The obstacle Problem for Quasilinear Stochastic PDE's

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Abstract

We prove an existence and uniqueness result for the obstacle problem of quasilinear parabolic stochastic PDEs. The method is based on the probabilistic interpretation of the solution by using the backward doubly stochastic differential equation.

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1 Introduction

We consider the following stochastic PDE, in $\mathbb{R}^d$,

$$\begin{aligned} du_t(x) + \Big[\ \frac{1}{2}\Delta u_t(x) + f(t,x,u_t(x),\nabla u_t(x)) + \mathrm{div} g_t\left(x,u_t\left(x\right),\nabla u_t\left(x\right)\right)\ \Big]\, dt \\ + h_t(x,u_t(x),\nabla u_t(x))\cdot \overleftarrow{dB}_t = 0, \end{aligned} \tag{1}$$

over the time interval $[0,T]$, with a given final condition $u_T = \Phi$ and $f, g = \big(g_1,\cdots,g_d\big)$, $h = \big(h_1,\cdots,h_{d^1}\big)$ non-linear random functions. The differential term with $\overleftarrow{dB}_t$ refers to the backward stochastic integral with respect to a d^1-dimensional Brownian motion on $\big(\Omega,\mathcal{F},\mathbb{P},(B_t)_{t\geq 0}\big)$. We use the backward notation beacause in the proof we will employ the doubly stochastic framework introduced by Pardoux and Peng [22].

In the case where f and g do not depend of u and ∇u, and if h is identically null, the equation (1) becomes a linear parabolic equation,

$$\partial_t u(t,x) + \frac{1}{2}\Delta u(t,x) + f(t,x) + \mathrm{div} g(t,x) = 0. \tag{2}$$

If $v : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}$ is a given function such that $v(T,x) \leq \Phi(x)$, we may roughly say that the solution of the obstacle problem for (2) is a function $u \in \mathbf{L}^2\big([0,T]; H^1(\mathbb{R}^d)\big)$ such that the following

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conditions are satisfied in $(0,T)\times\mathbb{R}^d$:

$$\begin{aligned}
&(i)\ \ u\geq v,\quad dt\otimes dx-\text{a.e.},\\
&(ii)\ \ \partial_t u+\ \frac{1}{2}\Delta u+f+\text{div}g\leq 0\\
&(iii)\ \ (u-v)\big(\partial_t u+\ \frac{1}{2}\Delta u+f+\text{div}g\big)=0.\\
&(iv)\ \ u_T=\Phi,\quad dx-\text{a.e.}
\end{aligned}\tag{3}$$

The relation (ii) means that the distribution appearing in the RHS of the inequality is a non-positive measure. The relation (iii) is not rigourously stated. We may roughly say that one has $\partial_t u+\ \frac{1}{2}\Delta u+f+\text{div}g=0$ on the set $\{u>v\}$.
If one expresses the obstacle problem for (2) in terms of variational inequalities one should also ask that the solution has a minimality property (see Mignot-Puel [18] or Bensoussan-Lions [3] p.250). The work of El Karoui et al [12] treats the obstacle problem for (2) within the framework of backward stochastic differential equations (BSDE in short). Namely the equation (2) is considered with f depending of u and ∇u, while the function g is null (as well h) and the obstacle v is continuous. The solution is represented stochastically as a process and the main new object of this BSDE framework is a continuous increasing process that controls the set $\{u=v\}$. This increasing process determines in fact the measure from the relation (ii). Bally et al [1] (see also [16]) point out that the continuity of this process allows one to extend the classical notion of strong variational solution (see Theorem 2.2 of [3] p.238) and express the solution to the obstacle as a pair (u,ν) where ν equals the RHS of (ii) and is supported by the set $\{u=v\}$. In the present paper we adopt this point of view which has the advantage of expressing the notion of solution independently of the double stochastic framework and without the minimality property of Mignot-Puel [18], which would be very difficult to manipulate in the case of the stochastic PDE. In section 2.2 we are going to examine the potential and the measure associated to a continuous increasing process. We call such potentials and measures, regular potentials, respectively regular measures.

Now let us consider the final condition to be a fixed function $\Phi\in\mathbf{L}^2\big(\mathbb{R}^d\big)$ and the obstacle v be a random continuous function, $v:\Omega\times[0,T]\times\mathbb{R}^d\longrightarrow\mathbb{R}$. Then the obstacle problem for the equation (1) is defined as a pair (u,ν), where ν is a random regular measure and $u\in\mathbf{L}^2\big(\Omega\times[0,T];H^1(R^d)\big)$ satisfies the following relations :

$$\begin{aligned}
&(i')\ \ u\geq v,\quad d\mathbb{P}\otimes dt\otimes dx-\text{a.e.},\\
&(ii')\ \ du_t(x)+\big[\ \frac{1}{2}\Delta u_t(x)+f(t,x,u_t(x),\nabla u_t(x))+\text{div}g_t\left(x,u_t\left(x\right),\nabla u_t\left(x\right)\right)\ \big]\,dt\\
&\qquad\qquad +h_t(x,u_t(x),\nabla u_t(x))\cdot\overleftarrow{dB}_t=-\nu(dt,dx),\quad a.s.,\\
&(iii')\ \ \nu\big(u>v\big)=0,\quad a.s.,\\
&(iv')\ \ u_T=\Phi,\quad d\mathbb{P}\otimes dx-\text{a.e.}.
\end{aligned}\tag{4}$$

In Section 2.4 we explain the rigorous sense of the relation (iii') which is based on the quasi-continuity of u . The main result of our paper is Theorem 4 which ensures the existence and uniqueness of the solution of the obstacle problem for (1). The method of proof is based on the penalization procedure and the doubly stochastic calculus which is essential, although the definition of the solution and the statement of the result avoids the doubly stochastic framework.
Similarly to the case treated in El Karoui et al [12], the most difficult point is to show that the approximating sequence converges uniformly on the trajectories over the coincidence set $\{u=v\}$. This is proven in Lemma 7. A useful tool in our paper is also the probabilistic representation of the divergence term obtained in [24] and the doubly stochastic representation corresponding to the divergence term of the stochastic PDE in [8].
Finally we would like to thank our friend Vlad Bally for a stimulating discussion on the obstacle problem we had "à la Gare de Montparnasse".

2 Preliminaries

The basic Hilbert space of our framework is $\mathbf{L}^2\left(\mathbb{R}^d\right)$ and we employ the usual notation for its scalar product and its norm,

$$(u,v)=\int_{\mathbb{R}^d} u\,(x)\,v\,(x)\,dx,\ \|u\|_2=\left(\int_{\mathbb{R}^d} u^2\,(x)\,dx\right)^{\frac{1}{2}}.$$

In general, we shall use the notation

$$(u,v)=\int_{\mathbb{R}^d} u(x)v(x)\,dx,$$

where u, v are measurable functions defined in $\mathbb{R}^d$ and $uv\in\mathbf{L}^1(\mathbb{R}^d)$.
Our evolution problem will be considered over a fixed time interval $[0,T]$ and the norm for a function $L^2\left([0,T]\times\mathbb{R}^d\right)$ will be denoted by

$$\|u\|_{2,2}=\left(\int_0^T\int_{\mathbb{R}^d}|u(t,x)|^2dxdt\right)^{\frac{1}{2}}.$$

Another Hilbert space that we use is the first order Sobolev space $H^1\left(\mathbb{R}^d\right)=H^1_0\left(\mathbb{R}^d\right)$. Its natural scalar product and norm are

$$(u,v)_{H^1(\mathbb{R}^d)}=(u,v)+(\nabla u,\nabla v)\,,\ \|u\|_{H^1(\mathbb{R}^d)}=\left(\|u\|_2^2+\|\nabla u\|_2^2\right)^{\frac{1}{2}}$$

where we denote the gradient by $\nabla u(t,x)=\big(\partial_1 u(t,x),\cdots,\partial_d u(t,x)\big)$.
Of special interest is the subspace $\widetilde{F}\subset\mathbf{L}^2\left([0,T];H^1\left(\mathbb{R}^d\right)\right)$ consisting of all functions $u(t,x)$ such that $t\longmapsto u_t=u(t,\cdot)$ is continuous in $\mathbf{L}^2(\mathbb{R}^d)$. The natural norm on $\widetilde{F}$ is

$$\|u\|_T=\sup_{0\le t\le T}\|u_t\|_2+\left(\int_0^T\|\nabla u_t\|_2dt\right)^{\frac{1}{2}}.$$

The Lebesgue measure in $\mathbb{R}^d$ will be sometimes denoted by m. The space of test functions which we employ in the definition of weak solutions of the evolution equations (1) or (2) is $\mathcal{D}_T=\mathcal{C}^\infty\,[0,T])\otimes\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$, where $\mathcal{C}^\infty\left([0,T]\right)$ denotes the space of real functions which can be extended as infinite differentiable functions in the neighborhood of $[0,T]$ and $\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ is the space of infinite differentiable functions with compact support in $\mathbb{R}^d$.

2.1 The probabilistic interpretation of the divergence term

The operator $\partial_t+\frac{1}{2}\Delta$, which represents the main linear part in the equation (1), is probabilistically interpreted by the Bownian motion in $\mathbb{R}^d$. We shall view the Brownian motion as a Markov process and therefore we next introduce some detailed notation for it. The sample space is $\Omega'=\mathcal{C}\left([0,\infty);\mathbb{R}^d\right)$, the canonical process $(W_t)_{t\ge0}$ is defined by $W_t(\omega)=\omega(t)$, for any $\omega\in\Omega'$, $t\ge0$ and the shift operator, $\theta_t:\Omega'\longrightarrow\Omega'$, is defined by $\theta_t(\omega)(s)=\omega(t+s)$, for any $s\ge0$ and $t\ge0$. The canonical filtration $\mathcal{F}_t^0=\sigma\left(W_s;s\le t\right)$ is completed by the standard procedure with respect to the probability measures produced by the transition function

$$P_t(x,dy)=q_t(x-y)dy,\quad t>0,\quad x\in\mathbb{R}^d,$$

where $q_t(x)=(2\pi t)^{-\frac{d}{2}}\exp\left(-|x|^2/2t\right)$ is the gaussian density. Thus we get a continuous Hunt process $\left(\Omega',W_t,\theta_t,\mathcal{F},\mathcal{F}_t^0,\mathbb{P}^x\right)$. We shall also use the backward filtration of the future events

$\mathcal{F}'_t = \sigma(W_s;\ s \geq t)$ for $t \geq 0$. $\mathbb{P}^0$ is the Wiener measure, which is supported by the set $\Omega'_0 = \{\omega \in \Omega',\ w(0) = 0\}$. We also set $\Pi_0(\omega)(t) = \omega(t) - \omega(0)$, $t \geq 0$, which defines a map $\Pi_0 : \Omega' \to \Omega'_0$. Then $\Pi = (W_0, \Pi_0) : \Omega' \to \mathbb{R}^d \times \Omega'_0$ is a bijection. For each probability measure on $\mathbb{R}^d$, the probability $\mathbb{P}^\mu$ of the Brownian motion started with the initial distribution μ is given by

$$\mathbb{P}^\mu = \Pi^{-1}\left(\mu \otimes \mathbb{P}^0\right).$$

In particular, for the Lebesgue measure in $\mathbb{R}^d$, which we denote by $m = dx$, we have

$$\mathbb{P}^m = \Pi^{-1}\left(dx \otimes \mathbb{P}^0\right).$$

These relations are saying that W_0 is independent of Π_0. It is known that each component $(W^i_t)_{t\geq 0}$ of the Brownian motion, $i = 1, \cdots, d$, is a martingale under any of the measures $\mathbb{P}^\mu$. The next lemma shows that $\left(W^i_{t-r}, \mathcal{F}'_{t-r}\right)$, $r \in (0,t]$ is a backward local martingale under $\mathbb{P}^m$.

Lemma 1. *Let $0 < s < t$. If $A \in \sigma(W_t)$ is such that $\mathbb{E}^m[|W_t|; A] < \infty$, then one has $\mathbb{E}^m[|W_s|; A] < \infty$. Moreover, for each $B \in \mathcal{F}'_t$, and $i = 1, \cdots, d$, one has*

$$\mathbb{E}^m\left[W^i_s; A \cap B\right] = \mathbb{E}^m\left[W^i_t; A \cap B\right].$$

Proof: We note that W_t is uniformly distributed, and consequently for each $c > 0$, the set $A_c = \{|W_t| \leq c\}$ satisfies

$$\mathbb{E}^m[|W_t|; A_c] < \infty.$$

This shows that the class of the sets to which applies the statement is rather large.
The vector $(W_0, W_s - W_0, W_t - W_s)$ has the distribution $m \otimes \mathcal{N}(0,s) \otimes \mathcal{N}(0, t-s)$, under the measure $\mathbb{P}^m$. Then one deduce that $(W_s, W_t - W_s)$ has the distribution $m \otimes \mathcal{N}(0, t-s)$ and we may write, for $\varphi_1, \varphi_2 \in \mathcal{C}_c\left(\mathbb{R}^d\right)$,

$$\begin{aligned}\mathbb{E}^m\left[\varphi_1(W_t - W_s)\varphi_2(W_t)\right] &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi_1(y)\varphi_2(x+y)q_{t-s}(y)dydx \\ &= \left(\int_{\mathbb{R}^d} \varphi_2(x)dx\right)\left(\int_{\mathbb{R}^d} \varphi_1(y)q_{t-s}(y)dy\right).\end{aligned}$$

This relation shows that the vector $(W_t - W_s, W_t)$ has the distribution $\mathcal{N}(0, t-s) \otimes m$, under $\mathbb{P}^m$.Then the obvious inequality $|W_s| \leq |W_t| + |W_t - W_s|\left(\mathbf{1}_{\{|W_t|\leq 1\}} + |W_t|\right)$ allows one to deduce the first assertion of the lemma.
In order to check the second assertion of the lemma we write

$$\mathbb{E}^m\left[W^i_s; A \cap B\right] = \mathbb{E}^m\left[W^i_t; A \cap B\right] - \mathbb{E}^m\left[W^i_t - W^i_s; A \cap B\right]$$

and all that it remains to check is that the last term is null. In order to shows this one first observe that the distribution of the vector $(W_t - W_s, W_t, W_{t^1} - W_t, W_{t^2} - W_{t^1}, \cdots, W_{t^n} - W_{t^{n-1}})$ is $\mathcal{N}(0, t-s) \otimes m \otimes \mathcal{N}(0, t^1 - t) \otimes \cdots \otimes \mathcal{N}(0, t_n - t_{n-1})$, for each system $s < t < t^1 < \cdots < t^n$. Then one has, for each $B \in \sigma\left(W_{t^1} - W_t, \cdots, W_{t^n} - W_{t^{n-1}}\right)$,

$$\mathbb{E}^m\left[W^i_t - W^i_s; A \cap B\right] = \mathbb{E}^0\left[W^i_t - W^i_s\right] m(A)\mathbb{P}^0(B) = 0,$$

which implies the assertion of the lemma. □

Now let us assume that f and $|g|$ belong to $\mathbf{L}^2\left([0,T] \times \mathbb{R}^d\right)$ and $u \in \widetilde{F}$ is a solution of the deterministic equation (2). Let us denote by

$$\int_s^t g_r * dW_r = \sum_{i=1}^d \left(\int_s^t g_i(r, W_r)dW^i_r + \int_s^t g_i(r, W_r)d\overleftarrow{W^i_r}\right). \tag{5}$$

Then one has the following representation (Theorem 3.1 in [24])

Theorem 1. *The following relation holds $\mathbb{P}^m$-a.s. for each $0 \le s \le t \le T$,*

$$u_t(W_t) - u_s(W_s) = \sum_{i=1}^{d} \int_s^t \partial_i u_r(W_r) dW_r^i - \int_s^t f_r(W_r) dr - \frac{1}{2} \int_s^t g_r * dW_r \tag{6}$$

In [24] one uses the backward martingale $\overleftarrow{M}^{\mu,i}$ defined under an arbitrary $\mathbb{P}^\mu$, with μ a probability measure in $\mathbb{R}^d$, in order to express the integral $\int_s^t g_r * dW_r$. Though formally the definition looks different, one easily sees that it is the same object.

