On Parameter Estimation of Threshold Autoregressive Models

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Abstract

This paper studies the threshold estimation of a TAR model when the underlying threshold parameter is a random variable. It is shown that the Bayesian estimator is consistent and its limit distribution is expressed in terms of a limit likelihood ratio. Furthermore, convergence of moments of the estimators is also established. The limit distribution can be computed via explicit simulations from which testing and inference for the threshold parameter can be conducted. The obtained results are illustrated with numerical simulations.

Key words and phrases: Bayesian estimator, continuous-time diffusion, compound Poisson process, limit distribution, limit likelihood ratio and nonlinear threshold models.

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1 Introduction

Since the publication of the seminal treatise of Tong [18], the field of nonlinear time series has been receiving considerable attention in the literature. Today, nonlinear time series has been widely applied to subjects such as ecology, engineering, chaos, finance and econometrics. From a statistical perspective, nonlinear time series also furnishes an exciting platform for nonstandard statistical inference both parametrically and nonparametrically. For a comprehensive survey on some of these recent developments, see Fan and Yao [9].

Among many different developments in nonlinear time series, estimation and testing of the threshold parameter constitute one of most challenging tasks. One of the main reasons of the difficulty arises from the fact that tricky and nonstandard asymptotic techniques are required to handle the threshold estimation, see Chan [3], Hansen [11] and [12]. A comprehensive theory for this type of problems seems to be lacking from the literature so far, however.

On the other hand, a relative complete theory for the statistical inference for diffusion processes in continuous time is available, see for example Kutoyants [14] and [15]. In particular, these two books demonstrate that both the maximum likelihood and the Bayesian approaches to diffusion processes can be put under a general context and an asymptotic theory can be developed, albeit to its non standard nature.

One of the main purposes of this paper is to make use of this general theory and apply it to the nonlinear time series context. Related contributions to continuous time ARMA and threshold ARMA models can be found, for example, in Brockwell [2], Chan and Tong [5], Stramer, Brockwell and Tweedie [17], and Tong [18] and the references therein.

Although likelihood inference for the threshold parameter of nonlinear time series was considered by Chan [3] and Hansen [11] previously, the asymptotic machineries employed were of special nature which cannot be easily generalized to other situations. For further background on likelihood tests of non-linearity, see Li and Li [16]. From a Bayesian perspective, Geweke and Teuri [10] considered a Bayesian threshold AR model and derived the posterior distribution of the threshold parameter. However, a detailed description of the asymptotic properties of the Bayesian estimator and its moment convergence were lacking.

By incorporating the developments in diffusion, this paper illustrates a general methodology to tackle both the maximum likelihood and Bayesian estimation problems from which simulations can be efficiently conducted. Moreover, the proposed approach is sufficiently transparent and can be easily adopted to other nonlinear time series context.

A second but equally important goal of this study is to develop an implementable scheme for simulating and computing the limit likelihood statistics. By linking the integral equation of the underlying invariant density of the nonlinear time series and the intensity of the limiting compound Commission process, one can compute the form of the limiting likelihood explicitly. To the best of our knowledge, this has never been conducted before and results obtained in this paper can greatly enhance the inference for the threshold parameter of a nonlinear time series and extend its applications.

This paper is organized as follows. Background introduction together with the statement of the problem and the main result are given in Section 2. Section 3 consists of simulations. Section 4 discusses the extension to cover the usual one-sided threshold setting while conclusions and possible extensions are given in Section 5.

2 Main result

Consider the model

$$X_{j+1} = \rho_1 \, X_j \, 1\!\!1_{\{|X_j|<\vartheta\}} + \rho_2 \, X_j \, 1\!\!1_{\{|X_j|\geq\vartheta\}} + \varepsilon_{j+1}, \qquad j = 0, \ldots, n-1, \tag{1}$$

where ε_j are i.i.d. $\mathcal{N}\left(0, \sigma^2\right)$, $\rho_1 \neq \rho_2$, $|\rho_1| < 1$ and $|\rho_2| < 1$. Note that model (1) appears to be different from the standard setting, where the threshold is usually partitioned as $1\!\!1_{\{X_j<\vartheta\}}$ and $1\!\!1_{\{X_j\geq\vartheta\}}$. We choose the current setting because it is more general and mathematically more convenient. Our results can be easily extended to encompass the standard setting as demonstrated in Section 4.1. We suppose that $\sigma^2 > 0, \rho_1, \rho_2$ are known and $\vartheta \in \Theta = (\alpha, \beta)$ is the unknown threshold parameter. Our goal is to estimate ϑ from observations $X^n = (X_0, X_1, \ldots, X_n)$ and to describe the asymptotic behavior of the estimators as $n \to \infty$. Recall that $(X_j)_{j\geq 1}$ is geometrically mixing (see Chen and Tsay [6]) and denote its stationary density function by $f\left(\vartheta, x\right)$, see also Fan and Yao [9].

In this paper, we consider both the maximum likelihood and Bayesian approaches. Recall that the likelihood function is written as

$$\begin{aligned} L\left(\vartheta, X^n\right) = f_0\left(X_0\right) \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\Bigg\{ -\frac{1}{2\sigma^2} \sum_{j=0}^{n-1} \Big(& X_{j+1} - \rho_1 \, X_j \, 1\!\!1_{\{|X_j|\leq\vartheta\}} \\ & -\rho_2 \, X_j \, 1\!\!1_{\{|X_j|>\vartheta\}} \Big)^2 \Bigg\}, \end{aligned}$$

and the maximum likelihood estimator (MLE) $\hat{\vartheta}_n$ is defined by the equation

$$\sup_{\vartheta\in\Theta} L\left(\vartheta, X^n\right) = \max\left[L\left(\hat{\vartheta}_n+, X^n\right), L\left(\hat{\vartheta}_n-, X^n\right)\right]. \tag{2}$$

If this equation has many solutions, we can, for example, call the MLE to be the value which is at the center of the gravity. Note that the function $L\left(\vartheta, X^n\right), \vartheta \in \Theta$ has jumps at the points

$$\vartheta_l = |X_j| \in \Theta, \qquad l = 1, \ldots, L,$$

where $L \leq n$. Clearly, if $\Theta = R$, then $L = n$.

To apply the Bayesian approach, suppose that the unknown parameter is a random variable with a known prior density $p(\theta), \theta \in \Theta$, which is continuous and positive. Using the quadratic loss function, the Bayesian estimator (which minimizes the mean squares error) is the conditional mathematical expectation

$$\tilde{\vartheta}_n = \int_\alpha^\beta \theta\, p(\theta)\, L\left(\vartheta, X^n\right)\, \mathrm{d}\theta = \frac{\int_\alpha^\beta \theta\, p(\theta)\, L\left(\vartheta, X^n\right)\, \mathrm{d}\theta}{\int_\alpha^\beta p(\theta)\, L\left(\vartheta, X^n\right)\, \mathrm{d}\theta}. \tag{3}$$

Properties of the least squares estimator (LSE) of ϑ were studied in Chan [3]. The LSE coincides with the MLE for Gaussian ε_j. We therefore recall properties of MLE and compare them with properties of the Bayesian estimators.

First, introduce the stochastic process

$$Z(u) = \begin{cases} \exp\left\{-\frac{\rho^2\vartheta^2}{2\sigma^2}\ N_+(u) - \frac{\rho\,\vartheta}{\sigma^2}\sum_{l=0}^{N_+(u)} \varepsilon_l^+\right\}, & u \geq 0, \\ \exp\left\{-\frac{\rho^2\vartheta^2}{2\sigma^2}\ N_-(-u) - \frac{\rho\,\vartheta}{\sigma^2}\sum_{l=0}^{N_-(-u)} \varepsilon_l^-\right\}, & u \leq 0, \end{cases}$$

where $N_+(\cdot)$ and $N_-(\cdot)$ are two independent Poisson processes of intensities $\lambda_+ = \lambda_- = 2f(\vartheta,\vartheta)$ ($f(\vartheta, x)$ is the stationary density function of X_j) and $\varepsilon_r^+, \varepsilon_l^-$ are independent Gaussian $\mathcal{N}\left(0, \sigma^2\right)$ random variables. It is easy to see that

$$Y_+(u) = \rho^2\,\vartheta^2\, N_+(u) + 2\rho\,\vartheta \sum_{l=0}^{N_+(u)} \varepsilon_l^+ = \sum_{l=0}^{N_+(u)} \left[\rho^2\,\vartheta^2 + 2\rho\,\vartheta\,\varepsilon_l^+\right], \qquad u \geq 0$$

$$Y_-(u) = \rho^2\,\vartheta^2\, N_-(u) + 2\rho\,\vartheta \sum_{l=0}^{N_-(u)} \varepsilon_l^- = \sum_{l=0}^{N_-(u)} \left[\rho^2\,\vartheta^2 + 2\rho\,\vartheta\,\varepsilon_l^-\right], \qquad u \geq 0$$

are compound Poisson processes.

The random process $Z(\cdot)$ is piecewise constant and as a result, the points u^* of the maximum of the process $Z(\cdot)$ is defined by

$$\sup_u Z(u) = Z(u^*)\ ,$$

where

$$\hat{u}_m < u^* < \hat{u}_M.$$

Here $\hat{u}_m$ and $\hat{u}_M$ are two consecutive events of the process $N_+(\cdot)$, or of the process $N_-(\cdot)$, or they are respectively the first event of $N_+(\cdot)$ and $N_-(\cdot)$. Simulated realizations of $Z(\cdot)$ are given in Section 3. The center of gravity of the interval is given by the point

$$\hat{u} = \frac{u_m + u_M}{2}. \tag{4}$$

Such a choice of $\hat{u}$ is explained in Section 4 below. It follows from the result of Chan [3] that the MLE $\hat{\vartheta}_n$ is consistent and

$$n\left(\hat{\vartheta}_n - \vartheta\right) \Longrightarrow \hat{u}.$$

Introduce the random variable

$$\tilde{u} = \frac{\int u\, Z(u)\, \mathrm{d}u}{\int Z(u)\, \mathrm{d}u}.$$

The main result is the following theorem.

