

L^p -Solutions for Reflected Backward Stochastic Differential Equations

Saïd Hamadène and Alexandre Popier
Laboratoire de statistiques et processus
Université du Maine
Avenue Olivier Messiaen
F-72085 Le Mans Cedex 9
France

March 27, 2009

Abstract

This paper deals with the problem of existence and uniqueness of a solution for a backward stochastic differential equation (BSDE for short) with one reflecting barrier in the case when the terminal value, the generator and the obstacle process are L^p -integrable with $p \in]1, 2[$. To construct the solution we use two methods: penalization and Snell envelope. As an application we broaden the class of functions for which the related obstacle partial differential equation problem has a unique viscosity solution.

2000 AMS subject classifications: Primary: 60G40, 60H10, 60H30; Secondary: 60G99

Key words and phrases: Reflected backward stochastic differential equations ; Snell envelope ; Penalization ; Obstacle PDE ; Viscosity solution.

1 Introduction

The notion of nonlinear backward stochastic differential equation (BSDE for short) was introduced by Pardoux and Peng (1990, [21]). A solution of this equation, associated with a terminal value ξ and a *generator or coefficient* $f(t, \omega, y, z)$, is a couple of adapted stochastic processes $(Y_t, Z_t)_{t \leq T}$ such that:

$$(1) \quad \forall t \leq T, \quad Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s,$$

where B is a Brownian motion and adaptation is related to natural filtration of B .

In their seminal work [21], Pardoux and Peng proved the existence and uniqueness of a solution under suitable assumptions, mainly square integrability of ξ and the process

$(f(t, \omega, 0, 0))_{t \leq T}$, on the one hand, and, the Lipschitz property w.r.t. (y, z) of the generator f , on the other hand. Since this first result, it has been widely recognized that BSDEs provide a useful framework for formulating a lot of mathematical problems in finance (see e.g. [2, 11, 13, 14],...), stochastic control and differential games (see e.g. [16, 17],...), partial differential equations (see e.g. [20, 22],...) and so on.

Another types of BSDEs, the one barrier reflected BSDEs, have been introduced by El-Karoui et al. [11]. For those BSDEs, one of the components of the solution is forced to stay above a given *barrier/obstacle* process $(L_t)_{t \leq T}$. Therefore a solution is a triple of adapted processes $(Y_t, Z_t, K_t)_{t \leq T}$ which satisfies:

$$(2) \quad \begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \\ Y_t &\geq L_t, \quad 0 \leq t \leq T \text{ and } \int_0^T (Y_s - L_s) dK_s = 0., \quad P - \text{a.s.}; \end{aligned}$$

here the process K is non-decreasing and its role is to push upward Y in order to keep it above the obstacle L .

These types of equations are connected with a wide range of applications especially the pricing of American options in markets, constrained or not, mixed control, partial differential variational inequalities, real options (see e.g. [7, 8, 11, 13, 15, 18, 19],...and the references therein). Once more under square integrability of the data and Lipschitz property of the coefficient f , the authors of [12] show existence and uniqueness of the solution.

For several years there have been a lot of works which deal with the issue of existence/uniqueness results under weaker assumptions than the ones of Pardoux-Peng [21] or El-Karoui et al [12]. However, for their own reasons, authors focus only on the weakness of the Lipschitz property of the coefficient and not on square integrability of the data ξ and $(f(t, \omega, 0, 0))_{t \leq T}$. Actually there have been relatively few papers which deal with the problem of existence/uniqueness of the solution for BSDEs in the case when the coefficients are not square integrable. Nevertheless we should point out that El-Karoui et al. (1997, [11]) and Briand et al. (2003, [3]) have proved existence and uniqueness of a solution for the standard BSDE (1) in the case when the data belong only to L^p for some $p \in]1, 2[$. Therefore the main objective of our paper is to complete those works and to study the reflected BSDE (2) in the case when the terminal condition ξ and the generator f are only p -integrable with $p \in]1, 2[$. The main motivation is that in several applications (finance, control, games, PDEs,...) the data are not square integrable and to assume them so is somehow restrictive.

In this article we show that if ξ , $\sup_{t \leq T} (L_t^+)$ and $\int_0^T |f(t, 0, 0)| dt$ belong to L^p for some $p \in]1, 2[$, then the BSDE (2) with one reflecting barrier associated with (f, ξ, L) has a unique solution. We prove existence and uniqueness of the solution in using penalization and Snell envelope of processes methods. We finally deal with the partial differential variational inequality (PDVI in short) associated with the reflected BSDE under consideration in the case when randomness comes from a diffusion process. Actually we show existence and uniqueness of the solution in viscosity sense for that PDVI in an appropriate space. This result broadens the class of functions for which there exists a unique solution for

this associated PDVI with obstacle. Another example of applications of our results is in connection with mixed control with coefficients only p -integrable (see e.g. [10]). Actually solving this latter problem turns into solving appropriate BSDEs of the types we consider in this work. Therefore we omit the proofs since they are the same, in their main steps, as the ones of [10].

The paper is organized as follows: the next section contains all the notations and basic estimates. In doing that, a key point is the use of the dual projection property for the process K in order to control its p -moments. We then prove uniqueness of the L^p -solution. In Sections 3 and 4 existence is established via two different methods. The first one studied in Section 3 uses a fixed point argument for an appropriate mapping and based also on arguments related to the Snell envelope of processes and optimal stopping. The second method, given in Section 4, is constructed by penalization of the constraint $Y \geq L$. In using Peng's monotonic limit [23], our proof is shorter and somehow different from the one of [12]. In Section 5, we show that, provided the problem is formulated within a Markovian framework, the solution of the reflected BSDE provides a probabilistic representation for the unique viscosity solution of an obstacle problem for a nonlinear parabolic partial differential equation with obstacle or variational inequality.

2 Notations, setting of the problem and preliminary results

Let (Ω, \mathcal{F}, P) be a fixed probability space on which is defined a standard d -dimensional Brownian motion $B = (B_t)_{t \leq T}$ whose completed, with the P -null sets of \mathcal{F} , natural filtration is denoted by $(\mathcal{F}_t)_{t \leq T}$, therefore it satisfies the usual conditions, i.e. it is right continuous and complete.

From now on stochastic processes will be defined for $t \in [0, T]$, where T is a positive real constant, and will takes their values in \mathbb{R}^n for some positive integer n . Finally for any $x, x' \in \mathbb{R}^k$, $|x|$ denotes the Euclidean norm of x and $\langle x, x' \rangle$ the inner product.

Next for any real constant $p \in]1, 2[$, let:

(i) $\mathcal{S}^p(\mathbb{R}^n)$ be the set of \mathbb{R}^n -valued, adapted and continuous processes $\{X_t\}_{t \in [0, T]}$ such that:

$$\|X\|_{\mathcal{S}^p} = \mathbb{E} \left[\sup_{t \in [0, T]} |X_t|^p \right]^{\frac{1}{p}} < +\infty.$$

The space $\mathcal{S}^p(\mathbb{R}^n)$ endowed with the norm $\|\cdot\|_{\mathcal{S}^p}$ is of Banach type.

(ii) $\mathcal{M}^p(\mathbb{R}^n)$ be the set of predictable processes $\{Z_t\}_{t \in [0, T]}$ with values in \mathbb{R}^n such that:

$$\|Z\|_{\mathcal{M}^p} = \mathbb{E} \left[\left(\int_0^T |Z_t|^2 dt \right)^{p/2} \right]^{1/p} < +\infty.$$

Once more $\mathcal{M}^p(\mathbb{R}^n)$ endowed with this norm $\|Z\|_{\mathcal{M}^p}$ is a Banach space.

Now let \mathcal{B}^p be the space $\mathcal{S}^p(\mathbb{R}) \times \mathcal{M}^p(\mathbb{R}^d)$. Let ξ be an \mathbb{R} -valued and \mathcal{F}_T -measurable random variable and let us consider a random function $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$

measurable with respect to $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)$ where \mathcal{P} denotes the σ -field of progressive subsets of $[0, T] \times \Omega$. Finally let $L := \{L_t\}_{t \in [0, T]}$ be a continuous progressively measurable \mathbb{R} -valued process. On the items ξ , L and f we make the following assumptions:

(H1) $\xi \in L^p(\Omega)$;

(H2) (i) the process $\{f(t, 0, 0), 0 \leq t \leq T\}$ satisfies $\mathbb{E} \left(\int_0^T |f(t, 0, 0)| dt \right)^p < +\infty$;

(ii) there exists a constant κ such that:

$$P - a.s., |f(t, y, z) - f(t, y', z')| \leq \kappa(|y - y'| + |z - z'|), \forall t, y, y', z, z'.$$

(H3) the barrier L is s.t. $L_T \leq \xi$ and $L^+ := L \vee 0 \in \mathcal{S}^p(\mathbb{R})$.

To begin with let us define the notion of a solution of the reflected BSDE associated with the triple (f, ξ, L) which we consider throughout this paper.

Definition 1 (of L^p -solutions) : We say that $\{(Y_t, Z_t, K_t), 0 \leq t \leq T\}$ is a L^p -solution of the reflected BSDE with one continuous lower reflecting barrier L , terminal condition ξ and generator f if the followings hold:

1. $\{(Y_t, Z_t), 0 \leq t \leq T\}$ belongs to \mathcal{B}^p ;
2. $K = \{K_t, 0 \leq t \leq T\}$ is an adapted continuous non decreasing process s.t. $K_0 = 0$ and $K_T \in L^p(\Omega)$;
3. $Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, 0 \leq t \leq T$ a.s.;
4. $Y_t \geq L_t, 0 \leq t \leq T$;
5. $\int_0^T (Y_s - L_s) dK_s = 0, P$ -a.s..

The following corollary is a minor modification of Corollary 2.3 in [3] and will be used several times later. The proof is the same as Lemma 2.2 in [3] and therefore we omit it.

Corollary 1 (Cor.2.3 in [3]) Assume that $(Y, Z) \in \mathcal{B}^p$ is a solution of the following BSDE:

$$Y_t = \xi + \int_t^T \tilde{f}(t, Y_s, Z_s) ds + A_T - A_t - \int_t^T Z_s dB_s, t \leq T$$

where:

- (i) \tilde{f} is a function which satisfies the same assumptions as f
- (ii) P -a.s. the process $(A_t)_{t \leq T}$ is of bounded variation type.

Then for any $0 \leq t \leq u \leq T$ we have:

$$\begin{aligned} & |Y_t|^p + c(p) \int_t^u |Y_s|^{p-2} \mathbf{1}_{Y_s \neq 0} |Z_s|^2 ds \\ & \leq |Y_u|^p + p \int_t^u |Y_s|^{p-1} \hat{Y}_s dA_s + p \int_t^u |Y_s|^{p-1} \hat{Y}_s \tilde{f}(s, Y_s, Z_s) ds - p \int_t^u |Y_s|^{p-1} \hat{Y}_s Z_s dB_s, \end{aligned}$$

where $c(p) = \frac{p(p-1)}{2}$ and $\hat{y} = \frac{y}{|y|} \mathbf{1}_{y \neq 0}$.

We are now going to provide some estimates on the bounds of the solution (Y, Z, K) w.r.t. the data (f, ξ, L) . These estimates will be used to obtain a control on the variation of solutions of those BSDEs induced by a variations of the data. As a by-product we have uniqueness. Actually we have:

Lemma 1 *Assume that $(Y, Z) \in \mathcal{B}^p$ is a solution of the following BSDE:*

$$Y_t = \xi + \int_t^T f(t, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, \quad t \leq T$$

where P -a.s. the process $(K_t)_{t \leq T}$ is continuous non decreasing, with $K_0 = 0$. Then there exists a real constant $C_{p,\kappa}$ depending only on p and κ (the Lipschitz constant of f) such that:

$$\mathbb{E} \left[\left(\int_0^T |Z_s|^2 ds \right)^{p/2} \right] \leq C_{p,\kappa} \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t|^p + \left(\int_0^T |f(s, 0, 0)| ds \right)^p \right].$$

Proof. Let α be a real constant and for each integer k let us define:

$$\tau_k = \inf \left\{ t \in [0, T], \int_0^t |Z_s|^2 ds \geq k \right\} \wedge T.$$

The sequence $(\tau_k)_{k \geq 0}$ is of stationary type since the process Z belongs to \mathcal{M}^p and then $\int_0^T |Z_s|^2 ds < \infty$, P -a.s.. Next using Itô's formula yields:

$$\begin{aligned} |Y_0|^2 + \int_0^{\tau_k} e^{\alpha s} |Z_s|^2 ds &= e^{\alpha \tau_k} |Y_{\tau_k}|^2 + \int_0^{\tau_k} e^{\alpha s} Y_s (2f(s, Y_s, Z_s) - \alpha Y_s) ds \\ &\quad + 2 \int_0^{\tau_k} e^{\alpha s} Y_s dK_s - 2 \int_0^{\tau_k} e^{\alpha s} Y_s Z_s dB_s \\ &\leq e^{\alpha \tau_k} |Y_{\tau_k}|^2 + \int_0^{\tau_k} e^{\alpha s} \{ 2|Y_s f(s, 0, 0)| + 2\kappa |Y_s|^2 + 2\kappa |Y_s| |Z_s| - \alpha |Y_s|^2 \} ds \\ &\quad + 2 \int_0^{\tau_k} e^{\alpha s} Y_s dK_s - 2 \int_0^{\tau_k} e^{\alpha s} Y_s Z_s dB_s \\ &\leq e^{\alpha \tau_k} |Y_{\tau_k}|^2 + 2 \sup_{s \leq \tau_k} e^{\alpha s} |Y_s| \times \int_0^{\tau_k} |f(s, 0, 0)| ds \\ &\quad + (2\kappa + \kappa \varepsilon^{-1} - \alpha) \int_0^{\tau_k} e^{\alpha s} |Y_s|^2 ds + \varepsilon \kappa \int_0^{\tau_k} e^{\alpha s} |Z_s|^2 ds \\ &\quad + 2 \int_0^{\tau_k} e^{\alpha s} Y_s dK_s - 2 \int_0^{\tau_k} e^{\alpha s} Y_s Z_s dB_s \end{aligned}$$

for any $\varepsilon > 0$. Therefore

$$\begin{aligned} |Y_0|^2 + (1 - \varepsilon \kappa) \int_0^{\tau_k} e^{\alpha s} |Z_s|^2 ds &\leq [e^{\alpha \tau_k} |Y_{\tau_k}|^2 + (1 + \frac{1}{\varepsilon}) \sup_{s \leq \tau_k} e^{2\alpha s} |Y_s|^2] \\ &\quad + \left(\int_0^{\tau_k} |f(s, 0, 0)| ds \right)^2 + (2\kappa + \kappa \varepsilon^{-1} - \alpha) \int_0^{\tau_k} e^{\alpha s} |Y_s|^2 ds \\ &\quad + \varepsilon K_{\tau_k}^2 - 2 \int_0^{\tau_k} e^{\alpha s} Y_s Z_s dB_s. \end{aligned}$$