2.2 Regular measures

In this section we shall be concerned with some facts related to the time -space Brownian motion, with the state space $[0,T[\times \mathbb{R}^d$, coresponding to the generator $\partial_t + \frac{1}{2}\Delta$. Its associated semigroup will be denoted by $\left(\widetilde{P}_t\right)_{t>0}$. We may express it in terms of the Gaussian density of the semigroup $(P_t)_{t>0}$ in the following way

$$\widetilde{P}_t \psi(s,x) = \begin{cases} \int_{\mathbb{R}^d} q_t(x,y) \psi(s+t,y) \, dy, & if \quad s+t < T, \\ 0, & otherwise, \end{cases}$$

where $\psi : [0,T[\times \mathbb{R}^d \to \mathbb{R}$ is a bounded Borel measurable function, $s \in [0,T[, x \in \mathbb{R}^d$ and $t > 0$. So we may also write $\left(\widetilde{P}_t \psi\right)_s = P_t \psi_{t+s}$ if $s+t < T$. The corresponding resolvent has a density expressed in terms of the density q_t too, as follows

$$\widetilde{U}_\alpha \psi(t,x) = \int_t^T \int_{\mathbb{R}^d} e^{-\alpha(s-t)} q_{s-t}(x-y) \psi(s,y) \, dy ds.$$

or

$$\left(\widetilde{U}_\alpha \psi\right)_t = \int_t^T e^{-\alpha(s-t)} P_{s-t} \psi_s \, ds.$$

In particular this ensures that the excessive functions with respect to the time-space Brownian motion are lower semicontinuous. In fact we will not use directly the time space process, but only its semigroup and resolvent. For related facts concerning excessive functions the reader is refered to [6] or [4]. Some further properties of this semigroup are presented in the next lemma.

Lemma 2. *The semigroup $\left(\widetilde{P}_t\right)_{t>0}$ acts as a strongly continuous semigroup of contractions on the spaces $\mathbf{L}^2\left([0,T[\times \mathbb{R}^d\right) = \mathbf{L}^2\left([0,T[; \mathbf{L}^2\left(\mathbb{R}^d\right)\right)$ and $\mathbf{L}^2\left([0,T[; H^1\left(\mathbb{R}^d\right)\right)$.*

Proof : Obviously it is enough to check the following relations

$$\lim_{r \to 0} \left(\int_0^{T-r} \|P_r u_{t+r} - u_t\|_2^2 \, dt + \int_{T-r}^T \|u_t\|_2^2 \, dt \right) = 0,$$

$$\lim_{r \to 0} \Big(\int_0^{T-r} \|\nabla\big(P_r u_{t+r} - u_t\big)\|_2^2 \, dt + \int_{T-r}^T \|\nabla u_t\|_2^2 \, dt \Big) = 0.$$

First we note that, for each function $u \in \mathbf{L}^2\left([0,T[\times \mathbb{R}^d\right)$ and $r > 0$, one has

$$\lim_{r \to 0} \int_0^{T-r} \|u_{t+r} - u_t\|_2^2 \, dt = 0.$$

This property is obvious for a function $u \in \mathcal{C}_c\left([0,T[\times \mathbb{R}^d\right)$ and then it is obtained by approximation for any function in $\mathbf{L}^2\left([0,T[\times \mathbb{R}^d\right)$. Then the relation

$$\lim_{r \to 0} \int_0^{T-r} \|P_r u_{t+r} - u_t\|_2^2 \, dt = 0,$$

easily follows. From it one deduces the strong continuity of $\left(\widetilde{P}_t\right)_{t>0}$ on $\mathbf{L}^2\left([0,T[\times\mathbb{R}^d\right)$.
In order to prove the same property in the space $L^2\left([0,T[\,;H^1\left(\mathbb{R}^d\right)\right)$ one should start with the relation

$$\lim_{r\to 0}\int_0^{T-r}\left\|\nabla\left(u_{t+r}-u_t\right)\right\|_2^2 dt=0,$$

which holds for each $u\in\mathcal{C}_c^\infty\left([0,T[\times\mathbb{R}^d\right)$ and then repeat with obvious modifications, the previous reasoning. □

The next definition restricts our attention to potentials belonging to $\widetilde{F}$, which is the class of potentials appearing in our parabolic case of the obstacle problem.

Definition 1. *(i) A function $\psi:[0,T]\times\mathbb{R}^d\to\overline{\mathbb{R}}$ is called quasicontinuous provided that that for each $\varepsilon>0$, there exists an open set, $D_\varepsilon\subset[0,T]\times\mathbb{R}^d$, such that ψ is finite and continuous on D_ε^c and*

$$\mathbb{P}^m\left(\{\omega\in\Omega'/\exists\, t\in[0,T]\ \ s.t.\,(t,W_t(\omega))\in D_\varepsilon\}\right)<\varepsilon.$$

(ii) A function $u:[0,T]\times\mathbb{R}^d\to[0,\infty]$ is called a regular potential, provided that its restriction to $[0,T[\times\mathbb{R}^d$ is excessive with respect to the time-space semigroup, it is quasicontinuous, $u\in\widetilde{F}$ and $\lim_{t\to T}u_t=0$ in $\mathbf{L}^2\left(\mathbb{R}^d\right)$.

Observe that if a function ψ is quasicontinuous, then the process $\left(\psi_t\left(W_t\right)\right)_{t\in[0,T]}$ is continuous. Next we will present the basic properties of the regular potentials. Do to the expression of the semigroup $\left(\widetilde{P}_t\right)_{t>0}$ in terms of the density, it follows that two excessive functions which represent the same element in $\widetilde{F}$ should coincide.

Theorem 2. *Let $u\in\widetilde{F}$. Then u has a version which is a regular potential if and only if there exists a continuous increasing process $A=(A_t)_{t\in[0,T]}$ which is $\left(\mathcal{F}_t\right)_{t\in[0,T]}$ -adapted and such that $A_0=0$, $\mathbb{E}^m\left[A_T^2\right]<\infty$ and*

$$u_t(W_t)=\mathbb{E}\left[A_T\left|\mathcal{F}_t\right.\right]-A_t,\ \mathbb{P}^m\text{–}a.s., \tag{i}$$

for each $t\in[0,T]$. The process A is uniquely determined by these properties. Moreover the following relations hold

$$u_t\left(W_t\right)=A_T-A_t-\sum_{i=1}^d\int_t^T\partial_i u_s\left(W_s\right)dW_s^i,\ \mathbb{P}^m-a.s.\,, \tag{ii}$$

$$\left\|u_t\right\|_2^2+\int_t^T\left\|\nabla u_s\right\|_2^2 ds=\mathbb{E}^m\left(A_T-A_t\right)^2\,, \tag{iii}$$

$$\left(u_0,\varphi_0\right)+\int_0^T\frac{1}{2}\left(\nabla u_s,\nabla\varphi_s\right)+\left(u_s,\partial_s\varphi_s\right)ds=\int_0^T\int_{\mathbb{R}^d}\varphi\left(s,x\right)\nu\left(dsdx\right), \tag{iv}$$

for each test function $\varphi\in\mathcal{D}$, where ν is the measure defined by

$$\nu\left(\varphi\right)=\mathbb{E}^m\int_0^T\varphi\left(t,W_t\right)dA_t,\ \varphi\in\mathcal{C}_c\left([0,T]\times\mathbb{R}^d\right). \tag{v}$$

Proof : We first remark that the uniqueness of the increasing process in the representation (i) follows from the uniqueness in the Doob -Meyer decomposition.
Let us now assume that $\overline{u}$ is a regular potential which is a version of u. We will use an approximation of $\overline{u}$ constructed with the resolvent. By the resolvent equation one has

$$\alpha\widetilde{U}_\alpha\overline{u}=\alpha\widetilde{U}_0\left(\overline{u}-\alpha\widetilde{U}_\alpha\overline{u}\right).$$

Let us set $f^n = n\left(\overline{u} - n\widetilde{U}_n\overline{u}\right)$ and $u^n = n\widetilde{U}_n\overline{u} = \widetilde{U}_0 f^n$. Since $\overline{u}$ is excessive, one has $f^n \geq 0$ and $u^n, n \in \mathbb{N}^*$, is an increasing sequence of excessive functions with limit $\overline{u}$. In fact $u^n, n \in \mathbb{N}^*$, are potentials and their trajectories are continuous. On the other hand, the trajectories $t \to \overline{u}_t(W_t)$ are continuous on $[0, T[$ by the quasi-continuity of $\overline{u}$. The process $(u_t(W_t))_{t\in[0,T[}$ is a super-martingale and, because $\lim_{t\to T} u_t = 0$ in $\mathbf{L}^2$, it is a potential and the trajectories have null limits at T. Therefore this approximation also holds uniformly on the trajectories, on the closed interval $[0, T]$,

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} |u_t^n(W_t) - \overline{u}_t(W)| = 0, \quad \mathbb{P}^m - a.s.$$

The function u^n solves the equation $(\partial_t + L)u^n + f^n = 0$ with the condition $u_T^n = 0$ and its backward representation is

$$u_t^n(W_t) = \int_t^T f_s^n(W_s)\,ds - \sum_{i=1}^d \int_t^T \partial_i u_s^n(W_s)\,dW_s^i.$$

If we set $A_t^n = \int_0^t f_s^n(W_s)\,ds$, after conditioning, this representation gives

$$u_t^n(W_t) = A_T^n - A_t^n - \sum_{i=1}^d \int_t^T \partial_i u_s^n(W_s)\,dW_s^i = \mathbb{E}^m\left[A_T^n/\mathcal{F}_t\right] - A_t^n. \qquad (*)$$

In particular one deduces

$$u_0^n(W_0) = \mathbb{E}^m\left[A_T^n/\mathcal{F}_0\right] = A_T^n - \sum_{i=1}^d \int_0^T \partial_i u_s^n(X_s)\,dW_s^i,$$

Also from the relation $(*)$, it follows that

$$\begin{aligned} \mathbb{E}^m\left(A_T^n - A_t^n\right)^2 &= \mathbb{E}^m\left(u_t^n(W_t) + \sum_{i=1}^d \int_t^T \partial_i u_s^n(W_s)dW_s^i\right)^2 \\ &= \|u_t^n\|_2^2 + \int_t^T \left\|\nabla u_s^n\right\|_2^2 ds. \qquad (**) \end{aligned}$$

A similar relation holds for differences, in particular one has

$$\mathbb{E}^m\left(A_T^n - A_T^k\right)^2 = \left\|u_0^n - u_0^k\right\|^2 + 2\int_0^T \left\|\nabla\left(u_s^n - u_s^k\right)\right\|_2^2 ds.$$

On the other hand, the preceding lemma ensures that $\lim_{\alpha\to\infty} \alpha\widetilde{U}_\alpha = I$, in the space $\mathbf{L}^2\left([0, T[\,; H^1\left(\mathbb{R}^d\right)\right)$, which implies

$$\lim_{n\to 0} \int_0^T \left\|\nabla\left(u_t^n - \overline{u}_t\right)\right\|_2^2 dt = 0.$$