Theorem 2.1. *The Bayesian estimator $\tilde{\vartheta}_n$ constructed by the observations X^n of the threshold autoregressive process is consistent, the normalized difference $n\left(\tilde{\vartheta}_n - \vartheta\right)$ converges in distribution :*

$$n\left(\tilde{\vartheta}_n - \vartheta\right) \Longrightarrow \tilde{u} \tag{5}$$

and for any $p > 0$

$$\lim_{n\to\infty} \mathbf{E}_\vartheta \left| n\left(\tilde{\vartheta}_n - \vartheta\right)\right|^p = \mathbf{E}_\vartheta \left|\tilde{u}\right|^p. \tag{6}$$

Proof. The proof of this theorem is based on the general result by Ibragimov and Khasminskii [13], Theorem 1.10.2. To apply it we study the normalized likelihood ratio process

$$Z_n(u) = \frac{L\left(\vartheta + \frac{u}{n}, X^n\right)}{L\left(\vartheta, X^n\right)}, \qquad u \in \mathbb{U}_n = \left[n\left(\alpha - \vartheta\right), n\left(\beta - \vartheta\right)\right],$$

where ϑ is the true value. Recall the main steps. Write the Bayesian estimator $\left(\theta_u = \vartheta + \frac{u}{n}\right)$ as

$$\begin{aligned}
\tilde{\vartheta}_n &= \frac{\int_\alpha^\beta \theta\, p(\theta)\, L\left(\theta, X^n\right)\, \mathrm{d}\theta}{\int_\alpha^\beta p(\theta)\, L\left(\theta, X^n\right)\, \mathrm{d}\theta} = \vartheta + \frac{1}{n}\frac{\int_{\mathbb{U}_n} u\, p(\theta_u)\, L\left(\theta_u, X^n\right)\, \mathrm{d}u}{\int_{\mathbb{U}_n} p(\theta_u)\, L\left(\theta_u, X^n\right)\, \mathrm{d}u} \\
&= \vartheta + \frac{1}{n}\frac{\int_{\mathbb{U}_n} u\, p(\theta_u)\, \frac{L(\theta_u, X^n)}{L(\vartheta, X^n)}\, \mathrm{d}u}{\int_{\mathbb{U}_n} p(\theta_u)\, \frac{L(\theta_u, X^n)}{L(\vartheta, X^n)}\, \mathrm{d}u} = \vartheta + \frac{1}{n}\frac{\int_{\mathbb{U}_n} u\, p(\theta_u)\, Z_n(u)\, \mathrm{d}u}{\int_{\mathbb{U}_n} p(\theta_u)\, Z_n(u)\, \mathrm{d}u}.
\end{aligned}$$

Suppose that we proved the convergence of the process $Z_n(\cdot)$ to the process $Z(\cdot)$ providing the convergence of these integrals. Then

$$n\left(\tilde{\vartheta}_n - \vartheta\right) \Longrightarrow \frac{\int u\, Z(u)\, \mathrm{d}u}{\int Z(u)\, \mathrm{d}u} = \tilde{u}.$$

This convergence together with an estimate on the large deviations of the tails of the process $Z_n(\cdot)$ allow us to prove the convergence of the moments (6).

Now check the conditions of the Theorem 1.10.2 in [13]. We need to prove

1. the convergence of the finite dimensional distributions of $Z_n(\cdot)$ to the finite dimensional distributions of $Z(\cdot)$, that is,
$$Z_n(\cdot) \to Z(\cdot) \quad \text{f.d.d.}, \tag{7}$$
2. to establish the estimate:
$$\mathbf{E}_\vartheta \left[Z_n^{1/2}(u_2) - Z_n^{1/2}(u_1) \right]^2 \leq C\, |u_2 - u_1|\,, \tag{8}$$
3. and to establish the estimate: for any $M > 0$
$$\mathbf{E}_\vartheta Z_n^{1/2}(u) \leq \frac{C_M}{|u|^M}\,. \tag{9}$$

The convergence of finite-dimensional distributions follows from the Proposition 2 of [3]. Instead of repeating a technical argument as in [3], we offer a different intuitive (but rigorous) explanation as follows. Rewrite the process (1) as

$$X_{j+1} = \rho_1\, X_j + \rho\, X_j\, 1\!\!\mathrm{I}_{\{|X_j| \geq \vartheta\}} + \varepsilon_{j+1}, \qquad j = 0, \ldots, n-1, \tag{10}$$

where we use $1\!\!\mathrm{I}_{\{|X_j| < \vartheta\}} = 1\!\!\mathrm{I} - 1\!\!\mathrm{I}_{\{|X_j| \geq \vartheta\}}$ and denote $\rho = \rho_2 - \rho_1$.

Put $Z_n(u) = \exp\left\{-\frac{1}{2\sigma^2} Y_n(u)\right\}$ and study the process $Y_n(u)$ for positive values of u.

$$\begin{aligned}
Y_n(u) &= \sum_{j=0}^{n-1} \left[\left(X_{j+1} - \rho_1\, X_j - \rho\, X_j\, 1\!\!\mathrm{I}_{\left\{|X_j| > \vartheta + \frac{u}{n}\right\}} \right)^2 \right. \\
&\qquad\qquad \left. - \left(X_{j+1} - \rho_1\, X_j - \rho\, X_j\, 1\!\!\mathrm{I}_{\{|X_j| > \vartheta\}} \right)^2 \right] \\
&= \sum_{j=0}^{n-1} \left(\rho\, X_j \left[1\!\!\mathrm{I}_{\{|X_j| > \vartheta\}} - 1\!\!\mathrm{I}_{\left\{|X_j| > \vartheta + \frac{u}{n}\right\}} \right] \right) \\
&\qquad \times \left(2X_{j+1} - 2\rho_1\, X_j - \rho\, X_j \left[1\!\!\mathrm{I}_{\{|X_j| > \vartheta\}} + 1\!\!\mathrm{I}_{\left\{|X_j| > \vartheta + \frac{u}{n}\right\}} \right] \right).
\end{aligned}$$

Note that
$$1\!\!\mathrm{I}_{\{|X_j| > \vartheta\}} - 1\!\!\mathrm{I}_{\left\{|X_j| > \vartheta + \frac{u}{n}\right\}} = 1\!\!\mathrm{I}_{\left\{\vartheta < |X_j| < \vartheta + \frac{u}{n}\right\}}.$$

Hence,

$$\begin{aligned}
Y_n(u) &= \sum_{j=0}^{n-1} \rho\, X_j \left[2X_{j+1} - 2\rho_1\, X_j - \rho\, X_j \right]\, 1\!\!\mathrm{I}_{\left\{\vartheta < |X_j| < \vartheta + \frac{u}{n}\right\}} \\
&= \sum_{j=0}^{n-1} \left(\rho^2\, X_j^2 + 2\rho\, X_j\, \varepsilon_{j+1} \right)\, 1\!\!\mathrm{I}_{\left\{\vartheta < |X_j| < \vartheta + \frac{u}{n}\right\}}.
\end{aligned} \tag{11}$$

Next introduce another process

$$Y_n^\circ(u) = \sum_{j=0}^{n-1} \left(\rho^2 \vartheta^2 + 2\rho\, \vartheta\, \mathrm{sgn}(X_j)\, \varepsilon_{j+1} \right)\, 1\!\!\mathrm{I}_{\left\{\vartheta < |X_j| < \vartheta + \frac{u}{n}\right\}} \tag{12}$$

and put $\mathbb{1}_{\left\{\vartheta<|X_j|<\vartheta+\frac{u}{n}\right\}}=\mathbb{1}_{\{\mathbb{B}_j(u)\}}$. We show that this process is asymptotically equivalent to the process $Y_n(u)$. We have

$$\begin{aligned}\mathbf{E}_\vartheta\left|Y_n(u)-Y_n^\circ(u)\right|\leq\rho^2\sum_{j=0}^{n-1}\mathbf{E}_\vartheta\left|X_j^2-\vartheta^2\right|\mathbb{1}_{\{\mathbb{B}_j(u)\}}\\+2\rho\sum_{j=0}^{n-1}\mathbf{E}_\vartheta\left|X_j-\vartheta\,\mathrm{sgn}\left(X_j\right)\right|\left|\varepsilon_{j+1}\right|\mathbb{1}_{\{\mathbb{B}_j(u)\}}.\end{aligned}$$

For the first term we write

$$\begin{aligned}\mathbf{E}_\vartheta\left|X_j^2-\vartheta^2\right|\mathbb{1}_{\{\mathbb{B}_j(u)\}}&=\int_{\vartheta\leq|x|\leq\vartheta+\frac{u}{n}}\left|x^2-\vartheta^2\right|f(\vartheta,x)\,\mathrm{d}x\\&\leq2\left(\vartheta+\frac{u}{n}\right)\left(\frac{u}{n}\right)^2\max_{\vartheta<|x|<\vartheta+\frac{u}{n}}f(\vartheta,x)\leq C\left(\frac{u}{n}\right)^2.\end{aligned}$$

The second term is (recall that X_j and ε_{j+1} are independent)

$$\begin{aligned}&\mathbf{E}_\vartheta\left|X_j-\vartheta\,\mathrm{sgn}\left(X_j\right)\right|\left|\varepsilon_{j+1}\right|\mathbb{1}_{\{\mathbb{B}_j(u)\}}\\&\qquad=\mathbf{E}_\vartheta\left|X_j-\vartheta\,\mathrm{sgn}\left(X_j\right)\right|\,\mathbb{1}_{\{\mathbb{B}_j(u)\}}\mathbf{E}_\vartheta\left|\varepsilon_{j+1}\right|\\&\qquad=\sqrt{\frac{2}{\pi}}\,\sigma\,\mathbf{E}_\vartheta\left|X_j-\vartheta\,\mathrm{sgn}\left(X_j\right)\right|\,\mathbb{1}_{\{\mathbb{B}_j(u)\}}\\&\qquad=2\sqrt{\frac{2}{\pi}}\,\sigma\int_{\vartheta\leq x\leq\vartheta+\frac{u}{n}}\left|x-\vartheta\right|f(\vartheta,x)\,\mathrm{d}x\leq C\left(\frac{u}{n}\right)^2.\end{aligned}$$

Hence,

$$\mathbf{E}_\vartheta\left|Y_n(u)-Y_n^\circ(u)\right|\leq C\,\frac{u^2}{n}\longrightarrow0$$

for any fixed u. Therefore, it is sufficient to study the limit distribution of the random function $Y_n^\circ(u)$ and to show the convergence

$$Y_n^\circ(u)\Longrightarrow Y_+(u)=\sum_{l=0}^{N_+(u)}\left[\rho^2\,\vartheta^2+2\rho\,\vartheta\,\varepsilon_l^+\right].\tag{13}$$

To see that the limit of $Y_n^\circ(u)$ is a compound Poisson process, first note that the characteristic function

$$\begin{aligned}\Phi(v)&=\mathbf{E}_\vartheta e^{\mathrm{i}vY_+(u)}=\mathbf{E}_\vartheta e^{\mathrm{i}v\sum_{l=0}^{N_+(u)}\left[\rho^2\,\vartheta^2+2\rho\,\vartheta\,\varepsilon_l^+\right]}\\&=\mathbf{E}_\vartheta\mathbf{E}_\vartheta\left(e^{\mathrm{i}v\sum_{l=0}^{N_+(u)}\left[\rho^2\,\vartheta^2+2\rho\,\vartheta\,\varepsilon_l^+\right]}\middle|\mathcal{F}_{N_+}\right)\\&=\mathbf{E}_\vartheta e^{\left[\mathrm{i}v\rho^2\,\vartheta^2-2v^2\rho^2\,\vartheta^2\,\sigma^2\right]N_+(u)}\\&=\exp\left\{u\left(e^{\mathrm{i}v\rho^2\,\vartheta^2-2v^2\rho^2\,\vartheta^2\,\sigma^2}-1\right)2f(\vartheta,\vartheta)\right\},\end{aligned}\tag{14}$$

where we denote $\mathcal{F}_{N_+}$ to be the σ-algebra related to the Poisson process and make use of the independence of ε_l^+ and $N_+(\cdot)$. The desired convergence will be proved if the convergence of the characteristic function of the process $Y_n^\circ(\cdot)$ to (14) is established.