But there exists a constant C_κ such that:

$$\begin{aligned} K_{\tau_k}^2 &\leq C_\kappa \left(|Y_0|^2 + |Y_{\tau_k}|^2 + \left(\int_0^{\tau_k} |f(s, 0, 0)| ds \right)^2 + \int_0^{\tau_k} |Y_s|^2 ds \right. \\ &\quad \left. + \int_0^{\tau_k} |Z_s|^2 ds + \left| \int_0^{\tau_k} Z_s dB_s \right|^2 \right). \end{aligned}$$

Plugging this last inequality in the previous one to get:

$$\begin{aligned} (1 - \varepsilon C_\kappa) |Y_0|^2 + (1 - \varepsilon \kappa) \int_0^{\tau_k} e^{\alpha s} |Z_s|^2 ds - \varepsilon C_\kappa \int_0^{\tau_k} |Z_s|^2 ds \\ \leq \{ (\varepsilon C_\kappa + e^{\alpha \tau_k}) |Y_{\tau_k}|^2 + (1 + \frac{1}{\varepsilon}) \sup_{s \leq \tau_k} e^{2\alpha s} |Y_s|^2 \} + \varepsilon C_\kappa \left| \int_0^{\tau_k} Z_s dB_s \right|^2 \\ + 2 \left| \int_0^{\tau_k} e^{\alpha s} Y_s Z_s dB_s \right| + (2\kappa + \kappa \varepsilon^{-1} - \alpha) \int_0^{\tau_k} e^{\alpha s} |Y_s|^2 ds \\ + (1 + \varepsilon C_\kappa) \left(\int_0^{\tau_k} |f(s, 0, 0)| ds \right)^2. \end{aligned}$$

Choosing now ε small enough and α such that $2\kappa + \kappa \varepsilon^{-1} - \alpha < 0$ we obtain:

$$\begin{aligned} \mathbb{E} \left(\int_0^{\tau_k} |Z_s|^2 ds \right)^{p/2} &\leq C(\kappa, p) \left(\left(\int_0^{\tau_k} |f(s, 0, 0)| ds \right)^p + \sup_{t \in [0, T]} |Y_t|^p \right) \\ &\quad + C(\kappa, p) \left| \int_0^{\tau_k} e^{\alpha s} Y_s Z_s dB_s \right|^{p/2}. \end{aligned}$$

Next thanks to BDG's inequality we have:

$$\begin{aligned} \mathbb{E} \left[\left| \int_0^{\tau_k} e^{\alpha s} Y_s Z_s dB_s \right|^{p/2} \right] &\leq \bar{C}_p \mathbb{E} \left[\left(\int_0^{\tau_k} |Y_s|^2 |Z_s|^2 ds \right)^{p/4} \right] \\ &\leq \bar{C}_p \mathbb{E} \left[\left(\sup_{t \in [0, T]} |Y_t| \right)^{p/2} \left(\int_0^{\tau_k} |Z_s|^2 ds \right)^{p/4} \right] \\ &\leq \frac{\bar{C}_p^2}{\eta} \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t|^p \right] + \eta \mathbb{E} \left(\int_0^{\tau_k} |Z_s|^2 ds \right)^{p/2} \end{aligned}$$

where η is an arbitrary positive real number. Finally plugging the last inequality in the previous one, choosing η small enough and finally using Fatou's Lemma to obtain the desired result. \square

We will now establish an estimate for the process Y . Actually we have:

Lemma 2 *We keep the notations of Lemma 1 and we assume moreover that P -a.s. $\int_0^T (Y_s - L_s)^+ dK_s = 0$. Then there exists a constant $C_{p, \kappa}$ such that:*

$$\mathbb{E} \sup_{t \in [0, T]} |Y_t|^p \leq C_{\kappa, p} \left[\mathbb{E} |\xi|^p + \mathbb{E} \left(\int_0^T |f(s, 0, 0)| ds \right)^p + \mathbb{E} \left(\sup_{t \in [0, T]} (L_s^+)^p \right) \right].$$

Proof. From Corollary 1 for any $\alpha \in \mathbb{R}$ and any $0 \leq t \leq u \leq T$ we have:

$$\begin{aligned} & e^{\alpha pt} |Y_t|^p + c(p) \int_t^u e^{\alpha ps} |Y_s|^{p-2} \mathbf{1}_{Y_s \neq 0} |Z_s|^2 ds \\ & \leq e^{\alpha pu} |Y_u|^p - p \int_t^u \alpha e^{\alpha ps} |Y_s|^p ds + p \int_t^u e^{\alpha ps} |Y_s|^{p-1} \operatorname{sgn}(Y_s) f(s, Y_s, Z_s) ds \\ & \quad + p \int_t^u e^{\alpha ps} |Y_s|^{p-1} \operatorname{sgn}(Y_s) dK_s - p \int_t^u e^{\alpha ps} |Y_s|^{p-1} \operatorname{sgn}(Y_s) Z_s dB_s \end{aligned}$$

where $\operatorname{sgn}(y) := \frac{y}{|y|} \mathbf{1}_{y \neq 0}$. But since f is Lipschitz then we have:

$$\begin{aligned} & e^{\alpha pt} |Y_t|^p + c(p) \int_t^u e^{\alpha ps} |Y_s|^{p-2} \mathbf{1}_{Y_s \neq 0} |Z_s|^2 ds \\ & \leq e^{\alpha pu} |Y_u|^p + p(\kappa - \alpha) \int_t^u e^{\alpha ps} |Y_s|^p ds + p \int_t^u e^{\alpha ps} |Y_s|^{p-1} |f(s, 0, 0)| ds \\ & \quad + p\kappa \int_t^u e^{\alpha ps} |Y_s|^{p-1} |Z_s| ds + p \int_t^u e^{\alpha ps} |Y_s|^{p-1} \operatorname{sgn}(Y_s) dK_s \\ & \quad - p \int_t^u e^{\alpha ps} |Y_s|^{p-1} \operatorname{sgn}(Y_s) Z_s dB_s. \end{aligned}$$

As we have

$$p\kappa |Y_s|^{p-1} |Z_s| \leq \frac{p\kappa^2}{(p-1)} |Y_s|^p + \frac{c(p)}{2} |Y_s|^{p-2} \mathbf{1}_{Y_s \neq 0} |Z_s|^2,$$

and by Young's inequality it holds true that:

$$p \int_t^u e^{\alpha ps} |Y_s|^{p-1} |f(s, 0, 0)| ds \leq (p-1) \gamma^{\frac{p}{p-1}} \left(\sup_{t \leq s \leq u} |Y_s|^p \right) + \gamma^{-p} \left(\int_t^u e^{\alpha ps} |f(s, 0, 0)| ds \right)^p$$

for any $\gamma > 0$. Then plug the two last inequalities in the previous ones to obtain:

$$\begin{aligned} & e^{\alpha pt} |Y_t|^p + \frac{c(p)}{2} \int_t^u e^{\alpha ps} |Y_s|^{p-2} \mathbf{1}_{Y_s \neq 0} |Z_s|^2 ds \\ & \leq e^{\alpha pu} |Y_u|^p + (p-1) \gamma^{\frac{p}{p-1}} \left(\sup_{t \leq s \leq u} |Y_s|^p \right) + \gamma^{-p} \left(\int_t^u e^{\alpha ps} |f(s, 0, 0)| ds \right)^p \\ & \quad + p \left(\kappa + \frac{\kappa^2}{(p-1)} - \alpha \right) \int_t^u e^{\alpha ps} |Y_s|^p ds \\ & \quad + p \int_t^u e^{\alpha ps} |Y_s|^{p-1} \operatorname{sgn}(Y_s) dK_s - p \int_t^u e^{\alpha ps} |Y_t|^{p-1} \operatorname{sgn}(Y_s) Z_s dB_s. \end{aligned}$$

Next let us deal with $\int_t^u e^{\alpha ps} |Y_s|^{p-1} \operatorname{sgn}(Y_s) dK_s$. Indeed the hypothesis related to increments of K and $Y - L$ implies that $dK_s = \mathbf{1}_{[Y_s \leq L_s]} dK_s$, for any $s \leq T$. Therefore we have:

$$\int_t^u e^{\alpha ps} |Y_s|^{p-1} \operatorname{sgn}(Y_s) dK_s = \int_t^u e^{\alpha ps} |Y_s|^{p-1} \operatorname{sgn}(Y_s) \mathbf{1}_{[Y_s \leq L_s]} dK_s \leq \int_t^u e^{\alpha ps} \theta(L_s) dK_s$$

where $\theta : x \in \mathbb{R} \mapsto \theta(x) = |x|^{p-1} \frac{x}{|x|} \mathbf{1}_{\{x \neq 0\}}$ which is actually a non-decreasing function. It follows that:

$$\begin{aligned} \int_t^u e^{\alpha ps} |Y_s|^{p-1} \text{sgn}(Y_s) dK_s &\leq \int_t^u e^{\alpha ps} |L_s|^{p-1} \text{sgn}(L_s) dK_s \\ &\leq \int_t^u e^{\alpha ps} (L_s^+)^{p-1} dK_s \leq \left(\sup_{t \in [0, T]} (L_s^+) \right)^{p-1} \int_t^u e^{\alpha ps} dK_s \\ &\leq \frac{(p-1)}{p} \frac{1}{\varepsilon^{\frac{p}{p-1}}} \left(\sup_{t \in [0, T]} ((L_s^+)^p) \right) + \frac{1}{p} \varepsilon^p \left(\int_t^u e^{\alpha ps} dK_s \right)^p \end{aligned}$$

for any $\varepsilon > 0$. Therefore choosing α such that

$$\kappa + \frac{\kappa^2}{p-1} \leq \alpha$$

and with $u = T$ we have:

$$\begin{aligned} (3) \quad e^{\alpha pt} |Y_t|^p + \frac{c(p)}{2} \int_t^T e^{\alpha ps} |Y_s|^{p-2} \mathbf{1}_{Y_s \neq 0} |Z_s|^2 ds &\leq e^{\alpha pT} |\xi|^p + (p-1) \gamma^{\frac{p}{p-1}} \left(\sup_{t \leq s \leq T} |Y_s|^p \right) \\ &+ \frac{1}{\gamma^p} \left(\int_t^T e^{\alpha ps} |f(s, 0, 0)| ds \right)^p + (p-1) \frac{1}{\varepsilon^{\frac{p}{p-1}}} \left(\sup_{t \in [0, T]} (L_s^+)^p \right) \\ &+ \varepsilon^p \left(\int_t^T e^{\alpha ps} dK_s \right)^p - p \int_t^T e^{\alpha ps} |Y_t|^{p-1} \text{sgn}(Y_s) Z_s dB_s. \end{aligned}$$

Next we focus on the control of the term $\int_t^T e^{\alpha ps} dK_s$. So using the predictable dual projection property (see e.g. [5]) we have: $\forall t \leq T$,

$$\begin{aligned} \mathbb{E}[(K_T - K_t)^p] &= \mathbb{E} \left[\int_t^T p(K_T - K_s)^{p-1} dK_s \right] = p \mathbb{E} \int_t^T \mathbb{E}[(K_T - K_s)^{p-1} | \mathcal{F}_s] dK_s \\ &\leq p \mathbb{E} \int_t^T [\mathbb{E}(K_T - K_s) | \mathcal{F}_s]^{p-1} dK_s, \quad \text{since } p \in]1, 2[. \end{aligned}$$

The last inequality holds true thanks to Jensen's conditional one. Recall now that

$$K_T - K_t = Y_t - \xi - \int_t^T f(s, Y_s, Z_s) ds + \int_t^T Z_s dB_s$$

then

$$\begin{aligned} \mathbb{E}[(K_T - K_t)^p] &\leq p \mathbb{E} \int_t^T \left[\mathbb{E} \left(Y_s - \xi - \int_s^T f(u, Y_u, Z_u) du \middle| \mathcal{F}_s \right) \right]^{p-1} dK_s \\ &\leq p \mathbb{E} \int_t^T \left[\mathbb{E} \left(2 \sup_{u \in [t, T]} |Y_u| + \int_s^T |f(u, Y_u, Z_u)| du \middle| \mathcal{F}_s \right) \right]^{p-1} dK_s \\ &\leq \frac{1}{2} \mathbb{E}[(K_T - K_t)^p] \\ &\quad + C_p \mathbb{E} \sup_{s \in [t, T]} \left[\mathbb{E} \left(2 \sup_{u \in [t, T]} |Y_u| + \int_t^T |f(u, Y_u, Z_u)| du \middle| \mathcal{F}_s \right) \right]^p. \end{aligned}$$

