ibragimov and Khasminskii starts from certain assumptions on Hellinger distances and from convergence of likelihood ratios 'uniformly in ϑ'. In several aspects, our approach is different. We develop limit theorems in diffusions with T–periodic semigroup which will be our key tool in view of convergence of likelihood ratios and estimators. Whereas these allow to check and exploit assumptions on Hellinger distances similiarly to the work quoted above, convergence of likelihood ratios 'uniformly in ϑ' is not suitable for our framework of inhomogeneity in time, and is systematically avoided. We are focussing on contiguous alternatives, make extensive use of 'LeCam's Third Lemma' (see [LY 90, pp. 22–23]), and exploit asymptotic equivariance of suitable estimator sequences with respect to contiguous alternatives. Our local asymptotic minimax bound controls a maximal quadratic risk on shrinking neighbourhoods of ϑ with radius proportional to $\frac{1}{n}$, i.e. under contiguous alternatives; a Bayes estimator sequence attains this bound.

We describe our setting in more detail. The observed diffusion process is inhomogeneous in time

(1) $$d\xi_t \;=\; [S(\vartheta,t) + b(\xi_t)]\, dt \;+\; \sigma(\xi_t)\, dW_t \;, \quad t \geq 0$$

and its drift involves a T-periodic signal

(2) $$S(\vartheta,t) \;=\; \lambda(t) \;+\; \left(\lambda^* 1_{(\vartheta,\vartheta+a)}\right)(i_T(t)) \;,\; t \geq 0 \;, \quad \text{with} \;\; i_T(t) \;:=\; t \,\text{modulo}\, T$$

depending on an unknown parameter ϑ. The function $\lambda(\cdot)$ is continuous and T-periodic on $[0,\infty)$;

$\lambda^*(\cdot)$ is strictly positive and continuous on $[0,T]$. The functions $b(\cdot)$ and $\sigma(\cdot)$ are Lipschitz; hence for all values of the parameter ϑ, we have Lipschitz and linear growth conditions for the time-dependent coefficients of the above SDE, and thus existence and pathwise uniqueness for its solution. The periodicity T is known and does not depend on ϑ. The duration a of the signal is fixed and known, and we put $\Theta := (0, T-a)$.

We are interested in convergence of local models and convergence of maximum likelihood (MLE) and Bayes (BE) estimators for the unknown parameter $\vartheta \in \Theta$ when a trajectory of ξ has been observed up to time nT. As $n \to \infty$, the right choice of local scale for local models at ϑ turns out to be $\frac{1}{n}$, at every point $\vartheta \in \Theta$. For the limit of local models at ϑ, we find likelihood ratios

$$(3) \qquad \widetilde{L}^{u/0} \;:=\; \exp\left\{ \widetilde{W}(uJ_\vartheta) - \frac{1}{2}\,|uJ_\vartheta| \right\} , \quad u \in I\!R$$

with double-sided Brownian motion $(\widetilde{W}_u)_{u\in I\!R}$ and with scaling constants $0 < J_\vartheta < \infty$. From Terent'yev [T 68] over Golubev [G 79] and Ibragimov and Khasminskii [IH 81] to Rubin and Song [RS 95] it has become evident that in experiments of type (3) with unknown parameter $u \in I\!R$, the Bayes estimator with respect to quadratic loss is strictly better than the maximum likelihood estimator. Note that the limit experiment (3) is not a 'quadratic' experiment, so we find features which are essentially different from the well known Gaussian, mixed Gaussian and – to less extent – quadratic limit experiments (e.g., LeCam [L 68], Strasser [S 85], Davies [D 85], LeCam and Yang [LY 90], Jeganathan [J 95], or also [H 08, sections VI+VII]). In particular, there is no analogue of a convolution theorem (due to Hájek [H 70] in the classical LAN case, and generalized by Jeganathan [J 82] to LAMN) which – together with a lemma stating that 'arbitrary estimator sequences are in some sense almost equivariant' – is the key tool to obtain a local asymptotic minimax theorem (see [LY 90, p. 83]) under LAN or LAMN. This theorem allows to compare arbitrary estimator sequences under arbitrary subconvex, bounded and continuous loss functions, under LAN or LAMN; estimator sequences whose rescaled estimation error is bound to the central sequence (which typically holds for MLE) turn out to be optimal, and this independently of the choice of the loss function. In our context, whereas the contiguity techniques known from LAN, LAMN or LAQ continue to work, there is no 'global' result like that. Our local asymptotic minimax theorem

$$(4) \qquad \lim_{C\uparrow\infty}\ \liminf_{n\to\infty}\ \inf_{\widetilde{\vartheta}_{nT}}\ \sup_{|u|\le C}\ E_{\vartheta+\frac{u}{n}}\left(\left[n\left(\widetilde{\vartheta}_{nT} - (\vartheta + \frac{u}{n}) \right) \right]^2 \right) \quad \ge \quad E\left([u^*]^2 \right)$$

is under squared loss. u^* is the Bayes estimator associated to quadratic loss in the limit experiment (3), and $\inf_{\widetilde{\vartheta}_{nT}}$ allows to compare all possible estimators based on observation of ξ up to time nT. We prove that a Bayes estimator sequence $(\vartheta^*_{nT})_n$ which corresponds to this choice of the loss function is asymptotically equivariant with respect to contiguous alternatives, and attains the local asymptotic minimax bound (4).

In view of asymptotic statistical properties, our model behaves exactly as the 'signal in white noise' setting of [IH 81, section VII.2] which corresponds to the special case $\sigma(\cdot) \equiv 1$, $b(\cdot) \equiv 1$ in (1) above. However, if limit theorems are the key tool to prove statistical properties (convergence of experiments, convergence of estimators, ...), these are radically different in our case. In their likelihoods thanks to $\sigma(\cdot) \equiv 1$, [IH 81] can work with very simple Gaussian processes where calculation of means and covariances is enough to determine the limiting behaviour. Similiarly, in the time homogeneous ergodic diffusion model of [K 04, section 3.4] with one discontinuity in the drift, well known limit theorems for convergence of martingales and of additive functionals for ergodic diffusions are at hand. Our time inhomogeneous T–periodic problem (1) with non-trivial $\sigma(\cdot)$ requires a completely new approach. So an essential part of the present paper is devoted to proving limit theorems which make statistical theories work in our setting. Also in view of the behaviour of estimators under contiguous alternatives, we have to go beyond what had been done earlier in order to obtain the local asymptotic minimax bound (4).

Our interest in periodicity structures in diffusions is linked to the following application. In some membrane potential data sets similiar to those investigated in [H 07] which we wish to interprete as realizations of certain SDE's (out of many references, we mention [LL 87], [T 89], [LS 99], [DL 05], [DL 06]), there is evidence for time-dependent 'input' in the drift which the modelization has to take into account. In analogy to the result of [BH 06, section 3.2] on large systems of neurons receiving identical time-dependent input, questions of periodic input received by a single neuron in an active network deserve to be studied. In particular, a discontinuity (2) with constants λ, λ^* can be interpreted as some stimulus switched on/off periodically, and is of biological relevance.

This paper is organized as follows. Section 1 states all statistical results and formulates the local asymptotic minimax bound (theorem 1.8). Section 2 deals with Harris properties of the chain of T–segments. Section 3 contains an exponential inequality adapted to our purposes from Brandt [B 05]). Section 4 works out the limit theorems which we need to do statistics in our problem (1)+(2). Sections 2–4 can be read independently, and are formulated in a slightly more general setting; the main results in this probabilistic part of the paper are theorems 2.1+4.1 (strong laws of large numbers for time inhomogeneous diffusions with periodicity structure) and theorem 4.3 (weak convergence of martingale terms which occur in the log-likelihood ratios of local models at ϑ). On this basis, section 5 contains the statistical part of work to be done, and collects all proofs for the results stated in section 1.

1 Outline of statistical results

In order to exploit 'ergodicity properties' of the time inhomogeneous diffusion $\xi = (\xi_t)_{t\geq 0}$ in (1) with T–periodic time-dependent input (2), our principal assumption will be

$(H1)$ the embedded chain $(\xi_{kT})_{k\in I\!N_0}$ is positive recurrent in the sense of Harris

for any fixed value of the parameter $\vartheta \in \Theta$. As an example, $(H1)$ always holds if piecewise continuous T-periodic input is added to an Ornstein-Uhlenbeck SDE, see 2.3 below. Under $(H1)$, there is a unique invariant probability $\mu^{(\vartheta)}$ for the chain $(\xi_{kT})_{k\in I\!N_0}$ under ϑ. We introduce the chain of T–segments in the path of ξ

$$X = (X_k)_k \quad \text{defined by} \quad X_k \ := \ (\xi_{(k-1)T+s})_{0\leq s\leq T} \ , \ k \geq 1$$

which – as a consequence of the T–periodicity in the drift of our SDE – is time homogeneous. X takes values in the path space $(C_T, \mathcal{C}_T)$ of continuous functions $[0,T] \to I\!R$. We deduce from assumption (H1) – see theorem 2.1 in section 2 below – that this T–segment chain is positive recurrent in the sense of Harris under ϑ, with a specified invariant probability $m^{(\vartheta)}$ on $(C_T, \mathcal{C}_T)$. Limit theorems for functionals of the process ξ which we need for our analysis of the statistical model (see theorems 4.1 and 4.3 in section 4 below) are then obtained through strong laws of large numbers in the Harris chain $X = (X_k)_{k\in I\!N_0}$.

Let Q^ϑ denote the law of the process $\xi = (\xi_t)_{t\geq 0}$ of (1)+(2) under $\vartheta \in \Theta$, a law on the canonical path space $(C, \mathcal{C})$ of continuous functions $[0,\infty) \to I\!R$ equipped with its canonical filtration $I\!\!G$. Our second major assumption

$(H2)$ $\sigma(\cdot)$ is bounded away from 0 and ∞ on $I\!R$

guarantees that for any pair of different values $\zeta' \neq \zeta$ in Θ, the laws Q^ζ and $Q^{\zeta'}$ are locally equivalent with respect to $\mathbb{G}$, and the likelihood ratio process of $Q^{\zeta'}$ to Q^{ζ} is

$$L_t^{\zeta'/\zeta} \quad = \quad \exp\left\{ \int_0^t \frac{S(\zeta',s) - S(\zeta,s)}{\sigma^2(\eta_s)}\, dM_s^{\zeta} \;-\; \frac{1}{2}\int_0^t \frac{(S(\zeta',s) - S(\zeta,s))^2}{\sigma^2(\eta_s)}\, ds \right\} \;, \qquad t \ge 0$$

where we write $\eta = (\eta_t)_{t\ge0}$ for the canonical process on $(C, \mathcal{C})$, and M^{ζ} for the $(Q^{\zeta},\mathbb{G})$–martingale part of η (see [JS 87], [LS 81], [Ku 04]). Let B denote a version of $\int_0^{\cdot} \frac{1}{\sigma(\eta_s)} dM_s^{\zeta}$ under some $\zeta \in \Theta$: then B is a $\mathbb{G}$–Brownian motion under Q^{ζ} for all $\zeta \in \Theta$. Since $S(\vartheta, \cdot)$ is – up to the continuous functions $\lambda(\cdot)$, $\lambda^*(\cdot)$, and up to T–periodic continuation – the indicator function $1_{(\vartheta,\vartheta+a)}$, the likelihood ratio $L_t^{\zeta'/\zeta}$ takes for ζ' sufficiently close to ζ the simple form

$$\exp\left\{ \left[-\int_0^t \frac{\lambda^*(s)}{\sigma(\eta_s)}\, 1_{(\zeta,\zeta')}(i_T(s))\, dB_s \;-\; \frac{1}{2}\int_0^t \left(\frac{\lambda^*(s))}{\sigma(\eta_s)}\right)^2 1_{(\zeta,\zeta')}(i_T(s))\, ds \right] \right.$$
$$\left. + \left[\int_0^t \frac{\lambda^*(s)}{\sigma(\eta_s)}\, 1_{(\zeta+a,\zeta'+a)}(i_T(s))\, dB_s \;-\; \frac{1}{2}\int_0^t \left(\frac{\lambda^*(s))}{\sigma(\eta_s)}\right)^2 1_{(\zeta+a,\zeta'+a)}(i_T(s))\, ds \right] \right\}$$

in case $\zeta < \zeta' < \zeta + a$; the same holds with ζ, ζ' in the intervals and sign $\pm$ in front of the stochastic integrals interchanged if $\zeta' < \zeta < \zeta' + a$. Our first main result – very easy to see in the 'trivial' case where $\sigma(\cdot)$ is constant (cf. [IH 81], lemma 2.4 on p. 334), essentially more difficult for Lipschitz functions $\sigma(\cdot)$ satisfying $(H1)$ where the argument has to go back to the chain of T–segments in the path of ξ (cf. theorems 4.1 and 4.3 in section 4 below) – is the following.

1.1 Theorem: Under Lipschitz and linear growth conditions on $b(\cdot)$ and $\sigma(\cdot)$, under $(H1)$ and $(H2)$, the following holds for every $\vartheta \in \Theta$:

a) we have convergence under Q^{ϑ} as $n \to \infty$ of

$$\left(L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta} \right)_{u\in\Theta_{\vartheta,n}} \quad , \quad \Theta_{\vartheta,n} := \{u \in I\!R : \vartheta + \frac{u}{n} \in \Theta\}$$

in the sense of finite dimensional distributions to

$$\text{(5)} \qquad \widetilde{L} = \left(\widetilde{L}^{u/0}\right)_{u\in I\!R} \quad , \quad \widetilde{L}^{u/0} \;:=\; \exp\left\{ \widetilde{W}(uJ_{\vartheta}) - \frac{1}{2}|uJ_{\vartheta}| \right\}$$

where $(\widetilde{W}_u)_{u\in I\!R}$ is two-sided standard Brownian motion, and J_{ϑ} the scaling constant

$$J_{\vartheta} \;:=\; \left\{ (\lambda^*(\vartheta))^2 \,(\mu^{(\vartheta)} P_{0,\vartheta}^{(\vartheta)}) + (\lambda^*(\vartheta{+}a))^2 \,(\mu^{(\vartheta)} P_{0,\vartheta+a}^{(\vartheta)}) \right\} (\frac{1}{\sigma^2}) \,.$$

Here $(P_{s,t}^{(\vartheta)})_{0\le s<t<\infty}$ denotes the semigroup of the process $(\xi_t)_{t\ge0}$ under ϑ, $\mu^{(\vartheta)}$ the invariant measure for $(\xi_{kT})_{k\in I\!N_0}$ according to $(H1)$, and we write for short $\widetilde{\mu}(f)$ for $\int f d\widetilde{\mu}$.

b) Let $\widetilde{W}$ in (5) be defined on some $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{P}_0)$. Then $(\widetilde{\Omega}, \widetilde{\mathcal{A}})$ carries a limit experiment

$$\text{(6)} \qquad \widetilde{\mathcal{E}} \;=\; \left\{ \widetilde{P}_u : u \in I\!R \right\} \;\text{ defined by }\; d\widetilde{P}_u := \widetilde{L}^{u/0} d\widetilde{P}_0 \;\text{ on }\; (\widetilde{\Omega}, \widetilde{\mathcal{A}})$$

such that we have convergence of experiments: local experiments at ϑ

$$\mathcal{E}_n^{(\vartheta)} \ := \ \left\{ Q^{\vartheta+\frac{u}{n}} \mid \mathcal{G}_{nT} \ : \ u \in \Theta_{\vartheta,n} \right\} \ , \quad n \geq 1$$

converge as $n \to \infty$ to the limit experiment $\mathcal{E}$ in the sense of Strasser ([S 85], p. 302).

We will prove this theorem, based on the results of sections 2 to 4, in 5.2 below.

1.2 Remark: We give an interpretation for the type of limit experiment in (6), putting $J_\vartheta = 1$ for short. Recall that two-sided standard Brownian motion $(\widetilde{W}_u)_{u \in I\!R}$ on $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{P}_0)$ means that two independent standard Brownian motions $(\widetilde{W}_v^+)_{v \geq 0}$ and $(\widetilde{W}_v^-)_{v \geq 0}$ exist on $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{P}_0)$ such that $\widetilde{W}_u$ is given by $\widetilde{W}_u^+$ if $u \geq 0$, and by $\widetilde{W}_{|u|}^-$ if $u \leq 0$. Define

$$(7) \qquad \widetilde{P}_u \ := \begin{cases} \mathcal{L}\left(\left(\widetilde{W}_v^+ + v \wedge u \ , \ \widetilde{W}_v^- \right)_{v \geq 0} \mid \widetilde{P}_0 \right) & \text{in case } \ u \geq 0 \\ \mathcal{L}\left(\left(\widetilde{W}_v^+ \ , \ \widetilde{W}_v^- + v \wedge |u| \right)_{v \geq 0} \mid \widetilde{P}_0 \right) & \text{in case } \ u \leq 0 \end{cases}$$

This means that constant drift 1 is added to $\widetilde{W}^+$ if $u > 0$ and to $\widetilde{W}^-$ if $u < 0$, and this drift is switched off at time $|u|$. Consider the likelihood ratio process of $\widetilde{P}_u$ to $\widetilde{P}_0$ relative to the filtration generated by the bivariate canonical process on $C([0,\infty), I\!R^2)$ (see proof of lemma 5.3 below, or use [JS 87], [LS 81]). If we are allowed to observe over the *infinite* time interval $[0,\infty)$, we end up with the likelihood ratios

$$\widetilde{L}^{u/0} \ = \ \exp\left\{ \ \widetilde{W}(u) - \frac{1}{2}|u| \ \right\}$$

given in (3). Up to the scaling factor J_ϑ, this is the situation of theorem 1.1. $\diamond$

A minor extension of theorem 1.1 – exactly analogous to [IH 81, p. 335] – is

1.3 Remark: If we replace $S(\vartheta, t) \ = \ \lambda(t) + \left(\lambda^* 1_{(\vartheta, \vartheta+a)} \right) (i_T(t))$ defined in (2) by

$$S(\vartheta, t) \ = \ \left(\lambda + \sum_{i=0}^{\ell} \lambda_{i+1}^* 1_{(\vartheta+a_i, \vartheta+a_{i+1})} \right) (i_T(t))$$

where $0 = a_0 < a_1 < \ldots < a_{\ell+1} = a$ are fixed and known, where λ_i^* are known continuous and strictly positive functions on $[0,T]$, for $1 \leq i \leq \ell$, and $\lambda_0^* \equiv 0 \equiv \lambda_{l+2}^*$, then theorem 1.1 remains valid with new definition

$$J_\vartheta \ := \ \left\{ \sum_{i=0}^{\ell+1} [(\lambda_{i+1}^* - \lambda_i^*)(\vartheta + a_i)]^2 \, (\mu^{(\vartheta)} P_{0,\vartheta+a_i}^{(\vartheta)}) \right\} (\frac{1}{\sigma^2}) \ .$$

of the scaling constants. This is a simple variant of the proof of theorem 1.1. $\diamond$

Based on theorem 1.1, we proceed using theorems 19–21 in appendix A I.4 of Ibragimov and Khasminskii [IH 81] (up to the uniform in ϑ part which can be omitted there both in the assertion and in the assumption of the theorem) and techniques developped in [IH 81, chapter I, theorems 5.1+5.2] to show

1.4 Theorem: Under Lipschitz and linear growth conditions on $b(\cdot)$ and $\sigma(\cdot)$, under $(H1)$ and $(H2)$, we have the following for every $\vartheta \in \Theta$:

a) For $K < \infty$ arbitrarily large, the likelihood ratios in local models $\mathcal{E}_n^{(\vartheta)}$ at ϑ

$$\left(L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta} \right)_{u\in[-K,K]} \quad \text{under } Q^\vartheta$$

converge weakly in $C([-K,K])$ as $n \to \infty$ to the likelihoods

$$\left(\widetilde{L}^{u/0} \right)_{u\in[-K,K]} \text{ under } \widetilde{P}_0$$

in the limit model $\widetilde{\mathcal{E}}$ of (5)+(6).

b) For arbitrary $p \in \mathbb{N}_0$ and $K_0 > 0$, there are constants $b_1(p,K_0)$ and b_2 such that

$$(8) \qquad Q^\vartheta \left(\sup_{|u|>K} |u|^p \, L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta} \geq 1 \right) \;\leq\; b_1(p,K_0)\, e^{-\,b_2\,K} \quad , \quad K_0 \leq K < \infty$$

holds for all $\vartheta \in \Theta$ and all $n \geq 1$, together with

$$(9) \qquad \widetilde{P}_0 \left(\sup_{|u|>K} |u|^p \, \widetilde{L}^{u/0} \geq 1 \right) \;\leq\; b_1(p,K_0)\, e^{-\,b_2\,K} \quad , \quad K_0 \leq K < \infty\,.$$

Here $b_1(p,K_0)$ does not depend on ϑ or n, and b_2 does not depend on ϑ, n, p, K_0.

c) For arbitrary $p \in \mathbb{N}_0$ and $K_0 > 0$, there are constants $\widetilde{b}_1(p,K_0)$ and $\widetilde{b}_2$ such that

$$(10) \qquad E_\vartheta \left(\int_{\{|u|>K\}} |u|^p \, \frac{L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta}}{\int_{\mathbb{R}} L_{nT}^{(\vartheta+\frac{u'}{n})/\vartheta} du'} \, du \right) \;\leq\; \widetilde{b}_1(p,K_0)\, e^{-\,\widetilde{b}_2\,K} \quad , \quad K_0 \leq K < \infty$$

holds for all $\vartheta \in \Theta$ and all $n \geq 1$, together with

$$(11) \qquad E_{\widetilde{P}_0} \left(\int_{\{|u|>K\}} |u|^p \, \frac{\widetilde{L}^{u/0}}{\int_{\mathbb{R}} \widetilde{L}^{u'/0} du'} \, du \right) \;\leq\; \widetilde{b}_1(p,K_0)\, e^{-\,\widetilde{b}_2\,K} \quad , \quad K_0 \leq K < \infty\,.$$

Again $\widetilde{b}_1(p,K_0)$ does not depend on ϑ or n, and $\widetilde{b}_2$ does not depend on ϑ, n, p, K_0.