Fix $u>0$, then as $n\to\infty$ the band $\left[\vartheta,\vartheta+\frac{u}{n}\right]$ becomes narrower and the events, when $|X_{j_l}|\in\left[\vartheta,\vartheta+\frac{u}{n}\right]$, become more rare. This means that the distance between two consecutive events $|X_{j_l}|\in\left[\vartheta,\vartheta+\frac{u}{n}\right]$ and $\left|X_{j_{l+1}}\right|\in\left[\vartheta,\vartheta+\frac{u}{n}\right]$ tends to infinity. As the process $\left(X_j\right)_{j\geq1}$ is geometrically mixing, these events become asymptotically independent. Under such circumstances, the characteristic function $\Phi_n(v)=\mathbf{E}_\vartheta e^{\mathrm{i}vY_n^\circ(u)}$ can be calculated explicitly as

$$\begin{aligned}\Phi_n(v)&=\mathbf{E}_\vartheta\left(\mathbf{E}_\vartheta\exp\left\{\sum_{j=0}^{n-1}\mathrm{i}v\left(\rho^2\vartheta^2+2\rho\,\vartheta\,\mathrm{sgn}\left(X_j\right)\,\varepsilon_{j+1}\right)\,1\!\!1_{\{\mathbb{B}_j(u)\}}\right\}\middle|\,\mathcal{F}_X\right)\\&=\mathbf{E}_\vartheta\exp\left\{\sum_{j=0}^{n-1}\left(\mathrm{i}v\rho^2\vartheta^2-2v^2\rho^2\,\vartheta^2\,\sigma^2\right)\,1\!\!1_{\{\mathbb{B}_j(u)\}}\right\}\\&=\left(\mathbf{E}_\vartheta\exp\left\{\left(\mathrm{i}v\rho^2\vartheta^2-2v^2\rho^2\,\vartheta^2\,\sigma^2\right)\,1\!\!1_{\{\mathbb{B}_1(u)\}}\right\}\right)^n.\end{aligned}$$

Further,

$$\begin{aligned}&\mathbf{E}_\vartheta\exp\left\{\left(\mathrm{i}v\rho^2\vartheta^2-2v^2\rho^2\,\vartheta^2\,\sigma^2\right)\,1\!\!1_{\{\mathbb{B}_1(u)\}}\right\}\\&\quad=\int e^{\left(\mathrm{i}v\rho^2\vartheta^2-2v^2\rho^2\,\vartheta^2\,\sigma^2\right)1\!\!1_{\{\mathbb{B}_1(u)\}}}f(\vartheta,x)\,\mathrm{d}x\\&\quad=\left(\int_{-\infty}^{-\vartheta-\frac{u}{n}}+\int_{\vartheta+\frac{u}{n}}^{\infty}+\int_{-\vartheta}^{\vartheta}\right)f(\vartheta,x)\,\mathrm{d}x\\&\qquad+\left(\int_{-\vartheta-\frac{u}{n}}^{-\vartheta}+\int_{\vartheta}^{\vartheta+\frac{u}{n}}\right)e^{\mathrm{i}v\rho^2\vartheta^2-2v^2\rho^2\,\vartheta^2\,\sigma^2}f(\vartheta,x)\,\mathrm{d}x\\&\quad=1-2\frac{u}{n}f(\vartheta,\vartheta)+2\frac{u}{n}e^{\mathrm{i}v\rho^2\vartheta^2-2v^2\rho^2\,\vartheta^2\,\sigma^2}f(\vartheta,\vartheta)+o\left(\frac{u}{n}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\ln\Phi_n(v)&=n\ln\left(1+\frac{u}{n}\left(e^{\mathrm{i}v\rho^2\vartheta^2-2v^2\rho^2\,\vartheta^2\,\sigma^2}-1\right)2f(\vartheta,\vartheta)+o\left(\frac{u}{n}\right)\right)\\&\longrightarrow u\left(e^{\mathrm{i}v\rho^2\vartheta^2-2v^2\rho^2\,\vartheta^2\,\sigma^2}-1\right)2f(\vartheta,\vartheta)=\ln\Phi(v)\ .\end{aligned}$$

That is, it coincides with (14) and as a result, (7) is proved. □

To establish conditions (8) and (9) we need the following two lemmas.

Lemma 2.1. *There exists a constant $C>0$ such that for all values $u_1,u_2\in\left(n\left(\alpha-\vartheta\right),n\left(\beta-\vartheta\right)\right)$ we have the inequality*

$$\mathbf{E}_\vartheta\left(Z_n^{1/2}(u_2)-Z_n^{1/2}(u_1)\right)^2\leq C\,\left|u_2-u_1\right|.$$

Proof. The first step is

$$\begin{aligned}\mathbf{E}_\vartheta\left(Z_n^{1/2}(u_2)-Z_n^{1/2}(u_1)\right)^2&=2-2\,\mathbf{E}_\vartheta\left[Z_n(u_2)\,Z_n(u_1)\right]^{1/2}\\&=2-2\,\mathbf{E}_{\vartheta+\frac{u_1}{n}}\left[\frac{Z_n(u_2)}{Z_n(u_1)}\right]^{1/2},\end{aligned}$$

where the measure is changed from $\mathbf{P}_\vartheta$ to $\mathbf{P}_{\vartheta+\frac{u_1}{n}}$. We have $(u_2 \geq u_1 > 0)$

$$
\begin{aligned}
&\mathbf{E}_{\vartheta+\frac{u_1}{n}} \left[\frac{Z_n\left(u_2\right)}{Z_n\left(u_1\right)}\right]^{1/2} \\
&\quad = \mathbf{E}_{\vartheta_1} \exp\left\{-\frac{1}{4\sigma^2}\sum_{j=0}^{n-1}\left[\rho^2\, X_j^2 + 2\rho\, X_j\, \varepsilon_{j+1}\right]\, \left[\mathbb{I}_{\{\mathbb{B}_j(u_2)\}} - \mathbb{I}_{\{\mathbb{B}_j(u_1)\}}\right]\right\}.
\end{aligned}
$$

Note that

$$
\mathbb{I}_{\{\mathbb{B}_j(u_2)\}} - \mathbb{I}_{\{\mathbb{B}_j(u_1)\}} = \mathbb{I}_{\left\{\vartheta+\frac{u_1}{n}\leq |X_j| \leq \vartheta+\frac{u_2}{n}\right\}} \equiv \mathbb{I}_{\{\mathbb{C}_j\}}.
$$

Further as $1 - e^{-x} \leq x$,

$$
\begin{aligned}
&\mathbf{E}_\vartheta \left|Z_n^{1/2}\left(u_2\right) - Z_n^{1/2}\left(u_1\right)\right|^2 \\
&\quad = 2 - 2\mathbf{E}_{\vartheta_1} \exp\left\{-\frac{1}{4\sigma^2}\sum_{j=0}^{n-1}\left[\rho^2\, X_j^2 + 2\rho\, X_j\, \varepsilon_{j+1}\right]\, \mathbb{I}_{\{\mathbb{C}_j\}}\right\} \\
&\quad \leq \frac{1}{2\sigma^2}\mathbf{E}_{\vartheta_1}\left\{\sum_{j=0}^{n-1}\left[\rho^2\, X_j^2 + 2\rho\, X_j\, \varepsilon_{j+1}\right]\, \mathbb{I}_{\{\mathbb{C}_j\}}\right\} \\
&\quad = \frac{n\rho^2}{2\sigma^2}\mathbf{E}_{\vartheta_1}\, X_j^2\, \mathbb{I}_{\{\mathbb{C}_j\}} = \frac{n\rho^2}{2\sigma^2}\int_{\mathbb{C}_j} x^2\, f\left(\vartheta + \frac{u_1}{n}, x\right)\, \mathrm{d}x \\
&\quad = \frac{n\rho^2}{2\sigma^2}\left(\vartheta + \frac{u_2}{n}\right)^2\, \frac{u_2 - u_1}{n}\, \left[f\left(\vartheta + \frac{u_1}{n}, \tilde{\vartheta}_+\right) + f\left(\vartheta + \frac{u_1}{n}, \tilde{\vartheta}_-\right)\right] \\
&\quad \leq C\, \left|u_2 - u_1\right|.
\end{aligned}
\tag{15}
$$

We see that (8) is fulfilled. □

Lemma 2.2. *For any $p > 0$ there exists a constant $C = C\left(p\right) > 0$ such that for all values $u \in \left(n\left(\alpha - \vartheta\right), n\left(\beta - \vartheta\right)\right)$ we have the inequality*

$$
\mathbf{E}_\vartheta Z_n^{1/2}\left(u\right) \leq \frac{C}{|u|^p}.
$$

Proof. We have to study the following expectation

$$
\mathbf{E}_\vartheta Z_n^{1/2}\left(u\right) = \mathbf{E}_\vartheta \exp\left\{-\frac{1}{4\sigma^2}\sum_{j=0}^{n-1}\left[\rho^2\, X_j^2 + 2\rho\, X_j\, \varepsilon_{j+1}\right]\, \mathbb{I}_{\{\mathbb{B}_j(u)\}}\right\}.
$$