The last inequality is obtained once more through Young's one. Thus using now Doob's maximal inequality to obtain:

$$\begin{aligned}
\frac{1}{2}\mathbb{E}[(K_T - K_t)^p] &\leq C_p \sup_{s \in [t, T]} \mathbb{E} \left[\mathbb{E} \left(2 \sup_{u \in [t, T]} |Y_u| + \int_t^T |f(u, Y_u, Z_u)| du \middle| \mathcal{F}_s \right) \right]^p \\
&\leq \tilde{C}_p \mathbb{E} \left[\sup_{u \in [t, T]} |Y_u|^p + \left(\int_t^T |f(u, Y_u, Z_u)| du \right)^p \right] \\
&\leq \bar{C}_{p, \kappa} \mathbb{E} \left[\sup_{u \in [t, T]} |Y_u|^p + \left(\int_t^T |f(u, 0, 0)| du \right)^p + \left(\int_t^T |Z_u| du \right)^p \right].
\end{aligned}$$

and then by Lemma 1 we have

$$(4) \quad \mathbb{E}[(K_T - K_t)^p] \leq C_{p, \kappa} \mathbb{E} \left[\sup_{u \in [t, T]} |Y_u|^p + \left(\int_t^T |f(u, 0, 0)| du \right)^p \right].$$

Hereafter $C_{\kappa, p}$ is a constant which depends on p , κ and possibly T which may change from a line to another. Now the local martingale $(\int_0^t e^{\alpha ps} |Y_t|^{p-1} \text{sgn}(Y_s) Z_s dB_s)_{t \leq T}$ is actually a martingale, therefore taking expectation in (3) and taking into account of (4) to obtain:

$$\begin{aligned}
(5) \quad \frac{c(p)}{2} \mathbb{E} \int_t^T e^{\alpha ps} |Y_s|^{p-2} \mathbf{1}_{Y_s \neq 0} |Z_s|^2 ds &\leq e^{\alpha p T} \mathbb{E} |\xi|^p + \frac{p-1}{\varepsilon^{\frac{p}{p-1}}} \mathbb{E} \left(\sup_{t \in [0, T]} (L_s^+)^p \right) \\
&+ C_{\kappa, p} \left\{ (\gamma^{\frac{p}{p-1}} + \varepsilon^p) \mathbb{E} \left(\sup_{t \leq s \leq T} |Y_s|^p \right) + \left(\frac{1}{\gamma^p} + \varepsilon^p \right) \mathbb{E} \left(\int_t^T e^{\alpha ps} |f(s, 0, 0)| ds \right)^p \right\}.
\end{aligned}$$

Next going back to (3) taking the supremum and then expectation we get after taking into account (4)

$$\begin{aligned}
(6) \quad \mathbb{E} \sup_{t \in [0, T]} e^{\alpha pt} |Y_t|^p &\leq e^{\alpha p T} \mathbb{E} |\xi|^p + C_{p, \kappa} \gamma^{-p} \left(\mathbb{E} \int_0^T e^{\alpha ps} |f(s, 0, 0)| ds \right)^p \\
&+ C_{p, \kappa} (\gamma^{\frac{p}{p-1}} + \varepsilon^p) \mathbb{E} \sup_{u \in [t, T]} |Y_u|^p + \frac{p-1}{\varepsilon^{\frac{p}{p-1}}} \mathbb{E} \left(\sup_{t \in [0, T]} (L_s^+)^p \right) \\
&+ p \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^T e^{\alpha ps} |Y_t|^{p-1} \text{sgn}(Y_s) Z_s dB_s \right|.
\end{aligned}$$

Next using the BDG inequality we have

$$\begin{aligned}
\mathbb{E} \sup_{t \in [0, T]} \left| \int_0^T e^{\alpha ps} |Y_t|^{p-1} \text{sgn}(Y_s) Z_s dB_s \right| &\leq 2 \mathbb{E} \left(\int_0^T e^{2\alpha ps} |Y_t|^{2(p-1)} \mathbf{1}_{Y_s \neq 0} |Z_s|^2 ds \right)^{1/2} \\
&\leq 2 \mathbb{E} \left[\left(\sup_{t \in [0, T]} e^{\alpha pt/2} |Y_t|^{p/2} \right) \left(\int_0^T e^{\alpha ps} |Y_t|^{p-2} \mathbf{1}_{Y_s \neq 0} |Z_s|^2 ds \right)^{1/2} \right] \\
&\leq \eta \mathbb{E} \left(\sup_{t \in [0, T]} e^{\alpha pt} |Y_t|^p \right) + \frac{1}{\eta} \mathbb{E} \left(\int_0^T e^{\alpha ps} |Y_t|^{p-2} \mathbf{1}_{Y_s \neq 0} |Z_s|^2 ds \right), \quad \forall \eta > 0.
\end{aligned}$$

We now plug this inequality in (6) and then we obtain:

$$\begin{aligned}
\mathbb{E} \sup_{t \in [0, T]} e^{\alpha p t} |Y_t|^p &\leq e^{\alpha p T} \mathbb{E} |\xi|^p + C_{p, \kappa} (\gamma^{-p} + \varepsilon^p) \left(\mathbb{E} \int_0^T e^{\alpha p s} |f(s, 0, 0)| ds \right)^p \\
&\quad + \frac{p-1}{\varepsilon^{\frac{p}{p-1}}} \mathbb{E} \left(\sup_{t \in [0, T]} (L_t^+)^p \right) + \{C_{p, \kappa} (\gamma^{\frac{p}{p-1}} + \varepsilon^p) + p\eta\} \mathbb{E} \sup_{u \in [t, T]} |Y_u|^p \\
&\quad + \frac{p}{\eta} \mathbb{E} \left(\int_0^T e^{\alpha p s} |Y_t|^{p-2} \mathbf{1}_{Y_s \neq 0} |Z_s|^2 ds \right). \\
&\leq \left(1 + \frac{2p}{c(p)\eta} \right) e^{\alpha p T} \mathbb{E} |\xi|^p + \left(1 + \frac{2p}{c(p)\eta} \right) \frac{p-1}{\varepsilon^{\frac{p}{p-1}}} \mathbb{E} \left(\sup_{t \in [0, T]} (L_t^+)^p \right) \\
&\quad + C_{p, \kappa} (\gamma^{-p} + \varepsilon^p) \left(1 + \frac{2p}{c(p)\eta} \right) \left(\mathbb{E} \int_0^T e^{\alpha p s} |f(s, 0, 0)| ds \right)^p \\
&\quad + \left\{ C_{p, \kappa} \left(1 + \frac{2p}{\eta c(p)} \right) (\gamma^{\frac{p}{p-1}} + \varepsilon^p) + p\eta \right\} \mathbb{E} \sup_{u \in [t, T]} |Y_u|^p.
\end{aligned}$$

The second inequality comes from (5). Finally it is enough to chose $\eta = \frac{1}{2p}$ and γ, ε small enough to obtain the desired result. \square

Lemma 3 *Assume that (f, ξ, L) and (f', ξ', L') are two triplets satisfying Assumptions (H). Suppose that (Y, Z, K) is a solution of the RBSDE (f, ξ, L) and (Y', Z', K') is a solution of the RBSDE (f', ξ', L') . Let us set:*

$$\begin{aligned}
\Delta f &= f - f', & \Delta \xi &= \xi - \xi' & \Delta L &= L - L' \\
\Delta Y &= Y - Y', & \Delta Z &= Z - Z' & \Delta K &= K - K'
\end{aligned}$$

and assume that $\Delta L \in \mathcal{S}^p(\mathbb{R})$. Then there exists a constant C such that

$$\begin{aligned}
\mathbb{E} \sup_{t \in [0, T]} |\Delta Y_t|^p &\leq C \mathbb{E} \left[|\Delta \xi|^p + \left(\int_0^T |\Delta f(s, Y_s, Z_s)| ds \right)^p \right] \\
&\quad + C (\Psi_T)^{1/p} \left[\mathbb{E} \sup_{t \in [0, T]} |\Delta L_t|^p \right]^{\frac{p-1}{p}},
\end{aligned}$$

with

$$\begin{aligned}
\Psi_T &= \mathbb{E} \left[|\xi|^p + \left(\int_0^T |f(u, 0, 0)| du \right)^p + \left(\sup_{t \in [0, T]} (L_t^+)^p \right) \right. \\
&\quad \left. + |\xi'|^p + \left(\int_0^T |f'(u, 0, 0)| du \right)^p + \left(\sup_{t \in [0, T]} ((L_t')^+)^p \right) \right].
\end{aligned}$$

Proof. Using Corollary 1, we have for all $0 \leq t \leq T$:

$$\begin{aligned}
& |\Delta Y_t|^p + c(p) \int_t^T |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} |\Delta Z_s|^2 ds \leq |\Delta \xi|^p \\
& + p \int_t^T |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) (f(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s)) ds \\
& + p \int_t^T \alpha |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) d(\Delta K_s) - p \int_t^u |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) \Delta Z_s dB_s \\
(7) \quad & \leq |\Delta \xi|^p + p\kappa \int_t^T |\Delta Y_s|^p ds \\
& + p\kappa \int_t^T |\Delta Y_s|^{p-1} |\Delta Z_s| ds + p \int_t^T |\Delta Y_s|^{p-1} |\Delta f(s, Y_s, Z_s)| ds \\
& + p \int_t^T |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) d(\Delta K_s) - p \int_t^T |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) \Delta Z_s dB_s.
\end{aligned}$$

First note that

$$p\kappa |\Delta Y_s|^{p-1} |\Delta Z_s| \leq \frac{p\kappa^2}{(p-1)} |\Delta Y_s|^p + \frac{c(p)}{2} |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} |\Delta Z_s|^2.$$

Next if we denote by $\tilde{\theta}$ the function $(x, a) \mapsto |x - a|^{p-2} \mathbf{1}_{x \neq a} (x - a)$, we have

$$\begin{aligned}
\int_t^T |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) dK_s &= \int_t^T |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) dK_s \\
&= \int_t^T \tilde{\theta}(Y_s, Y'_s) \mathbf{1}_{[Y_s=L_s]} dK_s = \int_t^T \tilde{\theta}(L_s, Y'_s) dK_s.
\end{aligned}$$

In the same way dealing with the other term as previously to obtain:

$$\begin{aligned}
\int_t^T |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) d(\Delta K_s) &= \int_t^T |L_s - Y'_s|^{p-2} \mathbf{1}_{L_s - Y'_s \neq 0} (L_s - Y'_s) dK_s \\
&\quad - \int_t^T |Y_s - L'_s|^{p-2} \mathbf{1}_{Y_s - L'_s \neq 0} (Y_s - L'_s) dK'_s.
\end{aligned}$$

But for any $x, a \in \mathbb{R}$, the functions $a \in \mathbb{R} \mapsto \theta(x, a)$ and $x \in \mathbb{R} \mapsto \theta(x, a)$ are respectively non-increasing and non-decreasing, therefore:

$$\begin{aligned}
& \int_t^T |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) d(\Delta K_s) \leq \\
& \int_t^T |\Delta L_s|^{p-2} \mathbf{1}_{\Delta L_s \neq 0} (\Delta L_s) dK_s - \int_t^T |\Delta L_s|^{p-2} \mathbf{1}_{\Delta L_s \neq 0} (\Delta L_s) dK'_s = \int_t^T |\Delta L_s|^{p-1} d(\Delta K_s)
\end{aligned}$$

since $Y \geq L$ and $Y' \geq L'$. Thus going back to (7) to get

$$\begin{aligned}
(8) \quad & |\Delta Y_t|^p + \frac{c(p)}{2} \int_t^T |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} |\Delta Z_s|^2 ds \leq \\
& |\Delta \xi|^p + p \int_t^T |\Delta Y_s|^{p-1} |\Delta f(s, Y_s, Z_s)| ds + \left(p\kappa + \frac{p\kappa^2}{(p-1)} \right) \int_t^T |\Delta Y_s|^p ds \\
& + p \int_t^T |\Delta L_s|^{p-1} d(\Delta K_s) - p \int_t^T |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) \Delta Z_s dB_s.
\end{aligned}$$

On the other hand the process $\left\{ \int_0^t |\Delta Y_s|^{p-1} \text{sgn}(\Delta Y_s) \Delta Z_s dB_s \right\}_{0 \leq t \leq T}$ is a martingale thanks to the Burkholder-Davis-Gundy and Young inequalities. With $t = 0$ and taking the expectation in (8) we have

$$\begin{aligned} \frac{c(p)}{2} \mathbb{E} \int_0^T |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} |\Delta Z_s|^2 ds &\leq \mathbb{E} |\Delta \xi|^p + \left(p\kappa + \frac{p\kappa^2}{(p-1)} \right) \mathbb{E} \int_0^T |\Delta Y_s|^p ds \\ &+ p \mathbb{E} \int_0^T |\Delta Y_s|^{p-1} |\Delta f(s, Y_s, Z_s)| ds + p \mathbb{E} \int_0^T |\Delta L_s|^{p-1} d(\Delta K_s). \end{aligned}$$

Coming once again back to (8), we also have

$$\begin{aligned} \mathbb{E} |\Delta Y_t|^p &\leq \mathbb{E} |\Delta \xi|^p + \left(p\kappa + \frac{p\kappa^2}{(p-1)} \right) \mathbb{E} \int_t^T |\Delta Y_s|^p ds \\ &+ p \mathbb{E} \int_0^T |\Delta Y_s|^{p-1} |\Delta f(s, Y_s, Z_s)| ds + p \mathbb{E} \int_0^T |\Delta L_s|^{p-1} d(\Delta K_s). \end{aligned}$$

With the Gronwall lemma we conclude that

$$\mathbb{E} \int_0^T |\Delta Y_s|^p ds \leq C_p \mathbb{E} \left(|\Delta \xi|^p + \int_0^T |\Delta Y_s|^{p-1} |\Delta f(s, Y_s, Z_s)| ds + \int_0^T |\Delta L_s|^{p-1} d(\Delta K_s) \right).$$