These last relations imply that there exists a limit $\lim_n A_T^n =: A_T$ in the sense of $\mathbf{L}^2(\mathbb{P}^m)$.
Let us denote by $M^n = (M_t^n)_{t\in[0,T]}$, $M = (M_t)_{t\in[0,T]}$ the martingales given by the conditional expectations $M_t^n = \mathbb{E}^m\left[A_T^n/\mathcal{F}_t\right]$, $M_t = \mathbb{E}^m\left[A_T/\mathcal{F}_t\right]$. Then one has $\lim_{n\to\infty} M^n = M$, in $\mathbf{L}^2(\mathbb{P}^m)$ and hence

$$\lim_{n\to\infty} \mathbb{E}^m \sup_{0\leq t\leq T} |M_t^n - M_t|^2 = 0.$$

Then the relation $u_t^n(W_t) = M_t^n - A_t^n$ shows that the processes $A^n, n \in \mathbb{N}^*$, also converge uniformly on the trajectories to a continuous process $A = (A_t)_{t\in[0,T]}$. The inequality

$$\sup_{0\leq t\leq T} |A_t^n - A_t| \leq A_T + |A_T^n - A_T|,$$

ensure the conditins to pass to the limit and get

$$\lim_{n\to\infty}\mathbb{E}^m\sup_{0\le t\le T}|A_t^n-A_t|^2=0.$$

Passing to the limit in the relations (*) and (**) one deduces the relations (i), (ii) and (iii).
In order to check the relation (iv) from the statement we observe that the relation is fulfilled by the functions u^n,

$$\begin{aligned}(u_0^n,\varphi_0)+\int_0^T\big(\frac{1}{2}(\nabla u_s^n,\nabla\varphi_s)+(u_s^n,\varphi_s)\big)\,ds &= \int_0^T\int_{\mathbb{R}^d}\varphi(s,x)\,f^n(s,x)\,dsdx\\ &= \mathbb{E}^m\int_0^T\varphi(s,W_s)\,dA_s^n,\end{aligned}$$

where φ is arbitrary in $\mathcal{D}_T$. In order to get the relation (iv) it would suffice to pass to the limit with $n\to\infty$ in this relation. The only term which poses problems is the last one. The uniform convergence on the trajectories implies that, $\mathbb{P}^m$ -a.s., the measures dA_t^n weakly converge to dA_t. Therefore one has

$$\lim_{n\to\infty}\int_0^T\varphi_t(W_t)\,dA_t^n=\int_0^T\varphi_t(W_t)\,dA_t,\quad \mathbb{P}^m-a.s.$$

On the other hand one has

$$\left|\int_0^T\varphi_t(W_t)\,dA_t^n\right|\le\sup_{0\le t\le T}\varphi_t^2(W_t)+A_T^2+|A_T^n-A_T|^2.$$

By Ito's formula and Doob's inequality one has

$$\begin{aligned}\mathbb{E}^m\Big(\sup_{0\le t\le T}\varphi^2(t,W_t)\Big)&\le 4\|\varphi_0\|^2+4\mathbb{E}^m\Big(\int_0^T\big|\partial_t\varphi(t,W_t)\big|\,dt\Big)^2+16\,\mathbb{E}^m\int_0^T\big|\nabla\varphi\big|^2(t,W_t)\,dt\\ &\quad+2\,\mathbb{E}^m\Big(\int_0^T\big|\Delta\varphi\big|(t,W_t)\,dt\Big)^2\\ &\le 4\|\varphi_0\|^2+4T\int_0^T\big\|\partial_t\varphi_t\big\|_2^2\,dt+16\int_0^T\big\|\nabla\varphi_t\big\|_2^2\,dt+2T\int_0^T\big\|\Delta\varphi_t\big\|_2^2\,dt<\infty.\end{aligned}$$

The preceding estimate ensures the possibility of passing to the limit and deducing that

$$\lim_n\mathbb{E}^m\int_0^T\varphi(s,W_s)\,dA_s^n=\mathbb{E}^m\int_0^T\varphi(s,W_s)\,dA_s.$$

and thus we obtain the relation (iv).
Let us now consider the converse. Assume that $u\in\widetilde{F}$ and A is a continuous increasing process adapted to $(\mathcal{F}_t)_{t\in[0,T]}$ and satisfying the relation (i). In order to simplify the subsequent notation it is convenient to extend our given function by putting $u_t=0$ for $t>T$. Now we shall show that

$$P_r(u_{t+r})\le u_t,\ \ t\in[0,T],\ r>0. \tag{7}$$

By the Markov property one gets

$$\begin{aligned}P_ru_{t+r}(W_t)&=\mathbb{E}^{W_t}\,[u_{t+r}(W_r)]=\mathbb{E}^m\,\Big[u_{t+r}(W_{r+t})\Big|\mathcal{F}_t\Big]\\ &=\mathbb{E}^m\,\Big[\mathbb{E}^m\,\Big[A_T\Big|\mathcal{F}_{t+r}\Big]-A_{t+r}\Big|\mathcal{F}_t\Big]=\mathbb{E}^m\,\Big[A_T\Big|\mathcal{F}_t\Big]-A_{t+r},\end{aligned}$$

where the last line comes from the relation (i). This shows that

$$P_ru_{t+r}(W_t)\le u_t(W_t),\ \mathbb{P}^m\!-a.s.$$

and as the distribution of W_t under $\mathbb{P}^m$ is m, we deduce the inequality (7). Moreover this inequality shows by iteration that if $r \leq r'$, then

$$P_{r'} u_{t+r'} \leq P_r u_{t+r}. \tag{8}$$

By the properties of the semigroup density and since $t \to u_t$ is continuous with values in $\mathbf{L}^2$, it follows that, for each $r > 0$, $P_r u_{t+r}, t \in [0,T]$, has a continuous version in $[0,T] \times \mathbb{R}^d$ defined by

$$\overline{u}^r (t,x) = \int_{\mathbb{R}^d} q_r (x,y) \, u_{t+r} (y) \, dy.$$

The inequality (8) shows in fact that $\overline{u}^r$ is supermedian with respect to $\left(\widetilde{P}_t\right)_{t>0}$ and, because of continuity, in fact it is excessive. Then $\overline{u} = \lim_{r\to 0} \overline{u}^r$ is also excessive and since $\lim_{r\to 0} P_r u_{t+r} = u_t$, in $\mathbf{L}^2$, clearly $\overline{u}$ is a version of u. The process $\left(\overline{u}_t (W_t)\right)_{t\in[0,T]}$ is a càdlàg supermartingale and more precisely a potential. By the relation (i) this process admits a continuous version. It follows that itself is continuous and, as a consequence, one has the following convergence, uniformly on the trajectories,

$$\lim_{r\to 0} \sup_{0\leq t\leq T} \left|\overline{u}_t^r (W_t) - \overline{u}_t (W_t)\right| = 0, \ \mathbb{P}^m - a.s.$$

On the other hand, by the representation (i) one has

$$\mathbb{E}^m \sup_{0\leq t\leq T} \left|\overline{u}_t (W_t)\right|^2 < \infty,$$

which leads to

$$\lim_{r\to 0} \mathbb{E}^m \sup_{0\leq t\leq T} \left|\overline{u}_t^r (W_t) - \overline{u}_t (W)\right|^2 = 0.$$

This relation implies that $\overline{u}$ is quasicontinuous, and hence it is a regular potential, completing the proof. □

It is known in the probabilistic potential theory that the regular potentials are associated to continous additive functionals (see [4], Section IV.3 or [13], Theorem 5.4.2). In the above theorem the additive aspect is not evident. In fact it is hiden in the relation (i). This relation implies that, for $t \leq s$, $A_s - A_t$ is measurable with respect to the completion $\sigma\big(W_r / r \in \left[t,s\right]\big)$. This can directly be proven but it also follows from the approximation of A by A^n. For the processes $A^n, n \in \mathbb{N}$, this measurability property obviously holds. And this measurability ensures the fact that A corresponds to an additive functional for the time -space process, which we are not explicitly using.

The measure ν from the theorem, expressed in the relation (v), is also completely determined by the relation (iv), because the test functions are dense in $\mathcal{C}_c\left([0,T] \times \mathbb{R}^d\right)$. A natural question now is whether one Radon measure on $[0,T] \times \mathbb{R}^d$ can be associated via the relation (iv) from the theorem to two distinct potentials. The answer is that there is only one such potential and more precisely it can be directly expressed with the density $q_t(x,y)$ in terms of the measure, as one can see from the next lemma.

Lemma 3. *Let u be a regular potential and ν a Radon measure on $[0,T] \times \mathbb{R}^d$ such that relation (iv) holds. Then one has*

$$(\phi, u_t) = \int_t^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \phi(x) \, q_{s-t} (x-y) \, dx \right) \nu (dsdy),$$

for each $\phi \in \mathbf{L}^2\left(\mathbb{R}^d\right)$ and $t \in [0,T]$.

Proof : We first remark that the relation (iv) is in fact equivalent to the following more explicit one

$$(u_t,\varphi_t)+\int_t^T\Big(\frac{1}{2}(\nabla u_s,\nabla\varphi_s)+(u_s,\partial_s\varphi_s)\Big)\,ds=\int_t^T\int_{\mathbb{R}^d}\varphi(s,x)\,\nu(dsdx)\,,$$

with any $\varphi\in\mathcal{D}$ and $t\in[0,T]$.
Clearly it is sufficient to prove the lemma for $\phi\in\mathcal{C}_c\left(\mathbb{R}^d\right)$ such that $\phi\geq 0$. Then we set $\psi(s,y)=\int_{\mathbb{R}^d}\phi(x)\,q_{s-t}(x-y)\,dx$, for $s\in[t,T]$ and $y\in\mathbb{R}^d$. Then $\psi_s=P_{s-t}\phi$ and the map $s\to\psi_s$ is in $\mathcal{C}^1\left(]t,T];L^2\left(\mathbb{R}^d\right)\right)$ and $\partial_s\psi=\frac{1}{2}\Delta\psi_s$. Let $\eta\in\mathcal{C}_c(\mathbb{R}_+)$ be a decreasing function such that $\eta=1$ on the interval $[0,1]$ and $\eta=0$ for $x\geq 2$. Set $\eta_n(x)=\eta\left(\frac{|x|}{n}\right)$, so that $(\eta_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{C}_c\left(\mathbb{R}^d\right)$ with limit $1_{\mathbb{R}^d}$. For each fixed n the function $\eta_n\psi$ can be approximated by convolution with smooth functions and then by test functions from $\mathcal{D}$, and consequently we may write the relation (iv) in the form

$$(u_t,\eta_n\psi_t)+\int_t^T\Big(\frac{1}{2}\left(\nabla u_s,\nabla\big(\eta_n\psi_s\big)\right)+(u_s,\eta_n\partial_s\psi_s)\Big)ds=\int_t^T\int_{\mathbb{R}^d}\eta_n(x)\,\psi(s,x)\,\nu(dsdx)\,.$$

Then it is easy to see that we may pass to the limit with $n\to\infty$, in this relation too. Then we get

$$(u_t,\psi_t)+\int_t^T\Big(\frac{1}{2}(\nabla u_s,\nabla\psi_s)+(u_s,\partial_s\psi_s)\Big)ds=\int_t^T\int_{\mathbb{R}^d}\psi(s,x)\,\nu(dsdx)\,,$$

which becomes the relation asserted by the lemma, on account of the relation $\partial_s\psi=\frac{1}{2}\Delta\psi_s$. □

We now introduce the class of measures which intervene in the notion of solution to the obstacle problem.

Definition 2. *A nonnegative Radon measure ν defined in $[0,T]\times\mathbb{R}^d$ is called regular provided that there exists a regular potential u such that the relation (iv) from the above theorem is satisfied.*

As a consequence of the preceding lemma we see that the regular measures are always represented as in the relation (v) of the theorem, with a certain increasing process. We also note the following properties of a regular measure, with the notation from the theorem.

1. A set $B\in\mathcal{B}\left([0,T]\times\mathbb{R}^d\right)$ satisfies the relation $\nu(B)=0$ if and only if $\int_0^T 1_B(t,W_t)\,dA_t=0,\ \mathbb{P}^m-a.s.$
2. If a set $B\in\mathcal{B}\left(]0,T[\times\mathbb{R}^d\right)$ is polar, in the sense that
$$\mathbb{P}^m\left(\{\omega\in\Omega'/\exists t\in[0,T],(t,W_t(\omega))\in B\}\right)=0,$$
then $\nu(B)=0$.
3. If $\psi^1,\psi^2:[0,T]\times\mathbb{R}^d\to\overline{\mathbb{R}}$ are Borel measurable and such that $\psi^1(t,x)\geq\psi^2(t,x),\ dt\otimes dx\ -a.e.$, and the processes $\left(\psi^i_t(W_t)\right)_{t\in[0,T]},i=1,2$, are a.s. continuous, then one has $\nu\left(\psi^1<\psi^2\right)=0$.

2.3 Hypotheses

Let $B=(B_t)_{t\geq0}$ be a standard d^1-dimentional Brownian motion on a probability space $\left(\Omega,\mathcal{F}^B,\mathbb{P}\right)$. So $B_t=\left(B_t^1,\cdots,B_t^{d_1}\right)$ takes values in $\mathbb{R}^{d^1}$. Over the time interval $[0,T]$ we define the backward filtration $\left(\mathcal{F}^B_{s,T}\right)_{s\in[0,T]}$ where $\mathcal{F}^B_{s,T}$ is the completion in $\mathcal{F}^B$ of $\sigma(B_r-B_s;s\leq r\leq T)$.

We denote by $\mathcal{H}_T$ the space of $H^1(\mathbb{R}^d)$-valued predictable and $\mathcal{F}^B_{t,T}$-adapted processes $(u_t)_{0\leq t\leq T}$ such that the trajectories $t \to u_t$ are in $\widetilde{F}$ a.s. and

$$\|u\|_T^2 < \infty.$$

In the reminder of this paper we assume that the final condition Φ is a given function in $\mathbf{L}^2(\mathbb{R}^d)$ and the functions appearing in the equation (1)

$$\begin{aligned} f &: [0,T]\times\Omega\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}\ ,\\ g &= (g_1,...,g_d)\ :\ [0,T]\times\Omega\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d\\ h &= (h_1,...,h_{d^1})\ :\ [0,T]\times\Omega\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^{d^1}\end{aligned}$$

are random functions predictable with respect to the backward filtration $\left(\mathcal{F}^B_{t,T}\right)_{t\in[0,T]}$. We set

$$f(\cdot,\cdot,\cdot,0,0) := f^0, \quad g(\cdot,\cdot,\cdot,0,0) := g^0 = (g^0_1,...,g^0_d) \quad \text{and} \quad h(\cdot,\cdot,\cdot,0,0) := h^0 = (h^0_1,...,h^0_{d^1}).$$

and assume the following hypotheses :

Assumption (H): There exist non-negative constants C, α, β such that

(i) $|f(t,\omega,x,y,z) - f(t,\omega,x,y',z')| \leq C\big(|y-y'| + |z-z'|\big)$

(ii) $\left(\sum_{j=1}^{d_1} |h_j(t,\omega,x,y,z) - h_j(t,\omega,x,y',z')|^2\right)^{\frac{1}{2}} \leq C\,|y-y'| + \beta\,|z-z'|,$

(iii) $\left(\sum_{i=1}^{d} |g_i(t,\omega,x,y,z) - g_i(t,\omega,x,y',z')|^2\right)^{\frac{1}{2}} \leq C\,|y-y'| + \alpha\,|z-z'|.$

(iv) the contraction property (as in [8]) : $\alpha + \dfrac{\beta^2}{2} < \dfrac{1}{2}$.