Start with the probability

$$
\begin{aligned}
\mathbf{P}_\vartheta\left\{\ln Z_n\left(u\right) > -c\left|u\right|\right\} &= \mathbf{P}_\vartheta\left\{-\frac{1}{4\sigma^2}Y_n\left(u\right) > -c\left|u\right|\right\} \\
&= \mathbf{P}_\vartheta\left\{-\frac{1}{4\sigma^2}\sum_{j=0}^{n-1}\left[\rho^2\, X_j^2 + 2\rho\, X_j\, \varepsilon_{j+1}\right]\, \mathbb{I}_{\{\mathbb{B}_j(u)\}} > -c\left|u\right|\right\}.
\end{aligned}
$$

Write

$$
\begin{aligned}
\mathbf{P}_\vartheta\left\{-\frac{1}{8\sigma^2}Y_n\left(u\right)>-\frac{c}{2}\left|u\right|\right\}&=\mathbf{P}_\vartheta\left\{-\frac{3}{32\sigma^2}\sum_{j=0}^{n-1}\rho^2\,X_j^2\,1\!\!1_{\{\mathbb{B}_j(u)\}}\right.\\
&\qquad\left.-\frac{1}{32\sigma^2}\sum_{j=0}^{n-1}\rho^2\,X_j^2\,1\!\!1_{\{\mathbb{B}_j(u)\}}-\frac{1}{4\sigma^2}\sum_{j=0}^{n-1}\rho\,X_j\,\varepsilon_{j+1}\,1\!\!1_{\{\mathbb{B}_j(u)\}}>-\frac{c}{2}\left|u\right|\right\}\\
&\leq\mathbf{P}_\vartheta\left\{-\frac{3}{32\sigma^2}\sum_{j=0}^{n-1}\rho^2\,X_j^2\,1\!\!1_{\{\mathbb{B}_j(u)\}}>-\frac{3c}{2}\left|u\right|\right\}\\
&\quad+\mathbf{P}_\vartheta\left\{-\frac{1}{32\sigma^2}\sum_{j=0}^{n-1}\rho^2\,X_j^2\,1\!\!1_{\{\mathbb{B}_j(u)\}}-\frac{1}{4\sigma^2}\sum_{j=0}^{n-1}\rho\,X_j\,\varepsilon_{j+1}\,1\!\!1_{\{\mathbb{B}_j(u)\}}>c\left|u\right|\right\}.
\end{aligned}
$$

For the last probability, by Markov inequality we have

$$
\mathbf{P}_\vartheta\left\{-\frac{1}{32\sigma^2}\sum_{j=0}^{n-1}\rho^2\,X_j^2\,1\!\!1_{\{\mathbb{B}_j(u)\}}-\frac{1}{4\sigma^2}\sum_{j=0}^{n-1}\rho\,X_j\,\varepsilon_{j+1}\,1\!\!1_{\{\mathbb{B}_j(u)\}}>c\left|u\right|\right\}\leq e^{-c|u|}
$$

because

$$
\mathbf{E}_\vartheta\exp\left\{-\frac{1}{32\sigma^2}\sum_{j=0}^{n-1}\rho^2\,X_j^2\,1\!\!1_{\{\mathbb{B}_j(u)\}}-\frac{1}{4\sigma^2}\sum_{j=0}^{n-1}\rho\,X_j\,\varepsilon_{j+1}\,1\!\!1_{\{\mathbb{B}_j(u)\}}\right\}=1.
$$

The last equality follows from the following property of the conditional expectation

$$
\begin{aligned}
&\mathbf{E}_\vartheta\left(\exp\left\{-\frac{\rho^2}{32\sigma^2}X_{n-1}^2\,1\!\!1_{\{\mathbb{B}_{n-1}(u)\}}-\frac{\rho}{4\sigma^2}\,X_{n-1}\,\varepsilon_n\,1\!\!1_{\{\mathbb{B}_{n-1}(u)\}}\right\}\middle|\,\mathcal{F}_{n-1}\right)\\
&=\exp\left\{-\frac{\rho^2X_{n-1}^2\,1\!\!1_{\{\mathbb{B}_{n-1}(u)\}}}{32\sigma^2}\right\}\mathbf{E}_\vartheta\left(\exp\left\{-\frac{\rho X_{n-1}\,1\!\!1_{\{\mathbb{B}_{n-1}(u)\}}}{4\sigma^2}\varepsilon_n\right\}\middle|\,\mathcal{F}_{n-1}\right)=1.
\end{aligned}
$$

Hence, it is sufficient to study the probability

$$
\mathbf{P}_\vartheta\left\{-\sum_{j=0}^{n-1}X_j^2\,1\!\!1_{\{\mathbb{B}_j(u)\}}>-c_1\,\left|u\right|\right\},
$$

where $c_1=16\,c\,\sigma^2\,\rho^{-2}$.

Fix some $\kappa\in(0,1)$ and consider first the *local values* u satisfying the condition $|u|<\,n^\kappa$.

Suppose that $u > 0$ (for $u < 0$ the consideration is similar). Then we have

$$
\begin{aligned}
&\mathbf{P}_\vartheta \left\{ \sum_{j=0}^{n-1} \left[\vartheta^2 - X_j^2\right] \mathbb{1}_{\{\mathbb{B}_j(u)\}} - \vartheta^2 \sum_{j=0}^{n-1} \mathbb{1}_{\{\mathbb{B}_j(u)\}} \geq -c_1\, u \right\} \\
&\quad = \mathbf{P}_\vartheta \left\{ \sum_{j=0}^{n-1} \left[X_j^2 - \vartheta^2\right] \mathbb{1}_{\{\mathbb{B}_j(u)\}} + \vartheta^2 \sum_{j=0}^{n-1} \mathbb{1}_{\{\mathbb{B}_j(u)\}} \leq c_1\, u \right\} \\
&\quad \leq \mathbf{P}_\vartheta \left\{ \sum_{j=0}^{n-1} \mathbb{1}_{\{\mathbb{B}_j(u)\}} \leq c_2\, u \right\}, \qquad c_2 = \frac{c_1}{\vartheta^2},
\end{aligned}
$$

because $\left[X_j^2 - \vartheta^2\right] \mathbb{1}_{\{\mathbb{B}_j(u)\}} \geq 0$. Recall that the sum $\sum_{j=0}^{n-1} \mathbb{1}_{\{\mathbb{B}_j(u)\}}$ converges to the Poisson process of intensity $\lambda = 2f\left(\vartheta, \vartheta\right)$, hence the last probability has to be small for the values $c_2 < \lambda$. Let $Y_j = \mathbb{1}_{\{\mathbb{B}_j(u)\}} - \mathbf{E}_\vartheta \mathbb{1}_{\{\mathbb{B}_j(u)\}}$ and note that

$$
\begin{aligned}
\mathbf{E}_\vartheta \mathbb{1}_{\{\mathbb{B}_j(u)\}} &= \int_{\vartheta \leq |x| \leq \vartheta + \frac{u}{n}} f\left(\vartheta, x\right)\, \mathrm{d}x = \frac{u}{n} \left[f\left(\vartheta, -\tilde{\vartheta}_1\right) + f\left(\vartheta, \tilde{\vartheta}_2\right) \right] \\
&= 2\, \frac{u}{n}\, f\left(\vartheta, \vartheta\right) \left(1 + o\left(1\right)\right) \geq \frac{u}{n}\, f\left(\vartheta, \vartheta\right)\,,
\end{aligned}
$$

where the last inequality holds for $n \geq n_1$. Recall that the function $f\left(\vartheta, x\right)$ is even and $f\left(\vartheta, -\vartheta\right) = f\left(\vartheta, \vartheta\right)$. Further,

$$
\begin{aligned}
\mathbf{P}_\vartheta \left\{ \sum_{j=0}^{n-1} \mathbb{1}_{\{\mathbb{B}_j(u)\}} \leq c_2\, u \right\} &\leq \mathbf{P}_\vartheta \left\{ \left| \sum_{j=0}^{n-1} Y_j \right| \geq \left(f\left(\vartheta, \vartheta\right) - c_2 \right)\, u \right\} \\
&= \mathbf{P}_\vartheta \left\{ \left| \sum_{j=0}^{n-1} Y_j \right| \geq c_3\, u \right\} \leq \frac{\mathbf{E}_\vartheta \left| \sum_{j=0}^{n-1} Y_j \right|^{2p}}{c_3^{2p}\, u^{2p}},
\end{aligned}
$$

where we chose such c that $c_3 = f\left(\vartheta, \vartheta\right) - 16 c \sigma^2 \rho^{-2} \vartheta^{-2} > 0$. To estimate the last expectation we apply the inequality of Dedeker and Doukhan (see (8.1) in [7]):

$$
\mathbf{E}_\vartheta \left| \sum_{j=0}^{n-1} Y_j \right|^{2p} \leq \left(4\, p\, n \sum_{j=0}^{n-1} \left[\mathbf{E}_\vartheta \left| Y_0\;\; \mathbf{E}_\vartheta \left(Y_j \middle| \mathcal{F}_0 \right) \right|^p \right]^{\frac{1}{p}} \right)^p . \tag{16}
$$

Write

$$
\mathbf{E}_\vartheta \left(Y_j \middle| \mathcal{F}_0 \right) = \mathbf{E}_\vartheta \left(\mathbf{E}_\vartheta \left(Y_j \middle| \mathcal{F}_{j-1} \right) \middle| \mathcal{F}_0 \right)
$$

and let $g(x)=\rho_1\, x\mathbb{1}_{\{|x|<\vartheta\}}+\rho_2\, x\mathbb{1}_{\{|x|\geq\vartheta\}}$, then

$$\begin{aligned}
\mathbf{E}_\vartheta\left(Y_j\middle|\,\mathcal{F}_{j-1}\right) &= \mathbf{E}_\vartheta\left(\mathbb{1}_{\{\mathbb{B}_j(u)\}}\middle|\,\mathcal{F}_{j-1}\right)-\mathbf{E}_\vartheta\mathbb{1}_{\{\mathbb{B}_j(u)\}}\\
&= \mathbf{E}_\vartheta\left(\mathbb{1}_{\left\{\vartheta\leq|g(X_{j-1})+\varepsilon_j|\leq\vartheta+\frac{u}{n}\right\}}\middle|\,\mathcal{F}_{j-1}\right)-\mathbf{E}_\vartheta\mathbb{1}_{\{\mathbb{B}_j(u)\}}\\
&= \int_{\vartheta\leq|g(X_{j-1})+x|\leq\vartheta+\frac{u}{n}}\varphi(x)\,\mathrm{d}x+\int_{\vartheta\leq|x|\leq\vartheta+\frac{u}{n}}f(\vartheta,x)\,\mathrm{d}x\\
&= \frac{u}{n}\left[\varphi\left(\vartheta-g(X_{j-1})+\frac{\tilde u}{n}\right)+\varphi\left(\vartheta+g(X_{j-1})-\frac{\tilde{\tilde u}}{n}\right)\right]\\
&\quad+\frac{u}{n}\left[f\left(\vartheta,\vartheta+\frac{\bar u}{n}\right)+f\left(\vartheta,\vartheta-\frac{\bar{\bar u}}{n}\right)\right]=\frac{u}{n}\,A(X_{j-1}),
\end{aligned}$$

where $\varphi(\cdot)$ is the density function of the Gaussian r.v. ε_j, i.e., $\varphi(\cdot)\sim\mathcal{N}\left(0,\sigma^2\right))$. Note that the function $A(x)$ defined by the last equality is bounded and $\mathbf{E}_\vartheta A(X_{j-1})=0$.