Now with Hölder's inequality

$$\mathbb{E} \int_0^T |\Delta L_s|^{p-1} d(\Delta K_s) \leq \left(\mathbb{E} \sup_{s \in [0, T]} |\Delta L_s|^p \right)^{\frac{p-1}{p}} \mathbb{E} (|\Delta K_T|^p)^{\frac{1}{p}}$$

and since $\mathbb{E} |\Delta K_T|^p \leq C_p (\mathbb{E} |K_T|^p + \mathbb{E} |K'_T|^p)$, using inequality (4) and Lemma 2, we deduce that

$$\mathbb{E} |\Delta K_T|^p \leq C \Psi_T.$$

Therefore we obtain

$$\begin{aligned} \mathbb{E} \int_0^T |\Delta Y_s|^p ds + \frac{c(p)}{2} \mathbb{E} \int_0^T |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} |\Delta Z_s|^2 ds \\ \leq C \mathbb{E} \left(|\Delta \xi|^p + \int_0^T |\Delta Y_s|^{p-1} |\Delta f(s, Y_s, Z_s)| ds \right) + C \left(\mathbb{E} \sup_{s \in [0, T]} |\Delta L_s|^p \right)^{\frac{p-1}{p}} (\Psi_T)^{\frac{1}{p}}. \end{aligned}$$

But

$$\begin{aligned} \int_0^T |\Delta Y_s|^{p-1} |\Delta f(s, Y_s, Z_s)| ds &\leq \sup_{s \leq T} |\Delta Y_s|^{p-1} \times \int_0^T |\Delta f(s, Y_s, Z_s)| ds \\ &\leq \rho^{\frac{p}{p-1}} \sup_{s \leq T} |\Delta Y_s|^p + \frac{1}{\rho^p} \left(\int_0^T |\Delta f(s, Y_s, Z_s)| ds \right)^p \end{aligned}$$

for any $\rho > 0$. Next with (8), BDG inequality, and the two previous inequalities, we obtain after having chosen ρ small enough:

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, T]} |\Delta Y_s|^p &\leq C \mathbb{E} \left(|\Delta \xi|^p + \left\{ \int_0^T |\Delta f(s, Y_s, Z_s)| ds \right\}^p \right) \\ &\quad + C \left(\mathbb{E} \sup_{s \in [0, T]} |\Delta L_s|^p \right)^{\frac{p-1}{p}} (\Psi_T)^{\frac{1}{p}}. \end{aligned}$$

The conclusion of the Lemma follows. \square

Theorem 1 *Under the assumptions [H1]-[H3], there is at most one L^p -solution for the reflected BSDE associated with (f, ξ, L) .*

Proof. Using the previous Lemma (since $L - L' = 0 \in \mathcal{S}^p$), we obtain immediately $Y = Y'$. Therefore we have also $Z = Z'$ and finally $K = K'$, whence uniqueness of the L^p -solution of the reflected BSDE associated with (f, ξ, L) . \square

3 Existence via the Snell Envelope Method

We now focus on the issue of existence. To begin with let us first assume that the function f does not depend on (y, z) .

Theorem 2 *The reflected BSDE associated with $(f(t), \xi, L)$ has a unique L^p -solution.*

Proof. We are going to prove the existence of a solution in using the Snell envelope of processes. The Snell envelope of a process X which belongs to class [D] is the smallest supermartingale of class [D] which is greater than X .

For $t \leq T$ let us set:

$$\tilde{Y}_t = \operatorname{esssup}_{\tau \geq t} \mathbb{E} \left[\int_0^\tau f(s) ds + L_\tau \mathbf{1}_{[\tau < T]} + \xi \mathbf{1}_{[\tau = T]} \middle| \mathcal{F}_t \right].$$

Here τ is a \mathcal{F}_t -stopping time. The processes \tilde{Y} verifies $\tilde{Y}_T = \xi$ and is called the Snell envelope of the process which is inside $\operatorname{esssup} \mathbb{E}(\dots)$.

Since the process $(\int_0^t |f(s)| ds)_{t \leq T}$ and ξ belong to $L^p(\Omega)$ and L^+ belongs to \mathcal{S}^p , then the process \tilde{Y} exists and belongs to \mathcal{S}^p . Furthermore thanks to Doob-Meyer decomposition there exists an increasing continuous process $(K_t)_{t \leq T}$ which belongs to \mathcal{S}^p ($K_0 = 0$) and a martingale $(M_t)_{t \leq T}$ which is also in \mathcal{S}^p (see e.g. [5], pp.221) such that:

$$\forall t \leq T, \tilde{Y}_t = M_t - K_t.$$

Next as $M \in \mathcal{S}^p$ then there exists a process $Z \in \mathcal{M}^p$ such that:

$$\forall t \leq T, M_t = M_0 + \int_0^t Z_s dB_s.$$

Now for $t \leq T$, let us set:

$$Y_t = \tilde{Y}_t - \int_0^t f(s)ds.$$

Therefore the triplet (Y, Z, K) verifies: for any $t \leq T$,

$$Y_t = \xi + \int_t^T f(s)ds + K_T - K_t - \int_t^T Z_s dB_s.$$

Moreover we obviously have $Y \geq L$. In order to complete the proof it remains to show that $(Y_t - L_t)dK_t = 0$ for any $t \leq T$. So let $\tau \leq T$ be a stopping time and let us set $L_t^\xi := L_t \mathbf{1}_{[t < T]} + \xi \mathbf{1}_{[t = T]}$ and D_τ the following stopping time:

$$D_\tau = \inf \left\{ s \geq \tau, \tilde{Y}_s = \int_0^s f(u)du + L_s^\xi \right\} \wedge T.$$

Since the process L is continuous on $[0, T[$ and may have a positive jump at T , then the stopping time D_τ is optimal after τ . It follows that the process $(\tilde{Y}_s)_{s \in [\tau, D_\tau]}$ is a martingale and $\tilde{Y}_{D_\tau} = L_{D_\tau}^\xi + \int_0^{D_\tau} f(s)ds$ (see e.g. [9], pp.129, pp.143). Henceforth we have $\int_\tau^{D_\tau} (\tilde{Y}_s - \int_0^s f(u)du - L_s^\xi)dK_s = 0$ which implies that $\int_0^T (\tilde{Y}_s - \int_0^s f(u)du - L_s^\xi)dK_s = 0$. If not, by continuity we can find a stopping time τ such that $\int_\tau^{D_\tau} (\tilde{Y}_s - \int_0^s f(u)du - L_s^\xi)dK_s > 0$, which is absurd. Now the definition of Y implies also that:

$$\int_0^T (Y_s - L_s)dK_s = 0.$$

Thus the proof is complete. □

We now deal with the general case of generator i.e. f depends on (y, z) and is Lipschitz w.r.t. those arguments. So for $(U, V) \in \mathcal{B}^p$ we define $(Y, Z, K) = \Phi(U, V)$ where (Y, Z) is the L^p -solution of the BSDE associated with $(f(t, U_t, V_t), \xi, L)$, i.e.,

$$\begin{aligned} (Y, Z) &\in \mathcal{B}^p, K \in \mathcal{S}^p \\ Y_t &= \xi + \int_t^T f(s, U_s, V_s)ds + K_T - K_t - \int_t^T Z_s dB_s, \quad t \leq T \\ Y_t &\geq L_t \text{ and } (Y_t - L_t)dK_t = 0. \end{aligned}$$

The solution of this equation exists and is unique thanks to Theorem 2.

Now for (U', V') in \mathcal{B}^p , we define in the same way $(Y', Z') = \Phi(U', V')$ and $(\Delta Y, \Delta Z)$ by $(Y - Y', Z - Z')$, $\Delta f_s = f(s, U_s, V_s) - f(s, U'_s, V'_s)$.

We are now going to prove that there exists a real constant $\alpha \in \mathbb{R}$ such that Φ is a contraction on \mathcal{B}^p , equipped with the equivalent norm:

$$\|(Y, Z)\| = \|e^{\alpha \cdot} Y\|_{\mathcal{S}^p} + \|e^{\alpha \cdot} Z\|_{\mathcal{M}^p}.$$

Actually we have:

Lemma 4 *There exists $\alpha \in \mathbb{R}$ and a constant C_α such that:*

$$(9) \quad \|e^\alpha \cdot \Delta Y\|_{\mathcal{S}^p} \leq C_\alpha (\|e^\alpha \cdot \Delta U\|_{\mathcal{S}^p} + \|e^\alpha \cdot \Delta V\|_{\mathcal{M}^p}).$$

Proof. Using Corollary 1, we have for all $0 \leq t \leq u \leq T$:

$$(10) \quad \begin{aligned} e^{\alpha p t} |\Delta Y_t|^p + c(p) \int_t^u e^{\alpha p s} |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} |\Delta Z_s|^2 ds \\ \leq e^{\alpha p u} |\Delta Y_u|^p + p \int_t^u e^{\alpha p s} |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) \Delta f_s ds - p \int_t^u \alpha e^{\alpha p s} |\Delta Y_s|^p ds \\ + p \int_t^u e^{\alpha p s} |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) d(\Delta K_s) - p \int_t^u e^{\alpha p s} |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) \Delta Z_s dB_s. \end{aligned}$$

Now for $\varepsilon > 0$, using Young's inequality

$$\begin{aligned} \int_t^u e^{\alpha p s} |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) \Delta f_s ds &\leq \int_t^u e^{\alpha p s} \left(\varepsilon^{-\frac{p}{p-1}} \frac{p-1}{p} |\Delta Y_s|^p + \frac{\varepsilon^p}{p} |\Delta f_s|^p \right) ds \\ &\leq \varepsilon^{-\frac{p}{p-1}} \frac{p-1}{p} \int_t^u e^{\alpha p s} |\Delta Y_s|^p ds + \frac{\kappa^p 2^{p-1} \varepsilon^p}{p} \int_t^u e^{\alpha p s} (|\Delta U_s|^p + |\Delta V_s|^p) ds \\ &\leq \varepsilon^{-\frac{p}{p-1}} \frac{p-1}{p} \int_t^u e^{\alpha p s} |\Delta Y_s|^p ds + \frac{\kappa^p 2^{p-1} T \varepsilon^p}{p} \left[\left(\sup_{s \in [t, u]} e^{\alpha p s} |\Delta U_s|^p \right) \right. \\ &\quad \left. + \left(\int_t^u e^{2\alpha s} |\Delta V_s|^2 ds \right)^{p/2} \right]. \end{aligned}$$

Moreover

$$\begin{aligned} \int_t^u e^{\alpha p s} |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) d(\Delta K_s) &= \int_t^u e^{\alpha p s} |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} (Y_s - L_s) dK_s \\ &\quad + \int_t^u e^{\alpha p s} |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} (Y'_s - L_s) dK'_s \\ &\quad - \int_t^u e^{\alpha p s} |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} (Y_s - L_s) dK'_s - \int_t^u e^{\alpha p s} |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} (Y'_s - L_s) dK_s \\ &\leq \int_t^u e^{\alpha p s} |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} (Y_s - L_s) dK_s + \int_t^u e^{\alpha p s} |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} (Y'_s - L_s) dK'_s \\ &= 0 \end{aligned}$$

since $dK_s = \mathbf{1}_{[Y_s=L_s]} dK_s$ and $dK'_s = \mathbf{1}_{[Y'_s=L_s]} dK'_s$, for any $s \in [0, T]$. Coming back to (10) we obtain:

$$(11) \quad \begin{aligned} e^{\alpha p t} |\Delta Y_t|^p + c(p) \int_t^u e^{\alpha p s} |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} |\Delta Z_s|^2 ds \\ \leq e^{\alpha p u} |\Delta Y_u|^p + \left(\varepsilon^{-\frac{p}{p-1}} \frac{p-1}{p} - p\alpha \right) \int_t^u e^{\alpha p s} |\Delta Y_s|^p ds \\ + \frac{\kappa^p 2^{p-1} T \varepsilon^p}{p} \left[\left(\sup_{s \in [t, u]} e^{\alpha p s} |\Delta Y_s|^p \right) + \left(\int_t^u e^{2\alpha s} |\Delta V_s|^2 ds \right)^{p/2} \right] \\ - p \int_t^u e^{\alpha p s} |\Delta Y_s|^{p-1} \operatorname{sgn}(\Delta Y_s) \Delta Z_s dB_s. \end{aligned}$$

But as in the proof of uniqueness, the process

$$\left\{ M_t = \int_0^t e^{\alpha ps} |\Delta Y_t|^{p-1} \text{sgn}(\Delta Y_s) \Delta Z_s dB_s \right\}_{0 \leq t \leq T}$$

is a uniformly integrable martingale. Therefore with (11), and by choosing α such that $\varepsilon^{-\frac{p}{p-1}} \frac{p-1}{p} - p\alpha \leq 0$, we obtain:

$$(12) \quad c(p) \mathbb{E} \left[\int_0^T e^{\alpha ps} |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} |\Delta Z_s|^2 ds \right] \leq \frac{\kappa^p 2^{p-1} T \varepsilon^p}{p} \mathbb{E} \left[\left(\sup_{s \in [t, u]} e^{\alpha ps} |\Delta U_s|^p \right) + \left(\int_t^u e^{2\alpha s} |\Delta V_s|^2 ds \right)^{p/2} \right]$$

and

$$(13) \quad \mathbb{E} \left[\sup_{t \in [0, T]} e^{\alpha pt} |\Delta Y_t|^p \right] \leq \frac{\kappa^p 2^{p-1} T \varepsilon^p}{p} \mathbb{E} \left[\left(\sup_{s \in [t, u]} e^{\alpha ps} |\Delta U_s|^p \right) + \left(\int_t^u e^{2\alpha s} |\Delta V_s|^2 ds \right)^{p/2} \right] + p \mathbb{E} \left[\langle M, M \rangle_T^{1/2} \right].$$