The proof of theorem 1.4 will be given in 5.5 below. As consequences of theorem 1.4, we will obtain convergence of maximum likelihood (MLE) and Bayes estimators (BE) for the unknown parameter when a trajectory of ξ is observed up to time nT, $n \to \infty$. The MLE sequence is

$$(12) \qquad \widehat{\vartheta}_{nT} \;=\; \underset{\zeta \in \overline{\Theta}}{\operatorname{argmax}} \; L_{nT}^{\zeta/\zeta_0} \;:=\; \min \left\{ \zeta \in \overline{\Theta} \;:\; L_{nT}^{\zeta/\zeta_0} = \max_{\zeta' \in \overline{\Theta}} L_{nT}^{\zeta'/\zeta_0} \right\} \quad , \quad n \in I\!N$$

with $\zeta_0 \in \Theta$ some fixed point, and $\overline{\Theta}$ the closure of Θ. Presence of 'min' in (12) guarantees for a measurable selection whenever the argmax is not unique. The MLE in the limit experiment

$$(13) \qquad \widehat{u} \;=\; \underset{u \in I\!R}{\operatorname{argmax}} \; \widetilde{L}^{u/0}$$

is finite–valued and uniquely determined almost surely, since (9) of theorem 1.4 controls the decrease of $u \to \widetilde{L}^{u/0}$ as u tends to $+\infty$ or $-\infty$, and by [IH 81, chapter VII, lemma 2.5]. The control (9) also guarantees that a BE 'with uniform prior on the entire real line'

$$(14) \qquad u^* \;:=\; \frac{\int_{-\infty}^{\infty} u \, \widetilde{L}^{u/0} \, du}{\int_{-\infty}^{\infty} \widetilde{L}^{u/0} \, du}$$

(sometimes called Pitman estimator) is well defined in the limit experiment. Correspondingly, we consider the BE sequence with uniform prior on $\Theta = (0, T{-}a)$

$$(15) \qquad \vartheta^*_{nT} \;:=\; \frac{\int_\Theta \zeta \; L_{nT}^{\zeta/\zeta_0} \; d\zeta}{\int_\Theta L_{nT}^{\zeta/\zeta_0} \; d\zeta} \quad , \quad n \in I\!N$$

for the unknown parameter $\vartheta \in \Theta$, based on observation of ξ up to time nT (in (15), we might as well use smooth and strictly positive prior densities on Θ, but this generalization turns out to be without interest in view of theorems 1.7 and 1.8 below).

1.5 Theorem: Under Lipschitz and linear growth conditions on $b(\cdot)$ and $\sigma(\cdot)$, under $(H1)$ and $(H2)$, we have the following properties of the MLE and the BE sequence, for every $\vartheta \in \Theta$:

a) weak convergence as $n \to \infty$:

$$\mathcal{L}\left(n \, (\widehat{\vartheta}_{nT} - \vartheta) \mid Q^\vartheta \right) \;\longrightarrow\; \mathcal{L}\left(\widehat{u} \mid \widetilde{P}_0 \right) \,,$$

$$\mathcal{L}\left(n \, (\vartheta^*_{nT} - \vartheta) \mid Q^\vartheta \right) \;\longrightarrow\; \mathcal{L}\left(u^* \mid \widetilde{P}_0 \right) \;;$$

b) finite moments of arbitrary order $p \in I\!N$, and convergence of moments as $n \to \infty$:

$$E_\vartheta \left(\left[n \, (\widehat{\vartheta}_{nT} - \vartheta) \right]^p \right) \;\longrightarrow\; E_{\widetilde{P}_0} \left(|\widehat{u}|^p \right) \,,$$

$$E_\vartheta \left(\left[n \, (\vartheta^*_{nT} - \vartheta) \right]^p \right) \;\longrightarrow\; E_{\widetilde{P}_0} \left(|u^*|^p \right) \,.$$

Theorem 1.5 will be proved in 5.6 and 5.7 below. Note that part a) of the theorem would be enough for convergence of risks with respect to loss functions which are continuous, subconvex and bounded, i.e. 'bowl-shaped' in the sense of LeCam and Yang [LY 90, p. 82], whereas part b) is useful e.g. for quadratic loss to be considered below.

1.6 Remark: For the 'signal in white noise' setting of remark 1.2 where J_ϑ equals 1, the following is known. Ibragimov and Khasminskii [IH 81, p. 342] calculated the variance of the MLE, whereas Rubin and Song [RS 95] calculated the variance of the BE: the ratio of MLE to BE variance is 26 to $16 \cdot \zeta(3) \approx 19.3$ where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is Riemann's zeta function. $\diamond$

In our setting with scaling factor J_ϑ in the limit experiment (5)+(6), the results quoted in remark 1.6 yield the following: the limit law for rescaled MLE errors has finite variance

$$J_\vartheta^{-2} \cdot 26 \tag{16}$$

at ϑ, different from the limit law for rescaled BE errors whose variance is

$$J_\vartheta^{-2} \cdot 16 \cdot \zeta(3) \;\approx\; J_\vartheta^{-2} \cdot 19.3\,. \tag{17}$$

Moreover, by definition of a Bayes estimator with respect to quadratic loss – this is the L^2-projection property of conditional expectations – we have at every finite stage n optimality of BE in the sense of integrated risk under quadratic loss functions: comparing all possible estimators $\widetilde{\vartheta}_{nT}$ based on observation of the trajectory of ξ up to time nT, a minimum

$$\min \left\{ \int_\Theta E_\vartheta \left(\left[n(\widetilde{\vartheta}_{nT} - \vartheta) \right]^2 \right) d\vartheta \;\mid\; \widetilde{\vartheta}_{nT} \text{ is } \mathcal{G}^\xi_{nT}\text{–measurable} \right\} \tag{18}$$

exists and is realized by the integrated risk of the BE (15) with respect to quadratic loss

$$\int_\Theta E_\vartheta \left([n(\vartheta^*_{nT} - \vartheta)]^2 \right) d\vartheta \,, \tag{19}$$

for every fixed n. This is an elementary and pre-asymptotic argument, averaging over the whole parameter space Θ, and does not contain much information on the behaviour of our estimator in small neighbourhoods of a parameter value ϑ. The following two results deal with shrinking neighbourhoods of fixed points in the parameter space, in the sense of contiguous alternatives, and fill this gap.

1.7 Proposition: Under Lipschitz and linear growth conditions on $b(\cdot)$ and $\sigma(\cdot)$, under $(H1)$ and $(H2)$, the following holds for every $\vartheta \in \Theta$.

a) The BE u^* in the limit experiment $\widetilde{\mathcal{E}}$ with 'uniform prior over $I\!R$' (14) is equivariant:

$$\text{for every } u \in I\!R \ : \quad \mathcal{L}\left(u^* - u \mid \widetilde{P}_u\right) \ = \ \mathcal{L}\left(u^* \mid \widetilde{P}_0\right) \ .$$

b) The sequence of BE ϑ^*_{nT} at stage n defined by (15) is asymptotically as $n \to \infty$ equivariant in the local models $\mathcal{E}_n^{(\vartheta)}$ at ϑ, in the following sense:

$$\lim_{n\to\infty} \sup_{|u|\le C} \left| E_{\vartheta+\frac{u}{n}}\left(\ell\left(n(\vartheta^*_{nT} - (\vartheta + \frac{u}{n}))\right)\right) \ - \ E_{\widetilde{P}_0}\left(\ell\left(u^*\right)\right) \right| \ = \ 0$$

for every continuous and subconvex loss function ℓ which admits a polynomial majorant, and for arbitrary choice of $C < \infty$.

Proposition 1.7 will be proved in 5.8 below. Now, for quadratic loss, we consider maximal risks over small neighbourhoods of arbitrary points $\vartheta \in \Theta$, in the sense of contiguous alternatives, and state a local asymptotic minimax theorem which allows to compare arbitrary sequences of $\mathcal{G}^\xi_{nT}$–measurable estimators $\widetilde{\vartheta}_{nT}$, $n \ge 1$, for the unknown parameter $\vartheta \in \Theta$.

1.8 Theorem: Under Lipschitz and linear growth conditions on $b(\cdot)$ and $\sigma(\cdot)$, under $(H1)$ and $(H2)$, the following holds for every $\vartheta \in \Theta$.

a) For squared loss, there is a local asymptotic minimax bound in terms of the BE u^* of (14)

$$\lim_{C\uparrow\infty} \liminf_{n\to\infty} \inf_{\widetilde{\vartheta}_{nT}} \sup_{|u|\le C} E_{\vartheta+\frac{u}{n}}\left(\left[n\left(\widetilde{\vartheta}_{nT} - (\vartheta + \frac{u}{n})\right)\right]^2\right) \quad \ge \quad E_{\widetilde{P}_0}\left([u^*]^2\right)$$

where at every stage $n \ge 1$, $\inf_{\widetilde{\vartheta}_{nT}}$ is with respect to all possible $\mathcal{G}^\xi_{nT}$–measurable estimators.

b) The BE sequence $(\vartheta^*_{nT})_n$ of (15) attains the bound given in a), in virtue of 1.7 b).

The proof will be given in 5.9 below. Note that our model thus exhibits one striking contrast with respect to the well studied LAN or LAMN situation. Under LAN or LAMN, simultaneously for a large class of loss functions, the same estimator (the MLE in the limit experiment, since in the Gaussian or mixed Gaussian limit experiment MLE signifies that estimation errors are equal to the central statistic) appears on the right hand side of the local asymptotic minimax bound. Here, we have made the particular choice of quadratic loss function, defined Bayes estimators correspondingly, and obtain in theorem 1.8 that this choice minimizes quadratic risk in the class of all possible estimator sequences over shrinking neighbourhoods of ϑ, in a local asymptotic sense. Different choice of a loss function, e.g. $\ell(x) = |x|^p$ with $p > 2$, would lead to different Bayes estimators, both at stage n and in the limit experiment. The definition [IH 81, theorem I.10.2] of BE with respect to loss functions $\ell(x) = |x|^p$ shows that in this case we would obtain an asymptotic equivariance property analogous to proposition 1.7 a). This would lead us to a local asymptotic minimax bound analogous to 1.8, stating that a BE sequence defined relative to $\ell(\cdot)$ is locally asymptotically minimax in the class of all estimator sequences provided we have decided to measure loss thoughout in terms of the loss function $\ell(\cdot)$. This means that our model does not allow to identify intrinsically optimal estimator sequences (as this is the case under LAN or LAMN where estimator sequences whose rescaled estimation errors are stochastically equivalent to the central sequence are locally asymptotically minimax for the class of subconvex, bounded, continuous loss functions, and where Le Cam's 'one step correction' allows to construct such estimators explicitely). In our model – this seems to be an important open question – we would probably end up with different Bayesians which do not compare among each other.

2 Ergodic properties for diffusions with T–periodic drift

In this section, we discuss ergodic properties for a diffusion (1)

$$d\xi_t \;=\; [S(t,\vartheta) + b(\xi_t)]\,dt \;+\; \sigma(\xi_t)\,dW_t$$

for fixed value of the parameter ϑ which we suppress from notation. Generalizing (2), we replace $S(t,\vartheta)$ by a deterministic time-dependent contribution $t \to S(t)$ to the drift such that

(20) $$S : [0,\infty) \to I\!R \quad \text{is } T\text{-periodic and piecewise continuous .}$$

We use the same notation $i_T(t) = t$ modulo T as in (2); the periodicity T is known. We need strong laws of large numbers as $t \to \infty$ for some class of functionals of the process $\xi = (\xi_t)_{t\geq 0}$. This class will be defined in (27) below; relevant examples are of the following type. Let $\Lambda_T(ds)$ denote some σ–finite measure on $[0,\infty)$ which is T–periodic in the sense that

(21) $$\Lambda_T(B) \;=\; \Lambda_T(B+kT) \quad \text{for all } B \in \mathcal{B}([0,\infty)) \text{ and all } k \in I\!N_0 \;.$$

For T–periodic measures $\Lambda_T(ds)$ and for suitable functions $f : I\!R \to I\!R$, define

(22) $$A = (A_t)_{t\geq 0} \quad , \quad A_t \;=\; \int_0^t f(\xi_s)\,\Lambda_T(ds) \;,\; t \geq 0 \;.$$

Functionals of this type allow to deal with the periodicity in the process (1)+(20). With fixed $0 < r < r' < T$, in view of the log-likelihoods in theorem 1.1, we may thus consider

$$(23) \qquad A_t \;=\; \sum_{k \in I\!N_0\,,\, kT+r \le t} f(\xi_{kT+r}) \quad , \quad \Lambda_T(ds) \;=\; \sum_{k \in I\!N_0} \epsilon_{(kT+r)}(ds)$$

(with ε_x Dirac measure at the point x) or

$$(24) \qquad A_t \;=\; \int_0^t f(\xi_s)\, 1_{(r,r')}(i_T(s))\, ds \quad , \quad \Lambda_T(ds) \;=\; \sum_{k \in I\!N_0} 1_{(kT+r, kT+r')}(i_T(s))\, ds$$

which both obviously are not additive functionals of the continuous-time process $(\xi_t)_{t \ge 0}$.

We fix notations and assumptions for this section in the following setting I)–III). Under Lipschitz conditions on $b(\cdot)$ and $\sigma(\cdot)$, for $t \to S(t)$ T-periodic and piecewise continuous, the process ξ of (1)+(20) satisfies all assumptions made in I)–III) below:

I) The process ξ is strongly Markov, with continuous paths, taking values in $(I\!R, \mathcal{B}(I\!R))$. The process is non-time homogeneous with semigroup $(P_{s,t})_{0 \le s < t < \infty}$, and measurable densities

$$P_{s,t}(x, dy) \;=\; p_{s,t}(x,y)\, dy \quad , \quad 0 \le s < t < \infty \;,\; x, y \in I\!R$$

exist with respect to Lebesgue measure. Moreover, the semigroup is T-periodic:

$$(25) \qquad p_{s,t}(x,y) \;=\; p_{kT+s, kT+t}(x,y) \quad \text{for all } k \in I\!N_0 \text{ and all } 0 \le s < t < \infty\,.$$

II) We take ξ as defined on some canonical path space $(\Omega, \mathcal{A}, I\!F, (P_x)_{x \in I\!R})$. 'Almost surely' means almost surely with respect to every P_x, $x \in I\!R$. An $I\!F^\xi$–increasing process is an $I\!F^\xi$–adapted càdlàg process $A = (A_t)_{t \ge 0}$ with nondecreasing paths and $A_0 = 0$, almost surely.

III) We write $(C_T, \mathcal{C}_T)$ for the space of all continuous functions $\alpha : [0,T] \to I\!R$ equipped with the metric of uniform convergence and its Borel σ-field $\mathcal{C}_T$. Then $(C_T, \mathcal{C}_T)$ is a Polish space, and $\mathcal{C}_T = \sigma(\pi_t : 0 \le t \le T)$, the σ–field generated by the coordinate projections π_t, $0 \le t \le T$.