Hence

$$\mathbf{E}_\vartheta\left(Y_j\middle|\,\mathcal{F}_0\right)=\frac{u}{n}\;\mathbf{E}_\vartheta\left(A(X_{j-1})\middle|\,\mathcal{F}_0\right)$$

and

$$\begin{aligned}
\sum_{j=0}^{n-1}\left[\mathbf{E}_\vartheta\left|Y_0\;\mathbf{E}_\vartheta\left(Y_j\middle|\,\mathcal{F}_0\right)\right|^p\right]^{\frac{1}{p}} &= \frac{u}{n}\sum_{j=0}^{n-1}\left[\mathbf{E}_\vartheta\left|Y_0\;\mathbf{E}_\vartheta\left(A(X_{j-1})\middle|\,\mathcal{F}_0\right)\right|^p\right]^{\frac{1}{p}}\\
\leq C\,\frac{u}{n}\sum_{j=0}^{n-1}\alpha(j-1) &= C\,\frac{u}{n}\sum_{j=0}^{n-1}\gamma^{j-1}\leq C\,\frac{u}{n},
\end{aligned}$$

where we used the geometrical ergodicity of $(X_j)_{j\geq1}$: $\alpha(j)\leq\gamma^j$, $0<\gamma<1$ (see [6]) and the inequality of Ibragimov

$$\begin{aligned}
&\left\|\mathbf{E}_\vartheta\left(A(X_j)\middle|\,\mathcal{F}_0\right)-\mathbf{E}_\vartheta A(X_j)\right\|_p\\
&\qquad\leq C\left\|\mathbf{E}_\vartheta\left(A(X_j)\middle|\,\mathcal{F}_0\right)-\mathbf{E}_\vartheta A(X_j)\right\|_1^{1/p}\leq C\,\alpha(j)^{1/p}
\end{aligned}$$

(see Bradley [1], Theorem 4.4, (a2)).

Finally, we obtain (for $|u|\leq n^\kappa$) the estimate

$$\mathbf{P}_\vartheta\left\{\sum_{j=0}^{n-1}\mathbb{1}_{\{\mathbb{B}_j(u)\}}\leq c_2\,|u|\right\}\leq\frac{C}{|u|^p}\,.$$

Consider now the case $|u|>n^\kappa$. Of course, $|u|\leq(\beta-\alpha)\,n$. We have

$$\begin{aligned}
&\mathbf{P}_\vartheta\left\{-\sum_{j=0}^{n-1}\mathbb{1}_{\{\mathbb{B}_j(u)\}}>-c_1\,|u|\right\}\\
&\qquad\leq\mathbf{P}_\vartheta\left\{\left|\sum_{j=0}^{n-1}\left(X_j^2\,\mathbb{1}_{\{\mathbb{B}_j(u)\}}-\mathbf{E}_\vartheta X_j^2\,\mathbb{1}_{\{\mathbb{B}_j(u)\}}\right)\right|\geq c_1\,|u|\right\},
\end{aligned}\tag{17}$$

because

$$\mathbf{E}_\vartheta X_j^2\,\mathbb{1}_{\{\mathbb{B}_j(u)\}}=\int_{\vartheta<|x|<\vartheta+\frac{|u|}{n}}x^2f(\vartheta,x)\,\mathrm{d}x\geq\alpha^2\,\frac{|u|}{n}\inf_{\alpha\leq x\leq\beta}f(\vartheta,x)\geq\frac{c_4}{n}\,|u|\,,$$

and the constant c is chosen such that $c_4>2c_1$. The last probability in (17) can be estimated with the help of the following lemma.

Lemma 2.3. (Rosenthal's moment inequality) *Let $(Z_j)_{j\geq1}$ be zero mean mixing series satisfying the condition: there exist $\varepsilon>0$ and $c\in2\mathbb{N}$, $c>2p>2$, such that*

$$\sum_{r=1}^{\infty}(r+1)^{c-2}\left[\alpha(r)\right]^{\frac{\varepsilon}{c+\varepsilon}}<\infty,\tag{18}$$

where $\alpha(r)$ is the α-mixing coefficient, then

$$\mathbf{E}\left|\sum_{j=1}^{n}Z_j\right|^{2p}\leq C\left[n\,\left(\mathbf{E}\left|Z_1\right|^{2p+\varepsilon}\right)^{\frac{2p}{2p+\varepsilon}}+n^p\left(\mathbf{E}Z_1^{2+\varepsilon}\right)^{\frac{2p}{2+\varepsilon}}\right].\tag{19}$$

For the proof see [8], p.26.

As the process X_j is geometrically mixing (see [9], Theorem 2.4), hence condition (18) is fulfilled with any $c>0$ and $\varepsilon>0$. We apply (19) with

$$Z_j=X_j^2\,\mathbb{1}_{\{\mathbb{B}_j(u)\}}-\mathbf{E}_\vartheta X_j^2\,\mathbb{1}_{\{\mathbb{B}_j(u)\}}.$$

Obviously, $e_n(u)\equiv\mathbf{E}_\vartheta X_j^2\,\mathbb{1}_{\{\mathbb{B}_j(u)\}}\leq\beta^2$ and $e_n(u)\leq C\,\frac{|u|}{n}$. We suppose for simplicity that $u>0$,

$$\begin{aligned}\mathbf{E}_\vartheta\left|Z_1\right|^{2p+\varepsilon}&=\int_{\vartheta<|x|<\vartheta+\frac{u}{n}}\left|x^2-e_n(u)\right|^{2p+\varepsilon}f(\vartheta,x)\,\mathrm{d}x\\&\quad+e_n(u)^{2p+\varepsilon}\left(1-\int_{\vartheta<|x|<\vartheta+\frac{u}{n}}f(\vartheta,x)\,\mathrm{d}x\right)\\&\leq C_1\,\frac{u}{n}+C_2\left|\frac{u}{n}\right|^{2p+\varepsilon}\end{aligned}$$

and similarly

$$\mathbf{E}_\vartheta\left|Z_1\right|^{2+\varepsilon}\leq C_3\,\frac{u}{n}+C_4\left|\frac{u}{n}\right|^{2+\varepsilon}.$$

Hence,

$$\begin{aligned}&\mathbf{E}_\vartheta\left|\sum_{j=0}^{n-1}\left(X_j^2\,\mathbb{1}_{\{\mathbb{B}_j(u)\}}-\mathbf{E}_\vartheta X_j^2\,\mathbb{1}_{\{\mathbb{B}_j(u)\}}\right)\right|^{2p}\\&\qquad\leq C\,\left(n\left|\frac{u}{n}\right|^{\frac{2p}{2p+\varepsilon}}+n\left|\frac{u}{n}\right|^{2p}+n^p\left|\frac{u}{n}\right|^{\frac{2p}{2+\varepsilon}}+n^p\left|\frac{u}{n}\right|^{2p}\right)\\&\qquad C\,\left(n^{\frac{\varepsilon}{2p+\varepsilon}}\left|u\right|^{1-\frac{\varepsilon}{2p+\varepsilon}}+n^{1-2p}\left|u\right|^{2p}+n^{\frac{p\varepsilon}{2+\varepsilon}}\left|u\right|^{p-\frac{\varepsilon}{2+\varepsilon}}+\frac{\left|u\right|^{2p}}{n^p}\right)\\&\qquad\leq C\,\left(\left|u\right|^{\frac{\varepsilon}{\kappa(2p+\varepsilon)}+1-\frac{\varepsilon}{2p+\varepsilon}}+\left|u\right|^{\frac{1-2p}{\kappa}+2p}+\left|u\right|^{\frac{p\varepsilon}{\kappa(2+\varepsilon)}+p-\frac{p\varepsilon}{2+\varepsilon}}+\left|u\right|^{p}\right)\leq C\,\left|u\right|^{p},\end{aligned}$$

where we have used the relations $n < |u|^{1/\kappa}$, $|u| \leq (\beta - \alpha)\, n$ and have chosen sufficiently small ε (or sufficiently large p).