For the last inequality we have made use of BDG's one. But

$$\begin{aligned} \mathbb{E} \left[\langle M, M \rangle_T^{1/2} \right] &\leq \mathbb{E} \left[\left(\sup_{t \in [0, T]} e^{\alpha pt/2} |\Delta Y_t|^{p/2} \right) \left(\int_0^T e^{\alpha ps} |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} |\Delta Z_s|^2 \right)^{1/2} \right] \\ &\leq \frac{1}{2p} \mathbb{E} \left[\sup_{t \in [0, T]} e^{\alpha pt} |\Delta Y_t|^p \right] + \frac{p}{2} \mathbb{E} \left[\int_0^T e^{\alpha ps} |\Delta Y_s|^{p-2} \mathbf{1}_{\Delta Y_s \neq 0} |\Delta Z_s|^2 \right]. \end{aligned}$$

Plugging now that inequality in (12) and (13) to obtain:

$$(14) \quad \begin{aligned} \frac{1}{2} \mathbb{E} \left[\sup_{t \in [0, T]} e^{\alpha pt} |\Delta Y_t|^p \right] &\leq \frac{\kappa^p 2^{p-1} T \varepsilon^p}{p} \mathbb{E} \left[\left(\sup_{s \in [0, T]} e^{\alpha ps} |\Delta U_s|^p \right) + \left(\int_t^u e^{2\alpha s} |\Delta V_s|^2 ds \right)^{p/2} \right] \\ &\quad + \frac{p \kappa^p 2^{p-1} T \varepsilon^p}{2c(p)} \mathbb{E} \left[\left(\sup_{s \in [0, T]} e^{\alpha ps} |\Delta U_s|^p \right) + \left(\int_t^u e^{2\alpha s} |\Delta V_s|^2 ds \right)^{p/2} \right]. \end{aligned}$$

Finally it is enough to choose

$$C_\alpha = \frac{2\kappa^p 2^{p-1} T \varepsilon^p}{p} \left(1 + \frac{p^2}{2c(p)} \right) \quad \text{and} \quad \varepsilon^{-\frac{p}{p-1}} \frac{p-1}{p} \leq p\alpha.$$

Thus the proof is complete. \square

We next focus on the same estimate for ΔZ .

Lemma 5 *There exists $\beta \in \mathbb{R}$ and a constant C'_β such that*

$$(15) \quad \|e^{\beta \cdot} \Delta Z\|_{\mathcal{M}^p} \leq C'_\beta (\|e^{\frac{\beta}{2} \cdot} \Delta U\|_{\mathcal{S}^p} + \|e^{\beta \cdot} \Delta V\|_{\mathcal{M}^p}).$$

Proof. For each integer $n \geq 1$ let us set:

$$\tau_n = \inf \left\{ t \in [0, T], \int_0^t |\Delta Z_s|^2 ds \geq n \right\} \wedge T.$$

Therefore using Itô's formula leads to

$$\begin{aligned} |\Delta Y_0|^2 + \int_0^{\tau_n} e^{\beta s} |\Delta Z_s|^2 ds &= e^{\beta \tau_n} |\Delta Y_{\tau_n}|^2 + 2 \int_0^{\tau_n} e^{\beta s} \Delta Y_s \Delta f_s ds - \beta \int_0^{\tau_n} e^{\beta s} |\Delta Y_s|^2 ds \\ &\quad + 2 \int_0^{\tau_n} e^{\beta s} \Delta Y_s d(\Delta K_s) - 2 \int_0^{\tau_n} e^{\beta s} \Delta Y_s \Delta Z_s dB_s. \end{aligned}$$

But for any $s \in [0, T]$, $\Delta Y_s d(\Delta K_s) \leq 0$ a.s. On the other hand since f is a Lipschitz function then for every $\nu > 0$

$$\begin{aligned} |\Delta Y_0|^2 + \int_0^{\tau_n} e^{\beta s} |\Delta Z_s|^2 ds &\leq e^{\beta \tau_n} |\Delta Y_{\tau_n}|^2 + \left(\frac{\kappa^2}{\nu} - \beta \right) \int_0^{\tau_n} e^{\beta s} |\Delta Y_s|^2 ds \\ &\quad + \nu \int_0^{\tau_n} e^{\beta s} (|\Delta U_s|^2 + |\Delta V_s|^2) ds - 2 \int_0^{\tau_n} e^{\beta s} \Delta Y_s \Delta Z_s dB_s. \end{aligned}$$

Now if $\frac{\kappa^2}{\nu} \leq \beta$ we obtain:

$$\begin{aligned} \int_0^{\tau_n} e^{\beta s} |\Delta Z_s|^2 ds &\leq e^{\beta \tau_n} |\Delta Y_{\tau_n}|^2 + (\nu T) \left[\sup_{t \in [0, T]} e^{\beta t} |\Delta U_t|^2 \right] \\ &\quad + \nu \int_0^{\tau_n} e^{\beta s} |\Delta V_s|^2 ds + 2 \left| \int_0^{\tau_n} e^{\beta s} \Delta Y_s \Delta Z_s dB_s \right|. \end{aligned}$$

It follows that

$$\begin{aligned} \left(\int_0^{\tau_n} e^{\beta s} |\Delta Z_s|^2 ds \right)^{p/2} &\leq 2^{(p-1)} \left\{ e^{\beta \tau_n p/2} |\Delta Y_{\tau_n}|^p + (\nu T)^{p/2} \left[\sup_{t \in [0, T]} e^{\beta t p/2} |\Delta U_t|^p \right] \right. \\ &\quad \left. + \nu^{p/2} \left(\int_0^{\tau_n} e^{\beta s} |\Delta V_s|^2 ds \right)^{p/2} + 2^{p/2} \left| \int_0^{\tau_n} e^{\beta s} \Delta Y_s \Delta Z_s dB_s \right|^{p/2} \right\}. \end{aligned}$$

But by the BDG inequality we have:

$$\begin{aligned} \mathbb{E} \left| \int_0^{\tau_n} e^{\beta s} \Delta Y_s \Delta Z_s dB_s \right|^{p/2} &\leq \bar{c}_p \mathbb{E} \left[\left(\int_0^{\tau_n} e^{2\beta s} |\Delta Y_s|^2 |\Delta Z_s|^2 ds \right)^{p/4} \right] \\ &\leq \bar{c}_p^2 2^{3p/2} \mathbb{E} \left[\sup_{t \in [0, T]} e^{\beta s p/2} |\Delta Y_s|^p \right] + 2^{-3p/2} \mathbb{E} \left[\left(\int_0^T e^{\beta s} |\Delta Z_s|^2 ds \right)^{p/2} \right]. \end{aligned}$$

Therefore plugging this inequality in the previous one to obtain:

$$\begin{aligned} \frac{1}{2} \mathbb{E} \left(\int_0^{\tau_n} e^{\beta s} |\Delta Z_s|^2 ds \right)^{p/2} &\leq 2^{3p-1} \bar{c}_p^2 \mathbb{E} \left[\sup_{t \in [0, T]} e^{\beta s p/2} |\Delta Y_s|^p \right] + \\ &2^{(p-1)} \mathbb{E} \left\{ e^{\beta \tau_n p/2} |\Delta Y_{\tau_n}|^p + (\nu T)^{p/2} \left[\sup_{t \in [0, T]} e^{\beta t p/2} |\Delta U_t|^p \right] \right. \\ &\quad \left. + \nu^{p/2} \left(\int_0^{\tau_n} e^{\beta s} |\Delta V_s|^2 ds \right)^{p/2} \right\}. \end{aligned}$$

Next using Fatou's Lemma yields:

$$\begin{aligned} \frac{1}{2} \mathbb{E} \left(\int_0^T e^{\beta s} |\Delta Z_s|^2 ds \right)^{p/2} &\leq \nu^{p/2} 2^{(p-1)} \mathbb{E} \left\{ T^{p/2} \left[\sup_{t \in [0, T]} e^{\beta t p/2} |\Delta U_t|^p \right] \right. \\ &\quad \left. + \left(\int_0^T e^{\beta s} |\Delta V_s|^2 ds \right)^{p/2} \right\} + 2^{3p-1} \bar{c}_p^2 \mathbb{E} \left[\sup_{t \in [0, T]} e^{\beta p/2 s} |\Delta Y_s|^p \right] \end{aligned}$$

Finally choosing β great enough (recall that $\beta > 0$) and using Lemma 4, to obtain :

$$(16) \quad \mathbb{E} \left(\int_0^T e^{\beta s} |\Delta Z_s|^2 ds \right)^{p/2} \leq C'_\beta \mathbb{E} \left[\sup_{t \in [0, T]} e^{\beta t p} |\Delta U_t|^p + \left(\int_0^T e^{\beta s} |\Delta V_s|^2 ds \right)^{p/2} \right]$$

with

$$C'_\beta = 2^{3p} (\bar{c}_p)^2 C_\beta + \nu^{p/2} 2^p \max(T^{p/2}, 1) \text{ and } \frac{\kappa^2}{\nu} \leq \beta.$$

□

As a result of Lemmas 1 & 2 we have:

Proposition 1 *There exist two constants γ and $\mathcal{C} < 1$ such that:*

$$\begin{aligned} &\mathbb{E} \left[\sup_{t \in [0, T]} e^{\gamma t p} |\Delta Y_t|^p \right] + \mathbb{E} \left(\int_0^T e^{\gamma s} |\Delta Z_s|^2 ds \right)^{p/2} \\ &\leq \mathcal{C} \left\{ \mathbb{E} \left[\sup_{t \in [0, T]} e^{\gamma t p} |\Delta U_t|^p \right] + \mathbb{E} \left(\int_0^T e^{\gamma s} |\Delta V_s|^2 ds \right)^{p/2} \right\}. \end{aligned}$$

Proof. Recall that in the proofs of Lemmas 4 and 5 we have required that the constants ε , α , ν and β should satisfy:

$$\varepsilon^{-\frac{p}{p-1}} \frac{p-1}{p} \leq p\alpha, \quad \frac{\kappa^2}{\nu} \leq \beta$$

$$C_\alpha = \frac{2\kappa^p 2^{p-1} T \varepsilon^p}{p} \left(1 + \frac{p^2}{4c(p)} \right),$$

$$C'_\beta = 2^{3p} \bar{c}_p^2 C_\alpha + \nu^{p/2} 2^p \max(T^{p/2}, 1).$$

So we can choose ε and ν sufficiently small in such a way that $C_\alpha < 1/2$ and $C'_\beta < 1/2$. Therefore the inequalities (9) and (15) still valid if we replace α and β with $\gamma = \max\{\alpha, \beta\}$. Also it is enough to choose $\mathcal{C} = C_\gamma + C'_\gamma < 1$ and the claim is proved. \square

We now give the main result of this section.

Theorem 3 *Under [H1]-[H3], there exists a unique L^p -solution for the reflected BSDE associated with $(f(t, y, z), \xi, L)$, i.e., there exists a triple of processes (Y, Z, K) such that:*

$$\begin{aligned} Y &\in \mathcal{S}^p, Z \in \mathcal{M}^p, K \in \mathcal{S}^p \text{ non-decreasing and } K_0 = 0 \\ Y_t &= \xi + \int_t^T f(r, Y_r, Z_r) dr + K_T - K_t - \int_t^T Z_r dB_r, \quad \forall t \leq T; \\ Y &\geq L \text{ and } (Y_t - K_t) dK_t = 0, \quad \forall t \leq T. \end{aligned}$$

Proof. Thanks to Proposition 1, the mapping Φ is a contraction in the Banach space \mathcal{B}^p endowed with the equivalent norm

$$\|(Y, Z)\|_{\gamma, p}^p = \mathbb{E} \left[\left(\sup_{t \leq T} e^{\gamma s} |Y_s| \right)^p \right] + \mathbb{E} \left[\left(\int_0^T e^{\gamma s} \|Z_s\|^2 ds \right)^{p/2} \right].$$

The constant γ is given by Proposition 1. Therefore it has a fixed point (Y, Z) which in combination with the associated K is the unique solution of the reflected BSDE associated with $(f(t, y, z), \xi, L)$. \square

4 Existence via Penalization

We are going now to deal with the issue of existence of the L^p -solution for the reflected BSDE associated with $(f(t, y, z), \xi, L)$ by using the penalization method. Actually for $n \geq 1$ let us consider $(Y^n, Z^n) \in \mathcal{B}^p$ the unique solution of the following BSDE:

$$\forall t \in [0, T], Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n) ds + n \int_t^T (Y_s^n - L_s)^- ds - \int_t^T Z_s^n dB_s.$$

Indeed thanks to the result by Briand et al. [3], this solution exists and is unique. Next let us define K^n by:

$$\forall t \in [0, T], K_t^n = n \int_0^t (Y_s^n - L_s)^- ds.$$

We first give some estimates for the processes Y^n, Z^n and K^n . Actually we have:

Proposition 2 *There exists some constants $\alpha \in \mathbb{R}$ and C which do not depend on n such that:*

$$\mathbb{E} \left[\sup_{t \in [0, T]} (e^{\alpha ps} |Y_s^n|^p) + \left(\int_0^T e^{2\alpha s} |Z_s^n|^2 \right)^{p/2} + |K_T^n|^p \right] \leq C.$$

Proof. It is obtained thanks to Lemmas 1 and 2, in combination with inequality (4). Indeed we have

$$\int_0^T (Y_s^n - L_s)^+ dK_s^n = n \int_0^T (Y_s^n - L_s)^+ (Y_s^n - L_s)^- ds = 0.$$

□

We are now going to show that the sequence of processes (Y^n, Z^n, K^n) converges to the L^p -solution of the BSDE associated with $(f(t, y, z), \xi, L)$.