The continuous-time Markov process $\xi = (\xi_t)_{t \ge 0}$ induces a $(C_T, \mathcal{C}_T)$-valued Markov chain

$$X = (X_k)_{k \in I\!N_0} \quad , \quad X_k := (\xi_{(k-1)T+s})_{0 \le s \le T} \;,\; k \ge 1 \;,\; X_0 = \alpha_0 \in C_T$$

which we call the chain of T-segments in the path of ξ. In virtue of (25), the chain $X = (X_k)_{k \in I\!N_0}$ is time homogeneous with one-step-transition kernel $Q(\cdot,\cdot)$ given by

$$Q(\alpha, F) \;:=\; P\left((\xi_s)_{0 \le s \le T} \in F \mid \xi_0 = \alpha(T) \right) \;, \quad \alpha \in C_T \;,\; F \in \mathcal{C}_T\,.$$

For Harris processes in discrete time, we refer to Revuz [R 75] or Meyn and Tweedie [MT 93]. For Harris processes in continuous time see Azéma, Duflo and Revuz [ADR 69] or Révuz and Yor [RY 91, Ch. 10.3]. Now we can formulate ergodicity properties for a non-time homogeneous diffusion $(\xi_t)_{t\ge 0}$ of type (1)+(20) with T-periodic drift:

2.1 Theorem: Under I)—III), assume that the embedded chain $(\xi_{kT})_{k\in\mathbb{N}_0}$ is positive recurrent in the sense of Harris, and write μ for its invariant probability on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

a) Then the chain $X = (X_k)_{k\in\mathbb{N}_0}$ of T-segments in the path of ξ is positive recurrent in the sense of Harris. Its invariant probability is the unique law m on $(C_T, \mathcal{C}_T)$ such that

$$(26)\qquad \begin{cases} \text{for arbitrary } 0 = t_0 < t_1 < \ldots < t_\ell < t_{\ell+1} = T \text{ and } A_i \in \mathcal{B}(\mathbb{R}) \;, \\ m\left(\{\pi_{t_i} \in A_i \,,\, 0 \le i \le \ell{+}1\}\right) \quad \text{is given by} \\ \int \ldots \int \mu(dx_0)\, 1_{A_0}(x_0) \prod_{i=0}^{\ell} P_{t_i,t_{i+1}}(x_i, dx_{i+1})\, 1_{A_{i+1}}(x_{i+1}) \;. \end{cases}$$

b) For every $\mathbb{F}^\xi$–increasing process $A = (A_t)_{t\ge 0}$ with the property

$$(27)\qquad \begin{cases} \text{there is some function } F : C_T \to \mathbb{R}, \text{ nonnegative, } \mathcal{C}_T\text{-measurable,} \\ \text{satisfying} \quad m(F) \;:=\; \int_{C_T} F\, dm \;<\; \infty \;, \quad \text{such that} \\ A_{kT} \;=\; \sum\limits_{j=1}^{k} F(X_k) \;=\; \sum\limits_{j=1}^{k} F\left((\xi_{(k-1)T+s})_{0\le s\le T} \right) \;,\; k \ge 1 \end{cases}$$

we have the strong law of large numbers

$$\lim_{t\to\infty} \frac{1}{t} A_t \;=\; \frac{1}{T}\, m(F) \quad \text{almost surely} \;.$$

Proof: 1) Harris recurrence of the process $(\xi_{kT})_{k\in\mathbb{N}_0}$ with invariant probability μ yields

$$A \in \mathcal{B}(\mathbb{R}) \;,\; \mu(A) > 0 \quad \text{implies} \quad \sum_k 1_A(\xi_{kT}) \;=\; \infty \quad \text{almost surely} \;,$$

and thus implies that the bivariate chain

$$\left(\xi_{kT} \;,\; \xi_{(k+1)T} \right)_{k\in\mathbb{N}_0}$$

is positive recurrent in the sense of Harris with invariant probability

$$\mu^{(2)}(dx, dy) \;:=\; \mu(dx) P_{0,T}(x, dy) \quad \text{on } \mathbb{R}{\times}\mathbb{R} \;.$$

2) Write m for the unique law on $(C_T, \mathcal{C}_T)$ whose finite dimensional distributions are given by (26). Since $(C_T, \mathcal{C}_T)$ is Polish, conditioning with respect to the pair of coordinate projections

(π_0, π_T), the probability m allows for a decomposition

$$m(F) \;=\; \int_{I\!R\times I\!R} \mu^{(2)}(dx,dy)\, K((x,y),F) \tag{28}$$

with $K(\cdot,\cdot)$ some transition probability from $I\!R\times I\!R$ to $\mathcal{C}_T$. Comparing with (26), $K((x,y),\cdot)$ is the law of the ξ–bridge from x at time 0 to y at time T

$$K((x,y),F) \;:=\; P\,(\,(\xi_s)_{0\le s\le T} \in F \mid \xi_0 = x\,,\, \xi_T = y\,)\;; \tag{29}$$

for $\mu^{(2)}$–almost all (x,y), with notations of (26), $K\,((x_0,x_{\ell+1}), \{\pi_{t_i}\in A_i\,,\, 0\le i\le \ell{+}1\}))$ equals

$$\frac{1}{p_{0,T}(x_0,x_{\ell+1})}\, 1_{A_0}(x_0) \int \ldots \int dx_1 \ldots dx_\ell \prod_{i=0}^{\ell} p_{t_i,t_{i+1}}(x_i,x_{i+1})\, 1_{A_{i+1}}(x_{i+1})$$

whenever $(x_0,x_{\ell+1})$ is in $\{p_{0,T}(\cdot,\cdot)>0\}$, with suitable default definition else.

3) From (28) we have for sets $F\in\mathcal{C}_T$

$$m(F)>0 \quad\Longrightarrow\quad \begin{cases} \text{there is some } \varepsilon>0 \text{ such that} \\ \{(x,y): K((x,y),F)>\varepsilon\} \;=:\; B(F,\varepsilon) \\ \text{has strictly positive measure under } \mu^{(2)}(dx,dy) \end{cases} \tag{30}$$

Hence the Harris property of the bivariate chain $\left(\xi_{kT}, \xi_{(k+1)T}\right)_{k\in I\!N_0}$ with invariant measure $\mu^{(2)}(dx,dy)$ gives in combination with (30) first

$$F\in\mathcal{C}_T\;,\; m(F)>0 \quad\Longrightarrow\quad \sum_k 1_{B(F,\varepsilon)}(\xi_{kT},\xi_{(k+1)T}) \;=\; \infty \quad \text{almost surely}$$

and then thanks to (28)

$$F\in\mathcal{C}_T\;,\; m(F)>0 \quad\Longrightarrow\quad \sum_k 1_F(X_k) \;=\; \infty \quad \text{almost surely}\,.$$

We have identified some probability measure m on $(C_t,\mathcal{C}_T)$ such that sets of positive m–measure are visited infinitely often by $X=(X_n)_n$: hence X is Harris. Every Harris chain admits a unique invariant measure. Periodicity (25) of the semigroup guarantees that m defined in (26) is invariant for X. We have proved that $X=(X_n)_n$ is positive recurrent in the sense of Harris with invariant measure m given by (26): this is part a) of the theorem.

5) Consider an $I\!F^\xi$–increasing process $A=(A_t)_{t\ge 0}$ related to a function $F: C_T\to[0,\infty)$ as in (27). Then $\Psi=(\Psi_k)_{k\in I\!N_0}$ defined by

$$\Psi_k \;:=\; A_{kT}\;,\; k\in I\!N_0$$

is an integrable additive functional of the chain $X = (X_k)_k$ of T–segments in ξ. Since X is Harris with invariant measure m, we have the ratio limit theorem

$$\lim_{k\to\infty} \frac{1}{k}\,\Psi_k \quad = \quad E_m\,(\Psi_1) \quad \text{almost surely .} \tag{31}$$

But $E_m(\Psi_1) = E_m(F(X_1)) = m(F)$, and

$$\lim_{k\to\infty} \frac{1}{k}\Psi_k \quad = \quad T \lim_{t\to\infty} \frac{1}{t} A_t$$

since A is increasing. This proves part b) of the theorem. $\diamond$

Now we come back to the class of examples (21)–(24) related to the likelihoods in our model:

2.2 Examples: Under all assumptions of theorem 2.1, we deduce in particular the following laws of large numbers for the functionals (22)–(24). For $0 < r < r' < T$ fixed,

a) if $f : I\!\!R \to I\!\!R$ is measurable and in $L^1(\,\mu P_{0,r}\,)$, then for $A = (A_t)_t$ considered in (23)

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k\in I\!\!N_0\,,\,kT+r\le t} f(\xi_{kT+r}) \quad = \quad \frac{1}{T}\,(\mu P_{0,r})(f) \quad \text{almost surely ;}$$

b) if $f : I\!\!R \to I\!\!R$ is measurable and in $L^1\left(\int_r^{r'} ds\,(\mu P_{0,s})\right)$, then for the functional in (24)

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(\xi_s)\, 1_{(r,r')}(i_T(s))\, ds \quad = \quad \frac{1}{T} \int_r^{r'} (\mu P_{0,s})(f)\, ds\ . \quad \text{almost surely}$$

c) if $f : I\!\!R \to I\!\!R$ is measurable and in $L^1\left(\int_0^T \Lambda_T(ds)\,(\mu P_{0,s})\right)$ for some T–periodic measure Λ_T as defined in (21), then we have for the functional $A = (A_t)_t$ in (22)

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(\xi_s)\, \Lambda_T(ds) \quad = \quad \frac{1}{T} \int_0^T \Lambda_T(ds)\,(\mu P_{0,s})(f) \quad \text{almost surely .}$$

This follows from theorem 2.1: put $F(\alpha) = f(\alpha(r))$ for part a), $F(\alpha) = \int_r^{r'} f(\alpha(s))ds$ for part b), and $F(\alpha) = \int_0^T f(\alpha(s))\Lambda(ds)$ for part c), with $\alpha \in C_T$. $\diamond$

Calculating explicitely the measures occurring in theorem 2.1 or example 2.2 requires to know the semigroup $(P_{s,t})_{0\le s<t<\infty}$ of the non-time homogeneous diffusion (1)+(2) with T–periodic drift. Sometimes this is possible; we mention one example.

2.3 Example: With $\sigma > 0$, $\gamma > 0$, and $S(\cdot)$ T-periodic and piecewise continuous, consider an Ornstein-Uhlenbeck type diffusion with T–periodic drift

$$d\xi_t \;=\; (S(t) - \gamma\,\xi_t)\,dt \;+\; \sigma\,dW_t\ , \quad t \ge 0\ .$$

The solution with initial value ξ_0 is

$$\xi_t \;=\; \xi_0\, e^{-\gamma\, t} \;+\; \int_0^t e^{-\gamma\,(t-s)}\, (\, S(s)\, ds + \sigma\, dW_s)$$

and the transition semigroup $(P_{s,t})_{0\le s<t<\infty}$ of ξ is

$$P_{s,t}(x,\cdot) \;=\; \mathcal{N}\left(x\, e^{-\gamma(t-s)} + \int_0^{t-s} e^{-\gamma v} S(t-v)\, dv \;,\; e^{-2\gamma(t-s)}\, \frac{e^{2\gamma(t-s)}-1}{2\gamma}\, \sigma^2 \right) \;.$$

Extending $S(\cdot)$ to a T–periodic function defined on the whole real axis, we can write

$$(32) \qquad P_{0,kT}(x,\cdot) \;\longrightarrow\; \mathcal{N}\left(\int_0^\infty e^{-\gamma v} S(-v)\, dv \;,\; \frac{\sigma^2}{2\gamma} \right) \;,\quad k\to\infty$$

with arbitrary $x \in I\!\!R$, and similiarly for fixed $0 < s < T$

$$P_{s,kT+s}(x,\cdot) \;\longrightarrow\; \mathcal{N}\left(\int_0^\infty e^{-\gamma v} S(s-v)\, dv \;,\; \frac{\sigma^2}{2\gamma} \right) \;,\quad k\to\infty\;.$$

It is easy to see that the measure on the right hand side of (32) is invariant for the chain $(\xi_{kT})_{k\in I\!\!N_0}$, and that sets of positive Lebesgue measure are visited infinitely often by $(\xi_{kT})_{k\in I\!\!N_0}$, for every choice of a starting point. Hence $(\xi_{kT})_{k\in I\!\!N_0}$ is positive Harris with invariant probability

$$\mu \;:=\; \mathcal{N}\left(\int_0^\infty e^{-\gamma v} S(-v)\, dv \;,\; \frac{\sigma^2}{2\gamma} \right) \;,$$

so all conditions of theorem 2.1 are satisfied. For $0 < s < T$, the measures occurring in 2.2 are

$$\mu\, P_{0,s} \;=\; \mathcal{N}\left(\int_0^\infty e^{-\gamma v} S(s-v)\, dv \;,\; \frac{\sigma^2}{2\gamma} \right) \;,\quad 0<s<T\;.$$

Hence in this example, all measures occcuring in 2.1 or 2.2 are known explicitely. Note that Laplace transforms of the periodic functions $v \to S(-v)$ or $v \to S(s-v)$ appear as expectations in the above limit laws, for $0 < s < T$. $\diamond$

3 An exponential inequality

In this section, we give an exponential inequality for a process $(\xi_t)_{t\ge 0}$ of type (1)+(20). Up to a minor modification which allows for time dependence in the drift, the result is due to Brandt ([B 05], Lemma 2.2.4) who extended the classical Bernstein inequality for local martingales (see formula (1.5) in Dzhaparidze and van Zanten [DvZ 01)]) to solutions of SDE. Periodicity will play no role in this section. We consider a time dependent diffusion

$$(33) \qquad d\xi_t \;=\; b(t,\xi_t)\, dt \;+\; \sigma(\xi_t)\, dW_t \;,\quad t\ge 0$$

where $(t,x) \to b(t,x)$ is continuous in restriction to segments $[d_n, d_{n+1}[\times I\!R$ for some given deterministic sequence $(d_n)_n$ with $d_0 = 0$ and $d_n \uparrow \infty$, under the assumptions

$$|b(t,x) - b(t,x')| + |\sigma(x) - \sigma(x')| \;\le\; L\,|x - x'| \quad \text{for all } x, x' \in I\!R \text{ and all } t \ge 0\ , \tag{34}$$

$$|b(t,x)| \;\le\; L\,(1 + |x|) \quad \text{for all } x \in I\!R \text{ and all } t \ge 0\ , \tag{35}$$

$$|\sigma(x)| \;\le\; M \quad \text{for all } x \in I\!R\ . \tag{36}$$

Then a strong solution exists for (33) – we construct it successively on $[d_n, d_{n+1}]$ as in Karatzas and Shreve [KS 91, theorems 5.2.9+5.2.13], taking for $n \ge 1$ the terminal value of the preceding step as starting value for the following one – and is pathwise unique. Clearly this setting includes as a special case equation (1)+(20) if we put $b(t,x) := S(t) + b(x)$, under Lipschitz conditions on $b(\cdot)$ and $\sigma(\cdot)$, if $\sigma(\cdot)$ is bounded.

3.1 Lemma: (Brandt [B 05]) Fix $0 < \lambda < \frac{1}{2}$ and $\frac{1}{2} < \eta < 1-\lambda$. For $(\xi_t)_{t\ge 0}$ of (33)–(36) there is some $\Delta_0 > 0$ (depending only on L of (34)+(35), on λ and on η) such that

$$P\left(\sup_{t_1 \le t \le t_1+\Delta} |\xi_t - \xi_{t_1}| > \Delta^\lambda\ ,\ |\xi_{t_1}| \le \left(\frac{1}{\Delta}\right)^\eta \right) \quad \le \quad c_1 \cdot \exp\left\{ -\, c_2 \left(\frac{1}{\Delta}\right)^{1-2\lambda} \right\} \tag{37}$$

holds for all $0 \le t_1 < \infty$ and all $0 < \Delta < \Delta_0$, with positive constants c_1 and c_2 which do not depend on $t_1 \ge 0$ or on $\Delta \in (0, \Delta_0)$.

Proof: The proof is from [B 05]. We start for $0 \le t_1 < t < t_1 + \Delta$ from

$$|\xi_t - \xi_{t_1}| \;\le\; |\int_{t_1}^t \sigma(\xi_s)\, dW_s| \;+\; \int_{t_1}^t |b(s,\xi_s)|\, ds\ .$$

Applying to $|b(s,\xi_s)| \le |b(s,\xi_{t_1})| + |b(s,\xi_s) - b(s,\xi_{t_1})|$ the conditions (34)+(35), this gives

$$|\xi_t - \xi_{t_1}| \;\le\; \left\{ \sup_{t_1 \le t \le t_1+\Delta} |\int_{t_1}^t \sigma(\xi_s)\, dW_s| \;+\; L(1 + |\xi_{t_1}|)\Delta \right\} \;+\; L \int_{t_1}^t |\xi_s - \xi_{t_1}|\, ds\ .$$

With Gronwall pathwise in ω we obtain (e.g. Bass [B 98, lemma I.3.3])

$$|\xi_t - \xi_{t_1}| \;\le\; \left\{ \sup_{t_1 \le t \le t_1+\Delta} |\int_{t_1}^t \sigma(\xi_s)\, dW_s| \;+\; L(1 + |\xi_{t_1}|)\Delta \right\} \cdot e^{Lt}$$

for $0 \le t_1 < t < t_1 + \Delta$. Hence

$$\begin{aligned} & P\left(\sup_{t_1 \le t \le t_1+\Delta} |\xi_t - \xi_{t_1}| > \Delta^\lambda\ ,\ |\xi_{t_1}| \le \left(\frac{1}{\Delta}\right)^\eta \right) \\ \le\ & P\left(\sup_{t_1 \le t \le t_1+\Delta} |\int_{t_1}^t \sigma(\xi_s)\, dW_s| \;>\; \left[\Delta^\lambda e^{-L\Delta} - L(1 + |\xi_{t_1}|)\Delta\right]^+\ ,\ |\xi_{t_1}| \le \Delta^{-\eta} \right) \\ \le\ & P\left(\sup_{t_1 \le t \le t_1+\Delta} |\int_{t_1}^t \sigma(\xi_s)\, dW_s| \;>\; \left[\Delta^\lambda e^{-L\Delta} - L\Delta - L\Delta^{1-\eta}\right]^+ \right) \end{aligned}$$

Exploiting assumption (36), the classical Bernstein inequality for continuous local martingales (formula (1.5) in [DvZ 01)]) gives

$$P\left(\sup_{t_1 \le t \le t_1+\Delta} |\int_{t_1}^{t} \sigma(\xi_s)\, dW_s| \ > \ z \right) \ \le \ c_1\, e^{-\tilde{c}_2\, z^2/\Delta} \ , \quad z > 0$$

with positive constants c_1 and $\tilde{c}_2$ which do not depend on t_1 or on Δ. We have also

$$z \ := \ \left[\Delta^\lambda e^{-L\Delta} - L\Delta - L\Delta^{1-\eta}\right]^+ \ = \ \left[\Delta^\lambda (e^{-L\Delta} - L\Delta^{1-\eta-\lambda} - L\Delta^{1-\lambda})\right]^+ \ \ge \ \ \frac{1}{2}\, \Delta^\lambda$$

provided Δ is sufficiently small, since $1 - \eta - \lambda > 0$ by assumption. The assertion follows. $\diamond$

4 Limit theorems related to log-likelihoods in local models

In this section, we prove the limit theorems (4.1 and 4.3 below) which allow to work with log-likelihoods in local models at ϑ. Again we keep ϑ fixed and suppressed from notation. As in the preceding sections 2–3, we consider the diffusion $\xi = (\xi_t)_{t \ge 0}$ of (1)+(20) with T-periodic contribution to the drift. We assume (34)+(35) and strengthen (36) to

$$\frac{1}{M} \ \le \ |\sigma(x)| \ \le \ M \quad \text{for all } x \in I\!R \tag{38}$$

for some large M. Systematically, we combine appropriate control of fluctuations in the process $(\xi_t)_t$, first with strong laws of large numbers for additive functionals of the chain of T-segments in ξ, second with sequences of auxiliary martingales to which we can apply a martingale limit theorem (cf. Jacod and Shiryaev [JS 87, VIII.3.22]). A combination of lemma 3.1 and theorem 1.1 will thus allow to investigate the asymptotics of the log-likelihood ratios in local models at ϑ.

4.1 Theorem: For ξ as above, assume that the embedded chain $(\xi_{kT})_k$ is positive Harris recurrent with invariant probability μ. Then we have for $0 < r < T$ fixed and arbitrary $h > 0$

$$\int_0^{nT} \frac{1}{\sigma^2(\xi_s)}\, 1_{(r, r+\frac{h}{n})}(i_T(s))\, ds \ \ = \ \ h\, \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\sigma^2(\xi_{jT+r})} \ + \ o_{P_x}(1) \tag{39}$$

$$\int_0^{nT} \frac{1}{\sigma^2(\xi_s)}\, 1_{(r-\frac{h}{n}, r)}(i_T(s))\, ds \ \ = \ \ h\, \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\sigma^2(\xi_{jT+r})} \ + \ o_{P_x}(1) \tag{40}$$

as $n \to \infty$, for all $x \in I\!R$. The leading term on the right hand side converges to

$$h\ (\mu P_{0,r})(\frac{1}{\sigma^2})$$

almost surely as $n \to \infty$, by theorem 1.1 applied to (23), or by 2.2 a).

Proof: The proof of the approximations (39)+(40) is in several steps. We take n large enough to have $r - \frac{h}{n}, r + \frac{h}{n}$ in $(0,T)$, and fix $0 < \lambda < \frac{1}{2}$ and $\frac{1}{2} < \eta < 1{-}\lambda$ as in lemma 3.1.