By Chebyshev inequality

$$\mathbf{P}_\vartheta \left\{ \left| \sum_{j=0}^{n-1} \left(X_j^2 \, \mathbb{1}_{\{\mathbb{B}_j(u)\}} - \mathbf{E}_\vartheta X_j^2 \, \mathbb{1}_{\{\mathbb{B}_j(u)\}} \right) \right| \geq c_1 \, |u| \right\} \leq \frac{C}{|u|^p}.$$

The estimates obtained above allow us to write the following expression: for any $p > 1$ and all $u \in (n\,(\alpha - \vartheta)\,, n\,(\beta - \vartheta))$, there exist constants $c > 0$ and $C > 0$ such that

$$\mathbf{P}_\vartheta \left\{ Z_n\,(u) > e^{-c|u|} \right\} \leq \frac{C}{|u|^p}. \tag{20}$$

For the expectation, note that

$$\begin{aligned} \mathbf{E}_\vartheta Z_n^{1/2}\,(u) &= \mathbf{E}_\vartheta Z_n^{1/2}\,(u)\, \mathbb{1}_{\left\{ Z_n^{1/2}(u) \geq e^{-\frac{c}{2}|u|} \right\}} + \mathbf{E}_\vartheta Z_n^{1/2}\,(u)\, \mathbb{1}_{\left\{ Z_n^{1/2}(u) < e^{-\frac{c}{2}|u|} \right\}} \\ &\leq \left(\mathbf{E}_\vartheta Z_n\,(u)\, \mathbf{P}_\vartheta \left\{ Z_n\,(u) > e^{-c|u|} \right\} \right)^{1/2} + e^{-\frac{c}{2}|u|} \leq \frac{C}{|u|^{p/2}}. \end{aligned}$$

Recall that this estimate is valid for any $p > 1$, hence (9) is verified. Therefore the required conditions are fulfilled and the Bayesian estimate satisfied all of the properties stipulated in Theorem 1 (see Theorem 1.10.2, [13]). □

3 Simulations

We obtain the density functions of limit distributions of the MLE and Bayesian estimators by the following simulations. The limit likelihood ratio is

$$Z\,(u) = \exp\left\{ -\frac{\rho^2 \vartheta^2}{2\sigma^2}\, N_+\,(u) - \frac{\rho\vartheta}{\sigma^2} \sum_{l=0}^{N_+(u)} \varepsilon_l^+ \right\}.$$

for $u \geq 0$ and

$$Z\,(u) = \exp\left\{ -\frac{\rho^2 \vartheta^2}{2\sigma^2}\, N_-\,(-u) - \frac{\rho\vartheta}{\sigma^2} \sum_{l=0}^{N_-(-u)} \varepsilon_l^- \right\}.$$

for $u \leq 0$. Here $N_+\,(\cdot)$ and $N_-\,(\cdot)$ are independent Poisson processes of intensity $\lambda = 2f\,(\vartheta, \vartheta)$ and the Gaussian random variables ε_l^+, $\varepsilon_l^- \sim \mathcal{N}\left(0, \sigma^2\right)$ are independent, $\varepsilon_0^\pm = 0$.

Denote

$$\gamma = \frac{(\rho_2 - \rho_1)\,\vartheta}{\sigma}, \qquad \epsilon_l^\pm = \frac{-\varepsilon_l^\pm}{\sigma}, \qquad u = \frac{v}{\lambda}, \quad \nu_\pm\,(v) = N_\pm \left(\frac{v}{\lambda} \right).$$

Then the Poisson processes $\nu_+\,(v)\,, v \geq 0$ and $\nu_-\,(v)\,, \geq 0$ have intensity 1 and the limit likelihood ratio

$$Z_\gamma\,(v) = \begin{cases} \exp\left\{ \gamma \sum_{l=0}^{\nu_-(-v)} \epsilon_l^- - \frac{\gamma^2}{2} \nu_-\,(-v) \right\}, & \text{if } v \leq 0, \\ \exp\left\{ \gamma \sum_{l=0}^{\nu_+(v)} \epsilon_l^+ - \frac{\gamma^2}{2} \nu_+\,(v) \right\}, & \text{if } v > 0. \end{cases}$$

Now the limit process $Z_\gamma(v)$ only depends on one parameter (γ) and the limit random variables $\hat{u}$ and $\tilde{u}$ can be written as

$$\hat{u} = \frac{\hat{u}_\gamma}{\lambda}, \quad \tilde{u} = \frac{\tilde{u}_\gamma}{\lambda}, \qquad \tilde{u}_\gamma = \frac{\int_{-\infty}^{\infty} v\, Z_\gamma(v)\, \mathrm{d}v}{\int_{-\infty}^{\infty} Z_\gamma(v)\, \mathrm{d}v},$$

in obvious notation.

The next problem is to find the function $f(\vartheta, x)$, where $f(\vartheta, x)$ is the stationary density function of X_j. As

$$X_{j+1} = \rho_1 X_j + \rho X_j \, 1\!\!1_{\{|X_j| > \vartheta\}} + \varepsilon_{j+1}$$

where X_j and ε_{j+1} are independent, we obtain the convolution equation

$$f(\vartheta, y) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} g(\vartheta, x)\, e^{-1/2\sigma^2 (y-x)^2} \mathrm{d}x\,.$$

Herein, we denote the density function of $\rho_1 X_j + \rho X_j \, 1\!\!1_{\{|X_j| > \vartheta\}}$ by $g(\vartheta, x)$. This density can be expressed as a function of $f(\cdot)$, which is a solution to a corresponding integral equation, see also Chan and Tong [4]. Specifically, observe that

$$g(\vartheta, x) = \frac{1}{\rho_1} f\left(\vartheta, \frac{x}{\rho_1}\right) 1\!\!1_{\left\{\frac{|x|}{\rho_1} < \vartheta\right\}} + \frac{1}{\rho_2} f\left(\vartheta, \frac{x}{\rho_2}\right) 1\!\!1_{\left\{\frac{|x|}{\rho_2} \geq \vartheta\right\}}.$$

Hence, the integral equation is

$$\begin{aligned} f(\vartheta, y) &= \int_{-\infty}^{\infty} \left[\frac{1}{\rho_1} f\left(\vartheta, \frac{x}{\rho_1}\right) 1\!\!1_{\left\{\frac{|x|}{\rho_1} < \vartheta\right\}} \right. \\ &\qquad \left. + \frac{1}{\rho_2} f\left(\vartheta, \frac{x}{\rho_2}\right) 1\!\!1_{\left\{\frac{|x|}{\rho_2} \geq \vartheta\right\}}\right] \varphi(y - x)\, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} f(\vartheta, x) \left[\varphi(y - x\rho_1)\, 1\!\!1_{\{|x| < \vartheta\}} + \varphi(y - x\rho_2)\, 1\!\!1_{\{|x| \geq \vartheta\}}\right] \mathrm{d}x. \end{aligned} \tag{21}$$

Solution to this equation at the point ϑ is the intensity $\lambda = 2f(\vartheta, \vartheta)$ of the corresponding Poisson processes. Therefore the value $f(\vartheta, \vartheta)$ satisfies the integral equation

$$f(\vartheta, \vartheta) = \int_{-\infty}^{\infty} f(\vartheta, x) \left[\varphi(\vartheta - x\rho_1)\, 1\!\!1_{\{|x| < \vartheta\}} + \varphi(\vartheta - x\rho_2)\, 1\!\!1_{\{|x| \geq \vartheta\}}\right] \mathrm{d}x\,,$$

where $\varphi(\cdot)$ is Gaussian $\mathcal{N}\left(0, \sigma^2\right)$ density. We see that $f(\vartheta, \vartheta) > 0$. To visualize the properties of the sample path $Z(u)$ and the invariant density $f(\theta, \vartheta)$, we conduct a simulation experiment by taking $\varepsilon_l^{\pm}$ to be i.i.d. standard normal random variables. The parameters used are $\vartheta = 2, \rho_1 = 0.15, \rho_2 = 0.95, \sigma = 1$ and $\lambda = 0.5$. A Gaussian kernel is used to estimate the form of $f(\theta, \vartheta)$ based on 50,000 observations of X_t generated from model (10) with ε_t being i.i.d. $\mathcal{N}(0, 1)$ random variables. The plots of $Z(u)$ and $f(\theta, \vartheta)$ are given in Figures 1 and 2 respectively.

Figure 1.
Figure 2.

For the maximum likelihood estimate, note that the maximum values of $Z(u)$ form an interval $[\hat{u}_m, \hat{u}_M]$ with length $|\hat{u}_M - \hat{u}_m| = \eta$, where η is an exponential random variable with probability density $2 f_\vartheta e^{-2 f_\vartheta x}, x \geq 0,\ f_\vartheta = f(\vartheta, \vartheta)$. We can take any value of u from this interval, the middle point $\hat{u} = \frac{\hat{u}_M + \hat{u}_m}{2}$, say. To have its density function we only need to simulate the exponential and the Gaussian independent random variables which will generate $\hat{u}_1, \ldots \hat{u}_N$. The historgram of $\hat{u}$ based on 20,000 simulated values of $\hat{u}$ is plotted in Figure 3. As can be seen clearly, the MLE performs reasonably well and converges to zero very fast. The sample mean of the simulated $\hat{u}$ is 0.01 with a standard deviation 0.033.

For the Bayesian estimators we first calculate the integral

$$
\begin{aligned}
J_+ = \int_0^\infty u\, Z(u)\, \mathrm{d}u &= \sum_{l=0}^{\infty} \int_{u_l}^{u_{l+1}} u\, e^{-\frac{\rho^2 \vartheta^2}{2\sigma^2}\, l - \frac{\rho\vartheta}{\sigma^2} \sum_{r=0}^{l} \varepsilon_r} \mathrm{d}u \\
&= \frac{1}{2} \sum_{l=0}^{\infty} e^{-\frac{\rho^2 \vartheta^2}{2\sigma^2}\, l - \frac{\rho\vartheta}{\sigma^2} \sum_{r=0}^{l} \varepsilon_r} \left(u_{l+1}^2 - u_l^2\right).
\end{aligned}
$$

Here $\varepsilon_r \sim \mathcal{N}\left(0, \sigma^2\right)$ and $u_l = \sum_{r=0}^{l} \eta_r$. By a similar way we have

$$
I_+ = \int_0^\infty Z(u)\, \mathrm{d}u = \sum_{l=0}^{\infty} e^{-\frac{\rho^2 \vartheta^2}{2\sigma^2}\, l - \frac{\rho\vartheta}{\sigma^2} \sum_{r=0}^{l} \varepsilon_r} \left(u_{l+1} - u_l\right).
$$

The limit random variable is

$$
\tilde{u} = \frac{J_- + J_+}{I_- + I_+},
$$

with obvious notation. To understand the behavior of the Bayesian estimator, we simulate the Bayesian estimator for 20,000 times with the histogram of $\tilde{u}$ given in Figure 4. From this figure, it is clearly seen that the Bayesian estimator converges to the expected value zero. The sample mean is -0.0026 with a standard deviation 0.028. It is interesting to see that this simulation results are consistent with the theory that the limit variances of the MLE and BE

$$
d_{MLE}^2 \sim \frac{1}{N} \sum_{q=1}^{N} \hat{u}_q^2 \quad \text{and} \quad d_{BE}^2 \sim \frac{1}{N} \sum_{q=1}^{N} \tilde{u}_q^2
$$

satisfy

$$
d_{MLE}^2 > d_{BE}^2.
$$

Note that it follows from the symmetry of the limit process, the random variables $\hat{u}$ and $\tilde{u}$ satisfy $\mathbf{E}_\vartheta \hat{u} = 0 = \mathbf{E}_\vartheta \tilde{u}$.