Assumption (HD2)

$$\mathbb{E}\left(\left\|f^0\right\|^2_{2,2} + \left\|g^0\right\|^2_{2,2;T} + \left\|h^0\right\|^2_{2,2}\right) < \infty,$$

Assumption (HO) : The obstacle $v(t,\omega,x)$ is a predictable random function with respect to the backward filtration $\left(\mathcal{F}^B_{t,T}\right)$. We also assume that $t \mapsto v(t,\omega,W_t)$ is $P\otimes P^m$-a.s. continuous on $[0,T]$ and satisfies

$$v(T,\cdot) \leq \Phi(\cdot).$$

We recall that a usual solution (non reflected one) of the equation (1) with final condition $u_T = \Phi$, is a processus $u \in \mathcal{H}_T$ such that for each test function $\varphi \in \mathcal{D}_T$ and any $\forall t\in[0,T]$, we have a.s.

$$\begin{aligned} &\int_t^T \big[(u_s,\partial_s\varphi_s) + \frac{1}{2}(\nabla u_s,\nabla\varphi_s) + \big(g_s,\nabla\varphi_s\big]ds - \big(\Phi,\varphi_T\big) + \big(u_t,\varphi_t\big)\\ &= \int_t^T (f_s,\varphi_s)\,ds + \int_t^T (h_s,\varphi_s)\cdot \overleftarrow{dB}_s. \end{aligned} \tag{9}$$

By Theorem 8 in [8] we have existence and uniqueness of the solution. Moreover, the solution belongs to $\mathcal{H}_T$. We denote by $\mathcal{U}(\Phi,f,g,h)$ this solution.

2.4 Quasi-continuity properties

In this section we are going to prove the quasi-continuity of the solution of the linear equation, i.e. when f, g, h do not depend of u and ∇u. To this end we first extend the double stochastic Ito's formula to our framework. We start by recalling the following result from [8] (stated for linear SPDE).

Theorem 3. *Let $u \in \mathcal{H}_T$ be a solution of the equation*

$$du_t + \frac{1}{2}\Delta u_t dt + \big(f_t + div g_t\big)\, dt + h_t \overleftarrow{dB}_t = 0,$$

where f, g, h are predictable processes such that

$$\mathbb{E}\int_0^T \left[\|f_t\|_2^2 + \|g_t\|_2^2 + \|h_t\|_2^2\right] dt < \infty \quad \text{and} \quad \|\Phi\|_2^2 < \infty.$$

Then, for any $0 \le s \le t \le T$, one has the following stochastic representation, $\mathbb{P}^m$-a.s.,

$$u\,(t, W_t) - u\,(s, W_s) = \sum_i \int_s^t \partial_i u\,(r, W_r)\, dW_r^i - \int_s^t f\,(r, W_r)\, dr - \frac{1}{2}\int_s^t g * dW - \int_s^t h\,(r, W_r) \cdot \overleftarrow{dB}_r. \quad (10)$$

We remark that $\mathcal{F}_T$ and $\mathcal{F}_{0,T}^B$ are independent under $\mathbb{P} \otimes \mathbb{P}^m$ and therefore in the above formula the stochastic integrals with respect to dW_t and $\overleftarrow{dW}_t$ act independently of $\mathcal{F}_{0,T}^B$ and similarly the integral with respect to $\overleftarrow{dB}_t$ acts independently of $\mathcal{F}_T$.

In particular the process $(u_t(W_t))_{t\in[0,T]}$ admits a continuous version which we usually denote by $Y = (Y_t)_{t\in[0,T]}$ and we introduce the notation $Z_t = \nabla u_t(W_t)$. As a consequence of this theorem we have the following result.

Corollary 1. *Under the hypothesis of the preceding theorem one has the following stochastic representation for u^2, $\mathbb{P} \otimes \mathbb{P}^m$-a.e., for any $0 \le t \le T$,*

$$\begin{aligned} u_t^2\,(W_t) - \Phi^2\big(W_T\big) &= 2\int_t^T \left[u_s f_s(W_s) - \frac{1}{2}|\nabla u_s|^2(W_s) - \langle \nabla u_s, g_s\rangle(W_s) + \frac{1}{2}|h_s|^2(W_s)\,\right] ds \\ &+ \int_t^T \big(u_r g_r\big)(W_r) * dW_r - 2\sum_i \int_t^T \big(u_r \partial_i u_r\big)\,(W_r)\, dW_r^i + 2\int_t^T \big(u_r h_r\big)\,(W_r) \cdot \overleftarrow{dB}_r. \end{aligned} \quad (11)$$

Moreover one has the estimate

$$\mathbb{E}\,\mathbb{E}^m\big(\sup_{t\le s\le T} |Y_s|^2\big) + \mathbb{E}\Big[\int_t^T \|\nabla u_s\|_2^2\, ds\Big] \le c\,\Big[\|\phi\|_2^2 + \mathbb{E}\int_t^T \left[\|f_s\|_2^2 + \|g_s\|_2^2 + \|h_s\|_2^2\,\right] ds\Big], \quad (12)$$

for each $t \in [0, T]$.

Remark 1. *With the notation introduced above one can write the relation (11) as*

$$\begin{aligned} |Y_t|^2 + \int_t^T |Z_r|^2 dr &= |Y_T|^2 + 2\int_t^T Y_r f_r(W_r) dr - 2\int_t^T \langle Z_r, g_r(W_r)\rangle\, dr + \int_t^T Y_r g_r(W_r) * dW_r \\ &- 2\sum_i \int_t^T Y_r Z_{i,r} dW_r^i + 2\int_t^T Y_r h_r(W_r) \cdot \overleftarrow{dB}_r + \int_t^T |h_r|^2(W_r) dr. \end{aligned} \quad (13)$$

Proof : Assume first that g is uniformly bounded and belongs to $(\mathcal{H}_T)^d$, so that $E\int_0^T \left\|divg_t\right\|_2^2 dt < \infty$. Then we may represent the solution in the form

$$u_t\left(W_t\right) - u_s\left(W_s\right) = \sum_i \int_s^t \partial_i u_r\left(W_r\right) dW_r^i - \int_s^t \left[f_r(W_r) + divg_r(W_r)\right] dr - \int_s^t h_r(W_r)\cdot \overleftarrow{dB}_r.$$

By Lemma 1.3 of [22] we may write

$$u_t^2\left(W_t\right) - u_s^2\left(W_s\right) = -2\int_s^t \left[u_r\left(f_r + divg_r\right)(W_r) - |\nabla u_r|^2(W_r) - |h_r|^2(W_r)\right] dr$$
$$+ 2\sum_i \int_s^t \left(u_r \partial_i u_r\right)\left(W_r\right) dW_r^i - 2\int_s^t \left(u_r h_r\right)\left(W_r\right)\cdot \overleftarrow{dB}_r.$$

On the other hand, by Lemma 3.1 of [24] one has

$$-2\int_s^t div(u_r\, g_r)(W_r)\, dr = \int_s^t u_r\, g_r(W_r) * dW_r,$$

so that the preceding relation immediately leads to the relation (11). Then the standard calculations of BDSDE involving Young's inequality, BDG inequality and Gronwal's lemma give the estimate (12).
Finally to obtain the result with general g one proceeds by approximation. □

In the deterministic case it was proven in [24] that the solution of a quasilinear equation has a quasicontinuous version. Here we shall prove the same property for the solution of an SPDE as is stated in the next proposition.

Proposition 1. *Under the hypothesis of Theorem 3, there exists a function* $\bar{u} : [0,T]\times\Omega\times\mathbb{R}^d \longrightarrow \mathbb{R}$ *which is a quasicontinuous version of* u*, in the sense that for each* $\epsilon > 0$*, there exits a predictable random set* $D^\epsilon \subset [0,T]\times\Omega\times\mathbb{R}^d$ *such that* $\mathbb{P}$*-a.s. the section* D_ω^ϵ *is open and* $\bar{u}\left(\cdot,\omega,\cdot\right)$ *is continuous on its complement* $\left(D_\omega^\epsilon\right)^c$ *and*

$$\mathbb{P}\otimes\mathbb{P}^m\ \left((\omega,\omega')\,\middle|\, \exists t\in[0,T]\ s.t.\ (t,\omega,W_t(\omega'))\in D^\epsilon\right) \le \epsilon.$$

In particular the process $\left(\overline{u}_t(W_t)\right)_{t\in[0,T]}$ *has continuous trajectories,* $\mathbb{P}\otimes\mathbb{P}^m$*-a.s.*

Proof : Let us choose $k\in\mathbb{N}$ with $k > \frac{d}{2}$, so that the Sobolev space $H^k(\mathbb{R}^d)$ is continuously imbeded in the space of Hölder continuous functions $\mathcal{C}^\gamma(\mathbb{R}^d)$, with $\gamma = 1 + [\frac{d}{2}] - \frac{d}{2}$. We first assume that $\phi\in H^k(\mathbb{R}^d)$ and $f,\ g_1,\cdots,g_d,\ h_1,\cdots,h_{d_1}$ belong to $\mathbf{L}^2\left([0,T]\times\Omega; H^k(\mathbb{R}^d)\right)$. By Theorem 8 in [8], applied with respect to the Hilbert space $H^k(\mathbb{R}^d)$, one deduces that the solution $u = \mathcal{U}\left(\Phi,f,g,h\right)$ has the trajectories $t\rightarrow u_t(\omega,\cdot)$ continuous in $H^k(\mathbb{R}^d)$ which implies that they are in $\mathcal{C}[[0,T]\times R^d)$. On the other hand, we have from (12) the following general estimate

$$\mathbb{E}\,\mathbb{E}^m\left(\sup_{0\le t\le T} u\left(t,W_t\right)^2\right) \le c\,\mathbb{E}\Big[\|\Phi\|_2^2 + \int_0^T \left(\|f_t\|_2^2 + \|g_t\|_2^2 + \|h_t\|_2^2\right)dt\Big].$$

Now, for general (Φ,f,g,h), one chooses an approximating sequence of data (Φ^n,f^n,g^n,h^n) which are $H^k(\mathbb{R}^d)$ -valued and such that

$$\mathbb{E}\left(\|\Phi_n - \Phi_{n+1}\|_2 + \int_0^T \left[\|f_t^n - f_t^{n+1}\|_2^2 + \|g_t^n - g_t^{n+1}\|_2^2 + \|h_t^n - h_t^{n+1}\|_2^2\right] dt\right) \le \frac{1}{2^n}$$

Let u^n be the sequence of $\mathbb{P}$-$a.s$ continuous solutions of the equation associated to (Φ^n, f^n, g^n, h^n). Then set $E_n^\epsilon = \{\left|u^n - u^{n+1}\right| > \epsilon\}$ and $D_k^\epsilon = \bigcup_{n\geq k} E_n^\epsilon$. Then we have

$$\epsilon^2 \mathbb{P}\otimes\mathbb{P}^m \left((\omega,\omega') \,\middle|\, \exists t\in[0,T]\ s.t.\ (t,\omega,W_t(\omega'))\in E_n^\epsilon\right) \leq \mathbb{E}\mathbb{E}^m\Big[\sup_{0\leq t\leq T}\left(u_t^n(W_t) - u_t^{n+1}(W_t)\right)^2\Big] \leq \frac{c}{2^n}.$$

Further one takes $\epsilon = \dfrac{1}{n^2}$ to get

$$\mathbb{P}\otimes\mathbb{P}^m \left((\omega,\omega') \,\middle|\, \exists t\in[0,T]\ s.t.\ (t,\omega,W_t(\omega'))\in D_k^\epsilon\right) \leq \sum_{n=k}^{\infty}\frac{cn^4}{2^n}.$$

This shows the statement. □

We also need the quasicontinuity of the solution associated to a random regular measure, as stated in the next proposition. We first give the formal definition of this object.

Definition 3. *We say that $u \in \mathcal{H}_T$ is a random regular potential provided that $u(\cdot,\omega,\cdot)$ has a version which is regular potential, $\mathbb{P}(d\omega)$-a.s. The random variable $\nu : \Omega \longrightarrow \mathcal{M}\left([0,T]\times\mathbb{R}^d\right)$ with values in the set of regular measures on $[0,T]\times\mathbb{R}^d$ is called a regular random measure, provided that there exits a random regular potential u such that the measure $\nu(\omega)(dtdx)$ is associated to the regular potential $u(\cdot,\omega,\cdot)$, $\mathbb{P}(d\omega)$-a.s.*

The relation between a random measure and its associated random regular potential is described by the following proposition.

Proposition 2. *Let u be a random regular potential and ν be the associated random regular measure. Let $\overline{u}$ be the excessive version of u, i.e. $\overline{u}(\cdot,\omega,\cdot)$ is a.s. an $\left(\widetilde{P}_t\right)_{t>0}$-excessive function which coincides with $u(\cdot,\omega,\cdot)$, $dtdx$-a.e. Then we have the following properties:*

(i) For each $\varepsilon > 0$, there exists a $\left(\mathcal{F}_{t,T}^B\right)_{t\in[0,T]}$-predictable random set $D^\varepsilon \subset [0,T]\times\Omega\times\mathbb{R}^d$ such that P -a.s. the section D_ω^ε is open and $\overline{u}(\cdot,\omega,\cdot)$ is continuous on its complement $(D_\omega^\varepsilon)^c$ and

$$\mathbb{P}\otimes\mathbb{P}^m\left((\omega,\omega')\,/\exists t\in[0,T]\quad s.t.\,(t,\omega,W_t(\omega))\in D_\omega^\varepsilon\right)\leq\varepsilon.$$

In particular the process $\left(\overline{u}_t(W_t)\right)_{t\in[0,T]}$ has continuous trajectories, $\mathbb{P}\otimes\mathbb{P}^m$-a.s.