1) With arbitrary constants K, the following auxiliary result will be needed frequently:

$$(41) \qquad \frac{1}{n}\sum_{j=0}^{n-1} 1_{\{\sup\limits_{0\le s\le T} |\xi_{jT+s}| > K\, n^\eta\}} \;\longrightarrow\; 0 \quad \text{almost surely as } n \to \infty\,.$$

This is easily derived from the strong law of large numbers in the chain $X = (X_k)_{k\in\mathbb{N}_0}$ of T-segments in the process ξ: for any fixed $c < \infty$, theorem 1.1 b) with $F(\alpha) = \sup\limits_{0\le s\le T} |\alpha(s)|$ (which is a continuous function on C_T) gives

$$\frac{1}{n}\sum_{j=0}^{n-1} 1_{\{\sup\limits_{0\le s\le T} |\xi_{jT+s}| > c\}} \;=\; \frac{1}{n}\sum_{k=0}^{n-1} 1_{\{F(X_k)>c\}} \;\longrightarrow\; m(\{\alpha\in C_T : \sup_{0\le s\le T}|\alpha(s)| > c\})$$

almost surely as $n \to \infty$. Since the limit on the right hand side decreases to 0 as $c \uparrow \infty$ and since n^η exceeds any fixed level as n tends to ∞, (41) is proved.

2) Next we prove for arbitrary $h > 0$ and arbitrary starting point x the approximation

$$(42) \qquad \frac{1}{n}\sum_{j=0}^{n-1} \frac{1}{\sigma^2(\xi_{jT+r-\frac{h}{n}})} \;=\; \frac{1}{n}\sum_{j=0}^{n-1} \frac{1}{\sigma^2(\xi_{jT+r})} \;+\; o_{P_x}(1) \quad , \quad n \to \infty\,.$$

Put $\Delta_n = \frac{h}{n}$. Since by assumption $\sigma(\cdot)$ is Lipschitz and bounded away from 0 and ∞, the j-th summand contributing to the difference in (42)

$$\left| \frac{1}{\sigma^2(\xi_{jT+r})} - \frac{1}{\sigma^2(\xi_{jT+r-\frac{h}{n}})} \right|$$

admits for every $j = 0, 1, \ldots, n{-}1$ fixed a bound of type

$$\begin{aligned} & d_1 \cdot 1_{\{\sup\limits_{0\le s\le T} |\xi_{jT+s}| \,>\, (\frac{1}{\Delta_n})^\eta\}} \\ +\;\; & d_1 \cdot 1_{\{\sup\limits_{jT+r-\frac{h}{n}\le t\le jT+r} |\xi_t - \xi_{jT+r-\frac{h}{n}}| \,>\, \Delta_n^\lambda \;,\; |\xi_{jT+r-\frac{h}{n}}| \,\le\, (\frac{1}{\Delta_n})^\eta\}} \\ +\;\; & d_2\, \Delta_n^\lambda \cdot 1_{\{\sup\limits_{jT+r-\frac{h}{n}\le t\le jT+r} |\xi_t - \xi_{jT+r-\frac{h}{n}}| \,\le\, \Delta_n^\lambda\}} \end{aligned}$$

with suitable constants d_1, d_2 (where d_2 involves the Lipschitz constant L). By the first type of bound combined with step 1, we see that

$$\frac{1}{n}\sum_{j=0}^{n-1} \left| \frac{1}{\sigma^2(\xi_{jT+r})} - \frac{1}{\sigma^2(\xi_{jT+r-\frac{h}{n}})} \right| \; 1_{\{\sup\limits_{0\le s\le T} |\xi_{jT+s}| \,>\, (\frac{1}{\Delta_n})^\eta\}}$$

vanishes almost surely as $n \to \infty$. Next, the exponential inequality in lemma 3.1 (applied to $t_1 = jT + r - \frac{h}{n}$ for $j = 0, 1, \ldots, n{-}1$) implies that

$$P\left(\sup_{jT+r-\frac{h}{n} \le t \le jT+r} |\xi_t - \xi_{jT+r-\frac{h}{n}}| \;>\; \Delta_n^\lambda \;\;,\;\; |\xi_{jT+r-\frac{h}{n}}| \;\le\; \left(\frac{1}{\Delta_n}\right)^\eta \;\;,\; \text{some } \; j = 0, 1, \ldots, n{-}1 \right)$$

vanishes as $n \to \infty$. Hence, by the second type of bound, the probability to find any strictly positive summand in

$$\frac{1}{n} \sum_{j=0}^{n-1} \left| \frac{1}{\sigma^2(\xi_{jT+r})} - \frac{1}{\sigma^2(\xi_{jT+r-\frac{h}{n}})} \right| \; 1_{\left\{ \sup_{jT+r-\frac{h}{n} \le t \le jT+r} |\xi_t - \xi_{jT+r-\frac{h}{n}}| \;>\; \Delta_n^\lambda \;,\; |\xi_{jT+r-\frac{h}{n}}| \;\le\; \left(\frac{1}{\Delta_n}\right)^\eta \right\}}$$

tends to 0 as $n \to \infty$: hence this sum vanishes in probability as $n \to \infty$. Finally, by the third type of bounds, we are left to consider averages

$$\frac{1}{n} \sum_{j=0}^{n-1} \left| \frac{1}{\sigma^2(\xi_{jT+r})} - \frac{1}{\sigma^2(\xi_{jT+r-\frac{h}{n}})} \right| \; 1_{\left\{ \sup_{jT+r-\frac{h}{n} \le t \le jT+r} |\xi_t - \xi_{jT+r-\frac{h}{n}}| \;\le\; \Delta_n^\lambda \right\}}$$

which are bounded by $d_2 \Delta_n^\lambda$, and thus vanish as $n \to \infty$. We have proved (42).

3) Next we show that for arbitrary $h > 0$ and arbitrary starting point x

$$\text{(43)} \qquad \sum_{j=0}^{n-1} \int_{jT+r-\frac{h}{n}}^{jT+r} \frac{1}{\sigma^2(\xi_s)}\, ds \;\; = \;\; h\, \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\sigma^2(\xi_{jT+r-\frac{h}{n}})} \;+\; o_{P_x}(1)$$

as $n \to \infty$. The proof of (43) follows the same scheme as the proof of (42): we consider for $j = 0, 1, \ldots, n{-}1$ summands

$$\text{(44)} \qquad \int_{jT+r-\frac{h}{n}}^{jT+r} \left| \frac{1}{\sigma^2(\xi_s)} - \frac{1}{\sigma^2(\xi_{jT+r-\frac{h}{n}})} \right| ds$$

and have for these – since $\sigma(\cdot)$ is Lipschitz and bounded away from 0 and ∞ – bounds of type

$$\begin{aligned} & \frac{1}{n}\, d_1 \cdot 1_{\left\{ \sup_{0 \le s \le T} |\xi_{jT+s}| \;>\; \left(\frac{1}{\Delta_n}\right)^\eta \right\}} \\ + \;\; & \frac{1}{n}\, d_1 \cdot 1_{\left\{ \sup_{jT+r-\frac{h}{n} \le t \le jT+r} |\xi_t - \xi_{jT+r-\frac{h}{n}}| \;>\; \Delta_n^\lambda \;,\; |\xi_{jT+r-\frac{h}{n}}| \;\le\; \left(\frac{1}{\Delta_n}\right)^\eta \right\}} \\ + \;\; & \frac{1}{n}\, d_2\, \Delta_n^\lambda \cdot 1_{\left\{ \sup_{jT+r-\frac{h}{n} \le t \le jT+r} |\xi_t - \xi_{jT+r-\frac{h}{n}}| \;\le\; \Delta_n^\lambda \right\}} \end{aligned}$$

which allow to proceed in complete analogy to step 2) above to establish (43).

4) Combining (42)+(43), we have proved (40). The proof of (39) is similiar, along the lines of steps 2)+3) above with obvious notational changes. $\diamond$

Now we make use of theorem 4.1 and (42) to establish limits for certain auxiliary martingales to which we can apply the martigale convergence theorem.

4.2 Proposition: For ξ as above, assume that the embedded chain $(\xi_{kT})_k$ is positive Harris recurrent with invariant probability μ. Fix points $0 < r_1 < r_2 < \ldots < r_m < T$ and some large $0 < H < \infty$. Consider n large enough to have $r_1 - \frac{H}{n}$ and $r_m + \frac{H}{n}$ in $(0,T)$, and also $r_j - r_{j-1} > \frac{2H}{n}$ for all $j = 1, \ldots, m$. Define processes with continuous paths in 'time' h

$$M_h^{n,j} \ := \ \sum_{k=0}^{n-1} \frac{1}{\sigma(\xi_{kT+r_j-\frac{H}{n}})} \left(W_{kT+r_j+\frac{h}{n}} - W_{kT+r_j} \right) \ , \quad 0 \le h \le H \ , \ j = 1, \ldots, m$$

$$M_h^{n,j} \ := \ \sum_{k=0}^{n-1} \frac{1}{\sigma(\xi_{kT+r_j-\frac{H}{n}})} \left(W_{kT+r_j} - W_{kT+r_j-\frac{h}{n}} \right) \ , \quad 0 \le h \le H \ , \ j = m+1, \ldots, 2m$$

with convention $r_j := r_{j-m}$ in case $j = m+1, \ldots, 2m$. Here W is the driving Brownian motion in equation (1). Define a filtration

$$I\!H^n \ = \ (\mathcal{H}_h^n)_{0 \le h \le H} \quad , \quad \mathcal{H}_h^n \ = \ \sigma\left(\xi_{kT+r_j-\frac{H}{n}}\right) \bigcup \bigcap_{h'>h} \mathcal{H}_{h'}^{n,0}$$

in 'time' h where

$$\mathcal{H}_{h'}^{n,0} \ = \ \sigma\left\{ (W_{s_2} - W_{s_1})_{kT+r_j-\frac{h'}{n} \le s_1 < s_2 \le kT+r_j+\frac{h'}{n}} \ , \ k = 0, 1, \ldots, n-1 \ , \ j = 1, \ldots, m \right\}$$

is generated by increments of W observed in balls with radius $\frac{h'}{n}$ centred at epochs

$$kT + r_j \ , \quad k = 0, 1, \ldots, n-1 \ , \quad j = 1, \ldots, m$$

of the time axis $[0, \infty)$. Then the following holds.

a) With respect to the filtration $I\!H^n$, the $\left(M_h^{n,j}\right)_{0 \le h \le H}$, $j = 1, \ldots, 2m$, are square integrable martingales with angle bracketts such that

$$\langle M^{n,j}, M^{n,j'} \rangle_h \ = \ 0 \quad \text{if} \ j \ne j'$$

$$\langle M^{n,j} \rangle_h \ = \ h \, \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\sigma^2(\xi_{jT+r_j-\frac{H}{n}})} \quad \longrightarrow \quad h \, (\mu P_{0,r_j})(\frac{1}{\sigma^2}) \quad \text{as } n \to \infty \, .$$

b) Let M^n denote the $2m$-dimensional $I\!H^n$–martingale with components $M^{n,1}, \ldots, M^{n,2m}$. Then

$$M^n \ \longrightarrow \ J^{1/2} \, B \quad \text{as } n \to \infty \ \ \text{(weak convergence in } C_{2m})$$

where $B = (B_h)_{h \ge 0}$ is $2m$-dimensional Brownian motion, J the $2m \times 2m$–diagonal matrix having entries $(\mu P_{0,r_{j'}})(\frac{1}{\sigma^2})$ in diagonal positions both j' and $m+j'$ for $j' = 1, \ldots, m$, and where

$(C_{2m}, \mathcal{C}_{2m})$ denotes the path space of continuous functions $[0,H] \to I\!R^{2m}$.

Proof: 1) The filtration $I\!H^n$ is by definition right-continuous, and the processes $(M^{n,j})_{0\le h\le H}$ are $I\!H^n$-adapted and continuous. We prove that $(M^{n,j})_{0\le h\le H}$ are martingales with respect to $I\!H^n = (\mathcal{H}^n_h)_{0\le h\le H}$. Consider first the case $m{+}1 \le j \le 2m$. For $h_1 < h_2$ in $[0,H]$, we have

$$M^{n,j}_{h_2} \;=\; M^{n,j}_{h_1} \;+\; \sum_{k=0}^{n-1} \frac{1}{\sigma(\xi_{kT+r_j-\frac{H}{n}})} \left(W_{kT+r_j-\frac{h_1}{n}} - W_{kT+r_j-\frac{h_2}{n}} \right) \tag{45}$$

and note that the second term on the right hand side is a limit in $L^2(P_x)$ of

$$\sum_{k=0}^{n-1} \frac{1}{\sigma(\xi_{kT+r_j-\frac{H}{n}})} \left(W_{kT+r_j-\frac{h_1+\delta_\ell}{n}} - W_{kT+r_j-\frac{h_2}{n}} \right) \tag{46}$$

as $\ell \to \infty$, for any sequence $\delta_\ell \downarrow 0$. By definition of $\mathcal{H}^n_{h_1}$, the conditional expectation of (46) given $\mathcal{H}^n_{h_1}$ equals 0 for every ℓ. This shows

$$E\left(M^{n,j}_{h_2} \mid \mathcal{H}^n_{h_1} \right) \;=\; M^{n,j}_{h_1} \quad \text{if } h_1 < h_2$$

in case $m{+}1 \le j \le 2m$. The proof in case $1 \le j \le m$ is similiar.

2) Consider the predictable quadratic variation of $M^{n,j}$ with respect to $I\!H^n$. We look to the case $m{+}1 \le j \le 2m$ first. For $h_1 < h_2$ in $[0,H]$, we have by definition of M^n and $I\!H^n$

$$E\left(\left(M^{n,j}_{h_2} - M^{n,j}_{h_1} \right)^2 \mid \mathcal{H}^n_{h_1} \right) \;=\; \sum_{k=0}^{n-1} \frac{1}{\sigma^2(\xi_{kT+r_j-\frac{H}{n}})} \, E\left(\left(W_{kT+r_j-\frac{h_1}{n}} - W_{kT+r_j-\frac{h_2}{n}} \right)^2 \right)$$

which equals

$$(h_2 - h_1)\, \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\sigma^2(\xi_{jT+r_j-\frac{H}{n}})} \; .$$

In case $1 \le j \le m$ we arrive at the same expression. Hence we have proved

$$\langle M^{n,j} \rangle_h \;=\; h\, \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\sigma^2(\xi_{jT+r_j-\frac{H}{n}})} \quad , \quad 0 \le h \le H \ , \ j \in \{1, \ldots, 2m\} \; .$$

Next, consider predictable quadratic covariation of $M^{n,j}$ and $M^{n,j'}$ with respect to $I\!H^n$. We look first to the case $1 \le j \le m$ and $j' = m{+}j$. Then we obtain for $h_1 < h_2$ in $[0,H]$

$$E\left(\left(M^{n,j}_{h_2} - M^{n,j}_{h_1} \right) \left(M^{n,m+j}_{h_2} - M^{n,m+j}_{h_1} \right) \mid \mathcal{H}^n_{h_1} \right) \;=\; 0$$

from the above representation (45) for $M^{n,j}$ and $M^{n,m+j}$ since

$$\left(W_{kT+r_j+\frac{h_2}{n}} - W_{kT+r_j+\frac{h_1}{n}} \right) \;,\; \left(W_{kT+r_j-\frac{h_1}{n}} - W_{kT+r_j-\frac{h_2}{n}} \right)$$

are increments over disjoint intervals and thus independent. The other cases $j \neq j'$ require similiar arguments. This gives

$$\langle M^{n,j}, M^{n,j'} \rangle_h \;=\; 0 \quad , \quad 0 \le h \le H \; , \; j \neq j' \in \{1, \ldots, 2m\} \; .$$

3) For $j \in \{1, \ldots, 2m\}$ and $0 \le h \le H$ fixed, we have convergence in probability as $n \to \infty$

$$\langle M^{n,j}, M^{n,j} \rangle_h \;\longrightarrow\; h \; (\mu P_{0,r_j})(\frac{1}{\sigma^2})$$

in virtue of step 2, (42) and theorem 4.1. Hence the martingale convergence theorem (Jacod and Shiryaev [JS 87], Theorem VIII.3.22) applies to M^n with respect to $I\!H^n$ and proves weak convergence of M^n to $J^{1/2}\, B$. This completes the proof of the proposition. $\diamond$

In proposition 4.2, we have considered an auxiliary martingale to which the martingale convergence theorem applies. The martingale terms which will arise in the log-likelihoods of local models at ϑ are slightly different. Their form will appear in the next theorem which is – together with 4.1 – the main result of this section.

4.3 Theorem: Under all assumptions and with all notations of 4.2, consider

$$Y^{n,j}_h \;:=\; \int_0^{nT} \frac{1}{\sigma(\xi_s)}\, 1_{\left(r_j\, ,\, r_j + \frac{h}{n}\right)}(i_T(s))\, dW_s \; , \quad 0 \le h \le H \; , \; j = 1, \ldots, m$$

$$Y^{n,j}_h \;:=\; \int_0^{nT} \frac{1}{\sigma(\xi_s)}\, 1_{\left(r_j - \frac{h}{n}\, ,\, r_j\right)}(i_T(s))\, dW_s \; , \quad 0 \le h \le H \; , \; j = m{+}1, \ldots, 2m$$

as processes in 'time' h, and write Y^n for the $2m$-dimensional process whose components are $Y^{n,1}, \ldots, Y^{n,2m}$. Then we can compare the processes Y^n to the $I\!H^n$-martingales M^n of proposition 4.2 as follows. For all $0 \le h \le H$ and $j = 1, 2, \ldots, 2m$, we have

$$Y^{n,j}_h \;=\; M^{n,j}_h \;+\; o_{P_x}(1) \; , \quad n \to \infty$$

for arbitrary starting points $x \in I\!R$, and as an immediate consequence

$$Y^n \;\longrightarrow\; J^{1/2}\, B \quad \text{as } n \to \infty \;\; \text{(in the sense of finite dimensional distributions)}$$

where $B = (B_h)_{h \ge 0}$ is $2m$-dimensional Brownian motion and J the $2m{\times}2m$–diagonal matrix having entries $(\mu P_{0,r_{j'}})(\frac{1}{\sigma^2})$ in diagonal positions both j' and $m{+}j'$, for $j' = 1, \ldots, m$.

Proof: 1) First we note the following, very similiar to (43) in the proof of theorem 4.1: for $h>0$ fixed and n large enough for $r-\frac{h}{n}, r+\frac{h}{n} \in (0,T)$, we have as $n\to\infty$

$$\sum_{j=0}^{n-1} \int_{jT+r-\frac{h}{n}}^{jT+r} \left| \frac{1}{\sigma(\xi_s)} - \frac{1}{\sigma(\xi_{jT+r-\frac{h}{n}})} \right|^2 ds \;=\; o_{P_x}(1) \tag{47}$$

for arbitrary starting point x. To show (47), we put $\Delta_n = \frac{h}{n}$, consider the summands $j = 0,1,\ldots,n-1$ in the sum (47) separately, and use – since $\sigma(\cdot)$ is Lipschitz and bounded away from 0 and ∞ – bounds

$$\begin{aligned} &\frac{1}{n}\, d_1 \cdot 1_{\{ \sup\limits_{0\le s\le T} |\xi_{jT+s}| \,>\, \left(\frac{1}{\Delta_n}\right)^\eta \}} \\ +\;& \frac{1}{n}\, d_1 \cdot 1_{\{ \sup\limits_{jT+r-\frac{h}{n}\le t\le jT+r} |\xi_t - \xi_{jT+r-\frac{h}{n}}| \,>\, \Delta_n^\lambda \;,\; |\xi_{jT+r-\frac{h}{n}}| \,\le\, \left(\frac{1}{\Delta_n}\right)^\eta \}} \\ +\;& \frac{1}{n}\, d_2\, \Delta_n^{2\lambda} \cdot 1_{\{ \sup\limits_{jT+r-\frac{h}{n}\le t\le jT+r} |\xi_t - \xi_{jT+r-\frac{h}{n}}| \,\le\, \Delta_n^\lambda \}} \end{aligned}$$

with suitable constants d_1, d_2. We then proceed with the summands in the sum (47) exactly as with the summands (44) in step 2) of the proof of theorem 4.1.

