On Cramér-von Mises test based on local time of switching diffusion process

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Abstract

We consider a Cramér-von Mises test for hypothesis that the observed diffusion process has sign-type trend coefficient. It is shown that the limit distribution of the proposed test statistic is defined by the integral type functional of continuous Gaussian process. We provide the Karhunen-Loève expansion on $[0, 1]$ of the corresponding limiting process. The representation for the limit statistic allow us to find the threshold.

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1 Introduction

We consider the Cramér-von Mises (C-vM) test, when the basic model is an ergodic diffusion process, i.e., the observations $X^T = \{X_t, 0 \le t \le T\}$ are form the stochastic differential equation

$$dX_t = S(X_t)\,dt + dW_t, \quad X_0, \quad 0 \le t \le T, \tag{1}$$

with initial value X_0, Wiener process $\{W_t, t \ge 0\}$ and unknown to the observer trend coefficient $S(\cdot)$. Diffusion process of this type is widely used as model in many different fields such as biology, physics, economics and finance. Let us introduce the stochastic differential equation

$$dX_t = S_0(X_t)\,dt + dW_t, \quad X_0, \quad 0 \le t \le T,$$

where $S_0(x)$ is some known function. We have two hypotheses:

$$\mathcal{H}_0: \qquad S(\cdot) = S_0(\cdot), \qquad \text{against alternative} \qquad \mathcal{H}_1: \qquad S(\cdot) \neq S_0(\cdot).$$

We observe the trajectory X^T of (1) and we test the hypothesis $\mathcal{H}_0$. Our goal is to study the C-vM test in this problem. The problem considered for this stochastic model is similar to the well-known in classical statistic C-vM test (see, e.g., Anderson and Darling [1, 2], Durbin and Knott [9], Durbin [10] and Stephens [24]). Indeed, in situation of i.i.d. observations $X^n = (X_1, \ldots, X_n)$ with c.d.f. $F(x)$ and the basic hypothesis $\mathcal{H}_0$: $F(x) = F_0(x)$. The C-vM statistic is,

$$W_n^2 = n \int_{-\infty}^{\infty} \left[\hat{F}_n(x) - F_0(x)\right]^2 dF_0(x), \qquad \hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{X_i < x\}},$$

where $\hat{F}_n(x)$ is the empirical distribution function. Anderson-Darling generalize the previous test by adding a (nonnegative) weight function. Let us denote by $\beta(t)$ a Brownian bridge, i.e., a continuous Gaussian process with

$$\mathbf{E}\beta(t) = 0, \qquad K(t,s) = \mathbf{E}\beta(t)\beta(s) = t \wedge s - ts.$$

Then the limit behavior of this statistic can be described with the help of this process as follows

$$W_n^2 = n \int_0^1 (\tilde{F}_n(t) - t)^2 dt = \int_0^1 \beta_n^2(t) dt \Longrightarrow \int_0^1 \beta^2(t) dt.$$

Hence the C-vM test $\psi_n(X^n) = 1_{\{W_n^2 > c_\alpha\}}$, with constant c_α defined by the equation

$$\mathbf{P}\left\{\int_0^1 \beta^2(t) dt > c_\alpha\right\} = \alpha.$$

is of asymptotic size α. The classical solution of this problem requires a preliminary computation of the sequence of eigenvalues $\lambda_1 > \lambda_2 > \ldots$ and eigenfunctions $f_1(t), f_2(t), \ldots$ of the Fredholm operator

$$f(t) \to \tilde{f}(t) = \int_0^1 K(t,s)\, f(s)\, ds.$$

By Mercer's theorem we have $K(t,s) = \sum_{k=1}^{\infty} \lambda_k f_k(s) f_k(t)$. Moreover, it is possible to expand $\beta(\cdot)$ into the Karhunen-Loève (KL) series (see, e.g., Kac and Siegert [15], Kac [14], Ash and Gardner [3])

$$\beta(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \xi_k\, f_k(t), \qquad W^2 = \sum_{k=1}^{\infty} \lambda_k\, \xi_k^2, \tag{2}$$

where $\{\xi_k : k \geq 1\}$ is a sequence of independent normal $\mathcal{N}(0,1)$ random variables. The relations (2) are used to approximate the value c_α.

For certain Gaussian processes, such expansions were obtained with the help of special functions. In particular, the trigonometric functions (case of Brownian bridge), Anderson-Darling process expansion has expression in Legendre polyonom functions. Recently, Bessel functions are used by Deheuvels and Martynov [7], Deheuvels [6] and Gassem [12]. More about Cramér-von Mises theory and the use of KL expansion can be found in [10], [21], [8].

In this present work we are interested by the C-vM test based on the observation X^T=$\{X_t, 0 \leq t \leq T\}$ solution of (1)

$$W_T^2 = T \int_{-\infty}^{\infty} \Big[f_T^\circ(x) - f_{S_0}(x) \Big]^2 dF_{S_0}(x), \qquad f_T^\circ(x) = \frac{\Lambda_T(x)}{T}.$$

where $f_T^\circ(x)$ is the local time type estimator of the density (LTE) (see [18] Section 1.1.3). This test was proposed in [4], but it is not distribution free.