For 20,000 simulated estimators, we obtain the limit variances as $d_{MLE}^2 = 22.83 \pm 0.68$ and $d_{BE}^2 = 16.79 \pm 0.39$. These values concur with the theoretical results that the Bayesian estimator outperforms the MLE.

Figure 3.
Figure 4.

To examine the finite sample performance of the test statistics, we computed the critical values of the limit distributions based on the MLE and the BE using the same set of parameters as given in Figures 3. The sizes are chosen for commonly used test statistics and the

	0.025	0.05	0.075	0.1	0.90	0.925	0.95	0.975
MLE	-9.66	-6.64	-5.28	-4.46	4.46	5.38	6.87	9.84
BE	-8.44	-6.29	-5.07	-4.27	4.21	5.09	6.26	8.43
Simulated Values	-9.70	-6.88	-5.48	-4.64	4.90	5.85	7.55	10.28

Table 1: Critical values for $\lambda = 0.5$, $\theta = 2$, $\sigma = 1$, $\rho_1 = 0.95$ and $\rho_2 = 0.15$.

limiting values are given in the first two rows of Table 1. As can be seen, both the MLE and BE procedures perform reasonably well and are in close agreement. Furthermore, the numbers in the last row of Table 1 are the critical values computed from the test statistics in (5), which are directly simulated from model (1) using the same set of parameters. It is seen that the critical values generated from the simulated statistics agree remarkably well with the critical values computed from the limit distributions in Table 1 based on MLE. In summary, Table 1 demonstrates the usefulness of the limit distributions in computing the critical values. If one needs to conduct a test for another set of parameters, then a similar table can be computed and the programming code is available from the authors upon request.

4 One-sided threshold

In the nonlinear time series literature, the threshold AR model usually takes the form (see, e.g., [3], [11] and [18])

$$X_{j+1} = \rho_1 X_j \mathbb{I}_{\{X_j < \vartheta\}} + \rho_2 X_j \mathbb{I}_{\{X_j \geq \vartheta\}} \varepsilon_{j+1} \tag{22}$$

Note that the study of these one-sided threshold models is no more complicated than (1) because the log-likelihood ratio

$$\ln Z_n(u) = \frac{L\left(\vartheta + \frac{u}{n}, X^n\right)}{L(\vartheta, X^n)} = -\frac{1}{2\sigma^2} Y_n(u)$$

depends on the stochastic process (for fixed $u > 0$)

$$Y_n(u) = \sum_{j=0}^{n-1} \left(\rho^2 X_j^2 + 2\rho X_j \varepsilon_{j+1}\right) \mathbb{I}_{\left\{\vartheta < X_j < \vartheta + \frac{u}{n}\right\}}$$

(see (11)), which is approximated by the process

$$Y_n^\circ(u) = \sum_{j=0}^{n-1} \left(\rho^2 \vartheta^2 + 2\rho\vartheta\varepsilon_{j+1}\right) \mathbb{I}_{\left\{\vartheta < X_j < \vartheta + \frac{u}{n}\right\}}.$$

Here we use the same notations as before and add the condition that $|\rho_1| < 1$. Comparison with (12) shows that the factor $\operatorname{sgn}(X_j)$ no longer exists and this simplifies matters much in the application of the limit theorems. Specifically, similar to (13), the corresponding limit for $Y_n^\circ(u)$ becomes

$$Y_n^\circ(u) \Longrightarrow Y_+(u) = \sum_{l=0}^{N_+(u)} \left[\rho^2\vartheta^2 + 2\rho\vartheta\varepsilon_l^+\right],$$

the only difference is: instead of $2f(\vartheta,\theta)$, the intensity of the Poisson process $N_+(u)$ is $\lambda_+ = f(\vartheta,\vartheta)$.

The inequalities for the process $Z_n(u)$ obtained in Lemmas 2.1 and 2.2 can be obtained for $Z_n(u)$ of the process (22) exactly the same way as in this paper. Consequently, the asymptotic behavior of the Bayesian estimator $\tilde{\vartheta}_n$ for model (22) is the same as that described in the Theorem 2.1 with the slightly difference due to the form of the limit likelihood ratio $Z(u)$, where the intensity of the Poisson process is now $f(\vartheta,\vartheta)$, not $2f(\vartheta,\theta)$.

5 Discussion

Let us explain heuristically why the choice (4) for the MLE is better than other types of the form $\hat{u}_\gamma = \gamma u_m + (1-\gamma) u_M$ with $\gamma \neq 1/2, \gamma \in [0,1]$. The interval $[u_m, u_M]$ can be on the positive, negative parts of R or it can be $\left[u_1^-, u_1^+\right]$, where u_1^- and u_1^+ are the first event of the Poisson processes $N_-(\cdot)$ and $N_+(\cdot)$ respectively. If $[u_m, u_M] = \left[u_1^-, u_1^+\right]$, then the random variables $-u_1^- = \zeta_1$ and $u_1^+ = \zeta_2$ are independent exponential with the parameter $\lambda = 2f(\vartheta,\vartheta)$.

If this interval is on the negative part, then $\left[u_{\hat{l}+1}^-, u_{\hat{l}}^-\right]$ (u_l^- is the l-th event of the Poisson process $N_-(\cdot)$) has random length $\hat{l}$ and

$$\hat{u}_\gamma = \gamma u_{\hat{l}+1}^- + (1-\gamma) u_{\hat{l}}^- = \gamma\left(u_{\hat{l}}^- - \zeta\right) + (1-\gamma) u_{\hat{l}}^- = u_{\hat{l}}^- - \gamma\zeta,$$

where ζ is exponential random variable with parameter λ. For the positive part

$$\hat{u}_\gamma = \gamma u_{\hat{l}}^+ + (1-\gamma) u_{\hat{l}+1}^+ = \gamma u_{\hat{l}}^+ + (1-\gamma)\left(u_{\hat{l}}^+ + \zeta\right) = u_{\hat{l}}^+ + (1-\gamma)\zeta.$$

Denote p_0 to be the probability that the maximum of the random process $Z(\cdot)$ is on the interval $\left[u_1^-, u_1^+\right]$. The positive and negative intervals are equiprobable, hence their probabilities p_-, p_+ satisfy the relations $p_- = p_+ = (1-p_0)/2 \equiv p$. We then write

$$\begin{aligned}\mathbf{E}_\vartheta \hat{u}_\gamma^2 &= p_-\,\mathbf{E}_\vartheta\left(u_{\hat{l}}^- - \gamma\zeta\right)^2 + p_0\,\mathbf{E}_\vartheta\left(\gamma\zeta_1 - (1-\gamma)\zeta_2\right)^2 \\ &\quad + p_+\,\mathbf{E}_\vartheta\left(u_{\hat{l}}^+ + (1-\gamma)\zeta\right)^2 = 2p\,\mathbf{E}_\vartheta\left(u_{\hat{l}}^+\right)^2 + 2p\,\mathbf{E}_\vartheta\left(u_{\hat{l}}^+\right)\,\mathbf{E}_\vartheta\zeta \\ &\quad + \frac{2p}{\lambda^2}\left(2\gamma^2 - 2\gamma + 1\right) + \frac{2p_0}{\lambda^2}\left(4\gamma^2 - 4\gamma + 1\right)\end{aligned}$$

and direct calculations show

$$\min_{\gamma\in[0,1]} \mathbf{E}_\vartheta\left(\hat{u}_\gamma\right)^2 = \mathbf{E}_\vartheta\left(\hat{u}\right)^2.$$

Note that in this problem of parameter estimation, it is possible to introduce the notion of asymptotic efficiency of estimators. The lower bound on the risk of all estimators $\bar{\vartheta}_n$ for the quadratic loss function is as follows:

$$\varliminf_{\delta\to 0}\varliminf_{n\to\infty}\sup_{|\vartheta-\vartheta_0|<\delta} n^2\,\mathbf{E}_\vartheta\left(\bar{\vartheta}_n - \vartheta\right)^2 \geq \mathbf{E}_{\vartheta_0}\left(\tilde{u}\right)^2. \tag{23}$$

This bound follows from the results of the Section 1.9 in [13]. We just note that the second moment $\mathbf{E}_\vartheta(\tilde{u})^2$ is a continuous function of ϑ.

As $\mathbf{E}_\vartheta\left(\tilde u\right)^2$ is the limit of the Bayesian estimator, we can think of these estimators having smaller limit error than the MLE (as in singular estimation problems). To prove this asymptotic efficiency of the Bayesian estimators, we need to show that the convergence of the second moments is uniform in ϑ on compacts. The corresponding uniform estimates on the process $Z_n\left(\cdot\right)$ can be easily verified and what remains to be done is to establish the uniform version of the convergence of finite dimensional distributions, which can also be verified.

Another possible generalization is to consider the case where $\{\varepsilon_j\}_{j\geq 1}$ are independent random variables with a known density function satisfying some regularity conditions. Then the estimator $\hat\vartheta_n$ (defined by (2)) becomes the least squares estimator and $\tilde\vartheta_n$ (defined by (3)) is no longer Bayesian, but becomes another estimator having desirable asymptotic properties. The behaviour of these estimators can be similarly studied and their limit distributions can be defined via the corresponding limit process $Z\left(\cdot\right)$ when $\varepsilon_l^{\pm}$ are no longer Gaussian.

Note that a continuous-time analogue of TAR model is prescribed by the following stochastic differential equation

$$\mathrm{d}X_t = -\varrho_1\, X_t\, 1\!\!1_{\{|X_t|<\vartheta\}}\mathrm{d}t - \varrho_2\, X_t\, 1\!\!1_{\{|X_t|\geq\vartheta\}}\mathrm{d}t + \sigma\,\mathrm{d}W_t,\quad 0\leq t\leq T,$$

where $\varrho_1\neq\varrho_2>0$. This model can be called *Threshold Ornstein-Uhlenbeck* (TOU) process and it can be considered as a continuous-time approximation of the discrete time model (1). The properties of the MLE and BE of the threshold ϑ can be studied with the help of the technique developed in [15].