2) Now we consider for $0<h<H$ and for $j=m{+}1,\ldots,2m$ the L^2–differences

$$E\left(\left| Y_h^{n,j} - M_h^{n,j} \right|^2 \right)$$

with respect to the $I\!H^n$–martingales $M^{n,j}$ of proposition 4.2. By definition of $Y_h^{n,j}$ and $M_h^{n,j}$ (usual stochastic integrals in time $t\ge 0$) for h fixed, the L^2–difference is

$$E\left(\sum_{j=0}^{n-1} \int_{jT+r-\frac{h}{n}}^{jT+r} \left| \frac{1}{\sigma(\xi_s)} - \frac{1}{\sigma(\xi_{jT+r-\frac{H}{n}})} \right|^2 ds \right) . \tag{48}$$

For $0<h<H$, replacing the squares in (48) up to multiplication with some constant by

$$\left| \frac{1}{\sigma(\xi_s)} - \frac{1}{\sigma(\xi_{jT+r-\frac{h}{n}})} \right|^2 + \left| \frac{1}{\sigma(\xi_{jT+r-\frac{h}{n}})} - \frac{1}{\sigma(\xi_{jT+r})} \right|^2 + \left| \frac{1}{\sigma(\xi_{jT+r-\frac{H}{n}})} - \frac{1}{\sigma(\xi_{jT+r})} \right|^2$$

and applying (47) to the first and an obvious analogue of (42)

$$\frac{1}{n} \sum_{j=0}^{n-1} \left| \frac{1}{\sigma(\xi_{jT+r-\frac{h}{n}})} - \frac{1}{\sigma(\xi_{jT+r})} \right|^2 \;=\; o_{P_x}(1) \quad , \quad n\to\infty$$

to the second and third contribution, we deduce that the sum inside the expectation in (48) vanishes in P_x-probability as $n\to\infty$. By dominated convergence the expectation (48) vanishes as $n\to\infty$. Thus we have for $m{+}1 \le j \le 2m$

$$Y_h^{n,j} - M_h^{n,j} \;\longrightarrow\; 0 \quad \text{in } L^2(P_x) \text{ as } n\to\infty\, . \tag{49}$$

The proof in case $1 \le j \le m$ is similiar. This is the first assertion of theorem 4.3.

3) From (49) with arbitrary j and h we see that the processes $(Y^n)_{0\le h\le H}$ and $(M^n)_{0\le h\le H}$ will have in the limit as $n \to \infty$ the same finite-dimensional distributions. Hence the second assertion of 4.3 is an immediate consequence of proposition 4.2 b), and the proof is complete. $\diamond$

5 Proofs for the results stated in section 1

In order to prove the results stated in section 1, we consider the process ξ of equation (1) with T-periodic signal (2) in the drift depending on parameter $\vartheta \in \Theta$. We assume the Harris condition $(H1)$ for all values of $\vartheta \in \Theta$, and Lipschitz and linear growth conditions on $b(\cdot)$ and $\sigma(\cdot)$ in combination with $(H2)$. This implies that the set of assumptions I)–III) of section 2 holds for all ϑ, as well as (34)+(35)+(38). Whenever we consider sequences of local models at ϑ, we will need the function $\lambda^*(\cdot)$ of (2) only on some fixed open set in $[0,T]$: hence we can take $\lambda^*(\cdot)$ as strictly positive, continuous and T-periodic on $I\!R$. As in section 1, Q^ϑ is the law of ξ under ϑ on the canonical path space $(C, \mathcal{C})$, η the canonical process on $(C, \mathcal{C})$, $I\!\!G$ the canonical filtration $I\!\!G = (\mathcal{G}_t)_{t\ge 0}$ with $\mathcal{G}_t = \bigcap_{t'>t} \sigma\{\eta_s : 0 \le \eta_s \le t'\}$. Whenever convenient, we use default definitions $Q^\vartheta := Q^0$ for $\vartheta \le 0$ and $Q^\vartheta := Q^{T-a}$ for $\vartheta \ge T-a$, in obvious extension of (2) with $\Theta = (0, T-a)$.

We restart from the representation of likelihood ratios $L_t^{\zeta'/\zeta}$ of $Q^{\zeta'}|\mathcal{G}_t$ with respect to $Q^\zeta|\mathcal{G}_t$ given before theorem 1.1: for $\zeta \in \Theta$ and ζ' close to ζ, $L_t^{\zeta'/\zeta}$ equals

$$(50)\qquad \exp\left\{ \left[-\int_0^t \frac{\lambda^*(s)}{\sigma(\eta_s)} 1_{(\zeta,\zeta')}(i_T(s))\, dB_s - \frac{1}{2}\int_0^t \left(\frac{\lambda^*(s))}{\sigma(\eta_s)}\right)^2 1_{(\zeta,\zeta')}(i_T(s))\, ds \right] \right.$$
$$\left. + \left[\int_0^t \frac{\lambda^*(s)}{\sigma(\eta_s)} 1_{(\zeta+a,\zeta'+a)}(i_T(s))\, dB_s - \frac{1}{2}\int_0^t \left(\frac{\lambda^*(s))}{\sigma(\eta_s)}\right)^2 1_{(\zeta+a,\zeta'+a)}(i_T(s))\, ds \right] \right\}$$

in case $\zeta < \zeta' < \zeta + a$, and

$$(51)\qquad \exp\left\{ \left[\int_0^t \frac{\lambda^*(s)}{\sigma(\eta_s)} 1_{(\zeta',\zeta)}(i_T(s))\, dB_s - \frac{1}{2}\int_0^t \left(\frac{\lambda^*(s))}{\sigma(\eta_s)}\right)^2 1_{(\zeta',\zeta)}(i_T(s))\, ds \right] \right.$$
$$\left. + \left[-\int_0^t \frac{\lambda^*(s)}{\sigma(\eta_s)} 1_{(\zeta'+a,\zeta+a)}(i_T(s))\, dB_s - \frac{1}{2}\int_0^t \left(\frac{\lambda^*(s))}{\sigma(\eta_s)}\right)^2 1_{(\zeta'+a,\zeta+a)}(i_T(s))\, ds \right] \right\}$$

in case $\zeta' < \zeta < \zeta' + a$. We state a lemma related to the geometry – in Hellinger sense – of the model determined by (50)+(51).

5.1 Lemma: Consider $\zeta, \zeta' \in \Theta$ such that $\zeta < \zeta' < \zeta + a$ or $\zeta' < \zeta < \zeta' + a$ holds. Then there are positive constants c_j and k not depending on ζ, ζ', t such that for all $t \geq 0$

$$E_\zeta \left(\sup_{0 \leq s \leq t} L_t^{\zeta'/\zeta} \right) \quad \leq \quad c_0 \, + \, c_1 \left(\left\lfloor \frac{t}{T} \right\rfloor + 1 \right) |\zeta' - \zeta| \, , \tag{52}$$

$$E_\zeta \left(\left[1 \, - \, \left(L_t^{\zeta'/\zeta} \right)^{1/2} \right]^2 \right) \quad \leq \quad \sum_{j=1}^{3} c_j \left(\left\lfloor \frac{t}{T} \right\rfloor + 1 \right)^j |\zeta' - \zeta|^j \, , \tag{53}$$

$$E_\zeta \left(\left[1 \, - \, \left(L_t^{\zeta'/\zeta} \right)^{1/4} \right]^4 \right) \quad \leq \quad \sum_{j=2}^{5} c_j \left(\left\lfloor \frac{t}{T} \right\rfloor + 1 \right)^j |\zeta' - \zeta|^j \, , \tag{54}$$

$$E_\zeta \left(\left[L_t^{\zeta'/\zeta} \right]^{1/2} \right) \quad \leq \quad \exp \left\{ - k \left\lfloor \frac{t}{T} \right\rfloor |\zeta' - \zeta| \right\} \, . \tag{55}$$

Proof: 1) For ζ, ζ' in Θ, the likelihood ratio process $(L_t^{\zeta'/\zeta})_{t \geq 0}$ under Q^ζ is the exponential

$$L^{\zeta'/\zeta} \;=\; \mathcal{E}_\zeta \left(\int_0^\cdot \delta_s \, dB_s \right) \quad \text{with} \quad \delta_s \;=\; \frac{S(\zeta', s) - S(\zeta, s)}{\sigma(\eta_s)} \, . \tag{56}$$

Hence $(L_t^{\zeta'/\zeta})_{t \geq 0}$ under Q^ζ solves the SDE

$$L_t^{\zeta'/\zeta} \;=\; 1 \,+\, \int_0^t L_s^{\zeta'/\zeta} \, \delta_s \, dB_s \, , \quad t \geq 0$$

from which one obtains by Ito formula under Q^ζ

$$V_t := \left(L_t^{\zeta'/\zeta} \right)^{1/4} \quad \text{satisfies} \quad V_t \;=\; 1 - \frac{3}{32} \int_0^t V_s \, \delta_s^2 \, ds + \frac{1}{4} \int_0^t V_s \, \delta_s \, dB_s \, , \tag{57}$$

$$V_t := \left(L_t^{\zeta'/\zeta} \right)^{1/2} \quad \text{satisfies} \quad V_t \;=\; 1 - \frac{1}{8} \int_0^t V_s \, \delta_s^2 \, ds + \frac{1}{2} \int_0^t V_s \, \delta_s \, dB_s \, . \tag{58}$$

2) From now on in this proof, we fix $\zeta \in \Theta$ and consider ζ' such that either $\zeta < \zeta' < \zeta + a$ or $\zeta' < \zeta < \zeta' + a$ holds. In the first case we have according to (50)

$$\delta_s \;:=\; \frac{\lambda^*(s)}{\sigma(\eta_s)} \left[- 1_{(\zeta, \zeta')} \,+\, 1_{(\zeta + a, \zeta' + a)} \right] (i_T(s))$$

with non-overlapping intervals, in the second case by (51)

$$\delta_s \;:=\; \frac{\lambda^*(s)}{\sigma(\eta_s)} \left[1_{(\zeta', \zeta)} \,-\, 1_{(\zeta' + a, \zeta + a)} \right] (i_T(s)) \, .$$

Let $s \to d(s)$ denote the deterministic T–periodic function

$$d(s) \;:=\; 1_{\{\delta \neq 0\}}(s) \;=\; \left[1_{(\zeta', \zeta)} + 1_{(\zeta' + a, \zeta + a)} \right] (i_T(s)) \, , \; s \geq 0 \, .$$

By our assumptions on $\lambda^*(\cdot)$ and $\sigma(\cdot)$, the process $(\frac{\lambda^*(s)}{\sigma(\eta_s)})_{s\geq 0}$ is bounded away from both 0 and ∞. Hence there are some $0 < \underline{c} < \overline{c} < \infty$ not depending on ζ, ζ', t such that

$$\text{(59)} \qquad \underline{c}\, d(s) \;\leq\; \delta_s^2 \;\leq\; \overline{c}\, d(s) \quad , \quad 2\left\lfloor \frac{t}{T} \right\rfloor |\zeta'-\zeta| \;\leq\; \int_0^t d(s)\, ds \;\leq\; 2\left(\left\lfloor \frac{t}{T} \right\rfloor + 1\right) |\zeta'-\zeta|$$

for all $t > 0$; here $\lfloor x \rfloor$ denotes the biggest integer stricly smaller than x.

3) We prove (52). The process $V_t := \left(L_t^{\zeta'/\zeta}\right)^{1/2}$ of (58) is nonnegative and satisfies $E_\zeta(V_s^2) = 1$ for all s. Thus (58) gives

$$0 \;\leq\; V_t \;\leq\; 1 \;+\; \frac{1}{2}\int_0^t V_s\, \delta_s\, dB_s$$

and allows to write (with [IW 89, p. 110])

$$\begin{aligned} E_\zeta\left(\sup_{0\leq s\leq t} V_t^2\right) \;&\leq\; c\left(1 \;+\; E_\zeta\left(\sup_{0\leq s\leq t}\left(\int_0^s V_r\, \delta_r\, dB_r\right)^2\right)\right) \\ &\leq\; c'\left(1 \;+\; E_\zeta\left(\int_0^t V_s^2\, \delta_s^2\, ds\right)\right) \\ &\leq\; c''\left(1 \;+\; E_\zeta\left(\int_0^t V_s^2\, d(s)\, ds\right)\right) \;\;\leq\;\; \widetilde{c}\left(1 \;+\; \int_0^t d(s)\, ds\right) \end{aligned}$$

which combined with (59) gives (52).

4) We prove (53). We start from $V_t := \left(L_t^{\zeta'/\zeta}\right)^{1/2}$ for which (58) yields the bound

$$\text{(60)} \qquad -\frac{1}{2}\int_0^t V_s\, \delta_s\, dB_s \;\;\leq\;\; 1 - V_t \;\;\leq\;\; \frac{1}{8}\int_0^t V_s\, \delta_s^2\, ds \;+\; \left(\frac{1}{2}\int_0^t V_s\, \delta_s\, dB_s\right)^-$$

where f^- denotes the negative part in $f = f^+ - f^-$. Combining (52)+(59) we have

$$\begin{aligned} E_\zeta\left(\left[\int_0^t V_s\, \delta_s^2\, ds\right]^2\right) \;\;&\leq\;\; E_\zeta\left(\sup_{0\leq s\leq t} V_s^2\right)\left[\overline{c}\int_0^t d(s)\, ds\right]^2 \\ &\leq\;\; c_2\left(\left\lfloor \frac{t}{T}\right\rfloor + 1\right)^2 |\zeta'-\zeta|^2 \;+\; c_3\left(\left\lfloor \frac{t}{T}\right\rfloor + 1\right)^3 |\zeta'-\zeta|^3 \end{aligned}$$

for suitable c_2, c_3. For the martingale term in (60), using again (59) and $E_\zeta(V_s^2) = 1$, we have

$$E_\zeta\left(\left[\int_0^t V_s\, \delta_s\, dB_s\right]^2\right) \;=\; E_\zeta\left(\int_0^t V_s^2\, \delta_s^2\, ds\right) \;\leq\; \overline{c}\int_0^t d(s)\, ds \;\;\leq\;\; c_1\left(\left\lfloor \frac{t}{T}\right\rfloor + 1\right) |\zeta'-\zeta|$$

for all $t \geq 0$. Squaring the bound (60), both inqualities together give (53).

5) To prove (54), we define $V_t := \left(L_t^{\zeta'/\zeta}\right)^{1/4}$ and have from (57) up to different constants again a bound of form (60) . We use

$$E_\zeta\left(\left[\int_0^t V_s\, \delta_s^2\, ds\right]^4\right) \;\;\leq\;\; E_\zeta\left(\sup_{0\leq s\leq t} V_s^4\right)\left[\overline{c}\int_0^t d(s)\, ds\right]^4$$

and with [IW 89, p. 110]

$$E_\zeta\left(\left[\int_0^t V_s\,\delta_s\,dB_s\right]^4\right) \;\le\; c\,E_\zeta\left(\left[\int_0^t V_s^2\,\delta_s^2\,ds\right]^2\right) \;\le\; c\,E_\zeta\left(\sup_{0\le s\le t} V_s^4\right)\left[\bar{c}\int_0^t d(s)\,ds\right]^2 .$$

Applying (52) to the first factor and (59) to the second factor appearing on the right hand side, the sum of the right hand sides of both last inequalities is at most

$$\sum_{j=2}^{5} c_j \left(\left\lfloor \frac{t}{T}\right\rfloor + 1\right)^j |\zeta'-\zeta|^j .$$

6) To prove (55), we deduce from the exponential representation (56) under Q^ζ that

$$\left(L_t^{\zeta'/\zeta}\right)^{1/2} \;=\; \mathcal{E}_\zeta\left(\int_0^{\cdot} (\frac{1}{2}\delta_s)\,dB_s\right)_t \,\exp\left\{-\frac{1}{8}\int_0^t \delta_s^2\,ds\right\} .$$

So the lower bound in (59) applies and shows

$$E_\zeta\left(\left(L_t^{\zeta'/\zeta}\right)^{1/2}\right) \;\le\; \exp\left\{-\frac{1}{8}\,\underline{c}\int_0^t d(s)\,ds\right\} \;\le\; \exp\left\{-\,k\left\lfloor\frac{t}{T}\right\rfloor|\zeta'-\zeta|\right\}$$

for suitable $k>0$. We have proved all assertions of lemma 5.1. ◇

From (53) in lemma 5.1, we have bounds for the squared Hellinger distance when we observe the trajectory of $(\xi)_{t\ge 0}$ up to time nT: for ζ' sufficiently close to ζ,

$$H^2\left(Q^\zeta|\mathcal{G}_{nT}\,,\,Q^{\zeta'}|\mathcal{G}_{nT}\right) \;=\; \frac{1}{2}\,E_\zeta\left(\left[1-\left(L_{nT}^{\zeta'/\zeta}\right)^{1/2}\right]^2\right) \;\le\; C\sum_{j=1}^{3} n^j\,|\zeta'-\zeta|^j \tag{61}$$

for all $n\ge 1$, with some constant C which does not depend on ζ,ζ',n. In particular, the parametrization in $\left\{Q^\zeta|\mathcal{G}_{nT} : \zeta\in\Theta\right\}$ – in Hellinger distance – is Hölder continuous with index $\frac{1}{2}$ at every point $\zeta\in\Theta$. In order to localize around some fixed point, the right hand side of (61) suggests to take local scale proportional to $\frac{1}{n}$. This leads to local models $\mathcal{E}_n^{(\vartheta)}$ at ϑ as in 1.1.