(ii) There exists a continuous increasing process $A = (A_t)_{t\in[0,T]}$ defined on $\Omega\times\Omega'$ such that $A_s - A_t$ is measurable with respect to the $\mathbb{P}\otimes\mathbb{P}^m$ -completion of $\mathcal{F}_{t,T}^B \vee \sigma(W_r/r\in[t,s])$, for any $0\leq s\leq t\leq T$, and such that the following relations are fulfilled a.s., with any $\varphi\in\mathcal{D}$ and $t\in[0,T]$,

$$(a)\quad (u_t,\varphi_t) + \int_t^T\left(\frac{1}{2}(\nabla u_s,\nabla\varphi_s) + (u_s,\partial_s\varphi_s)\right)ds = \int_t^T\int_{\mathbb{R}^d}\varphi(s,x)\,\nu(dsdx),$$

$$(b)\quad u_t(W_t) = \mathbb{E}\left[A_T\left|\mathcal{F}_t\vee\mathcal{F}_{t,T}^B\right.\right] - A_t,$$

$$(c)\quad u_t(W_t) = A_T - A_t - \sum_{i=1}^d\int_t^T\partial_i u_s(W_s)\,dW_s^i,$$

$$(d)\quad \|u_t\|_2^2 + \int_t^T\left\|\nabla u_s\right\|_2^2 ds = \mathbb{E}^m(A_T - A_t)^2,$$

$$(e)\quad \nu(\varphi) = \mathbb{E}^m\int_0^T\varphi(t,W_t)\,dA_t.$$

Proof: The proof of this proposition results from the approximation procedure used in the proof of Theorem 2.

(i) Let $r > 0$. The process $\bar{u}^r = (\bar{u}^r_t)_{t\in[0,T]}$, defined by $\bar{u}^r_t = P_r u_{t+r}$, has the property that $(t,x) \longrightarrow \bar{u}^r_t$ is jointly continuous $\mathbb{P}$-*a.s.* We also have

$$\lim_{r\to 0} \mathbb{E}\mathbb{E}^m \sup_{0\le t\le T} \left|\bar{u}^r_t(W_t) - \bar{u}_t(W_t)\right|^2 = 0,$$

by the arguments used at the end of the proof of Theorem 2. The one concludes as in the proof of the preceding proposition.

(ii) The construction of the increasing process described in Theorem 2 holds globally for a random regular potential producing on *a.e.* trajectory $\omega \in \Omega$, the increasing process corresponding to $u(\cdot,\omega,\cdot)$. □

We remark that, taking the expectation of the relation (ii-d) of this proposition one gets

$$\mathbb{E}\mathbb{E}^m\left(A_T^2\right) = \mathbb{E}\big(\left\|u_0\right\|_2^2 + \int_0^T \left\|\nabla u_t\right\|_2^2 dt\big).$$

3 Existence and uniqueness of the solution of the obstacle problem

3.1 The weak solution

We now precise the definition of the solution of our obstacle problem. We recall that the data satisfy the hypotheses of Section 2.3.

Definition 4. *We say that a pair (u,ν) is a weak solution of the obstacle problem for the SPDE* (1) *associated to (Φ, f, g, h, v), if*

(i) $u \in \mathcal{H}_T$ and $u(t,x) \ge v(t,x)$, $d\mathbb{P}\otimes dt\otimes dx$ a-e. and $u(T,x) = \Phi(x)$, $d\mathbb{P}\otimes dx$ a-e..

(ii) ν is a random regular measure on $(0,T)\times\mathbb{R}^d$.

(iii) for each $\varphi \in \mathcal{D}_T$, and $t \in [0,T]$,

$$\begin{aligned}\int_t^T &\left[(u_s, \partial_s\varphi_s) + \frac{1}{2}(\nabla u_s, \nabla\varphi_s)\right] ds - \big(\Phi, \varphi_T\big) + \big(u_t, \varphi_t\big)\\ &= \int_t^T \left[\,\big(f\big(s, u_s, \nabla u_s\big), \varphi_s\big) - (g(s,.), \nabla\varphi_s)\,\right] ds \\ &+ \int_t^T \left(h\left(s, u_s, \nabla u_s\right), \varphi_s\right)\cdot \overleftarrow{dB}_s + \int_t^T\int_{\mathbb{R}^d} \varphi_s(x)\,\nu(dx, ds).\end{aligned} \tag{14}$$

(iv) If $\overline{u}$ is a quasicontinuous version of u, then one has

$$\int_0^T \int_{\mathbb{R}^d} \left(\overline{u}_s\left(x\right) - v_s\left(x\right)\right) \nu\left(dsdx\right) = 0, \ a.s.$$

We note that a given solution u can be writen as a sum $u = u_1 + u_2$, where u_1 satisfies a linear equation $u_1 = \mathcal{U}\big(\Phi, f(u,\nabla u), g(u,\nabla u), h(u,\nabla u)\big)$ with f, g, h determined by u, while u_2 is the random regular potential corresponding to the measure ν. By the Propositions 1 and 2, the conditions (ii) and (iii) imply that the process u always admits a quasicontinuous version, so that the condition (iv) makes sense. We also note that if $\overline{u}$ is a quasicontinuous version of u, then the trajectories of W do not visit the set $\{\overline{u} < v\}$, $\mathbb{P}\otimes\mathbb{P}^m$-*a.s.*
Here it is the main result of our paper

Theorem 4. *Assume that the assumptions **(H)**, **(HD2)** and **(HO)** hold. Then there exists a unique weak solution of the obstacle problem for the SPDE* (1) *associated to* (Φ, f, g, h, v).

In order to solve the problem we will use the backward stochastic differential equation technics. In fact, we shall follow the main steps of the second proof in [12], based on the penalization procedure. The uniqueness assertion of Theorem 4 results from the following comparison result :

Theorem 5. *Let* Φ', f', v' *be similar to* Φ, f, v *and let* (u, ν) *be the solution of the obstacle problem corresponding to* (Φ, f, g, h, v) *and* (u', ν') *the solution corresponding to* (Φ', f', g, h, v'). *Assume that the following conditions hold*

(i) $\Phi \leq \Phi'$, $dx \otimes d\mathbb{P}$ *-a.e.*

(ii) $f(u, \nabla u) \leq f'(u, \nabla u)$, $dtdx \otimes \mathbb{P}$ *-a.e.*

(iii) $v \leq v'$, $dtdx \otimes \mathbb{P}$ *-a.e.*

Then one has $u \leq u'$, $dtdx \otimes \mathbb{P}$ *-a.e.*

Proof : The proof is identical to that of the similar result of El Karoui et al ([12], Theorem 4.1). One starts with the following version of Ito's formula, written with some quasicontinuous versions $\overline{u}, \overline{u'}$ of the solutions u, u' in the term involving the regular measures ν, ν',

$$\begin{aligned}
&\mathbb{E}\left\|(u_t - u'_t)^+\right\|_2^2 + \mathbb{E}\int_t^T \left\|\nabla\big(u_s - u'_s\big)^+\right\|_2^2 ds = \mathbb{E}\left\|(\Phi - \Phi')^+\right\|_2^2 \\
&+ 2\mathbb{E}\int_t^T \Big(\left(u_s - u'_s\right)^+, f_s\left(u_s, \nabla u_s\right) - f'_s\left(u'_s, \nabla u'_s\right)\Big) ds + 2\mathbb{E}\int_t^T \int_{\mathbb{R}^d} \left(\overline{u}_s - \overline{u'}_s\right)^+(x)\left(\nu - \nu'\right)(dsdx) \\
&+ 2\mathbb{E}\int_t^T \Big(\nabla\left(u_s - u'_s\right)^+, g_s\left(u_s, \nabla u_s\right) - g'_s\left(u'_s, \nabla u'_s\right)\Big) ds + \mathbb{E}\int_t^T \left\|h_s\left(u_s, \nabla u_s\right) - h'_s\left(u'_s, \nabla u'_s\right)\right\|_2^2 ds.
\end{aligned}$$

We remark that the inclusion $\{\overline{u} > \overline{u'}\} \subset \{\overline{u} > v\} \cup \{v > v'\} \cup \{v' > \overline{u'}\}$ and the fact that the set $\{v > v'\} \cup \{v' > \overline{u'}\}$ is not visited by W, imply that $\nu\big(\overline{u} > \overline{u'}\big) = 0$, *a.s.*. Therefore

$$\int_t^T \int_{\mathbb{R}^d} \left(\overline{u}_s - \overline{u'}_s\right)^+(x)\left(\nu - \nu'\right)(dsdx) \leq 0, \quad a.s.$$

and then one concludes the proof by Gronwall's lemma. □

3.2 Approximation by the penalization method

For $n \in \mathbb{N}$, let u^n be a solution of the following SPDE

$$\begin{aligned}
&du^n_t(x) + \frac{1}{2}\Delta u^n_t(x)dt + f(t, x, u^n_t(x), \nabla u^n_t(x))dt + n(u^n_t(x) - v_t(x))^- \, dt \\
&+ \operatorname{div}\big(g\left(t, x, u^n_t(x), \nabla u^n_t(x)\right)\big)\, dt + h(t, x, u^n_t(x), \nabla u^n_t(x))\, \overleftarrow{dB}_t = 0
\end{aligned} \tag{15}$$

with final condition $u^n_T = \Phi$.
Now set $f_n(t, x, y, z) = f(t, x, y, z) + n(y - v_t(x))^-$ and $\nu^n(dt, dx) := n\big(u^n_t(x) - v_t(x)\big)^- dtdx$. Clearly for each $n \in \mathbb{N}$, f_n is Lipschitz continuous in (y, z) uniformly in (t, x) with Lipschitz coefficient $C + n$. For each $n \in \mathbb{N}$, Theorem 8 in [8] ensures the existence and uniqueness of a weak solution $u^n \in \mathcal{H}_T$ of the SPDE (15) associated with the data (Φ, f_n, g, h). We denote by $Y^n_t = u^n(t, W_t)$,

$Z_n = \nabla u^n(t, W_t)$ and $S_t = v(t, W_t)$. We shall also assume that u^n is quasi-continuous, so that Y^n is $\mathbb{P}\otimes\mathbb{P}^m$-*a.e.* continuous. Then $\big(Y^n, Z^n\big)$ solves the BSDE associated to the data (Φ, f_n, g, h)

$$\begin{aligned} Y_t^n = \Phi\left(W_T\right) + \int_t^T f_r\left(X_r, Y_r^n, Z_r^n\right) dr + n\int_t^T \left(Y_r^n - S_r^n\right)^- dr + \frac{1}{2}\int_t^T g_r\left(W_r, Y_r^n, Z_r^n\right) * dW \\ + \int_t^T h_r\left(W_r, Y_r^n, Z_r^n\right)\cdot \overleftarrow{dB}_r - \sum_i \int_t^T Z_{i,r}^n dW_r^i . \end{aligned} \tag{16}$$

We define $K_t^n = n\int_0^t (Y_s^n - S_s)^- ds$ and establish the following lemma.

Lemma 4. *The triple (Y^n, Z^n, K^n) satisfies the following estimates*

$$\begin{aligned} \mathbb{E}\mathbb{E}^m\left|Y_t^n\right|^2 + \lambda_\epsilon \mathbb{E}\mathbb{E}^m \int_t^T \left|Z_r^n\right|^2 dr \le c\mathbb{E}\mathbb{E}^m\big[|\Phi(W_T)|^2 + \int_t^T \left(|f_s^0(W_s)|^2 + |g_s^0(W_s)|^2 + |h_s^0(W_s)|^2\right) ds\big] \\ + c_\epsilon\, \mathbb{E}\,\mathbb{E}^m \int_t^T \left|Y_r^n\right|^2 dr + c_\delta\, \mathbb{E}\,\mathbb{E}^m\,\big(\sup_{t\le r\le T}|S_r|^2\big) + \delta\,\mathbb{E}\,\mathbb{E}^m\left(K_T^n - K_t^n\right)^2 \end{aligned} \tag{17}$$

where $\lambda_\epsilon = 1 - 2\alpha - \beta^2 - \epsilon$, c_ϵ, c_δ are a positive constants and $\epsilon > 0$, $\delta > 0$ can be chosen small enough such that $\lambda_\epsilon > 0$.

Proof : By using Ito's formula (13) for (Y^n, Z^n) we get

$$\begin{aligned} |Y_t^n|^2 + \int_t^T |Z_r^n|^2 dr = |\Phi(W_T)|^2 + 2\int_t^T Y_s^n f_s(W_s, Y_s^n, Z_s^n) ds + 2\int_t^T Y_s^n dK_s^n \\ - 2\int_t^T \langle Z_s^n, g_s\big(W_s, Y_s^n, Z_s^n\big)\rangle\, ds + \int_t^T Y_s^n g_s(W_s, Y_s^n, Z_s^n) * dW - 2\sum_i \int_t^T Y_s^n Z_{i,s}^n dW_s^i \\ + 2\int_t^T Y_s^n h_s(W_s, Y_s^n, Z_s^n)\cdot \overleftarrow{dB}_s + \int_t^T |h_s(W_s, Y_s^n, Z_s^n|^2 ds. \end{aligned} \tag{18}$$

Using assumption **(H)** and taking the expectation in the above equation under $\mathbb{P}\otimes\mathbb{P}^m$ we get

$$\begin{aligned} \mathbb{E}\mathbb{E}^m\left|Y_t^n\right|^2 + \mathbb{E}\mathbb{E}^m\int_t^T \left|Z_s^n\right|^2 ds \le \mathbb{E}\left|\Phi(W_T)\right|^2 + c_\varepsilon \mathbb{E}\mathbb{E}^m \int_t^T \left[|f_s^0(W_s)|^2 + |g_s^0(W_s)|^2 + |h_s^0(W_s)|^2\right] ds \\ + c_\varepsilon\, \mathbb{E}\mathbb{E}^m \int_t^T \left|Y_s^n\right|^2 ds + \left(2\alpha + \beta^2 + \varepsilon\right)\, \mathbb{E}\mathbb{E}^m \int_t^T \left|Z_s^n\right|^2 ds \\ + \frac{1}{\gamma}\mathbb{E}\mathbb{E}^m\big[\sup_{t\le s\le T}|S_s|^2\big] + \gamma\mathbb{E}\mathbb{E}^m[(K_T^n - K_t^n)^2] \end{aligned}$$

where $\varepsilon > 0$, $\gamma > 0$ are a arbitrary constants and c_ε is a constant which can be different from line to line. We have used the inequality $\int_t^T Y_s^n dK_s^n \ge \int_t^T S_s^n dK_s^n$ and then we have applied Schwartz's inequality. We also have used the fact that under the measure $\mathbb{P}^m$ the forward-backward integral $\int Y_r^n g(r, W_r, Y_r^n, Z_r^n) * dW$ as well the other stochastic integrals with respect to the brownian terms have null expectation under $\mathbb{P}\otimes\mathbb{P}^m$. Finally Gronwall's lemma leads to the desired inequality. $\square$