First thanks to comparison theorem we have $Y^n \leq Y^{n+1}$, for any $n \geq 0$. Note that here comparison can be obtained as usual (see e.g. [11]). Therefore there exists a \mathcal{P} -measurable process Y such that for any $t \leq T$, $Y_t = \lim_{n \rightarrow \infty} \nearrow Y_t^n$. Additionally thanks to Fatou's Lemma we have

$$\mathbb{E}[|Y_t|^p] < \infty, \forall t \leq T$$

since $\mathbb{E} \sup_{t \leq T} |Y_t^n|^p \leq C, \forall n \geq 0$.

We now focus on the following:

Lemma 6 : P -a.s., $Y \geq L$, $Y \in \mathcal{S}^p$ and $\mathbb{E}[(\sup_{s \leq T} (Y_s^n - L_s)^-)^p] \rightarrow 0$ as $n \rightarrow \infty$.

Proof. For any $n \geq 0$ and $t \leq T$, we have:

$$Y_t^n - Y_t^0 = \int_t^T \{a^n(s)(Y_s^n - Y_s^0) + b^n(s)(Z_s^n - Z_s^0)\} ds + (K_T^n - K_t^n) - \int_t^T (Z_s^n - Z_s^0) dB_s$$

where the processes $(a^n(s))_{s \leq T}$ and $(b^n(s))_{s \leq T}$ are \mathcal{P} -measurable and uniformly bounded by the Lipschitz constant of f . But through Proposition 2, there exists a constant C such that:

$$\forall n \geq 0, \mathbb{E} \left[\int_0^T |a_s^n(Y_s^n - Y_s^0) + b_s^n(Z_s^n - Z_s^0)|^p ds \right] + \mathbb{E} \left[\int_0^T |Z_s^n - Z_s^0|^p ds \right] \leq C.$$

Therefore there exist subsequences and processes $(g_t)_{t \leq T}$ and $(z_t)_{t \leq T}$ which are the weak limits in $L_{\mathbb{R}}^p([0, T] \times \Omega, dt \otimes dP, \mathcal{P})$ of $(g_s^n := (a_s^n(Y_s^n - Y_s^0) + b_s^n(Z_s^n - Z_s^0)))_{s \leq T}$ and $(z_s^n := (Z_s^n - Z_s^0))_{s \leq T}$ respectively. Henceforth for any stopping time $\tau \leq T$, the following weak convergence holds :

$$\int_0^\tau z_s^n dB_s \rightarrow \int_0^\tau z_s dB_s \text{ and } \int_0^\tau g_s^n ds \rightarrow \int_0^\tau g_s ds.$$

But

$$K_\tau^n = -(Y_\tau^n - Y_\tau^0) + (Y_0^n - Y_0^0) - \int_0^\tau g_s^n ds + \int_0^\tau z_s^n dB_s$$

thus we have also the weak convergence

$$K_\tau^n \rightarrow K_\tau := -(Y_\tau - Y_\tau^0) + (Y_0 - Y_0^0) - \int_0^\tau g_s ds + \int_0^\tau z_s dB_s \text{ and } \mathbb{E}(K_\tau)^p < \infty.$$

Now for any stopping times $\sigma \leq \tau \leq T$ it holds true that $K_\sigma \leq K_\tau$ since $K_\sigma^n \leq K_\tau^n$. It follows that the process $(K_t)_{t \leq T}$ is non-decreasing. Additionally we have $E[(K_T)^p] \leq \liminf_{n \rightarrow \infty} E[(K_T^n)^p] \leq C$ since the norm is *lsc* with respect to the weak topology. Henceforth thanks to the monotonic limit of S.Peng ([23], Lemma 2.2, pp.481) the processes $Y - Y^0$ and K are RCLL and so is Y since Y^0 is continuous.

Next from $E[(K_T^n)^p] \leq C$ for any $n \geq 0$ we deduce, in taking the limit as $n \rightarrow \infty$, that:

$$\mathbb{E} \int_0^T (L_s - Y_s)^- ds = 0$$

and then *P*-a.s., $Y_t \geq L_t$ for any $t < T$. As $\xi \geq L_T$ it follows that $Y \geq L$. Using now Dini's theorem and the Lebesgue dominated convergence one obtains:

$$\mathbb{E}[(\sup_{s \leq T} (L_s - Y_s^n)^-)^p] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now for any $t \leq T$ we have,

$$Y_t^0 \leq Y_t^n \leq \sup_{t \leq t} (L_t - Y_t^n)^- + (L_t^\xi)^+.$$

Taking the limit as $n \rightarrow \infty$ to get that $Y \in \mathcal{S}^p$ since Y^0 and L^ξ (see Section 3 for its definition) are so. \square

Remark 1 *Note that the process Y is RCLL and its jumps are negative since $\Delta Y = -\Delta K \leq 0$.*

Theorem 4 *The sequence of processes $((Y^n, Z^n, K^n))_{n \geq 0}$ converges to the L^p -solution (Y, Z, K) of the BSDE (2) associated with $(f(t, y, z), \xi, L)$.*

Proof. For $k \geq 0$, let us set:

$$\tau_k := \inf\{t \geq 0, Y_t \geq k \text{ or } Y_t^0 \leq -k \text{ or } |L_t| \geq k\} \wedge T.$$

The sequence of stopping times $(\tau_k)_{k \geq 0}$ is increasing, of stationary type converging to T since the process Y is RCLL and Y^0, L continuous. Additionally we have:

$$\max\{\sup_{t \leq \tau_k} |L_t|, \sup_{t \leq \tau_k} |Y_t|, \sup_{t \leq \tau_k} |Y_t^n|\} \leq k$$

since L and Y^0 are continuous, Y has only negative jumps and finally $Y^0 \leq Y^n \leq Y$. Next for any $k \geq 0$ and $n \geq 0$ we have:

$$(17) \quad Y_{t \wedge \tau_k}^n = Y_{\tau_k}^n + \int_{t \wedge \tau_k}^{\tau_k} f(s, Y_s^n, Z_s^n) ds + K_{\tau_k}^n - K_{t \wedge \tau_k}^n - \int_{t \wedge \tau_k}^{\tau_k} Z_s^n dB_s, \quad \forall t \leq T.$$

Then for any n, m and $t \leq T$, it holds true that:

$$\begin{aligned}
Y_{t \wedge \tau_k}^n - Y_{t \wedge \tau_k}^m &= (Y_{\tau_k}^n - Y_{\tau_k}^m) + \int_{t \wedge \tau_k}^{\tau_k} \{f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m)\} ds \\
&\quad + (K_{\tau_k}^n - K_{\tau_k}^m) - (K_{t \wedge \tau_k}^n - K_{t \wedge \tau_k}^m) - \int_{t \wedge \tau_k}^{\tau_k} (Z_s^n - Z_s^m) dB_s \\
&= (Y_{\tau_k}^n - Y_{\tau_k}^m) + \int_{t \wedge \tau_k}^{\tau_k} \{a^{n,m}(s)(Y_s^n - Y_s^m) + b^{n,m}(s)(Z_s^n - Z_s^m)\} ds \\
&\quad + (K_{\tau_k}^n - K_{\tau_k}^m) - (K_{t \wedge \tau_k}^n - K_{t \wedge \tau_k}^m) - \int_{t \wedge \tau_k}^{\tau_k} (Z_s^n - Z_s^m) dB_s.
\end{aligned}$$

where $a^{n,m}$ and $b^{n,m}$ are \mathcal{P} -measurable processes uniformly bounded by the Lipschitz constant of f . Using now Itô's formula to obtain:

$$\begin{aligned}
(Y_{t \wedge \tau_k}^n - Y_{t \wedge \tau_k}^m)^2 &+ \int_{t \wedge \tau_k}^{\tau_k} |Z_s^n - Z_s^m|^2 ds = (Y_{\tau_k}^n - Y_{\tau_k}^m)^2 \\
&+ 2 \int_{t \wedge \tau_k}^{\tau_k} \{a^{n,m}(s)(Y_s^n - Y_s^m)^2 + b^{n,m}(s)(Y_s^n - Y_s^m)(Z_s^n - Z_s^m)\} ds \\
&+ 2 \int_{t \wedge \tau_k}^{\tau_k} (Y_s^n - Y_s^m)(dK_t^n - dK_t^m) - 2 \int_{t \wedge \tau_k}^{\tau_k} (Y_s^n - Y_s^m)(Z_s^n - Z_s^m) dB_s.
\end{aligned}$$

Next using localization and then taking expectation to obtain:

$$\begin{aligned}
&\mathbb{E} \int_0^{t \wedge \tau_k} |Z_s^n - Z_s^m|^2 ds \\
&\leq \mathbb{E}(Y_{\tau_k}^n - Y_{\tau_k}^m)^2 + C \mathbb{E} \int_{t \wedge \tau_k}^{\tau_k} (Y_s^n - Y_s^m)^2 ds + 2 \mathbb{E} \int_{t \wedge \tau_k}^{\tau_k} (Y_s^n - Y_s^m)(dK_t^n - dK_t^m) \\
&\leq \mathbb{E}(Y_{\tau_k}^n - Y_{\tau_k}^m)^2 + C \mathbb{E} \int_{t \wedge \tau_k}^{\tau_k} (Y_s^n - Y_s^m)^2 ds + 2 \mathbb{E} \{(K_{\tau_k}^n)^p\}^{1/p} \mathbb{E} \{(\sup_{t \leq \tau_k} (L_t - Y_t^n)^+)^q\}^{1/q} \\
&\quad + 2 \mathbb{E} \{(K_{\tau_k}^m)^p\}^{1/p} \mathbb{E} \{(\sup_{t \leq \tau_k} (L_t - Y_t^m)^+)^q\}^{1/q}
\end{aligned}$$

where q is the conjugate of p . Next using dominated convergence theorem and Proposition 2 to deduce that:

$$\mathbb{E} \int_0^{\tau_k} |Z_s^n - Z_s^m|^2 ds \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Now thanks to Lemma 1, there exists a constant C such that

$$\mathbb{E} \left\{ \int_0^T |Z_s^n|^p ds \right\} \leq C.$$

Therefore there exists a subsequence and a \mathcal{P} -measurable process Z which is the weak limit of that subsequence in $L_{\mathbb{R}}^p([0, T] \times \Omega, dt \otimes dP, \mathcal{P})$. It follows that for any $k \geq 0$ we have:

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^{\tau_k} |Z_s^n - Z_s|^p ds \right] = 0.$$

Further we can argue as in [12] to obtain that:

$$\mathbb{E} \left[\sup_{t \leq T} (|Y_{s \wedge \tau_k}^n - Y_{s \wedge \tau_k}^m|^2 + |K_{s \wedge \tau_k}^n - K_{s \wedge \tau_k}^m|^2) \right] \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

It follows that for any $k \geq 0$, the process $(Y_{t \wedge \tau_k})_{t \leq T}$ is continuous and since $(\tau_k)_{k \geq 0}$ is of stationary type then Y is also a continuous process. Moreover thanks to Dini's theorem and monotonic convergence theorem we have:

$$E[\sup_{s \leq T} |Y_s^n - Y_s|^p] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Finally for any $t \leq T$, let us set:

$$K_t = Y_0 - Y_t - \int_0^t f(s, Y_s, Z_s) ds + \int_0^t Z_s dB_s.$$

Then the process K is continuous, belongs to \mathcal{S}^p and for any $k \geq 0$ we have:

$$\mathbb{E} \left[\sup_{t \leq T} |K_{s \wedge \tau_k}^n - K_{s \wedge \tau_k}|^p \right] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

As K^n is increasing and the sequence $(\tau_k)_k$ is of stationary type then K is also increasing. Now going back to (17) taking the limit as $n \rightarrow \infty$ to obtain that:

$$(18) \quad Y_{t \wedge \tau_k} = Y_{\tau_k} + \int_{t \wedge \tau_k}^{\tau_k} f(s, Y_s, Z_s) ds + K_{\tau_k} - K_{t \wedge \tau_k} - \int_{t \wedge \tau_k}^{\tau_k} Z_s dB_s, \quad \forall t \leq T.$$

Additionally we can argue as in [12] to obtain that:

$$\int_0^{T \wedge \tau_k} (Y_s - L_s) dK_s = 0.$$

Taking now the limit w.r.t. k in the two previous equalities to obtain that:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, \quad \forall t \leq T.$$

and

$$\int_0^T (Y_s - L_s) dK_s = 0.$$

Henceforth the process (Y, Z, K) is the unique L^p -solution of the BSDE associated with $(f(t, y, z), L, \xi)$. \square

Remark 2 *We have also the following result whose proof is classical and then we omit it:*

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t^n - Y_t|^p + \left(\int_0^T |Z_t^n - Z_t|^2 dt \right)^{p/2} + \sup_{t \in [0, T]} |K_t^n - K_t|^p \right] = 0.$$

5 Viscosity solutions

Let $b : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ be two globally Lipschitz functions and let us consider the following SDE:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, t \leq T.$$

We denote by $(X_s^{t,x})_{s \geq t}$ the unique solution of the previous SDE starting from x at time $s = t$. Now we are given three continuous functions:

$$g : \mathbb{R}^d \rightarrow \mathbb{R}, \quad f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}, \quad h : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$$

such that f is Lipschitz w.r.t. y and z , uniformly w.r.t. t and x . Moreover there exists $p \in]1, 2[$ s.t. for every $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$(19) \quad \mathbb{E} \int_0^T (|f(s, X_s^{t,x}, 0, 0)|^p + |h(s, X_s^{t,x})|^p) ds + \mathbb{E}|g(X_T^{t,x})|^p < +\infty.$$

As said in the introduction, this condition relaxes the standard polynomial growth assumption and will enlarge the class of solutions. It is satisfied if for example $\sigma\sigma^*$ is uniformly elliptic, i.e., there exists $\lambda > 0$ such that

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad \forall \zeta \in \mathbb{R}^d \setminus \{0\}, \quad \zeta(\sigma\sigma^*)(t, x)\zeta^* \geq \lambda|\zeta|^2,$$

and if for some constant $A > 0$ (depending on b and σ , and T) such that uniformly w.r.t. $t \in [0, T]$

$$(20) \quad \lim_{|x| \rightarrow +\infty} [|f(t, x, 0, 0)| + |g(x)| + |h(t, x)|] \exp(-A(\ln |x|)^2) = 0.$$

The geometric Brownian motion (or Black-Scholes model in finance) is an example for which such a conditions are satisfied. Note that if we have stronger conditions on b or σ , we can have weaker growth ones on f , g and h .