5.2 Proof of theorem 1.1: From (50)+(51) we have for n sufficiently large

$$\begin{aligned} L_{nT}^{(\vartheta+\frac{h}{n})/\vartheta} \;=\; \exp\Bigg\{ &\int_0^{nT} \frac{\lambda^*(s)}{\sigma(\eta_s)}\left[1_{(\vartheta+a,\vartheta+a+\frac{h}{n})} - 1_{(\vartheta,\vartheta+\frac{h}{n})}\right](i_T(s))\,dB_s \\ &- \frac{1}{2}\int_0^{nT}\left(\frac{\lambda^*(s))}{\sigma(\eta_s)}\right)^2\left[1_{(\vartheta+a,\vartheta+a+\frac{h}{n})} + 1_{(\vartheta,\vartheta+\frac{h}{n})}\right](i_T(s))\,ds \Bigg\} \end{aligned} \tag{62}$$

in case $u=h>0$, where all intervals are disjoint, and in case $u=-h<0$

$$\begin{aligned} L_{nT}^{(\vartheta-\frac{h}{n})/\vartheta} \;=\; \exp\Bigg\{ &\int_0^{nT} \frac{\lambda^*(s)}{\sigma(\eta_s)}\left[1_{(\vartheta-\frac{h}{n},\vartheta)} - 1_{(\vartheta+a-\frac{h}{n},\vartheta+a)}\right](i_T(s))\,dB_s \\ &- \frac{1}{2}\int_0^{nT}\left(\frac{\lambda^*(s))}{\sigma(\eta_s)}\right)^2\left[1_{(\vartheta+a-\frac{h}{n},\vartheta+a)} + 1_{(\vartheta-\frac{h}{n},\vartheta)}\right](i_T(s))\,ds \Bigg\} . \end{aligned} \tag{63}$$

a) We have to prove convergence as $n\to\infty$ of the finite dimensional distributions of likelihood ratios in the local model at ϑ

$$\left(L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta} \right)_{u\in\Theta_{\vartheta,n}} \text{ under } Q^\vartheta \quad , \quad \Theta_{\vartheta,n} := \{ u \in I\!R : \vartheta + \frac{u}{n} \in \Theta \}$$

to the finite dimensional distributions of the likelihoodratios (5)+(6)

$$\left(\widetilde{L}^{u/0} \right)_{u\in I\!R} \text{ under } \widetilde{P}_0 \quad , \quad \widetilde{L}^{u/0} \;:=\; \exp\left\{ \widetilde{W}(uJ_\vartheta) - \frac{1}{2}|uJ_\vartheta| \right\}$$

in the limit model $\widetilde{\mathcal{E}}$, with $(\widetilde{W}_u)_{u\in I\!R}$ two-sided standard Brownian motion and

$$J_\vartheta \;=\; \left\{ (\lambda^*(\vartheta))^2\,(\mu^{(\vartheta)}P^{(\vartheta)}_{0,\vartheta}) + (\lambda^*(\vartheta{+}a))^2\,(\mu^{(\vartheta)}P^{(\vartheta)}_{0,\vartheta+a}) \right\} (\frac{1}{\sigma^2})\ .$$

By the continuous mapping theorem, it is sufficient to consider only the exponents in (62), (63) and (5), and to prove convergence of the finite dimensional distributions for these.

ii) We consider the martingale parts first. Since $\lambda^*(\cdot)$ is continuous and since all intervals under consideration have length $O(\frac{1}{n})$, it is sufficient to put

$$r_1 = r_3 = \vartheta\ , \quad r_2 = r_4 = \vartheta{+}a$$

and to replace the martingale part in the exponent of $L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta}$ in (62)+(63) by

$$\lambda^*(r_2)\,\widetilde{Y}_h^{n,2} \;-\; \lambda^*(r_1)\,\widetilde{Y}_h^{n,1} \text{ in case } u>0\ , \quad \lambda^*(r_3)\,\widetilde{Y}_h^{n,3} \;-\; \lambda^*(r_4)\,\widetilde{Y}_h^{n,4} \text{ in case } u<0$$

where we write throughout $h=|u|$, and in exact analogy to the notation of theorem 4.3

$$\begin{aligned} \widetilde{Y}_h^{n,j} &:= \int_0^{nT} \frac{1}{\sigma(\eta_s)}\, 1_{(r_j, r_j+\frac{h}{n})}(i_T(s))\, dB_s \quad \text{for } h>0\ , \quad j=1,2 \\ \widetilde{Y}_h^{n,j} &:= \int_0^{nT} \frac{1}{\sigma(\eta_s)}\, 1_{(r_j-\frac{h}{n}, r_j)}(i_T(s))\, dB_s \quad \text{for } h>0\ , \quad j=3,4\ . \end{aligned}$$

These are viewed as processes in 'time' h. By theorem 4.3, finite dimensional distributions of

$$\widetilde{Y}^n \;=\; \left(\widetilde{Y}_h^{n,1}, \widetilde{Y}_h^{n,2}, \widetilde{Y}_h^{n,3}, \widetilde{Y}_h^{n,4} \right)_{h\geq 0}$$

converge as $n\to\infty$ to the finite dimensional distributions of

$$\widetilde{Y} \;=\; \left(\gamma_1\,\widetilde{B}_h^1\,,\ \gamma_2\,\widetilde{B}_h^2\,,\ \gamma_3\,\widetilde{B}_h^3\,,\ \gamma_4\,\widetilde{B}_h^4 \right)_{h\geq 0}$$

where $\widetilde{B}^i$, $i=1,\ldots,4$, are independent Brownian motions and

$$\gamma_1 = \gamma_3 = \sqrt{(\mu^{(\vartheta)}P^{(\vartheta)}_{0,\vartheta})(\frac{1}{\sigma^2})} \quad , \quad \gamma_2 = \gamma_4 = \sqrt{(\mu^{(\vartheta)}P^{(\vartheta)}_{0,\vartheta+a})(\frac{1}{\sigma^2})}\ .$$

From this, convergence in the sense of finite dimensional distributions of the martingale parts in the exponents in (62)+(63) to those in the exponent of (5) follows: since $(\lambda^*(r_1))^2\,\gamma_1^2 + (\lambda^*(r_2))^2\,\gamma_2^2$ equals J_ϑ, we have on the 'positive branch' $u > 0$ convergence

$$\left(\lambda^*(r_2)\,\widetilde{Y}_h^{n,2} \;-\; \lambda^*(r_1)\,\widetilde{Y}_h^{n,1}\right)_{h\geq 0} \quad \text{to} \quad \left(J_\vartheta^{1/2}\,\widetilde{W}_h^+\right)_{h\geq 0}$$

as $n \to \infty$, and similiarly on the 'negative branch' $u < 0$

$$\left(\lambda^*(r_3)\,\widetilde{Y}_h^{n,3} \;-\; \lambda^*(r_4)\,\widetilde{Y}_h^{n,4}\right)_{h\geq 0} \quad \text{to} \quad \left(J_\vartheta^{1/2}\,\widetilde{W}_h^-\right)_{h\geq 0}$$

where W^+, W^- are independent Brownian motions.

ii) For the BV parts in the exponents in (62)+(63) or in the exponent of (5), the argument is similiar and follows directly from theorem 4.1.

b) Convergence of local models $\mathcal{E}_n^{(\vartheta)}$ in the sense of [S 85, p. 302] as $n \to \infty$ to the limit model $\widetilde{\mathcal{E}}$ reduces to finite dimensional convergence of likelihoods as proved in a) since all probability measures in $\left\{Q^{\vartheta+\frac{h}{n}}|\mathcal{F}_{nT} : h \in \Theta_{\vartheta,n}\right\}$ and in $\left\{\widetilde{P}_h : h \in I\!R\right\}$ are equivalent. $\diamond$

We remark that the same scheme of proof based on theorem 4.3 allows to consider finitely many discontinuities in the signal shifted by the same one-dimensional parameter ϑ, as in remark 1.3, or more generally finitely many discontinuities which arise at epochs $0 < \vartheta_1 < \vartheta_2 < \ldots < \vartheta_d < T$. The last case would lead to d-dimensional parameter $\vartheta = (\vartheta_1, \ldots, \vartheta_d)$ and to limits of local models at ϑ which are d-fold products of independent experiments $\widetilde{\mathcal{E}}$ with different scaling factors.

Now we turn to the limit experiment $\widetilde{\mathcal{E}}$ in theorem 1.1 b) or in remark 1.2.

5.3 Lemma: In the limit experiment $\widetilde{\mathcal{E}}$ of (6)+(5), we have for every $u \in I\!R$ fixed

(64) $$\widetilde{L}^{(u+h)/u} \;=\; \exp\left\{\left(\widetilde{W}_{u+h} - \widetilde{W}_u\right) - \frac{1}{2}\,|h|\right\} \quad \text{for all } h \in I\!R\,.$$

Note that $\left(\widetilde{W}_{u+h} - \widetilde{W}_u\right)_{h\in I\!R}$ is again a two-sided Brownian motion under $\widetilde{P}_u$. Hence for every $u \in I\!R$ fixed, $\left\{\widetilde{P}_{u+h} : h \in I\!R\right\}$ is statistically the same experiment as $\widetilde{\mathcal{E}}$.

Proof: We will consider the likelihood ratio $\widetilde{L}^{u/u_0}$ separately in all possible cases $0 < u_0 < u$, $0 < u < u_0$, $u_0 < u < u$, $u < 0 < u_0$, ... We give the detailed proof in case $u < 0 < u_0$. We put for simplicity $J_\vartheta = 1$, and use notation (7) of remark 1.2. Write $(\eta^{(1)}, \eta^{(2)})$ for the canonical process on $C([0,\infty), I\!R^2)$. By (7), the canonical process satisfies under $\widetilde{P}_u$ with $u<0$

$$d\eta_t^{(1)} \;=\; d\widetilde{W}_t^+ \quad , \quad d\eta_t^{(2)} \;=\; 1_{(0,|u|)}(t)\,dt \;+\; d\widetilde{W}_t^- \;.$$

Again by (7), under $\widetilde{P}_{u_0}$ with $u_0>0$, the canonical process satisfies

$$d\eta_t^{(1)} \;=\; 1_{(0,u_0)}(t)\,dt \;+\; d\widetilde{W}_t^+ \quad , \quad d\eta_t^{(2)} \;=\; d\widetilde{W}_t^- \;.$$

Since $\widetilde{W}^+$ and $\widetilde{W}^-$ are independent, the likelihood ratio process of $\widetilde{P}_u$ to $\widetilde{P}_{u_0}$ relative to the canonical filtration on the path space $C([0,\infty), I\!R^2)$ is given by

$$\exp\left\{\int_0^t (-1_{(0,u_0)}(t))\,d\widetilde{W}_t^+ \;+\; \int_0^t 1_{(0,|u|)}(t)\,d\widetilde{W}_t^- \;-\; \frac{1}{2}(u_0\wedge t) \;-\; \frac{1}{2}(|u|\wedge t)\right\} \quad , \quad t\geq 0 \;.$$

Letting $t\to\infty$, we get by definition of $\widetilde{W}$ in 1.2

$$\widetilde{L}^{u/u_0} \;=\; \exp\left\{\left(\widetilde{W}_u - \widetilde{W}_{u_0}\right) \;-\; \frac{1}{2}\,|u-u_0|\right\} \quad , \quad u<0<u_0 \;.$$

We have proved (64) in case $u<0<u_0$. The remaining cases are proved similiarly. $\diamond$

5.4 Lemma: In the limit experiment $\widetilde{\mathcal{E}}$ of (6)+(5), there are constants c,k (not depending on u,u') such that the following holds for all u,u' in $I\!R$:

(65) $$E_u\left(\left[1-\left(\widetilde{L}^{u'/u}\right)^{1/2}\right]^2\right) \;\leq\; c\,|u'-u| \;,$$

(66) $$E_u\left(\left[1-\left(\widetilde{L}^{u'/u}\right)^{1/4}\right]^4\right) \;\leq\; c\,|u'-u|^2 \;,$$

(67) $$E_u\left(\left[\widetilde{L}^{u'/u}\right]^{1/2}\right) \;\leq\; \exp\{-\,k\,|u'-u|\} \;.$$

Proof: We use (64) to write for $u'\neq u$

$$\left[\widetilde{L}^{u'/u}\right]^{1/2} \;=\; \exp\left\{\frac{1}{2}\left(\widetilde{W}_{u'}-\widetilde{W}_u\right) \;-\; \frac{1}{8}\,|u'-u|\right\} e^{-\frac{1}{8}|u'-u|} \;.$$

By lemma 5.3, the expectation of the first term on the right hand side under $\widetilde{P}_u$ equals 1: this proves (67). At the same time, this gives

$$E_u\left(\left[1-\left(\widetilde{L}^{u'/u}\right)^{1/2}\right]^2\right) \;=\; 2\left(1-e^{-\frac{1}{8}|u'-u|}\right)$$

and thus (65). Similiar calculations considering $\left[\widetilde{L}^{u'/u}\right]^{j/4}$ for $j=1,2,3$ give

$$E_u\left(\left[1-\left(\widetilde{L}^{u'/u}\right)^{1/4}\right]^4\right) \;=\; 2 \;+\; 6\,e^{-\frac{1}{8}|u'-u|} \;-\; 8\,e^{-\frac{3}{32}|u'-u|}$$

which behaves as $cst \cdot |u' - u|^2$ as $u' \to u$. This proves (66). $\diamond$

During the proof of (65) above, we have calculated $2\,H^2\left(\widetilde{P}_{u'}, \widetilde{P}_u\right)$, so we have 'smoothness with index $\frac{1}{2}$' of the parametrization in the sense of Hellinger distance

$$(68) \qquad \lim_{u' \to u} \frac{1}{\sqrt{|u' - u|}}\, H\left(\widetilde{P}_{u'}, \widetilde{P}_u\right) \;=\; \sqrt{\frac{1}{8}}$$

at every point u of the limit experiment $\widetilde{\mathcal{E}} = \left\{\widetilde{P}_u : u \in I\!R\right\}$.

The next step is to pass from convergence of local models at ϑ in the sense of finite dimensional distributions of likelihoods – established in theorem 1.1 – to weak convergence in $C([-K, K])$ of likelihoods as asserted in theorem 1.4, for K arbitrarily large. To do this, we shall follow Ibragimov and Khasminskii [IH 81, theorems 5.1+5.2 in section 1 and 19–21 in appendix A1.4].

5.5 Proof of theorem 1.4: Fix $\vartheta \in \Theta$. In analogy to [IH 81], we use notations

$$Z_{n,\vartheta}(u) \;:=\; L_{nT}^{(\vartheta + \frac{u}{n})/\vartheta}\;,\; u \in \Theta_{\vartheta,n} \quad , \quad \widetilde{Z}(u) \;:=\; \widetilde{L}^{u/0}\;,\; u \in I\!R$$

and extend $\Theta_{\vartheta,n}$ to $I\!R$ by using the default definition indicated in the beginning of this section whenever necessary. For local models $\mathcal{E}_n^{(\vartheta)}$ at $\vartheta \in \Theta$, (54) in lemma 5.1 shows that there is some $q' \in I\!N$ and some constant C (both do not depend on ϑ or n) such that

$$(69) \qquad E_\vartheta\left(\left[\,Z_{n,\vartheta}^{1/4}(u_1) \;-\; Z_{n,\vartheta}^{1/4}(u_2)\,\right]^4\right) \;\le\; C\,(1 + K^{q'})\,|u_1 - u_2|^2 \quad \text{if } \; |u_1|, |u_2| \le K$$

holds for all $K \in I\!N$, $\vartheta \in \Theta$, $n \in I\!N$. By theorem 1.1 a), we have convergence as $n \to \infty$

$$(70) \qquad \left(Z_{n,\vartheta}^{1/4}(u)\right)_{u \in I\!R} \;\longrightarrow\; \left(\widetilde{Z}^{1/4}(u)\right)_{u \in I\!R}$$

in the sense of finite dimensional distributions. (55) in lemma 5.1 shows that there is some k (not depending on ϑ, n) such that

$$(71) \qquad E_\vartheta\left(Z_{n,\vartheta}^{1/2}(u)\right) \;\le\; e^{-k\,|u|} \quad , \quad u \in I\!R$$

for all $\vartheta \in \Theta$ and $n \in I\!N$. In the limit experiment $\widetilde{\mathcal{E}}$, we have the corresponding assertions

$$(72) \qquad E_{\widetilde{P}_0}\left(\left[\,\widetilde{Z}^{1/4}(u_1) \;-\; \widetilde{Z}^{1/4}(u_2)\,\right]^4\right) \;\le\; C\,|u_1 - u_2|^2 \quad , \quad u_1, u_2 \in I\!R$$

$$(73) \qquad E_{\widetilde{P}_0}\left(\widetilde{Z}^{1/2}(u)\right) \;\le\; e^{-k\,|u|} \quad , \quad u \in I\!R$$

from (66)+(67) in lemma 5.4.

1) Fix $K < \infty$ arbitrarily large. In virtue of [IH 81, p. 378], the assertions (69)+(70) imply weak convergence in $C([-K,K])$ of $Z_{n,\vartheta}^{1/4}$ under Q^ϑ to $\widetilde{Z}^{1/4}$ under $\widetilde{P}_0$ as $n \to \infty$, and thus weak convergence in $C([-K,K])$ of $Z_{n,\vartheta}$ under Q^ϑ to $\widetilde{Z}$ under $\widetilde{P}_0$ as $n \to \infty$. This proves part a) of the theorem.

2) According to Ibragimov and Khasminskii, certain bounds on maxima of the likelihood ratios over spherical regions with center ϑ in the parameter space guarantee convergence of both maximum likelihood and Bayes estimators. We formulate such bounds in (74) and (75) below. Exactly on the lines of the arguments given in [IH 81, p. 372 and p. 42–44], we deduce from assertions (69)+(71) above the following: there is some $q \in I\!N$ and – with respect to any $K_0 > 0$ fixed – suitable constants $b_3, b_2 > 0$ such that

$$(74) \qquad \begin{cases} Q^\vartheta \left(\sup_{K+r \le |u| < K+r+1} Z_{n,\vartheta}(u) \ \geq \varepsilon \right) \ \leq \ \frac{1}{\sqrt{\varepsilon}}\, b_3\, (K{+}r{+}1)^q\, e^{-\,b_2(K+r)} \\ \text{for any choice of } \ \varepsilon > 0\ ,\ r \in I\!N_0\ ,\ K \geq K_0\ . \end{cases}$$

We remark that the constants b_3, b_2, q in (74) do not depend on n or ϑ (since they come from the right hand sides in (69)+(71) and from choice of K_0). Similiarly, at the level of the limit experiment $\widetilde{\mathcal{E}}$, we deduce in the same way from (72)+(73) above

$$(75) \qquad \begin{cases} \widetilde{P}_0 \left(\sup_{K+r \le |u| < K+r+1} \widetilde{Z}(u) \ \geq \varepsilon \right) \ \leq \ \frac{1}{\sqrt{\varepsilon}}\, b_3\, (K{+}r{+}1)^q\, e^{-\,b_2(K+r)} \\ \text{for any choice of } \ \varepsilon > 0\ ,\ r \in I\!N_0\ ,\ K \geq K_0\ . \end{cases}$$

3) From (74) with $\varepsilon = 1$, we draw a first conclusion: we have for all $\vartheta \in \Theta$ and $n \geq 1$

$$(76) \qquad Q^\vartheta \left(\sup_{|u|>K} Z_{n,\vartheta}(u) \ \geq 1 \right) \ \leq \ b_1\, e^{-\,b_2\, K}\ , \quad K \geq K_0$$

with constants which do not depend on ϑ, n or $K \geq K_0$. This is seen similiar to [IH 81, p. 43] after summation over $r \in I\!N_0$ in (74) with $\varepsilon = 1$. As a second conclusion from (74), we prove the assertion (8) in theorem 1.4: for $p \in I\!N$ arbitrarily large

$$Q^\vartheta \left(\sup_{|u|>K} |u|^p\, Z_{n,\vartheta}(u) \ \geq 1 \right) \ \leq \ b_1(p)\, e^{-\,b_2\, K}\ , \quad K \geq K_0$$

for all $\vartheta \in \Theta$ and $n \geq 1$, where the constants do not depend on ϑ, n or $K \geq K_0$. This follows from

$$Q^\vartheta \left(\sup_{K+r \le |u| < K+r+1} |u|^p\, Z_{n,\vartheta}(u) \ \geq 1 \right) \ \leq \ Q^\vartheta \left(\sup_{K+r \le |u| < K+r+1} Z_{n,\vartheta}(u) \ \geq \frac{1}{(K+r+1)^p} \right)$$

and (74) where ε is replaced by $\varepsilon_r := \frac{1}{(K+r+1)^p}$, after summation over $r \geq 0$. At the level of the limit experiment $\widetilde{\mathcal{E}}$, we start from (75) and obtain in the same way

$$\widetilde{P}_0 \left(\sup_{|u|>K} \widetilde{Z}(u) \ \geq 1 \right) \ \leq \ b_1 \, e^{-\, b_2 \, K} \ , \quad K \geq K_0 \tag{77}$$

together with the assertion (9) in theorem 1.4

$$\widetilde{P}_0 \left(\sup_{|u|>K} |u|^p \, \widetilde{Z}(u) \ \geq 1 \right) \ \leq \ b_1(p) \, e^{-\, b_2 \, K} \ , \quad K \geq K_0$$

for arbitrary $p \in I\!\!N$. Part b) of theorem 1.4 is proved.