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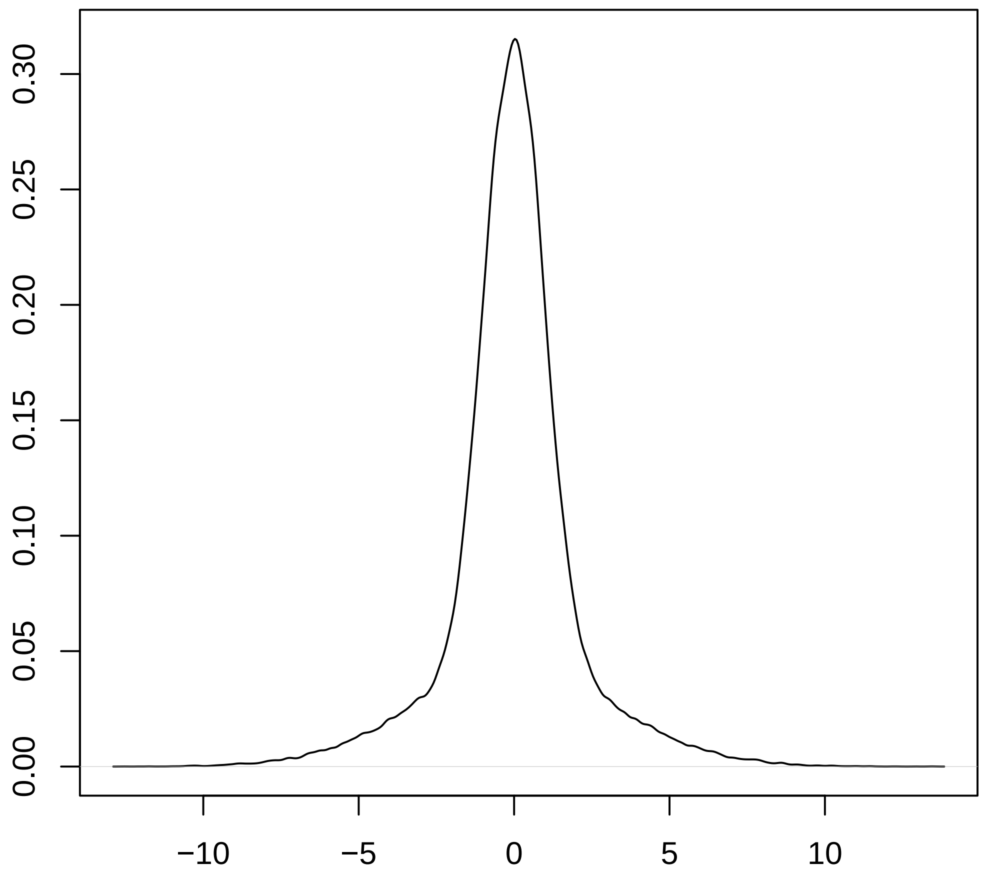

Figure 1: The kernel density function of $f(\theta)$.

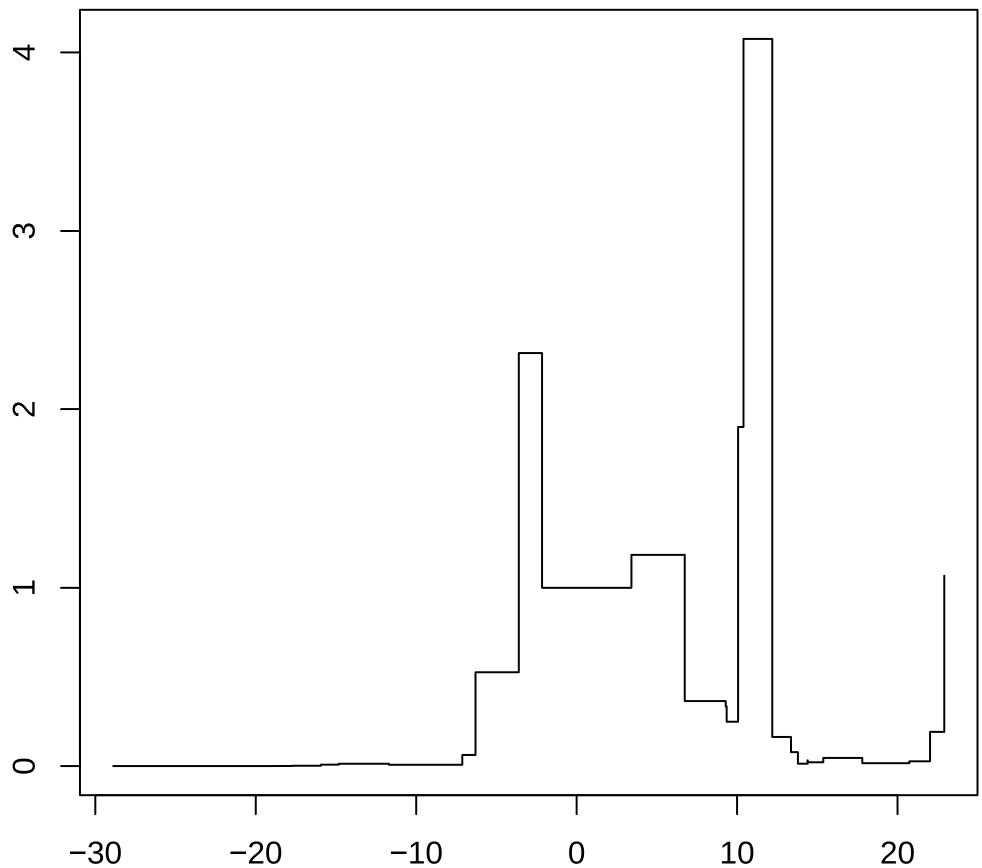

Figure 2: A sample path of $Z(u)$.

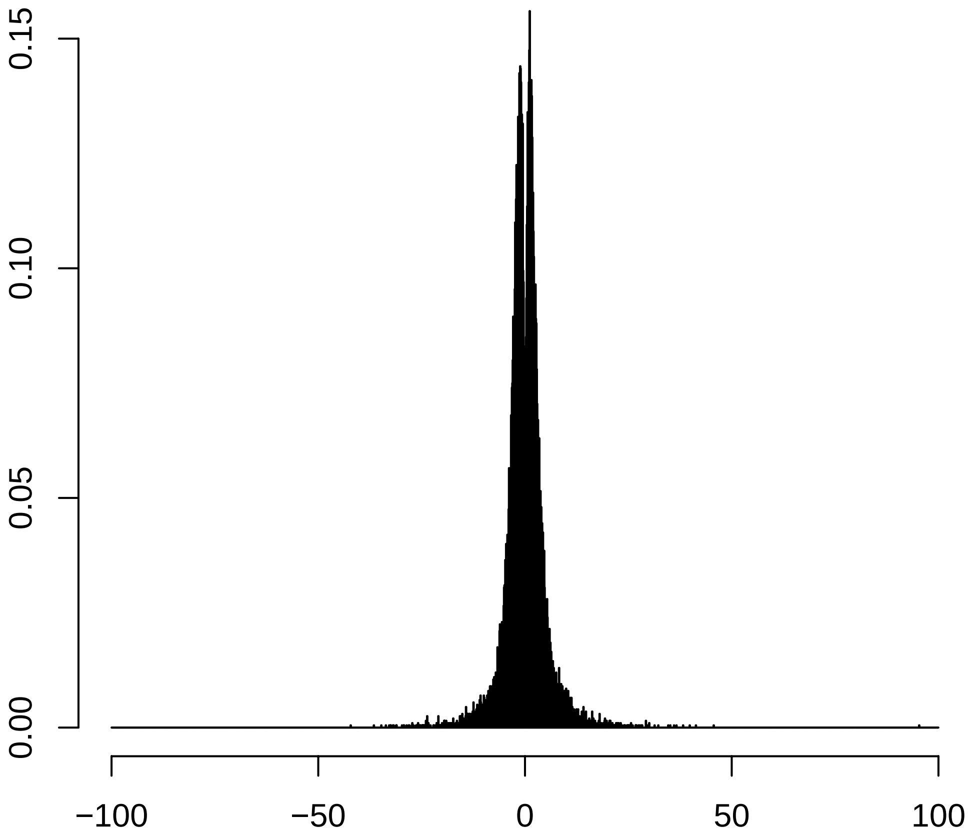

Figure 3: Histogram of the MLE $\hat{u}$.

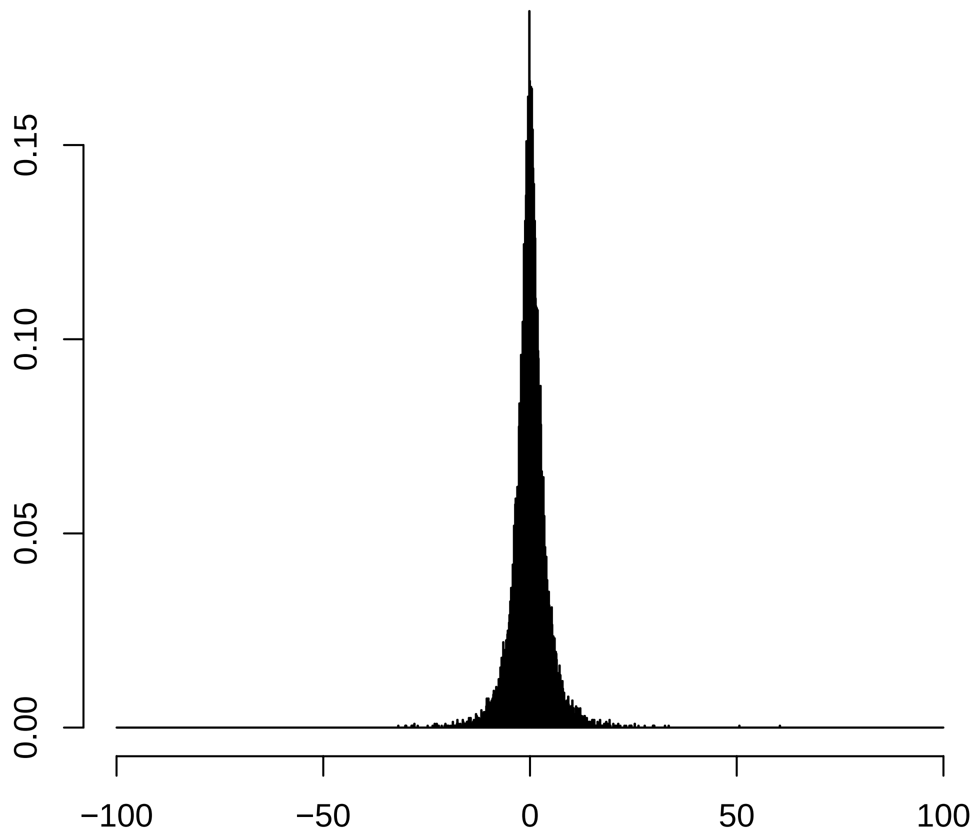

Figure 4: Histogram of the Bayesian estimator $\tilde{u}$.