Lemma 5.

$$\begin{aligned} \mathbb{E}\mathbb{E}^m[(K_T^n - K_t^n)^2] \le c'\Big[\mathbb{E}\mathbb{E}^m\left|Y_t^n\right|^2 + \|\Phi\|_2^2\Big] + c_\varepsilon\Big[\mathbb{E}\mathbb{E}^m \int_t^T \Big[\left|Y_s^n\right|^2 + \left|Z_s^n\right|^2\Big]\, ds \\ + \mathbb{E}\int_t^T \left[\|f_s^0\|_2^2 + \|g_s^0\|_2^2 + \|h_s^0\|_2^2\right] ds\Big]. \end{aligned} \tag{19}$$

Proof : Let now $(\widetilde{u}^n)_{n\in\mathbb{N}}$ be the weak solutions of the following linear type equations

$$d\widetilde{u}_t^n + \frac{1}{2}\Delta \widetilde{u}_t^n + div\, g_t\left(u_t^n, \nabla u_t^n\right)\, dt + h_t\left(u_t^n, \nabla u_t^n\right) \cdot \overleftarrow{dB}_t = 0,$$

with final condition $\widetilde{u}_T^n = 0$. Set $\widetilde{Y}_t^n = \widetilde{u}^n(t, W_t)$ and $\widetilde{Z}_n = \nabla \widetilde{u}^n(t, W_t)$. Then by the estimate (12) one has

$$\mathbb{E}\mathbb{E}^m\Big[\big|\widetilde{Y}_t^n\big|^2 + \int_0^T \big|\widetilde{Z}_s^n\big|\, ds\Big] \leq \tilde{c}\Lambda \tag{20}$$

where $\Lambda = \mathbb{E}\mathbb{E}^m \int_0^T \big[|g_s(W_s, Y_s^n, Z_s^n)|^2 + |h_s(W_s, Y_s^n, Z_s^n)|^2\big]\, ds$. Since $u^n - \widetilde{u}^n$ verifies the equation

$$\partial_t(u_t^n - \widetilde{u}_t^n) + \frac{1}{2}\Delta(u^n - \widetilde{u}_t^n) + f_t(u_t^n, \nabla u_t^n) + n(u_t^n - v_t)^-\, dt = 0,$$

we have the stochastic representation

$$Y_t^n - \widetilde{Y}_t^n = \Phi\left(W_T\right) + \int_t^T f_r\left(W_r, Y_r^n, Z_r^n\right) dr + K_T^n - K_t^n - \sum_i \int_t^T \big(Z_{i,r}^n - \widetilde{Z}_{i,r}^n\big)\, dW_r^i.$$

from which one easily obtains the estimate

$$\mathbb{E}\mathbb{E}^m[(K_T^n - K_t^n)^2] \leq c\,\mathbb{E}\mathbb{E}^m\Big[\,|Y_t^n|^2 + |\widetilde{Y}_t^n|^2 + \big|\Phi\left(W_T\right)\big|^2 + \int_t^T \big(|f_s^0(W_s)|^2 + |Y_s^n|^2 + |Z_s^n|^2\,\big)\, ds + \int_t^T |\widetilde{Z}_s^n|^2\, ds\Big].$$

Hence, using (20), we get

$$\mathbb{E}\mathbb{E}^m[(K_T^n - K_t^n)^2] \leq c'\,\mathbb{E}\mathbb{E}^m\,\Big[|Y_t^n|^2 + |\Phi(W_T)|^2\Big] + c'_\varepsilon \mathbb{E}\mathbb{E}^m\Big[\int_t^T \Big(|Y_s^n|^2 + |Z_s^n|^2\Big)\, ds + \int_t^T \big[|f_s^0(W_s)|^2 + |g_s^0(W_s)|^2 + |h_s^0(W_s)|^2\big]\, ds\Big],$$

which gives our assertion. □

Lemma 6. *The triple (Y^n, Z^n, K^n) satisfies the following estimate*

$$\mathbb{E}\,\mathbb{E}^m\big(\sup_{0\leq s\leq T}|Y_s^n|^2\big) + \mathbb{E}\mathbb{E}^m \int_0^T |Z_s^n|^2\, ds + \mathbb{E}\,\mathbb{E}^m\left(K_T^n\right)^2 \leq\; c\,\Big[\|\Phi\|_2^2 + \mathbb{E}\,\mathbb{E}^m\big(\sup_{0\leq s\leq T}|S_s|^2\big) + \mathbb{E}\int_0^T \Big[\|f_s^0\|_2^2 + \|g_s^0\|_2^2 + \|h_s^0\|_2^2\Big]\, ds\Big]$$

where $c > 0$ is a constant.

Proof : From (17) and (19) we get

$$(1 - \delta c')\,\mathbb{E}\,\mathbb{E}^m\; |Y_s^n|^2 + \big(1 - 2\alpha - \beta^2 - \epsilon - \delta c'_\varepsilon\big)\,\mathbb{E}\mathbb{E}^m \int_s^T |Z_r^n|^2\, dr \leq\; (1 + c'\delta)\,\|\Phi\|_2^2 + (c_\epsilon + \delta c'_\varepsilon)\,\Lambda$$
$$(c_\epsilon + \delta c'_\epsilon)\;\mathbb{E}\,\mathbb{E}^m \int_s^T |Y_r^n|^2\, ds + c_\delta\,\mathbb{E}\,\mathbb{E}^m\big(\sup_{t\leq r\leq T}|S_r|^2\big)$$

where $\Lambda = \mathbb{E}\,\mathbb{E}^m \int_t^T \Big[|f_s^0(W_s)|^2 + |g_s^0(W_s)\|^2 + |h_s^0(W_s)|^2 \Big]\, ds$. It then follows from Gronwall's lemma that

$$\sup_{0\le s\le T} \mathbb{E}\,\mathbb{E}^m \left(|Y_s^n|^2\right) + \mathbb{E}\mathbb{E}^m \int_s^T |Z_r^n|^2\, dr + \mathbb{E}\,\mathbb{E}^m\,(K_T^n)^2 \le c_1 \Big[\|\Phi\|_2^2 + \mathbb{E}\,\mathbb{E}^m \big(\sup_{0\le r\le T} |S_r|^2\big) + \mathbb{E} \int_s^T \Big[\|f_r^0\|_2^2 + \|g_r^0\|_2^2 + \|h_r^0\|_2^2 \Big]\, dr \Big].$$

Coming back to the equation (16) and using Bukholder-Davis-Gundy inequality and the last estimates we get our statement. □

In order to prove the strong convergence of the sequence (Y^n, Z^n, K^n) we shall need the following result.

Lemma 7. *(The essential step)*

$$\lim_{n\to\infty} \mathbb{E}\mathbb{E}^m \left[\sup_{0\le t\le T} \left((Y_t^n - S_t)^- \right)^2 \right] = 0 \tag{21}$$

Proof: Let $(u^n)_{n\in\mathbb{N}}$ be the sequence of solutions of the penalized SPDE defined in (15). From Lemma 6, it follows that the sequence $\left(f(u^n,\nabla u^n), g(u^n,\nabla u^n), h(u^n,\nabla u^n)\right)_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^2\left([0,T]\times\Omega\times\mathbb{R}^d;\mathbb{R}^{1+d+d_1}\right)$. We may choose then a subsequence which is weakly convergent to a system of predictable processes $(\bar f, \bar g, \bar h)$ and, on account of the Lemma 13 in the Appendix, we obtain a sequence of families of coefficients of convex combinations, $(a^k)_{k\in\mathbb{N}}$, such that the sequences

$$\hat f^k = \sum_{i\in I_k} \alpha_i^k f(u^i,\nabla u^i), \quad \hat g^k = \sum_{i\in I_k} \alpha_i^k g(u^i,\nabla u^i) \quad \text{and} \quad \hat h^k = \sum_{i\in I_k} \alpha_i^k h(u^i,\nabla u^i)$$

converge strongly, i.e.

$$\lim_{k\to\infty} \mathbb{E} \int_0^T \|\hat f_t^k - \bar f_t\|_2^2 dt = 0$$

and similarly for $\hat g^k$, $\bar g$ and $\hat h^k$, $\bar h$.
Now for $i \ge n$ we denote by $u^{i,n}$ the solution of the equation

$$du_t^{i,n} + [\frac{1}{2}\Delta u_t^{i,n} - n u_t^{i,n} + n v_t + f_t(u^i,\nabla u^i) + div\, g_t(u^i,\nabla u^i)\,]\, dt + h_t(u^i,\nabla u^i)\cdot \overleftarrow{dB}_t = 0 \tag{22}$$

with final condition $u_T^{i,n} = v_T$. By comparison (Theorem 5) we have that $u^{i,n} \le u^i$. Further we set $\hat u^k = \sum_{i\in I_k} \alpha_i^k u^{i,n_k}$, where $n_k = \inf I_k$ and we deduce that

$$\hat u^k \le \sum_{i\in I_k} \alpha_i^k u^i \le \lim_{n\to\infty} u^n, \tag{23}$$

where the last inequality comes from the monotonicity of the sequence u^n. Moreover we observe that $\hat u^k$ is a solution of the equation

$$d\hat u_t^k + [\frac{1}{2}\Delta \hat u_t^k - n_k \hat u_t^k + n_k v_t + \hat f_t^k + div\, \hat g_t^k\,]\, dt + \hat h_t^k \cdot \overleftarrow{dB}_t = 0 \tag{24}$$

with final condition $\hat u_T^k = v_T$.
Now we are going to take the advantage of the fact that the equations satisfied by the sequence of solutions $\hat u^k$ have strongly convergent coefficients. Let us denote by $\widehat{Y}^k$ the continuous version

on $[0,T]$ of the process $\left(\hat{u}^k(W_t)\right)_{t\in[0,T]}$, for any $k \in \mathbb{N}$. We will prove now that there exists a subsequence such that

$$\lim_{k\to\infty} \sup_{0\le t\le T} |\widehat{Y}_t^k - S_t| = 0, \ \ \mathbb{P}\otimes\mathbb{P}^m\text{-}a.s. \tag{25}$$

Since the equation (24) is linear the solution decomposes as a sum of four terms each corresponding to one of the coefficients $\hat{f}^k$, $\hat{g}^k$, $\hat{h}^k$, v. So it is enough to treat separately each term.

a) In the case where $f \equiv 0$, $g \equiv 0$, $h \equiv 0$ one obtains the term corresponding to v. Then the relation (25) is a direct consequence of the Lemma 11.

b) In the case where $v \equiv 0$, $g \equiv 0$, $h \equiv 0$, the representation of $\widehat{Y}^k$ is given by

$$\widehat{Y}_t^k = \int_t^T e^{-n_k(s-t)} \hat{f}_s^k(W_s)ds - \sum_{i=1}^d \int_t^T e^{-n_k(s-t)} \partial_i \hat{u}_r^k\,(W_s)\, dW_s^i.$$

Thus we have

$$\left|\int_t^T e^{-n_k(s-t)} \hat{f}_s^k(W_s)ds\right| \le \frac{1}{\sqrt{2n_k}} \left(\int_t^T \left(\hat{f}_s^k(W_s)\right)^2 ds\right)^{1/2}.$$

This shows that $\lim_{k\to\infty} \sup_{0\le t\le T} \left|\int_t^T e^{-n_k(s-t)} \hat{f}_s^k(W_s)ds\right| = 0$, $\mathbb{P}\otimes\mathbb{P}^m$-$a.s.$, on some subsequence. For the second term in the expression of $\hat{Y}^k$ we make an integration by parts formula to get

$$\int_t^T e^{-n_k(s-t)} \partial_i \hat{u}_s^k\,(W_s)\, dW_s^i = e^{-n_k(T-t)} U_T^{i,k} - U_t^{i,k} + n_k \int_t^T U_s^{i,k} e^{-n_k(s-t)} ds$$

where $U_s^{i,k} = \int_0^s \partial_i \hat{u}_r^k\,(W_r)\, dW_r^i$. By the Corollary 3 of section 4 we know that the martingales $U^{i,k}$, $k \in \mathbb{N}$ converges to zero in $\mathbf{L}^2$, and hence on a subsequence we have $\lim_{k\to\infty} \sup_{0\le t\le T} |U_t^{i,k}| = 0$, $\mathbb{P}\otimes\mathbb{P}^m$-$a.s.$ Then by Lemma 12 we see that for that subsequence

$$\lim_{k\to\infty} \sup_{0\le t\le T} \left|\int_t^T e^{-n_k(s-t)} \partial_i \hat{u}_s^k\,(W_s)\, dW_s^i\right| = 0, \ \mathbb{P}\otimes\mathbb{P}^m\text{-}a.s.$$

Therefore the desired result (25) holds also in this case. This time we get $\lim_{k\to\infty} \sup_{0\le t\le T} |Y_t^k| = 0$, $\mathbb{P}\otimes\mathbb{P}^m$-$a.s.$.

c) In the case where $f \equiv 0$, $h \equiv 0$, $v \equiv 0$, the representation of $\widehat{Y}^k$ is given by

$$\widehat{Y}_t^k = \int_t^T e^{-n_k(s-t)} \hat{g}_s^k * dW - \sum_{i=1}^d \int_t^T e^{-n_k(s-t)} \partial_i \hat{u}_r^k\,(W_s)\, dW_s^i$$

$$= \sum_i \int_t^T e^{-n_k(s-t)} \hat{g}_s^k(W_s) dW_s^i + \sum_i \int_t^T e^{-n_k(s-t)} \hat{g}_s^k(W_s) \overleftarrow{dM}_s^i - \sum_{i=1}^d \int_t^T e^{-n_k(s-t)} \partial_i \hat{u}_r^k\,(W_s)\, dW_s^i.$$

Now the proof is similar to that of the preceding case. We treat only the second term in the last expression. We set $\overleftarrow{U}_s^{i,k} = \int_s^T \hat{g}_r^k(W_r) \overleftarrow{dW}_r^{m,i}$. Integration by parts formula gives

$$\int_t^T e^{-n_k(s-t)} d\overleftarrow{U}_s^{i,k} = \overleftarrow{U}_t^{i,k} - e^{-n_k(T-t)} \overleftarrow{U}_T^{i,k} - n_k \int_t^T \overleftarrow{U}_s^{i,k} e^{-n_k(s-t)} ds.$$

On the other hand the convergence $\hat{g}^k \to \bar{g}$ implies that the backward martingale $\big(\overleftarrow{U}_t^{i,k}\big)_{t\in[0,T]}$ converges to $\big(\int_t^T \bar{g}_{i,r}(W_r)\overleftarrow{dW}_r^{m,i}\big)_{t\in[0,T]}$ in $\mathbf{L}^2\big(\mathbb{P}\otimes\mathbb{P}^m\big)$. The other terms in the above expression of $\hat{Y}^k$ may be handled similarly by integration by parts and taking into account Corollary 4. Using again Lemma 12, as in the preceding case we get the relation (25) in the form $\lim_{k\to\infty}\sup_{0\le t\le T}|Y_t^k| = 0$, $\mathbb{P}\otimes\mathbb{P}^m$-*a.s.*..

d) In the case where $f\equiv 0$, $g\equiv 0$, $v\equiv 0$, the representation of $\widehat{Y}^k$ is given by

$$\widehat{Y}_t^k = -\sum_{i=1}^d \int_t^T e^{-n_k(s-t)}\partial_i\hat{u}_r^k(W_s)\,dW_s^i + \int_t^T e^{-n_k(s-t)}\hat{h}_s^k\cdot\overleftarrow{dB}_s$$

On account of Lemma 10, the same arguments used in the previous cases work again.