4) To prepare for part c) of theorem 1.4, we consider first – following [IH 81, p. 45] – the denominators in (10) or (11): there is some $D > 0$ such that

$$Q^\vartheta \left(\int_{I\!\!R} Z_{n,\vartheta}(u')du' \ < \ \frac{\delta}{2} \right) \ < \ D\sqrt{\delta} \quad , \quad \widetilde{P}_0 \left(\int_{I\!\!R} \widetilde{Z}(u')du' \ < \ \frac{\delta}{2} \right) \ < \ D\sqrt{\delta}$$

holds for all $0 < \delta < 1$ and for all n, ϑ. As in [IH 81, p. 45-46], this is seen from

$$Q^\vartheta \left(\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} Z_{n,\vartheta}(u')du' \ < \ \frac{\delta}{2} \right) \ = \ Q^\vartheta \left(\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} [Z_{n,\vartheta}(u') - 1]du' \ < \ -\frac{\delta}{2} \right)$$

$$\leq \ Q^\vartheta \left(\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} |Z_{n,\vartheta}(u') - 1|du' \ > \ \frac{\delta}{2} \right) \ \leq \ 2 \cdot \frac{1}{\delta} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} E_\vartheta \left(|Z_{n,\vartheta}(u') - 1| \right) du'$$

where the last integrand is a total variation distance, thus smaller than Hellinger distance, hence we use (53) for values $|u'| \leq \frac{1}{2}$ of the local parameter to have $E_\vartheta \left(|Z_{n,\vartheta}(u') - 1| \right) \leq cst\sqrt{|u'|}$. This proves the first of the two assertions; the second follows in the same way from (65).

5) Next we show on the lines of [IH 81, p. 47] that for K_0 fixed and all $K \geq K_0$, $r \in I\!\!N_0$

$$E_\vartheta \left(\int_{\{K+r \leq |u| < K+r+1\}} \frac{Z_{n,\vartheta}(u)}{\int_{I\!\!R} Z_{n,\vartheta}(u')du'} \, du \right) \ \leq \ cst \cdot e^{-\, \frac{1}{4} \, b_2 (K+r)} \tag{78}$$

$$E_{\widetilde{P}_0} \left(\int_{\{K+r \leq |u| < K+r+1\}} \frac{\widetilde{Z}(u)}{\int_{I\!\!R} \widetilde{Z}(u')du'} \, du \right) \ \leq \ cst \cdot e^{-\, \frac{1}{4} \, b_2 (K+r)} \tag{79}$$

with respect to the same spherical sections as in step 2), and with constants which do not depend on n or ϑ. Write for short

$$I_r \ := \ \int_{\{K+r \leq |u| < K+r+1\}} Z_{n,\vartheta}(u) \, du \quad , \quad Q_r \ := \ \int_{\{K+r \leq |u| < K+r+1\}} \frac{Z_{n,\vartheta}(u)}{\int_{I\!\!R} Z_{n,\vartheta}(u')du'} \, du \ .$$

Obviously $\{I_r > 2\varepsilon\}$ is a subset of $\{ \sup_{K+r \leq |u| < K+r+1} Z_{n,\vartheta}(u) \geq \varepsilon \}$, so we apply (74) of step 2) above with $b_2 > 0$ as there, with $\varepsilon = \varepsilon_r = \frac{1}{2} e^{-b_2(K+r)}$ depending on r, and obtain

$$Q^\vartheta \left(\ I_r > e^{-b_2(K+r)} \ \right) \ \leq \ \widetilde{b}_3 \, (K+r+1)^q \, e^{-\, \frac{1}{2} \, b_2 \, (K+r)} \quad , \quad r \in I\!\!N_0 \ , \ K \geq K_0 \ .$$

From this we get thanks to $Q_r \le 1$ and step 4) for any $0 < \delta < 1$

$$\begin{aligned} E_\vartheta\,(\,Q_r\,) \;&\le\; Q^\vartheta\left(\int_{I\!R} Z_{n,\vartheta}(u')du' \,<\, \frac{\delta}{2}\right) \,+\, Q^\vartheta\left(\,I_r > e^{-b_2(K+r)}\,\right) \,+\, 2\,\frac{1}{\delta}\,e^{-\,b_2\,(K+r)} \\ &\le\; cst\cdot\left(\sqrt{\delta} \,+\, (K{+}r{+}1)^q\, e^{-\,\frac{1}{2}\,b_2\,(K+r)} \,+\, \frac{1}{\delta}\,e^{-\,b_2\,(K+r)}\right) \end{aligned}$$

with $\delta = \delta_r = e^{-\,\frac{1}{2}\,b_2(K+r)}$ the assertion (78). The proof of (79) uses the same arguments.

6) To finish the proof of part c) of theorem 1.4, it is sufficient to bound for arbitrary $p \in I\!N$ and $K \ge K_0$ the integrals

$$E_\vartheta\left(\int_{\{|u|>K\}} |u|^p\,\frac{Z_{n,\vartheta}(u)}{\int_{I\!R} Z_{n,\vartheta}(u')du'}\,du\right) \quad\text{or}\quad E_{\widetilde{P}_0}\left(\int_{\{|u|>K\}} |u|^p\,\frac{\widetilde{Z}(u)}{\int_{I\!R} \widetilde{Z}(u')du'}\,du\right)$$

by sums

$$\sum_{r\in I\!N_0} (K+r+1)^p\; E_\vartheta\left(\int_{\{K+r\le|u|<K+r+1\}} \frac{Z_{n,\vartheta}(u)}{\int_{I\!R} Z_{n,\vartheta}(u')du'}\,du\right)$$

or

$$\sum_{r\in I\!N_0} (K+r+1)^p\; E_{\widetilde{P}_0}\left(\int_{\{K+r\le|u|<K+r+1\}} \frac{\widetilde{Z}(u)}{\int_{I\!R} \widetilde{Z}(u')du'}\,du\right)$$

and to apply (78) and (79). This finishes the proof of part c) of theorem 1.4. $\diamond$

Now we can apply this to convergence of maximum likelihood estimators (12)+(13) and of Bayes estimators (15)+(14). The core of the argument is well known, see the exposition in [K 08].

5.6 Proof of theorem 1.5 a): 1) Due to Ibragimov and Khasminskii, the following is known for the MLE in the limit experiment $\widetilde{\mathcal{E}}$. By [IH 81, lemma 2.5 on p. 335–336], the law $\mathcal{L}(\widehat{u}|\widetilde{P}_0)$ has no point masses, and the argmax in (13) is unique almost surely. Symmetry in law of two-sided Brownian motion around 0 implies that $\mathcal{L}(\widehat{u}|\widetilde{P}_0)$ is symmetric around 0, cf. (5)+(6).

2) With explicit reference to the 'min' in definition (12), we start from

$$Q^\vartheta\left(n\,(\widehat{\vartheta}_{nT} - \vartheta) \le x\right) \;=\; Q^\vartheta\left(\sup_{u\le x} L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta} \;\ge\; \sup_{u\ge x} L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta}\right) \tag{80}$$

and

$$\widetilde{P}_0\,(\,\widehat{u} \le x\,) \;=\; \widetilde{P}_0\left(\sup_{u\le x} \widetilde{L}^{u/0} \;\ge\; \sup_{u\ge x} \widetilde{L}^{u/0}\right)\;. \tag{81}$$

Fix $\delta > 0$ arbitrarily small. By (8) in theorem 1.4 with $p = 1$, there is some large K such that the right hand side in (80) can be approximated first by

$$Q^\vartheta\left(\sup_{u\in[-K,x]} L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta} \;\ge\; \sup_{u\in[x,K]} L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta}\;,\; \sup_{|u|>K} L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta} < \frac{1}{K}\right) \quad\pm\quad \frac{\delta}{2}$$

and second by

(82) $$Q^{\vartheta}\left(\sup_{u\in[-K,x]} L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta} \;\geq\; \sup_{u\in[x,K]} L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta} \right) \quad \pm \quad \delta\ ,$$

up to errors which independently of n and ϑ are bounded by δ. Using (9) in theorem 1.4 with $p=1$, the same argument in the limit experiment $\widetilde{\mathcal{E}}$ shows that the right hand side of (81) is

(83) $$\widetilde{P}_0\left(\sup_{u\in[-K,x]} \widetilde{L}^{u/0} \geq \sup_{u\in[x,K]} \widetilde{L}^{u/0} \right) \quad \pm \quad \delta$$

up to an error bounded by δ. Since $\alpha \to \sup\limits_{u\in[-K,x]} \alpha(u)$ and $\alpha \to \sup\limits_{u\in[x,K]} \alpha(u)$ are continuous functions from $C([-K,K])$ to $I\!R$, the continuous mapping theorem combined with part a) of theorem 1.4 shows that pairs

$$\left(\sup_{u\in[-K,x]} L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta}\ ,\ \sup_{u\in[x,K]} L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta} \right) \quad \text{under } Q^{\vartheta}$$

converge weakly in $I\!R^2$ as $n\to\infty$ to

$$\left(\sup_{u\in[-K,x]} \widetilde{L}^{u/0}\ ,\ \sup_{u\in[x,K]} \widetilde{L}^{u/0} \right) \quad \text{under } \widetilde{P}_0$$

Again by [IH 81, lemma 2.5 on p. 335–336], 0 is a continuity point for the law of the difference

$$\mathcal{L}\left(\sup_{u\in[-K,x]} \widetilde{L}^{u/0} - \sup_{u\in[x,K]} \widetilde{L}^{u/0} \mid \widetilde{P}_0 \right) .$$

Since δ was arbitrary, we see that the right hand side of (80) converges as $n\to\infty$ to the right hand side of (81). We have proved weak convergence of MLE.

3) As an auxiliar remark towards weak convergence of BE with the help of theorem 1.4 a), we recall (8) in theorem 1.4 with $p=3$

$$Q^{\vartheta}\left(L_{nT}^{(\vartheta+\frac{u}{n})\,/\,\vartheta} \;\geq\; |u|^{-3} \text{ for some } |u|\geq K \right) \;\leq\; b_1\, e^{-b_2 K}\ ,\quad K\geq K_0$$

with constants which do not depend on n or ϑ; using this, we obtain

$$Q^{\vartheta}\left(\int_{\{|u|>K\}} |u|\, L_{nT}^{(\vartheta+\frac{u}{n})\,/\,\vartheta}\, du \;>\; \frac{1}{K} \right) \;\leq\; b_1\, e^{-b_2 K}\ ,\quad K\geq K_0\ .$$

Similiarly, in the limit experiment, (9) in theorem 1.4 with $p=3$ gives

$$\widetilde{P}_0\left(\int_{\{|u|>K\}} |u|\, \widetilde{L}^{u\,/\,0}\, du \;>\; \frac{1}{K} \right) \;\leq\; b_1\, e^{-b_2 K}\ ,\quad K\geq K_0\ .$$

4) To prove weak convergence of BE, we write the rescaled estimation errors at ϑ as

$$n\,\frac{\int_\Theta (\zeta-\vartheta)\, L_{nT}^{\zeta/\zeta_0}\, d\zeta}{\int_\Theta L_{nT}^{\zeta/\zeta_0}\, d\zeta} \;=\; n\,\frac{\int_\Theta (\zeta-\vartheta)\, L_{nT}^{\zeta/\vartheta}\, d\zeta}{\int_\Theta L_{nT}^{\zeta/\vartheta}\, d\zeta} \;=\; \frac{\int_{\Theta_{\vartheta,n}} u\, L_{nT}^{\vartheta+\frac{u}{n}/\vartheta}\, du}{\int_{\Theta_{\vartheta,n}} L_{nT}^{\vartheta+\frac{u}{n}/\vartheta}\, du}$$

according to definition (15), and consider

$$(84)\qquad Q^\vartheta\left(n\,(\vartheta^*_{nT}-\vartheta)\le x\right) \;=\; Q^\vartheta\left(\frac{\int_{\Theta_{\vartheta,n}} u\, L_{nT}^{\vartheta+\frac{u}{n}/\vartheta}\, du}{\int_{\Theta_{\vartheta,n}} L_{nT}^{\vartheta+\frac{u}{n}/\vartheta}\, du}\le x\right) .$$

By definition in (14), the analogue in the limit experiment $\widetilde{\mathcal{E}}$ is

$$(85)\qquad \widetilde{P}_0\left(\, u^* \le x\,\right) \;=\; \widetilde{P}_0\left(\frac{\int_{-\infty}^{\infty} u\, \widetilde{L}^{u/0}\, du}{\int_{-\infty}^{\infty} \widetilde{L}^{u/0}\, du}\le x\right) .$$

For $\delta>0$ arbitrarily small, step 3) gives $K=K(\delta)$ large enough to approximate

$$(86)\qquad \left(\int_{\Theta_{\vartheta,n}} u\, L_{nT}^{\vartheta+\frac{u}{n}/\vartheta}\, du\;,\;\int_{\Theta_{\vartheta,n}} L_{nT}^{\vartheta+\frac{u}{n}/\vartheta}\, du\right)\quad \text{under } Q^\vartheta$$

independently of n or ϑ by

$$(87)\qquad \left(\int_{[-K,K]} u\, L_{nT}^{\vartheta+\frac{u}{n}/\vartheta}\, du\;,\;\int_{[-K,K]} L_{nT}^{\vartheta+\frac{u}{n}/\vartheta}\, du\right)\quad \text{under } Q^\vartheta$$

(in the sense that the Q^ϑ–probability to have differences bigger than $\frac{1}{K}$ in the first or in the second component is at most δ, independently of n or ϑ). In the limit experiment we approximate

$$(88)\qquad \left(\int_{-\infty}^{\infty} u\, \widetilde{L}^{u/0}\, du\;,\;\int_{-\infty}^{\infty} \widetilde{L}^{u/0}\, du\right)\quad \text{under } \widetilde{P}_0$$

in the same sense by

$$(89)\qquad \left(\int_{[-K,K]} u\, \widetilde{L}^{u/0}\, du\;,\;\int_{[-K,K]} \widetilde{L}^{u/0}\, du\right)\quad \text{under } \widetilde{P}_0\,.$$

Since the function $\alpha \longrightarrow \left(\int_{[-K,K]} u\,\alpha(u)\,du\,,\,\int_{[-K,K]}\alpha(u)\,du\right)$ is continuous on $C([-K,K])$, we have weak convergence in $I\!R^2$ of (87) under Q^ϑ to (89) under $\widetilde{P}_0$. Since δ was arbitrary, we have also weak convergence in $I\!R^2$ of (86) under Q^ϑ to (88) under $\widetilde{P}_0$. The limit variable (88) has its second component strictly positive, $\widetilde{P}_0$–almost surely. Thus the continous mapping theorem applies to the ratio of the first to the second component in (86)+(88). We have proved weak convergence of Bayes estimators. This concludes the proof of theorem 1.5 a). $\diamond$

5.7 Proof of theorem 1.5 b): 1) In a first step, we deduce from theorem 1.4 c) the following: For arbitrary $H_0>0$ fixed, there are constants $\widetilde{b}_1$ and $\widetilde{b}_2$ such that

$$(90)\qquad Q^\vartheta\left(\,|n\,(\vartheta^*_{nT}-\vartheta)|>H\,\right)\;\le\;\widetilde{b}_1\, e^{-\widetilde{b}_2\, H}\quad,\quad H>H_0$$

holds, where the right hand side does not depend on n or ϑ. Using the trivial inequality

$$\int_{\{|u|\le K\}} |u| \, \frac{L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta}}{\int_{I\!R} L_{nT}^{(\vartheta+\frac{u'}{n})/\vartheta} du'} \, du \;\; \le \;\; K$$

assertion (90) follows as in [IH 81, p. 45] via

$$Q^\vartheta \left(\left| \int_{I\!R} u \, L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta} du \right| \, > \, 2K \int_{I\!R} L_{nT}^{(\vartheta+\frac{u'}{n})/\vartheta} du' \right) \;\; \le \;\; Q^\vartheta \left(\int_{I\!R} |u| \, \frac{L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta}}{\int_{I\!R} L_{nT}^{(\vartheta+\frac{u'}{n})/\vartheta} du'} \, du \, > \, 2K \right)$$

$$\le \;\; Q^\vartheta \left(\int_{\{|u|>K\}} |u| \, \frac{L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta}}{\int_{I\!R} L_{nT}^{(\vartheta+\frac{u'}{n})/\vartheta} du'} \, du \, > \, K \right) \;\; \le \;\; \frac{1}{K} E_\vartheta \left(\int_{\{|u|>K\}} |u| \, \frac{L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta}}{\int_{I\!R} L_{nT}^{(\vartheta+\frac{u'}{n})/\vartheta} du'} \, du \right)$$

from the exponential bound (10) in theorem 1.4 c), valid for all $K \ge K_0$, and not depending on n or ϑ. (90) is the analogue of (76) above for MLE

$$Q^\vartheta \left(\left| n \, (\widehat{\vartheta}_{nT} - \vartheta) \right| > H \right) \; \le \; Q^\vartheta \left(\sup_{|u|>H} \; Z_{n,\vartheta}(u) \; \ge 1 \right) \;\; \le \;\; b_1 \, e^{-\, b_2 \, H} \;\; , \quad H \ge H_0 \; .$$

By weak convergence in virtue of theorem 1.5 a) combined with monotonicity properties in H, these bounds (90)+(76) carry over to the limit experiment:

$$\text{(91)} \qquad \widetilde{P}_0 \left(|u^*| > H \right) \; \le \; \widetilde{b}_1 \, e^{-\widetilde{b}_2 \, H} \quad , \quad H \ge H_0 \; ,$$

$$\text{(92)} \qquad \widetilde{P}_0 \left(|\widehat{u}| > H \right) \; \le \; b_1 \, e^{-b_2 \, H} \quad , \quad H \ge H_0 \; .$$

2) Now convergence of moments of order $p \in I\!N$ for MLE and BE

$$\int_0^\infty H^{p-1} \, Q^\vartheta \left(\left| n \, (\widehat{\vartheta}_{nT} - \vartheta) \right| > H \right) \, dH \quad , \quad \int_0^\infty H^{p-1} \, Q^\vartheta \left(\, | n \, (\vartheta^*_{nT} - \vartheta) | > H \, \right) \, dH$$

as $n \to \infty$ is a consequence of weak convergence established in theorem 1.5 a) together with dominated convergence guaranteed by the exponential bounds (90)+(76). The limits as $n \to \infty$

$$\int_0^\infty H^{p-1} \, \widetilde{P}_0 \left(\, |\widehat{u}| > H \, \right) \, dH \quad , \quad \int_0^\infty H^{p-1} \, \widetilde{P}_0 \left(\, |u^*| > H \, \right) \, dH$$

are the moments of order p of MLE and BE in the limit experiment $\widetilde{\mathcal{E}}$. ◇

Now we discuss equivariance properties of the BE sequence and prove proposition 1.7.

5.8 Proof of proposition 1.7: 1) We have seen in lemma 5.3 that

$$\widetilde{L}^{(u+h)/u} \;=\; \exp\left\{ \left(\widetilde{W}_{u+h} - \widetilde{W}_u \right) - \frac{1}{2} \, |h| \right\} \quad \text{for all } h \in I\!R$$

where $\left(\widetilde{W}_{u+h} - \widetilde{W}_u\right)_{h\in I\!R}$ is two-sided Brownian motion under $\widetilde{P}_u$. As a first consequence, we have noted in 5.3 that $\left\{\widetilde{P}_{u+h} : h \in I\!R\right\}$ is statistically the same experiment as $\widetilde{\mathcal{E}}$, for every u fixed. As a second consequence, the law of the estimation error of a Bayes estimator 'with uniform prior over $I\!R$' as defined in (14) does not depend on $u \in I\!R$ since

$$\mathcal{L}\left(u^* - u \mid \widetilde{P}_u\right) \;=\; \mathcal{L}\left(\frac{\int_{-\infty}^{\infty} u' \, \widetilde{L}^{u'/0} \, du'}{\int_{-\infty}^{\infty} \widetilde{L}^{u'/0} \, du'} - u \;\mid \widetilde{P}_u \right) \;=\; \mathcal{L}\left(\frac{\int_{-\infty}^{\infty} h \, \widetilde{L}^{(u+h)/u} \, dh}{\int_{-\infty}^{\infty} \widetilde{L}^{(u+h)/u} \, dh} \;\mid \widetilde{P}_u \right) .$$

This is equivariance of the BE in the limit experiment as asserted in 1.7 a).