From now on we assume that $1 < p < 2$ and that for every $(t, x) \in [0, T] \times \mathbb{R}^d$ let us define $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{s \in [t, T]}$ the unique solution of the reflected BSDE

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_t^T f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})du + K_T^{t,x} - K_t^{t,x} - \int_t^T Z_u^{t,x} dB_u$$

with

$$a.s. \forall s \in [t, T], \quad h(s, X_s^{t,x}) \leq Y_s^{t,x}.$$

Moreover on $[0, t]$, we set $Y_s^{t,x} = Y_t^{t,x}$, $Z_s^{t,x} = Z_t^{t,x} = 0$.

For every (t, x) , we will show that $Y_t^{t,x}$ is deterministic and we define a function

$$(21) \quad u(t, x) = Y_t^{t,x}.$$

In a first part we will prove that u is continuous and is a viscosity solution of the following obstacle problem

$$(22) \quad \min \left[u(t, x) - h(t, x), \right. \\ \left. -\frac{\partial u}{\partial t}(t, x) - \mathcal{L}u(t, x) - f(t, x, u(t, x), \sigma(t, x)\nabla u(t, x)) \right] = 0 \\ (t, x) \in [0, T] \times \mathbb{R}^d, \\ u(T, x) = g(x), \quad x \in \mathbb{R}^d,$$

with the second order partial differential operator

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^d ((\sigma\sigma^*)(t, x))_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d (b(t, x))_i \frac{\partial}{\partial x_i}.$$

In a second part we will prove that this is the unique continuous viscosity solution that belongs to a specific class of functions. However first let us recall the following definitions:

Definition 2 *Let u be a function that belongs to $C([0, T]) \times \mathbb{R}^d$. It is said to be:*

(i) *a viscosity subsolution of (22) if $u(T, x) \leq g(x)$, $x \in \mathbb{R}^d$, and for any function $\phi \in C^{1,2}((0, T) \times \mathbb{R}^d)$, if $u - \phi$ has a local maximum at (t, x) then*

$$\min(u(t, x) - h(t, x), -\frac{\partial \phi}{\partial t} - \mathcal{L}\phi(t, x) - f(t, x, u(t, x), \sigma\nabla\phi(t, x))) \leq 0.$$

(ii) *a viscosity supersolution of (22) if $u(T, x) \geq g(x)$, $x \in \mathbb{R}^d$, and for any function $\phi \in C^{1,2}((0, T) \times \mathbb{R}^d)$, if $u - \phi$ has a local minimum at (t, x) then*

$$\min(u(t, x) - h(t, x), -\frac{\partial \phi}{\partial t} - \mathcal{L}\phi(t, x) - f(t, x, u(t, x), \sigma\nabla\phi(t, x))) \geq 0.$$

(iii) *a viscosity solution of (22) if it is both a viscosity sub- and supersolution.*

5.1 Continuity and viscosity solution

We have the following result:

Proposition 3 *For every (t, x) , $Y_t^{t,x}$ is deterministic and the function*

$$u(t, x) = Y_t^{t,x}$$

is continuous and satisfies

$$(23) \quad \lim_{|x| \rightarrow +\infty} |u(t, x)| \exp(-A(\ln|x|)^2) = 0.$$

Proof. It suffices to show that whenever $(t_n, x_n) \rightarrow (t, x)$,

$$(24) \quad \mathbb{E} \left(\sup_{s \in [0, T]} |Y_s^{t_n, x_n} - Y_s^{t, x}|^p \right) \rightarrow 0.$$

Indeed, this will show that $(s, t, x) \mapsto Y_s^{t, x}$ is L^p -continuous, and so is $(t, x) \mapsto Y_t^{t, x}$. But $Y_t^{t, x}$ is deterministic, since it is \mathcal{F}_t^t measurable. Recall that $\{\mathcal{F}_s^t, t \leq s \leq T\}$ denotes the natural filtration of the Brownian motion $\{B_s - B_t, t \leq s \leq T\}$ augmented with the \mathbb{P} null sets of \mathcal{F} . Now (24) is a consequence of Lemma 3 and the following convergences

$$\begin{aligned} & \mathbb{E} |g(X_T^{t, x}) - g(X_T^{t_n, x_n})|^p \rightarrow 0, \\ & \mathbb{E} \left(\sup_{s \in [0, T]} |h(s, X_s^{t, x}) - h(s, X_s^{t_n, x_n})|^p \right) \rightarrow 0, \\ & \mathbb{E} \left(\int_0^T |\mathbf{1}_{[t, T]}(s) f(s, X_s^{t, x}, Y_s^{t, x}, Z_s^{t, x}) - \mathbf{1}_{[t_n, T]}(s) f(s, X_s^{t_n, x_n}, Y_s^{t, x}, Z_s^{t, x})| ds \right)^p \rightarrow 0, \end{aligned}$$

which follow from the continuity assumptions and the growths of f , g and h .

The condition (23) follows from Lemma 2 and condition (20). \square

In order to prove that u is a viscosity solution of the obstacle problem (22) we need a comparison result concerning the solutions of reflected BSDE. For $\tilde{\xi}$, \tilde{f} and \tilde{L} , let us denote by $(\tilde{Y}, \tilde{Z}, \tilde{K})$ the unique solution of

$$\tilde{Y}_t = \tilde{\xi} + \int_t^T \tilde{f}(s, \tilde{Y}_s, \tilde{Z}_s) ds + \tilde{K}_T - \tilde{K}_t - \int_t^T \tilde{Z}_s dB_s$$

with

$$P - a.s. \forall s \in [0, T], \tilde{L}_s \leq \tilde{Y}_s \text{ and } \int_0^T (\tilde{Y}_t - \tilde{L}_t) d\tilde{K}_t = 0.$$

Proposition 4 *If a.s. $\tilde{\xi} \geq \xi$, $\tilde{f} \geq f$ and $\tilde{L} \geq L$, then a.s. $\tilde{Y}_t \geq Y_t$ for every $t \in [0, T]$.*

Proof. Let $(\tilde{Y}^n, \tilde{Z}^n)$ and (Y^n, Z^n) be the sequence constructed by penalization (see Section 4). Using the standard comparison result for BSDE (see for example [20]), then for every $n \in \mathbb{N}$, $\tilde{Y}_t^n \geq Y_t^n$. Since the sequences \tilde{Y}^n and Y^n converge to resp. \tilde{Y} and Y , the conclusion follows. \square

Theorem 5 *The function $u : (t, x) \mapsto u(t, x) = Y_t^{t, x}$ defined in (21) is a viscosity solution of the obstacle problem (22).*

Proof. We are going to use the approximation of the RBSDE (2) by penalization, which was studied in Section 4. For each $(t, x) \in [0, T] \times \mathbb{R}^d$, let $(Y^{t, x, n}, Z^{t, x, n})$ denote the solution of the BSDE

$$\begin{aligned} \forall s \in [t, T], Y_s^{t, x, n} = & g(X_T^{t, x}) + \int_s^T f(u, Y_u^{t, x, n}, Z_u^{t, x, n}) du \\ & + n \int_s^T (Y_u^{t, x, n} - h(u, X_u^{t, x}))^- du - \int_s^T Z_u^{t, x, n} dB_u. \end{aligned}$$

From [20] it is known that

$$u_n(t, x) = Y_t^{t, x, n}, \quad (t, x) \in [0, T] \times \mathbb{R}^d,$$

is the viscosity solution of the parabolic PDE: for every $(t, x) \in [0, T] \times \mathbb{R}^d$

$$\frac{\partial u_n}{\partial t}(t, x) + \mathcal{L}u_n(t, x) + f_n(t, x, u_n(t, x), \sigma(t, x)\nabla u_n(t, x)) = 0$$

with

$$f_n(t, x, y, z) = f(t, x, y, z) + n(y - h(t, x))^-$$

and $u_n(T, x) = g(x)$ for each $x \in \mathbb{R}^d$. To be more precise, we have to say that in [20] p is supposed to be equal to 2. But with straightforward modifications in the proof of Theorem 3.2 in [20], we can show that the result holds also for $1 < p < 2$.

However, from the results of the previous section, for each $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$u_n(t, x) \uparrow u(t, x), \quad \text{as } n \rightarrow +\infty.$$

Since u_n and u are continuous, it follows from Dini's theorem that the above convergence is uniform on compacts.

We now show that u is a subsolution of (22). Let (t, x) be a point at which $u(t, x) > h(t, x)$, and let ϕ be a $C^{1,2}$ function such that $u - \phi$ has a maximum at point (t, x) . From Lemma 6.1 in [4], there exists sequences $n_j \rightarrow +\infty$, $(t_j, x_j) \rightarrow (t, x)$ such that

$$\frac{\partial \phi}{\partial t}(t_j, x_j) + \mathcal{L}\phi(t_j, x_j) + f_{n_j}(t_j, x_j, u_{n_j}(t_j, x_j), \sigma(t_j, x_j)\nabla \phi(t_j, x_j)) \leq 0.$$

From the assumption that $u(t, x) > h(t, x)$ and the uniform convergence of u_n , it follows that for j large enough $u_{n_j}(t_j, x_j) > h(t_j, x_j)$. Therefore in taking the limit as $j \rightarrow +\infty$, the above inequality yields:

$$\frac{\partial \phi}{\partial t}(t, x) + \mathcal{L}\phi(t, x) + f(t, x, u(t, x), \sigma(t, x)\nabla \phi(t, x)) \leq 0.$$

and we have proved that u is a subsolution of (22).

The same arguments (with converse inequalities) prove that u is also a supersolution of (22). \square

5.2 Uniqueness of the solution

In order to establish the uniqueness of the solution of equation (22), we need to impose the following additional assumption on f . For each $R > 0$, there exists a continuous function $m_R : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $m_R(0) = 0$ and

$$(25) \quad |f(t, x, r, p) - f(t, y, r, p)| \leq m_R(|x - y|(1 + |p|)),$$

for all $t \in [0, T]$, $|x| \leq R$, $|y| \leq R$, $|r| \leq R$, and $p \in \mathbb{R}^d$.

Theorem 6 *Under the above assumptions and (25), the obstacle problem (22) has at most one solution satisfying (23).*

The proof is similar to the uniqueness proof given in [1]. We just have to take into account the obstacle h .

Let u and v be two viscosity solutions of (22). As in [1], the proof consists in two steps. We first show that $u - v$ and $v - u$ are subsolutions of a specific partial differential equation, then we build a suitable sequence of smooth supersolutions of this equation to show that $|u - v| = 0$ in $[0, T] \times \mathbb{R}^d$. Hereafter we denote by $\|\cdot\|$ the sup norm in \mathbb{R}^d .

Lemma 7 *Let u be a subsolution and v a supersolution of (22). Then the function $w = u - v$ is a viscosity subsolution of*

$$(26) \quad \min(w, -\frac{\partial w}{\partial t} - \mathcal{L}w - \kappa\|w\| - \kappa\|\sigma\nabla w\|) = 0,$$

where κ is the Lipschitz constant of f in (y, z) .

Proof. First remark that $(u - v)(T, x) \leq 0$. Next let $\phi \in C^{1,2}((0, T) \times \mathbb{R}^d)$ and let $(t_0, x_0) \in (0, T) \times \mathbb{R}^d$ be a strict global maximum point of $w - \phi$ and we suppose that $w(t_0, x_0) > 0$. Our aim is to prove that at (t_0, x_0)

$$-\frac{\partial \phi}{\partial t} - \mathcal{L}\phi - \kappa|w| - \kappa|\sigma\nabla\phi| \leq 0.$$

We sketch the proof of Lemma 3.7 in [1]. We introduce the function

$$\psi_{\varepsilon, \alpha}(t, x, s, y) = u(t, x) - v(s, y) - \frac{|x - y|^2}{\varepsilon^2} - \frac{|t - s|^2}{\alpha^2} - \phi(t, x),$$

where ε and α are positive parameters which are devoted to tend to zero. Since (t_0, x_0) is a strict global maximum point of $u - v - \phi$, there exists a sequence $(\bar{t}, \bar{x}, \bar{s}, \bar{y})$ such that

- $(\bar{t}, \bar{x}, \bar{s}, \bar{y})$ is a global maximum point of $\psi_{\varepsilon, \alpha}$ in $([0, T] \times \bar{B}_R)^2$ where B_R is a ball with a large radius R ;
- $(\bar{t}, \bar{x}), (\bar{s}, \bar{y}) \rightarrow (t_0, x_0)$ as $(\varepsilon, \alpha) \rightarrow 0$;
- $\frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2}$ and $\frac{|\bar{t} - \bar{s}|^2}{\alpha^2}$ are bounded and tend to zero when $(\varepsilon, \alpha) \rightarrow 0$.