Now it is easy to see that the relation (25) holds for the general case. On the other hand (25) and (23) clearly imply the relation

$$\lim_{n\to\infty}\sup_{0\le t\le T}(Y_t^n - S_t)^- = 0,\ \mathbb{P}\otimes\mathbb{P}^m\text{-}a.s.$$

and then, since Y^n is bounded in $\mathbf{L}^2$, one gets the relation of our statement. □

We have also the following result

Lemma 8. *There exists a progressively measurable triple of processes* $(Y_t, Z_t, K_t)_{t\in[0,T]}$ *such that*

$$\mathbb{E}\,\mathbb{E}^m\left[\sup_{0\le s\le T}|Y_t^n - Y_t|^2 + \int_0^T |Z_t^n - Z_t|^2dt + \sup_{0\le t\le T}|K_t^n - K_t|^2\right] \longrightarrow 0 \quad as\ n\to\infty. \tag{26}$$

Moreover we have that $(Y_t, Z_t, K_t)_{t\in[0,T]}$ *satisfies* $Y_t \ge S_t,\ \forall\, t\in[0,T]$ *and* $\int_0^T (Y_s - S_s)dK_s = 0$, $\mathbb{P}\otimes\mathbb{P}^m$*-a.e.*

Proof : From the monotonicity of the sequence $(f_n)_{n\in\mathbb{N}}$ and the comparison theorem 5 we get that $u^n(t,x)\le u^{n+1}(t,x)$, $\ dtdx\otimes\mathbb{P}$ -a.e., therefore one has $Y_t^n\le Y_t^{n+1}$, for all $t\in[0,T]$, $\mathbb{P}\otimes\mathbb{P}^m$ -a.s. Thus, there exists a predictable real valued process $Y=(Y_t)_{t\in[0,T]}$ such that $Y_t^n\uparrow Y_t$, for all $t\in[0,T]$ a.s. and by Lemma 6 and Fatou's lemma , one gets

$$\mathbb{E}\,\mathbb{E}^m\left(\sup_{0\le s\le T}|Y_t|^2\right)\le c.$$

Moreover, from the dominated convergence theorem one has

$$\mathbb{E}\,\mathbb{E}^m\int_0^T |Y_t^n - Y_t|^2dt \longrightarrow 0 \quad \text{as } n\to\infty. \tag{27}$$

The relation (13) gives , for $n \geq p$,

$$
\begin{aligned}
&|Y_t^n - Y_t^p|^2 + \int_t^T |Z_s^n - Z_s^p|_a^2 ds = 2\int_t^T (Y_s^n - Y_s^p)\,[f_s(W_s, Y_s^n, Z_s^n) - f_s(W_s, Y_s^p, Z_s^p)]\; ds \\
&+ 2\int_t^T (Y_s^n - Y_s^p)\, d\,(K_s^n - K_s^p) - 2\int_t^T \langle Z_s^n - Z_s^p, g_s\big(W_s, Y_s^n, Z_s^n\big) - g_s\big(W_s, Y_s^p, Z_s^p\big)\rangle\, ds \\
&+ \int_t^T (Y_s^n - Y_s^p)\,[g_s(X_s, Y_s^n, Z_s^n) - g_s(W_s, Y_s^p, Z_s^p\,)]\, * dW \\
&- 2\sum_i \int_t^T (Y_s^n - Y_s^p)\left(Z_{i,s}^n - Z_{i,s}^p\right) dW_s^i + 2\int_t^T (Y_s^n - Y_s^p)\,[h_s(W_s, Y_s^n, Z_s^n\,) - h_s(W_s, Y_s^p, Z_s^p\,]\cdot \overleftarrow{dB}_s \\
&+ \int_t^T |h_s(W_s, Y_s^n, Z_s^n) - h_s(W_s, Y_s^p, Z_s^p)|^2 ds.
\end{aligned}
\tag{28}
$$

By standard calculation one deduces that

$$
\begin{aligned}
\mathbb{E}\,\mathbb{E}^m \int_t^T |Z_s^n - Z_s^p|^2 ds \leq c\mathbb{E}\,\mathbb{E}^m \int_t^T |Y_s^n - Y_s^p|^2 \;+ 4\mathbb{E}\,\mathbb{E}^m \int_t^T (Y_s^n - S_s)^-\; dK_s^p \\
+ 4\mathbb{E}\,\mathbb{E}^m \int_t^T (Y_s^p - S_s)^-\; dK_s^n
\end{aligned}
\tag{29}
$$

Therefore from Lemma 7, (27) and (29) one gets

$$
\mathbb{E}\,\mathbb{E}^m \int_0^T |Y_t^n - Y_t^p|^2 dt + \mathbb{E}\,\mathbb{E}^m \int_0^T |Z_t^n - Z_t^p|^2 dt \longrightarrow 0 \quad \text{as } n, p \to \infty. \tag{30}
$$

The rest of the proof is the same as in El Karoui et al ([12] p.721-722), in particular we get that there exists a pair (Z, K) of progressively measurable processes with values in $\mathbb{R}^d \times \mathbb{R}$ such that

$$
\mathbb{E}\,\mathbb{E}^m \left[\sup_{0 \leq s \leq T} |Y_t^n - Y_t|^2 + \int_0^T |Z_t^n - Z_t|^2 dt + \sup_{0 \leq t \leq T} |K_t^n - K_t|^2 \right] \longrightarrow 0 \quad \text{as } n \to \infty.
$$

It is obvious that $(K_t)_{t\in[0,T]}$ is an increasing continuous process. On the other hand since from Lemma 7 we have $\lim_{n\to\infty} \mathbb{E}\,\mathbb{E}^m \big[\sup_{0\leq t\leq T} ((Y_t^n - S_t)^-)^2 \big] = 0$, then, $\mathbb{P} \otimes \mathbb{P}^m$*−a.s.*,

$$
Y_t \geq S_t, \quad \forall\, t \in [0, T], \tag{31}
$$

which yields that $\int_0^T (Y_s - S_s) dK_s \geq 0$. Finally we also have $\int_0^T (Y_s - S_s) dK_s = 0$ since on the other hand the sequences $(Y^n)_{n\geq 0}$ and $(K^n)_{n\geq 0}$ converge uniformly (at least for a subsequence) respectively to Y and K and

$$
\int_0^T (Y_s^n - S_s) dK_s^n = -n \int_0^T ((Y_s^n - S_s)^-)^2 ds \leq 0.
$$

□

As a consequence of the last proof we obtain the following generalization of the RBSDE introduced in [12] :

Corollary 2. *The limiting triple of processes $(Y_t, Z_t, K_t)_{t\in[0,T]}$ is a solution of the following reflected backward doubly stochastic differential equation (in short RBDSDE) :*

$$\begin{aligned} Y_t = \Phi(W_T) + \int_t^T f_r(W_r, Y_r, Z_r)\, dr + K_T - K_t + \frac{1}{2}\int_t^T g_r(W_r, Y_r, Z_r) * dW \\ + \int_t^T h_r(W_r, Y_r, Z_r^n) \cdot \overleftarrow{dB}_r - \sum_i \int_t^T Z_{i,r} dW_r^i \end{aligned} \tag{32}$$

with $Y_t \geq S_t$, $\forall\, t \in [0,T]$, $(K_t)_{t\in[0,T]}$ is an increasing continuous process, $K_0 = 0$ and

$$\int_0^T (Y_s - S_s) dK_s = 0. \tag{33}$$

Proof of Theorem 4 : Since

$$\int_0^T \left(\left\|u_t^n - u_t^p\right\|_2^2 + \left\|\nabla u_t^n - \nabla u_t^p\right\|_2^2 dt = \mathbb{E}^m \int_0^T \left(\left|Y_t^n - Y_t^p\right|^2 + \left|Z_t^n - Z_t^p\right|^2\right) dt,$$

by the preceding lemma one deduces that the sequence $\left(u^n\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbf{L}^2\big(\Omega \times [0,T]; H^1(\mathbb{R}^d)\big)$ and hence has a limit u in this space. Also from the preceding lemma it follows that dK_t^n weakly converges to dK_t, $\mathbb{P}\otimes\mathbb{P}^m$-*a.e.* This implies that

$$\lim_n \int_0^T \int_{\mathbb{R}^d} n\big(u^n - v\big)^- \varphi(t,x)\, dtdx = \lim_n \mathbb{E}^m \int_0^T \varphi_t(W_t)\, dK_t^n = \int_0^T \int_{\mathbb{R}^d} \varphi(t,x)\, \nu\big(dtdx\big),$$

where ν is the regular measure defined by

$$\int_0^T \int_{\mathbb{R}^d} \varphi(t,x)\, \nu\big(dtdx\big) = E^m \int_0^T \varphi_t(W_t)\, dK_t.$$

Writing the equation (15) in the weak form and passing to the limit one obtains the equation (14) with u and this ν. The arguments we have explained after Definition 4 ensure that u admits a quasicontinuous version $\bar{u}$. Then one deduces that $\big(\bar{u}_t(W_t)\big)_{t\in[0,T]}$ should coincide with $(Y_t)_{t\in[0,T]}$, $\mathbb{P}\otimes\mathbb{P}^m$-*a.e.* Therefore the inequality $Y_t \geq S_t$ implies $u \geq v$, $dt\otimes\mathbb{P}\otimes dx$-*a.e.* and the relation $\int_0^T \big(Y_t - S_t\big)\, dK_t = 0$ implies the relation (iv) of Definition 4. □

4 Some technical lemmas

Lemma 9. *Let $f \in \mathbf{L}^2\big([0,T]\times\mathbb{R}^d;\mathbb{R}\big)$ and denote by $(u^n)_{n\in\mathbb{N}}$ the sequence of solutions of the equations*

$$\big(\partial_t + \frac{1}{2}\Delta\big)u^n - nu^n + f = 0, \quad \forall n \in \mathbb{N},$$

with final condition $u_T^n = 0$. Then we have

$$\int_0^T \|\nabla u_t^n\|_2^2 dt \leq c\left[\frac{1}{n}\int_0^T \|f_t\|_2^2 dt + \int_0^T e^{-2n(T-t)}\|f_t\|_2^2 dt\right]. \tag{34}$$

Proof : It is well known that the solution $(u^n)_{n\in\mathbb{N}}$ is expressed in terms of the semigroup P_t by

$$u_t^n = \int_t^T e^{-n(s-t)} P_{s-t} f_s ds.$$

A direct calculation shows that one has

$$n\int_t^T e^{-n(s-t)}P_{s-t}u_s^0 ds = u_t^0 - u_t^n,$$

which leads to

$$u_t^n = e^{-n(T-t)}u_t^0 + n\int_t^T e^{-n(s-t)}\left(u_t^0 - P_{s-t}u_s^0\right) ds. \tag{35}$$

The function $\bar{u}_t^n = e^{-n(T-t)}u_t^0$ is a solution of the equation $\left(\partial_t + \frac{1}{2}\Delta\right)\bar{u}^n - n\bar{u}^n + \bar{f} = 0$ where $\bar{f}_t = e^{(T-t)}f_t$. Therefore one has the following estimate for the gradient of the first term in the expression of u^n

$$\int_0^T e^{-n(T-t)}\|\nabla u_t^0\|_2^2 dt \le c\int_0^T e^{-2n(T-t)}\|f_t\|_2^2 dt \tag{36}$$

(see Lemma 5 of [8] for details). In order to estimate the gradient of the second term of the expression of u^n we first remark that

$$u_t^0 - P_{s-t}u_s^0 = \int_t^s P_{r-t}f_r dr,$$

so that one has

$$\begin{aligned}\left\| n\nabla\int_t^T e^{-n(s-t)}\left(u_t^0 - P_{s-t}u_s^0\right) ds\right\|_2 &\le n\int_0^{T-t} e^{-ns}\int_0^s \|\nabla P_r f_{t+r}\|_2\, drds\\ &\le nc\int_0^{T-t} e^{-ns}\int_0^s \frac{1}{\sqrt{r}}\|f_{t+r}\|_2\, drds,\end{aligned}$$

where we have used the well known inequality

$$\|\nabla P_r\varphi\|_2 \le \frac{c}{\sqrt{r}}\|\varphi\|_2, \quad \text{for } \varphi\in\mathbf{L}^2.$$

Then we estimate the time integral of the norm of the gradient, which is the expression we are interested in,

$$\begin{aligned}&\int_0^T \left\| n\nabla\int_t^T e^{-n(s-t)}\left(u_t^0 - P_{s-t}u_s^0\right) ds\right\|_2^2 dt \le c^2\int_0^T \left[\int_0^{T-t} ne^{-ns}\int_0^s \frac{1}{\sqrt{r}}\|f_{t+r}\|_2\, drds\right]^2 dt\\ &= c^2\int_0^T\int_0^s\int_0^T\int_0^{s'}\int_0^{T-s\vee s'} ne^{-ns}ne^{-ns'}\frac{1}{\sqrt{r}}\|f_{t+r}\|_2\frac{1}{\sqrt{r'}}\|f_{t+r'}\|_2\, dt\, dr'\, ds'\, dr\, ds\\ &\le \int_0^T \|f_t\|_2^2 dt\left(\int_0^T \frac{1}{2}\sqrt{s}\, n\, e^{-ns}\, ds\right)^2 \le \frac{c}{n}\int_0^T \|f_t\|_2^2 dt.\end{aligned}$$

This estimate together with (36) imply the statement (34). □

Obviously the lemma implies that $\lim_{n\to\infty}\int_0^T \|\nabla u_t^n\|_2^2 dt = 0$. We need a strengthened version of this relation, which is presented in the next corollary whose proof is easy, so you omit it.