2) In order to prove 1.7 b), we shall use 'LeCam's Third Lemma' for contiguous alternatives (see [LY 90, pp. 22–23], or [H 08, 3.6+3.16]) in combination with the above equivariance property of the BE in the limit experiment. We have seen in the proof 5.6 for theorem 1.5 a) that – as a consequence of weak convergence as $n \to \infty$ of likelihood ratios in $C([-K, K])$, for K arbitrarily large – pairs (86) under Q^ϑ

$$\left(\int_{\Theta_{\vartheta,n}} u \, L_{nT}^{\vartheta + \frac{u}{n} / \vartheta} \, du \, , \, \int_{\Theta_{\vartheta,n}} L_{nT}^{\vartheta + \frac{u}{n} / \vartheta} \, du \right)$$

converge weakly in $I\!R^2$ as $n \to \infty$ to the pair (88) under $\widetilde{P}_0$

$$\left(\int_{-\infty}^{\infty} u \, \widetilde{L}^{u/0} \, du \, , \, \int_{-\infty}^{\infty} \widetilde{L}^{u/0} \, du \right) .$$

Obviously, for $u_0 \in I\!R$ fixed, the same argument also yields joint convergence of triplets

$$(93) \qquad \left(L_{nT}^{\vartheta + \frac{u_0}{n} / \vartheta} \, , \, \int_{\Theta_{\vartheta,n}} u \, L_{nT}^{\vartheta + \frac{u}{n} / \vartheta} \, du \, , \, \int_{\Theta_{\vartheta,n}} L_{nT}^{\vartheta + \frac{u}{n} / \vartheta} \, du \right) \quad \text{under } Q^\vartheta$$

weakly in $I\!R^3$ as $n \to \infty$ to

$$\left(\widetilde{L}^{u_0/0} \, , \, \int_{-\infty}^{\infty} u \, \widetilde{L}^{u/0} \, du \, , \, \int_{-\infty}^{\infty} \widetilde{L}^{u/0} \, du \right) \quad \text{under } \widetilde{P}_0 \, .$$

For any convergent sequence $(u_n)_n$ tending to the limit u_0, this convergence remains valid if we place $L_{nT}^{\vartheta + \frac{u_n}{n} / \vartheta}$ instead of $L_{nT}^{\vartheta + \frac{u_0}{n} / \vartheta}$ into the first component of (93). From joint convergence with the sequence of likelihood ratios, LeCam's Third Lemma deduces weak convergence under the corresponding contiguous alternatives: thus

$$\left(L_{nT}^{\vartheta + \frac{u_n}{n} / \vartheta} \, , \, \int_{\Theta_{\vartheta,n}} u \, L_{nT}^{\vartheta + \frac{u}{n} / \vartheta} \, du \, , \, \int_{\Theta_{\vartheta,n}} L_{nT}^{\vartheta + \frac{u}{n} / \vartheta} \, du \right) \quad \text{under } Q^{\vartheta + \frac{u_n}{n}}$$

converges weakly in $I\!R^3$ as $n \to \infty$ to

$$\left(\widetilde{L}^{u_0/0} \, , \, \int_{-\infty}^{\infty} u \, \widetilde{L}^{u/0} \, du \, , \, \int_{-\infty}^{\infty} \widetilde{L}^{u/0} \, du \right) \quad \text{under } \widetilde{P}_{u_0} \, .$$

Using the continuous mapping theorem we obtain weak convergence of ratios

$$\frac{\int_{\Theta_{\vartheta,n}} (u - u_n)\, L_{nT}^{\vartheta+\frac{u}{n}/0}\, du}{\int_{\Theta_{\vartheta,n}} L_{nT}^{\vartheta+\frac{u}{n}/0}\, du} \quad \text{under } Q^{\vartheta+\frac{u_n}{n}}$$

as $n \to \infty$ to

$$\frac{\int_{-\infty}^{\infty} (u - u_0)\, \widetilde{L}^{u/0}\, du}{\int_{-\infty}^{\infty} \widetilde{L}^{u/0}\, du} \quad \text{under } \widetilde{P}_{u_0}\ .$$

Thus we have proved weak convergence as $n \to \infty$ of rescaled BE errors

$$\mathcal{L}\left(n(\vartheta^*_{nT} - (\vartheta + \frac{u_n}{n})) \mid Q^{\vartheta+\frac{u_n}{n}} \right) \quad\longrightarrow\quad \mathcal{L}\left(u^* - u_0 \mid \widetilde{P}_{u_0} \right) \tag{94}$$

under contiguous alternatives, for arbitrary convergent sequences $u_n \to u_0$.

3) Consider a loss function $\ell(\cdot)$ which is continuous, subconvex and bounded. Then (94) gives

$$E_{\vartheta+\frac{u_n}{n}} \left(\ell \left(n(\vartheta^*_{nT} - (\vartheta + \frac{u_n}{n})) \right) \right) \quad\longrightarrow\quad E_{\widetilde{P}_0} \left(\ell \left(u^* - u_0 \right) \right) \tag{95}$$

for arbitrary convergent sequences $u_n \to u_0$. Via selection of convergent subsequences in compacts $[-C, C]$, for C arbitrarily large, (95) shows

$$\lim_{n\to\infty} \sup_{|u|\le C} \left| E_{\vartheta+\frac{u}{n}} \left(\ell \left(n(\vartheta^*_{nT} - (\vartheta{+}\frac{u}{n})) \right) \right) \;-\; E_{\widetilde{P}_u} \left(\ell \left(u^* - u \right) \right) \right| \quad=\quad 0 \tag{96}$$

for loss functions $\ell(\cdot)$ which are continuous, subconvex and bounded. This is part b) of proposition 1.7, in the special case of bounded loss functions. However, this is sufficient to prove part b) of 1.7 in general. We have exponential decrease in the bounds (90) – where the constants are independent of ϑ and n – and polynomial bounds for ℓ, thus contributions

$$\sup_{n\ge n_0} \sup_{\vartheta'\in\Theta} \int_{H>K} H^{p-1}\, Q^{\vartheta'} \left(\left| n\left(\vartheta^*_{nT} - \vartheta' \right) \right| > H \right)\ dH$$

can be made arbitrarily small by suitable choice of K. At the level of the limit experiments, we use (91). This remark combined with (96) completes the proof of proposition 1.7. $\diamond$

We turn to the proof of theorem 1.8.

5.9 Proof of theorem 1.8: Fix $\vartheta \in \Theta$. Throughout this proof, we consider only $\ell(x) = x^2$. Since part b) of theorem 1.8 is immediate from proposition 1.7 b), we only have to prove the local asymptotic minimax bound proposed in part a).

1) In a first step, we prove for fixed $C < \infty$ a preliminary bound

$$\liminf_{n\to\infty} \inf_{\widetilde\vartheta_{nT}} \sup_{|u|\le C} E_{\vartheta+\frac{u}{n}}\left(\left[n\left(\widetilde\vartheta_{nT} - (\vartheta+\frac{u}{n})\right)\right]^2 \right)$$

(97)
$$\ge \int_{[-C,C]} \frac{du}{2C}\, E_{\widetilde P_u}\left(\left[\frac{\int_{[-C,C]} (u'-u)\, \widetilde L^{u'/u}\, du'}{\int_{I\!R} \widetilde L^{u'/u}\, du'} \right]^2 \right)$$

where $\inf_{\widetilde\vartheta_{nT}}$ is with respect to all possible $\mathcal{G}^{\xi}_{nT}$–measurable estimators.
To prove (97), introduce an auxiliary estimator in the limit experiment $\widetilde{\mathcal{E}}$

$$u^{**} := \frac{\int_{[-C,C]} u'\, \widetilde L^{u'/0}\, du'}{\int_{[-C,C]} \widetilde L^{u'/0}\, du'}$$

and auxiliary random objects related to the BE of (15)

$$\vartheta^{**}_{nT} := \frac{\int_{V_{n,\vartheta}(C)} \zeta\, L^{\zeta/0}_{nT}\, d\zeta}{\int_{V_{n,\vartheta}(C)} L^{\zeta/0}_{nT}\, d\zeta} = \frac{\int_{[-C,C]} (\vartheta + \frac{u'}{n})\, L^{(\vartheta+\frac{u'}{n})/\vartheta}_{nT}\, du'}{\int_{[-C,C]} L^{(\vartheta+\frac{u'}{n})/\vartheta}_{nT}\, du'} \quad , \quad n \ge 1$$

where we write $V_{n,\vartheta}(C)$ for the closed ball of radius $\frac{C}{n}$ centred at ϑ. After replacing $\sup_{|u|\le C} \ldots$ on the left hand side of (97) by an integral $\int_{[-C,C]} \frac{du}{2C} \ldots$ which obviously yields a lower bound, the L^2-projection property of conditional expectations shows that a minimum

$$\min_{\widetilde\vartheta_{nT}} \int_{[-C,C]} \frac{du}{2C}\, E_{\vartheta+\frac{u}{n}}\left(\left[n\left(\widetilde\vartheta_{nT} - (\vartheta+\frac{u}{n})\right)\right]^2 \right)$$

exists for every $n \ge 1$ fixed, and is realized by

$$\int_{[-C,C]} \frac{du}{2C}\, E_{\vartheta+\frac{u}{n}}\left(\left[n\left(\vartheta^{**}_{nT} - (\vartheta+\frac{u}{n})\right)\right]^2 \right) .$$

Next, as $n \to \infty$, we use weak convergence under contiguous alternatives very similiar to step 2) in the proof 5.8 of proposition 1.7 to show that for every $u \in I\!R$ fixed, we have weak convergence

$$\mathcal{L}\left(n\left(\vartheta^{**}_{nT} - (\vartheta+\frac{u}{n})\right) \mid Q^{\vartheta+\frac{u}{n}} \right) \longrightarrow \mathcal{L}\left(u^{**} - u \mid \widetilde P_u \right) \quad , \quad n \to \infty .$$

As a consequence, for any bounded loss function $\ell \wedge N(\cdot)$ obtained by truncating $\ell(x) = x^2$,

$$E_{\vartheta+\frac{u}{n}}\left(\ell \wedge N\left(n\left(\vartheta^{**}_{nT} - (\vartheta+\frac{u}{n})\right)\right)\right) \longrightarrow E_{\widetilde P_u}\left(\ell \wedge N\left(u^{**} - u\right)\right) , \quad n \to \infty .$$

Now, for $u \in [-C, C]$, since u^{**} as well as $n(\vartheta^{**}_{nT} - \vartheta)$, $n \ge 1$, take their values in $[-C, C]$, we see that truncation by N in the last line is without effect whenever N is large enough. This has the following two consequences: i) we can state the last line with $\ell(x) = x^2$, without truncation; ii) the convergence is dominated, thus we have convergence of integrals

$$\int_{[-C,C]} \frac{du}{2C}\, E_{\vartheta+\frac{u}{n}} \left(\left[n \left(\vartheta^{**}_{nT} - (\vartheta + \frac{u}{n}) \right) \right]^2 \right) \quad \longrightarrow \quad \int_{[-C,C]} \frac{du}{2C}\, E_{\widetilde{P}_u} \left([u^{**} - u]^2 \right)$$

as $n \to \infty$. Note that C is fixed in the present step. Since obviously

$$[u^{**} - u]^2 \;=\; \left[\frac{\int_{[-C,C]} (u' - u)\, \widetilde{L}^{u'/0}\, du'}{\int_{[-C,C]} \widetilde{L}^{u'/0}\, du'} \right]^2 \;\geq\; \left[\frac{\int_{[-C,C]} (u' - u)\, \widetilde{L}^{u'/u}\, du'}{\int_{I\!R} \widetilde{L}^{u'/u}\, du'} \right]^2$$

we have proved (97).

2) Now we consider the object inside the square brackett on the right hand side of the last inequality. We introduce a representation

$$\text{(98)} \qquad \frac{\int_{[-C,C]} (u' - u)\, \widetilde{L}^{u'/u}\, du'}{\int_{I\!R} \widetilde{L}^{u'/u}\, du'} \;=\; (u^* - u) \;+\; \varrho^{(1)}_u(C) \;+\; \varrho^{(2)}_u(C)$$

with remainder terms $\varrho^{(i)}_u(C)$ which vanish as $C \to \infty$ in a sense to be considered below. For $u \in I\!R$ and $C < \infty$, introduce functions $\Phi_{C,u}$ on $I\!R$ by

$$\Phi_{C,u}(h) \;:=\; 1_{[-C-u, C-u]}(h) - 1_{[-C,C]}(h) \quad , \quad h \in I\!R \,.$$

With this notation, the left hand side of (98)

$$\int_{I\!R} 1_{[-C-u, C-u]}(h)\; h\; \frac{\widetilde{L}^{(u+h)/u}}{\int_{I\!R} \widetilde{L}^{(u+h)/u}\, dh}\, dh$$

can be written using the definition (14) of the BE in the limit experiment in the form (98) where

$$\text{(99)} \qquad \varrho^{(1)}_u(C) \;=\; \int_{I\!R} \Phi_{C,u}(h)\; h\; \frac{\widetilde{L}^{(u+h)/u}}{\int_{I\!R} \widetilde{L}^{(u+h)/u}\, dh}\, dh$$

$$\text{(100)} \qquad \varrho^{(2)}_u(C) \;=\; -\int_{I\!R} 1_{\{|h|>C\}}\; h\; \frac{\widetilde{L}^{(u+h)/u}}{\int_{I\!R} \widetilde{L}^{(u+h)/u}\, dh}\, dh$$

By lemma 5.3 and as in the beginning of the proof 5.8 for proposition 1.7, the law under $\widetilde{P}_u$ of the first term and of the third term on the right hand side of (98) does not depend on $u \in I\!R$.

3) Now consider $K < \infty$ arbitrarily large but fixed, arbitrary $C > K$, and $u \in [-C, C]$. Since

$$\text{(101)} \qquad 1_{\{|h| \leq K\}}\, \Phi_{C,u}(h) \;=\; 0 \quad \text{for all } h \in I\!R \text{ provided } C > K + |u|$$

we obtain bounds as follows. If $u \in [C{-}K, C]$ or $u \in [-C, -C{+}K]$, we put

$$\text{(102)} \qquad \left| \varrho^{(i)}_u(C) \right| \;\leq\; \int_{I\!R} |h|\, \frac{\widetilde{L}^{(u+h)/u}}{\int_{I\!R} \widetilde{L}^{(u+h)/u}\, dh}\, dh \quad , \quad i = 1, 2 \,.$$

If $u \in (-C+K, C-K)$, introducing into (99) indicator functions $1_{\{|h|>K\}}$ and $1_{\{|h|\le K\}}$ and exploiting (101), we get

$$\left|\varrho_u^{(i)}(C)\right| \;\le\; \int_{\{|h|>K\}} |h| \, \frac{\widetilde{L}^{(u+h)/u}}{\int_{I\!R} \widetilde{L}^{(u+h)/u}\, dh}\, dh \quad , \quad i=1,2 \,. \tag{103}$$

4) We look to moments under $\widetilde{P}_u$ of the quantites considered in step 3) and define

$$D(K) \;:=\; 4\, E_{\widetilde{P}_0}\left(\int_{\{|h|>K\}} |h|^2 \, \frac{\widetilde{L}^{h/0}}{\int_{I\!R} \widetilde{L}^{h/0}\, dh}\, dh \right) \quad , \quad K \in I\!N_0 \,.$$

By theorem 1.4 c), $D(0)$ is finite, and $D(K) \to 0$ as $K \to \infty$. From (102)+(103) – bounds whose laws under $\widetilde{P}_u$ do not depend on $u \in I\!R$ – we have for $K < \infty$ fixed, for $C > K$ and $u \in [-C, C]$

$$E_{\widetilde{P}_u}\left(\left| \varrho_u^{(1)}(C) + \varrho_u^{(2)}(C) \right|^2 \right) \;\le\; D(0) \quad \text{if } u \in [C-K, C] \text{ or } u \in [-C, -C+K] \,,$$

$$E_{\widetilde{P}_u}\left(\left| \varrho_u^{(1)}(C) + \varrho_u^{(2)}(C) \right|^2 \right) \;\le\; D(K) \quad \text{if } u \in [-C+K, C-K] \,.$$

From (98) and Cauchy-Schwarz and $E_{\widetilde{P}_0}\left(|u^*|^2\right) \le \frac{1}{4} D(0)$, the difference

$$\left| E_{\widetilde{P}_u}\left(\left[\frac{\int_{[-C,C]} (u'-u)\, \widetilde{L}^{u'/u}\, du'}{\int_{I\!R} \widetilde{L}^{u'/u}\, du'} \right]^2 \right) \;-\; E_{\widetilde{P}_0}\left(|u^*|^2 \right) \right| \tag{104}$$

is bounded by

$$\begin{cases} 2D(0) & \text{if } u \in [C-K, C] \text{ or } u \in [-C, -C+K] \,, \\ D(K) \,+\, \sqrt{D(K)D(0)} & \text{if } u \in [-C+K, C-K] \,. \end{cases} \tag{105}$$

This holds for $K < \infty$ fixed and arbitrarily large, for arbitrary $C > K$, and for all $u \in [-C, C]$.

5) Now we conclude the proof of theorem 1.8. For K arbitrarily large but fixed, we combine the preliminary bound (97) with (104)+(105) which yields

$$\begin{aligned}
&\lim_{C\to\infty} \; \liminf_{n\to\infty} \; \inf_{\widetilde{\vartheta}_{nT}} \; \sup_{|u|\le C} \; E_{\vartheta+\frac{u}{n}}\left(\left[n\left(\widetilde{\vartheta}_{nT} - (\vartheta+\frac{u}{n}) \right) \right]^2 \right) \\
&\ge \; \lim_{C\to\infty} \int_{[-C,C]} \frac{du'}{2C}\, E_{\widetilde{P}_u}\left(\left[\frac{\int_{[-C,C]} (u'-u)\, \widetilde{L}^{u'/u}\, du'}{\int_{I\!R} \widetilde{L}^{u'/u}\, du'} \right]^2 \right) \\
&\ge \; E_{\widetilde{P}_0}\left(|u^*|^2 \right) \;-\; \lim_{C\to\infty} \left(2D(0)\,\frac{K}{C} \;+\; \left[D(K) + \sqrt{D(K)D(0)} \right] \left[1 - \frac{K}{C} \right] \right) .
\end{aligned}$$

We can make the right hand side of this inequality arbitrarily close to $E_{\widetilde{P}_0}\left(|u^*|^2\right)$ by choosing K large. This ends the proof of theorem 1.8. ◇

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Yury A. Kutoyants

Laboratoire de Statistique et Processus, Université du Maine, F–72085 Le Mans Cedex 9

kutoyants@univ-lemans.fr

http://www.univ-lemans.fr/sciences/statist/pages_persos/kuto.html

Reinhard Höpfner

Institut für Mathematik, Universität Mainz, D–55099 Mainz

hoepfner@mathematik.uni-mainz.de

http://joguinf.informatik.uni-mainz.de/~hoepfner