Moreover there exists two symmetric matrices X and Y such that since u is a subsolution, at point (\bar{t}, \bar{x})

$$(27) \quad \min(u - h, \bar{a} - \frac{\partial \phi}{\partial t} - \frac{1}{2}\text{Tr}(\sigma\sigma^*X) - \langle b, (\bar{p} + \nabla\phi) \rangle - f(\bar{t}, \bar{x}, u, \sigma(\bar{p} + \nabla\phi))) \leq 0,$$

and since v is a supersolution, at point (\bar{s}, \bar{y})

$$(28) \quad \min(v - h, \bar{a} - \frac{1}{2}\text{Tr}(\sigma\sigma^*Y) - \langle b, \bar{p} \rangle - f(\bar{s}, \bar{y}, v, \sigma\bar{p})) \geq 0,$$

where $\bar{a} = \frac{2(\bar{t} - \bar{s})}{\alpha^2}$ and $\bar{p} = \frac{2(\bar{x} - \bar{y})}{\varepsilon^2}$. Now we want to subtract these inequalities. Then from the Lipschitz continuity of σ and b we obtain:

$$\text{Tr}(\sigma\sigma^*(\bar{t}, \bar{x})X) - \text{Tr}(\sigma\sigma^*(\bar{s}, \bar{y})Y) \leq C \frac{|\bar{x} - \bar{y}|^2 + |\bar{t} - \bar{s}|^2}{\varepsilon^2} + \text{Tr}(\sigma\sigma^*(\bar{t}, \bar{x})D^2\phi(\bar{t}, \bar{x}));$$

and

$$|\langle b(\bar{t}, \bar{x}), \bar{p} \rangle - \langle b(\bar{s}, \bar{y}), \bar{p} \rangle| \leq C \frac{|\bar{x} - \bar{y}|^2 + |\bar{t} - \bar{s}|^2}{\varepsilon^2}.$$

We now consider the difference between the nonlinear terms:

$$\begin{aligned} & f(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), \sigma(\bar{t}, \bar{x})(\bar{p} + \nabla\phi(\bar{t}, \bar{x}))) - f(\bar{s}, \bar{y}, v(\bar{s}, \bar{y}), \sigma(\bar{s}, \bar{y})\bar{p}) \\ & \leq \rho_\varepsilon(|\bar{t} - \bar{s}|) + m(|\bar{x} - \bar{y}|(1 + |\bar{p}\sigma(\bar{s}, \bar{y})|)) + \kappa|u(\bar{t}, \bar{x}) - v(\bar{s}, \bar{y})| \\ & \quad + \kappa|\bar{p}(\sigma(\bar{t}, \bar{x}) - \sigma(\bar{s}, \bar{y})) + \sigma(\bar{t}, \bar{x})\nabla\phi(\bar{t}, \bar{x})|. \end{aligned}$$

Note that

$$|\bar{p}(\sigma(\bar{t}, \bar{x}) - \sigma(\bar{s}, \bar{y}))| \leq C \frac{|\bar{x} - \bar{y}|^2 + |\bar{t} - \bar{s}|^2}{\varepsilon^2}$$

and

$$|\bar{x} - \bar{y}||\bar{p}\sigma(\bar{s}, \bar{y})| \leq C \frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2}.$$

Now let us go back to (27) and (28). We claim that $u(\bar{t}, \bar{x}) - h(\bar{t}, \bar{x}) > 0$ (taking a subsequence if necessary). If not, there exists a subsequence such that $u(\bar{t}, \bar{x}) - h(\bar{t}, \bar{x}) \leq 0$. Passing to the limit we get $u(t_0, x_0) - h(t_0, x_0) \leq 0$. But from the assumption $u(t_0, x_0) - v(t_0, x_0) > 0$, we deduce that $0 \geq u(t_0, x_0) - h(t_0, x_0) > v(t_0, x_0) - h(t_0, x_0)$. Therefore we have $v(\bar{s}, \bar{y}) - h(\bar{s}, \bar{y}) < 0$, which leads to a contradiction with (28). Henceforth (27) becomes

$$\bar{a} - \frac{\partial\phi}{\partial t} - \frac{1}{2}\text{Tr}(\sigma\sigma^*X) - \langle b, (\bar{p} + \nabla\phi) \rangle - f(\bar{t}, \bar{x}, u, \sigma(\bar{p} + \nabla\phi)) \leq 0.$$

Thus we obtain

$$\begin{aligned} & -\frac{\partial\phi}{\partial t} - \frac{1}{2}\text{Tr}(\sigma\sigma^*X) - \langle b, (\bar{p} + \nabla\phi) \rangle - f(\bar{t}, \bar{x}, u, \sigma(\bar{p} + \nabla\phi)) \\ & \quad + \frac{1}{2}\text{Tr}(\sigma\sigma^*Y) + \langle b, (\bar{p} + \nabla\phi) \rangle + f(\bar{s}, \bar{y}, v, \sigma\bar{p}) \leq 0. \end{aligned}$$

With all previous estimates we get:

$$\begin{aligned} & -\frac{\partial\phi}{\partial t} - \frac{1}{2}\text{Tr}(\sigma\sigma^*X) - \langle b, \nabla\phi \rangle - \kappa|u(\bar{t}, \bar{x}) - v(\bar{s}, \bar{y})| - \kappa|\sigma(\bar{t}, \bar{x})\nabla\phi(\bar{t}, \bar{x})| \\ & \leq \rho_\varepsilon(|\bar{t} - \bar{s}|) + C \frac{|\bar{x} - \bar{y}|^2 + |\bar{t} - \bar{s}|^2}{\varepsilon^2} + m(|\bar{x} - \bar{y}|(1 + |\bar{p}\sigma(\bar{s}, \bar{y})|)). \end{aligned}$$

Next let first α goes to zero. Since $\frac{|\bar{t} - \bar{s}|^2}{\alpha^2}$ is bounded then $|\bar{t} - \bar{s}|$ goes to zero. We remove the first term and the term $|\bar{t} - \bar{s}|^2$ of the right-hand side above. Then we let $\varepsilon \rightarrow 0$ and since $(\bar{t}, \bar{x}) \rightarrow (t_0, x_0)$ we finally have

$$-\frac{\partial\phi}{\partial t}(t_0, x_0) - \mathcal{L}\phi(t_0, x_0) - \kappa|w(t_0, x_0)| - \kappa|\sigma(t_0, x_0)\nabla\phi(t_0, x_0)| \leq 0.$$

Therefore w is a subsolution of the equation (26). \square

Recall now Lemma 3.8 in [1].

Lemma 8 For any $A > 0$, there exists $C > 0$ such that the function

$$\chi(t, x) = \exp[(C(T - t) + A)\psi(x)]$$

where

$$\psi(x) = [\ln((|x|^2 + 1)^{1/2}) + 1]^2,$$

satisfies

$$-\frac{\partial \chi}{\partial t} - \mathcal{L}\chi - \kappa\chi - \kappa|\sigma \nabla \chi| > 0, \quad \text{in } [t_1, T] \times \mathbb{R}^d,$$

where $t_1 = T - A/C$.

Proof. The proof of this result is given in [1]. Note that here, w.r.t. the setting of this latter article, we have the same assumptions without the jump process, i.e. $B \equiv 0$. Now the function χ is positive therefore it is a supersolution of (22) with $f(t, x, y, \sigma z) = \kappa y + \kappa|\sigma z|$ and $h \equiv 0$. \square

The rest of the proof of Theorem 6 is the same as in [1]. For any $\alpha > 0$,

$$|u(t, x) - v(t, x)| \leq \alpha \chi(t, x), \quad \text{in } [t_1, T] \times \mathbb{R}^d.$$

The sketch of the proof is the following.

- Using the growth condition on u and v , $\lim_{n \rightarrow +\infty} |u - v|(t, x)/\chi(t, x) = 0$ uniformly for $t \in [t_1, T]$, for some $A > 0$. This implies that $|u - v| - \alpha \chi$ is bounded from above in $[t_1, T] \times \mathbb{R}^d$. Hence

$$M = \max_{[t_1, T] \times \mathbb{R}^d} (|u - v| - \alpha \chi) e^{K(T-t)}$$

is achieved at some point (t_0, x_0) . Without loss of generality we may assume that $w(t_0, x_0) = (u - v)(t_0, x_0) > 0$.

- This means that the function $w - \phi$ has a global maximum point at (t_0, x_0) , where

$$\phi(t, x) = \alpha \chi(t, x) + (w - \alpha \chi)(t_0, x_0) e^{K(t-t_0)}.$$

We use the fact that w is a subsolution of (26), i.e.,

$$-\frac{\partial \phi}{\partial t}(t_0, x_0) - \mathcal{L}\phi(t_0, x_0) - \kappa|w(t_0, x_0)| - \kappa|\sigma(t_0, x_0) \nabla \phi(t_0, x_0)| \leq 0.$$

But since $w(t_0, x_0) = |w(t_0, x_0)|$, the left-hand side is

$$\alpha \left[-\frac{\partial \chi}{\partial t}(t_0, x_0) - \mathcal{L}\chi(t_0, x_0) - \kappa|\chi(t_0, x_0)| - \kappa|\sigma(t_0, x_0) \nabla \chi(t_0, x_0)| \right].$$

- This leads to a contradiction. Then $t_0 = T$ and since $w(T, x) = 0$, we have $|w(t, x)| - \alpha \chi(t, x) \leq 0$ on $[t_1, T] \times \mathbb{R}^d$. Letting α tending to zero, we obtain $u = v$ in $[t_1, T] \times \mathbb{R}^d$.
- Repeat recursively this argument on $[t_2, t_1]$ with $t_2 = (t_1 - A/C)^+$, etc., we obtain $u = v$ on $[0, T] \times \mathbb{R}^d$.

References

- [1] BARLES, G., BUCKDAHN, R. and PARDOUX, E., *Backward stochastic differential equations and integral-partial differential equations*, Stochastics and Stochastics Reports, vol.60, pp. 57-83
- [2] BARRIEU, P. and EL-KAROUI, N., *Inf-convolution of risk measures and optimal risk transfer*, Finance and stochastics, 9 (2). pp. 269-298
- [3] BRIAND, PH., DELYON, B., HU, Y., PARDOUX, E. and STOICA, L. (2003). *L^p solutions of backward stochastic differential equations*, Stochastic Process. Appl., **108**, 109–129.
- [4] CRANDALL, M. G., ISHII, H. and LIONS, P.-L. (1992). *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), **27**, 1, 1–67.
- [5] DELLACHERIE, C. and MEYER, P.A. (1980). Probabilités et Potentiel V-VIII, *Hermann, Paris*.
- [6] DJEHICHE, B. and HAMADÈNE, S. (2009). *On a finite horizon starting and stopping problem with risk of abandonment* to appear in Internat. Journ. of Theoretical & Applied Finance (IJTAF).
- [7] DJEHICHE, B., HAMADÈNE, S. and POPIER, A. (2007) *A Finite Horizon Optimal Multiple Switching Problem*, Available online at arXiv:0707.2663 (July 2007).
- [8] EL-ASRI, B. and HAMADÈNE, S. (2009). *The Finite Horizon Optimal Multi-Modes Switching Problem: the Viscosity Solution Approach*, Appl. Math. Optim. DOI 10.1007/s00245-009-9071-3
- [9] EL KAROUI, N. (1979). *Les aspects probabilistes du contrôle stochastique*. Ecole d'été de St-Flour, Lect. Notes in Math. 876, (Springer, Berlin), pp.73-238.
- [10] EL KAROUI, N., HAMADÈNE, S., MATOUSSI, A. (2009). *Backward SDEs and Applications*. In Indifference Pricing: Theory and Applications, edited by R. Carmona, Princeton Series in Financial Engineering, pp.267-303.
- [11] EL KAROUI, N., PENG, S. and QUENEZ, M.-C. (1997). *Backward stochastic differential equations in finance*, Math. Finance, **7**, 1, 1–71.
- [12] EL KAROUI, N., KAPOUDJIAN, C., PARDOUX, E., PENG, S., and QUENEZ, M.-C. (1997). *Reflected solutions of backward SDE's, and related obstacle problems for PDE's*, Ann. Probab., **25**, 2, 702–737.
- [13] EL KAROUI, N., PARDOUX, E., and QUENEZ, M.-C. (1997). *Reflected backward SDEs and American options*. In Numerical methods in finance, L. Robers, D. Talay (eds.), Cambridge University Press, p. 215-231.

- [14] EL KAROUI, N. and ROUGE, R. (2000). *Pricing via utility maximization and entropy*. Mathematical Finance, April 2000, vol. 10, no. 2, pp. 259-276(18).
- [15] HAMADÈNE, S. and JEANBLANC, M. (2007). *On the stopping and starting problem: application to reversible investment*. Mathematics of Operations Research, 32, pp.182-192.
- [16] HAMADÈNE, S. and LEPELTIER, J. P. (1995). *Zero-sum stochastic differential games and backward equations*. Systems Control Lett. 24 259-263.
- [17] HAMADÈNE, S. and LEPELTIER, J. P. (1995). *Backward equations, stochastic control and zero-sum stochastic differential games*, Stochastics and stochastic Reports, vol.54, pp.221-231.
- [18] HAMADÈNE, S. and ZHANG, J. (2007). *The Starting and Stopping Problem under Knightian Uncertainty*. Available online at arXiv:0710.0908 (October 2007).
- [19] LEPELTIER, J.-P., MATOUSSI, A. and XU, M. (2005). *Reflected backward stochastic differential equations under monotonicity and general increasing growth conditions*, Adv. Appl. Probab., **37**, 1, 134–159.
- [20] PARDOUX, E. *BSDEs, weak convergence and homogenization of semilinear PDEs*, Nonlinear analysis, differential equations and control, (Montreal, QC, 1998), Kluwer Acad. Publ., Dordrecht, 503–549
- [21] PARDOUX, E. and PENG, S. (1990). *Adapted solutions of a backward stochastic differential equation*. Systems Control Letters 14, 55-61.
- [22] PARDOUX, E. and PENG, S. (1992). *Backward stochastic differential equations and quasilinear parabolic partial differential equations*. In: B.L. Rozovskii and R.B. Sowers, Editors, Stochastic partial differential equations and their applications Lecture Notes in Control and Inform. Sci. 176, Springer, Berlin, Heidelberg New York, pp. 200-217.
- [23] PENG, S. (1999). *Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer's type*. Probability Theory and Related Fields 113, 473-499.