Corollary 3. *Let $f,\ f^n, \in \mathbf{L}^2\left([0,T]\times\mathbb{R}^d;\mathbb{R}\right),\ n\in\mathbb{N}$, be such that $\lim_{n\to\infty}\int_0^T \|f_t^n - f_t\|_2^2 dt = 0$. Then the solutions $(u^n)_{n\in\mathbb{N}}$ of the equations*

$$\left(\partial_t + \frac{1}{2}\Delta\right)u^n - nu^n + f^n = 0,$$

with final condition $u_T^n = 0$, satisfy the relation $\lim_{n\to\infty}\int_0^T \|\nabla u_t^n\|_2^2 dt = 0$.

Corollary 4. *Let $g^n,\ g \in \mathbf{L}^2\left([0,T]\times\mathbb{R}^d;\mathbb{R}^d\right)$ be such that $\lim_{n\to\infty}\int_0^T \|g_t^n - g_t\|_2^2 dt = 0$. Then the solutions $(u^n)_{n\in\mathbb{N}}$ of the equations*

$$(\partial_t + \frac{1}{2}\Delta)u^n - nu^n + div\, g^n = 0$$

with final condition $u_T^n = 0$, satisfy the relation $\lim_{n\to\infty}\int_0^T \|\nabla u_t^n\|_2^2 dt = 0$.

Proof : We regularize g by setting $g_{i,t}^\epsilon = P_\epsilon g_{i,t}$, for $i = 1,\dots,d$, $\epsilon > 0$, $t \in [0,T]$. Then $g_i^\epsilon \in H_0^1(\mathbb{R}^d)$ and $f^\epsilon = \mathrm{div} g^\epsilon$ is in $\mathbf{L}^2\left([0,T]\times\mathbb{R}^d;\mathbb{R}\right)$. Moreover we have $\lim_{\epsilon\to 0}\int_0^T \|g_t^\epsilon - g_t\|_2^2 dt = 0$. Let $u^{\epsilon,n}$ be the solution of the equation

$$\left(\partial_t + \frac{1}{2}\Delta\right)u^{\epsilon,n} - nu^{\epsilon,n} + f^\epsilon = 0,$$

with final condition $u_T^{\epsilon,n} = 0$. By Lemma 5 of [8] one has

$$\int_0^T \|\nabla u_t^n - \nabla u_t^{\epsilon,n}\|_2^2 dt \le c\int_0^T \|g_t^n - g_t^\epsilon\|_2^2\, dt \le c\int_0^T \left(\|g_t^n - g_t\|_2^2 + \|g_t^\epsilon - g_t\|_2^2\right) dt.$$

On the other hand, Lemma 9 implies, for ϵ fixed, $\lim_{n\to\infty}\int_0^T \|\nabla u_t^{\epsilon,n}\|_2^2 dt = 0$. From these facts one easily concludes the proof. □

Lemma 10. *Let $h,\ h^n,\ n\in\mathbb{N}$ be $\mathbf{L}^2(\mathbb{R}^d;\mathbb{R}^{d^1})$ -valued predictable processes on $[0,T]$ with respect to $\left(\mathcal{F}_{t,T}^B\right)_{t\ge 0}$ and such that*

$$\mathbb{E}\int_0^T \|h_t\|_2^2 dt < \infty,\quad \mathbb{E}\int_0^T \|h_t^n\|_2^2 dt < \infty \quad \textit{and} \lim_{n\to\infty}\mathbb{E}\int_0^T \|h_t^n - h_t\|_2^2 dt = 0.$$

Let $(u^n)_{n\in\mathbb{N}}$ be the solutions of of the equations

$$du_t^n + [\frac{1}{2}\Delta u_t^n - nu_t^n]\, dt + h_t^n \cdot \overleftarrow{dB}_t = 0,$$

with final condition $u_T^n = 0$, for each $n\in\mathbb{N}$. Then one has $\lim_{n\to\infty}\int_0^T \|\nabla u_t^n\|_2^2 dt = 0$.

Proof : We regularize the process h by setting $\bar{h}_{i,t}^\epsilon = P_\epsilon h_{i,t}$, for $i = 1,\dots,d_1$, $\epsilon > 0$, $t\in[0,T]$. Then $\bar{h}_{i,t}^\epsilon \in H_0^1(\mathbb{R}^d)$ and $\mathbb{E}\int_0^T \left\|\nabla\bar{h}_t^\epsilon\right\|_2^2 dt < \infty$ and $\lim_{\epsilon\to 0}\mathbb{E}\int_0^T \|\bar{h}_t^\epsilon - h_t\|_2^2 dt = 0$. Let $u^{\epsilon,n}$ be the solution of the equation

$$du_t^{\epsilon,n} + \frac{1}{2}\Delta u_t^{\epsilon,n} - nu_t^{\epsilon,n} + \bar{h}_t^\epsilon \cdot \overleftarrow{dB}_t = 0$$

with final condition $u_T^{\epsilon,n} = 0$, for each $n\in\mathbb{N}$. The relation (iii) of Proposition 6 in [8] written with respect to the Hilbert space $H = H_0^1(R^d)$ takes the form

$$\mathbb{E}[\left\|\nabla u_t^{\epsilon,n}\right\|_2^2 + \int_t^T \|\frac{1}{2}\Delta u_s^{\epsilon,n}\|^2 ds + n\int_t^T \left\|\nabla u_s^{\epsilon,n}\right\|_2^2 ds\,] = \mathbb{E}\int_t^T \left\|\nabla\overline{h}_s^\epsilon\right\|_2^2 ds\,.$$

In particular one has

$$\int_t^T \|\nabla u_s^{\epsilon,n}\|^2 ds \le \frac{1}{n}\int_t^T \left\|\nabla\overline{h}_s^\epsilon\right\|_2^2 ds.$$

Now we write the relation (iii) of Proposition 6 in [8] for the solution $u^n - u^{\epsilon,n}$ with respect to the Hilbert space $H = \mathbf{L}^2(R^d)$,

$$\mathbb{E}\left[\|u_0^n - u_0^{\epsilon,n}\|^2 + \int_0^T \left\|\nabla u_s^n - \nabla u_s^{\epsilon,n}\right\|_2^2 ds + n\int_0^T \|u_s^n - u_s^{\epsilon,n}\|_2^2 ds\right] = \mathbb{E}\int_0^T \|\bar{h}_s^n - \bar{h}_s^\epsilon\|_2^2 ds\,.$$

In particular one obtains

$$\mathbb{E}\int_0^T \|\nabla u_s^n - \nabla u_s^{\epsilon,n}\|_2^2 ds \le \mathbb{E}\int_0^T \|\bar{h}_s^n - \bar{h}_s^\epsilon\|_2^2 ds.$$

From this and the preceding inequality one deduces

$$\limsup_{n\to\infty} \mathbb{E}\int_0^T \|\nabla u_s^n\|_2^2 ds \le \mathbb{E}\int_0^T \|\bar{h}_s - \bar{h}_s^\epsilon\|_2^2 ds.$$

Letting $\epsilon \to 0$, one deduces the relation from the statement. □

Lemma 11. *Let $v : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}$ be a function such that the process $(v_t(W_t))_{t\in[0,T]}$ admits a version $S = (S_t)_{t\in[0,T]}$ with continuous trajectories on $[0,T]$ and such that the random variable $S^* = \sup_{0\le t\le T} S_t$ satisfies the condition $\mathbb{E}^m\left[S^*\right]^2 < \infty$. Let u^n be the solution of the equation*

$$\big(\partial_t + \frac{1}{2}\Delta\big)\, u^n - nu^n + nv = 0,$$

with the terminal condition $u_T^n = v_T$. Let $Y^n = (Y_t^n)_{t\in[0,T]}$ be a continuous version of the process $(u_t^n(W_t))_{t\in[0,T]}$, for each $n\in\mathbb{N}$. Then the following holds

$$\lim_{n\to\infty} \mathbb{E}^m\left[\sup_{0\le t\le T} |Y_t^n - S_t|^2\right] = 0.$$

Proof : Let us set $\bar{u}_t^n = e^{-nt}u_t^n$ and observe that this function is a solution of the equation

$$\big(\partial_t + \frac{1}{2}\Delta\big)\,\bar{u}^n + \bar{v} = 0,$$

with $\bar{v}_t = e^{-nt}v_t$ and terminal condition $\bar{u}_T^n = \bar{v}_T$. Writting the representation of Theorem 3 with $g = h = 0$ for $\bar{u}^n(W_t)$ one obtains

$$\bar{u}_t^n\,(t, W_t) = e^{-nT}v_T - \sum_{i=1}^d \int_t^T \partial_i \bar{u}_r^n\,(W_r)\,dW_r^i + n\int_t^T e^{-nr}v_r(W_r)dr,$$

and this leads to the representation of our process Y^n, given by

$$Y_t^n = \mathbb{E}^m\left[e^{-n(T-t)}S_T + n\int_t^T e^{-n(r-t)}S_r dr\,\Big|\mathcal{F}_t\right].$$

Then one has

$$|S_t - Y_t| \le \mathbb{E}^m\left[\left|S_t - e^{-n(T-t)}S_T - n\int_t^T e^{-n(r-t)}S_r dr\right|\,\Big|\mathcal{F}_t\right].$$

Let us denote by

$$V^n = \sup_{0\le t\le T}\left|S_t - e^{-n(T-t)}S_T - n\int_t^T e^{-n(r-t)}S_r dr\right|$$

Obviously one has $V^n \le 2S^*$. On the other hand one has for any fixed $\delta > 0$,

$$V^n \le \sup_{|t-s|\le\delta}|S_t - S_s| + 2e^{-n\delta}S^*. \tag{37}$$

This follows from Lemma 12. From the inequality (37) one deduces that $\lim_{n\to\infty} V^n = 0$, $\mathbb{P}^m$-*a.s.*, and hence from the dominated convergence Theorem, one gets $\lim_{n\to\infty}\mathbb{E}^m\left[V^n\right]^2 = 0$. Since

$$|S_t - Y_t^n| \le \mathbb{E}^m\left[V^n \middle| \mathcal{F}_t\right],$$

Doob's Theorem implies the assertion of the lemma. □

Finally, we mention the following calculus lemma.

Lemma 12. *Let $\varphi \in C([0,1];\mathbb{R})$ and $\delta \in (0,T)$, $\lambda > 0$. Then one has*

$$\left|\lambda\int_0^\delta e^{-\lambda t}\varphi(t)dt + e^{-\lambda\delta}\varphi(\delta) - \varphi(0)\right| \le \sup_{0\le t\le\delta}|\varphi(t)-\varphi(0)|.$$

and

$$\left|\lambda\int_t^T e^{-\lambda(s-t)}\varphi(s)ds + e^{-\lambda(T-t)}\varphi(T) - \varphi(t)\right| \le \sup_{|s-r|\le\delta, s\ge 0}|\varphi(s)-\varphi(r)| + 2e^{-\lambda\delta}\|\varphi\|_\infty .$$

Proof : The first inequality follows from the relation $\lambda\int_0^\delta e^{-\lambda t}dt + e^{-\lambda\delta} = 1$. In order to check the second relation one dominates the expression of the right hand side by

$$\left|\lambda\int_t^{t+\delta} e^{-\lambda(s-t)}\varphi(s)ds + e^{-\lambda\delta}\varphi(t+\delta) - \varphi(t)\right|$$
$$+ e^{-\lambda\delta}\left|\lambda\int_{t+\delta}^T e^{-\lambda(s-(t+\delta))}\varphi(s)ds + e^{-\lambda(T-(t+\delta))}\varphi(T) - \varphi(t+\delta)\right|$$

and then apply the first relation to dominate the first term. □

5 Appendix

The next lemma is a classical result in convex analysis, known as Mazur's Theorem (see [5], Remark 5 p. 38). We state here the result with some notation that is useful for our proof. Let X be a Banach space and $(x_n)_{n\in\mathbb{N}}$ a sequence of elements in X. We call finite family of coefficients of a convex combination a family $a = \{\alpha_i | i \in I\}$ where I is a finite subset of $\mathbb{N}$, $\alpha_i > 0$ for each $i \in I$ and $\sum_{i\in I}\alpha_i = 1$. The convex combination that corresponds to such a family of coefficients is the point expressed in terms of our sequence by $\sum_{i\in I}\alpha_i x_i$.

Lemma 13. *Let $(x_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence of elements in X with limit x. Then there exits a sequence $(a^k)_{k\in\mathbb{N}}$ of families of coefficients of convex combinations, $a^k = \{\alpha_i^k | i \in I_k\}$, such that the corresponding convex combinations $x^k = \sum_{i\in I_k}\alpha_i^k x_i$, $k\in\mathbb{N}$, converge strongly to x : $\lim_{k\to\infty}\|x^k - x\| = 0$.*

□

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