Despite the fact that this statistic converges in distribution (under hypothesis) to a functional of a Gaussian process [18], the choice of the threshold c_α for the test $\Phi_T(X^T) = 1_{\{W_T^2 > c_\alpha\}}$ is not easy due to the structure of the covariance. To avoid such difficulty, a weighting of this statistic was introduced to make this test asymptotically distribution free (see kutoyants [19]). The purpose of the paper is to study the C-vM test for hypothesis that the observed process (1) is switching diffusion, i.e., the trend coefficient $S_0(x) = -\text{sgn}(x)$ is discontinuous function and taking just two values $+1$ and -1, the observed process has the form

$$dX_t = -\text{sgn}(x)\, dt + dW_t, \quad X_0, \quad 0 \leq t \leq T, \tag{3}$$

This work is a continuation of a study started by Gassem in [12], where the limit distribution of the test for this model was studied. Here we provide the KL expansion of the corresponding limiting process and we present the threshold c_α of the test.

2 Model of observation

Suppose that the observed process is (1), where the trend coefficient $S(x)$ satisfies the conditions of the existence and uniqueness of the solution and this solution has ergodic properties, i.e., there exists an invariant probability distribution $F_S(x)$, and for any integrable w.r.t. this distribution function

$h(x)$ the law of large numbers holds

$$\frac{1}{T}\int_0^T h(X_t)\,dt \longrightarrow \int_{-\infty}^{\infty} h(x)\,dF_S(x).$$

These conditions can be found, in Durrett [11] and Kutoyants [18].

The invariant density function $f_S(x)$ is defined by the equality

$$f_S(x) = \frac{1}{G(S)}\exp\left\{2\int_0^x S(v)\,dv\right\}, \qquad x \in \mathbb{R},$$

where $G(S)$ is the normalizing constant. This density function can be estimated by the LTE $f_T^\circ(x) = \Lambda_T(x)/T$. The local time $\Lambda_T(x)$ admits the Tanaka-Meyer representation (see [22])

$$\Lambda_T(x) = |X_T - x| - |X_0 - x| - \int_0^T \operatorname{sgn}(X_t - x)\,dX_t,$$

and recently started to play an important role in statistical inference [18], [5], [16]. Note that this estimator is $\sqrt{T}$ asymptotically normal as T$\to\infty$ (see [18], Proposition 1.51)

$$\sqrt{T}\left(f_T^\circ(x) - f_S(x)\right) \Longrightarrow \mathcal{N}\left(0, R_{f_S}(x,x)\right).$$

Moreover, this normed difference converge weakly to the limit Gaussian process (see [18], Theorem 4.13) with zero mean and covariance function

$$R_{f_S}(x,y) = 4f_S(x)f_S(y)\mathbf{E}\left(\frac{[1_{\{\xi>x\}} - F_S(\xi)][1_{\{\xi>y\}} - F_S(\xi)]}{f_S(\xi)^2}\right).$$

3 Choice of the threshold

3.1 C-vM test

It is easy to see that (3) is an ergodic diffusion process with the stationary density $f_{S_0}(x) = e^{-2|x|}$, for $x \in \mathbb{R}$.

Suppose that we observed the trajectory $X^T = \{X_t, 0 \le t \le T\}$ of (1) then to test the hypothesis $\mathcal{H}_0$ we can use the estimator LTE for construction of goodness-of-fit test based on the C-vM statistic

$$W_T^2 = T\int_{-\infty}^{\infty}\left[f_T^\circ(x) - f_{S_0}(x)\right]^2 dF_{S_0}(x) = \int_0^1 \eta_T^2(s)\,ds$$

The transformation $Y_t = F_{S_0}(X_t)$ simplifies the writing, because by the Itô formula the diffusion process Y_t satisfies the differential equation

$$Y_t = f_{S_0}(X_t)\left[2S_0(X_t)dt + dW_t\right], \qquad Y_0 = F_{S_0}(X_0)$$

with reflecting bounds in 0 and 1 and under hypothesis $\mathcal{H}_0$ has uniform on $[0,1]$ invariant distribution.

Fix a number $\alpha \in (0,1)$ and define the class $\mathcal{K}_\alpha$ of tests of asymptotic level $1-\alpha$ as follows:

$$\mathcal{K}_\alpha = \{\phi_T : \overline{\lim_{T\to\infty}}\, \mathbf{E}_{S_0}\phi_T(X^T) \leq \alpha\}.$$

Let us denote by c_α the value defined by the equation

$$\mathbf{P}\left\{\int_0^1 \eta_f^2(s)\, ds > c_\alpha\right\} = \mathbf{P}\left(W^2 > c_\alpha\right) = \alpha, \tag{4}$$

We have the following

Proposition 3.1. *The C-vM test* $\Phi_T(X^T) = 1_{\{W_T^2 > c_\alpha\}}$, *belongs to* $\mathcal{K}_\alpha$.

Proof. The main idea of the proof is to show (under $\mathcal{H}_0$) by using the method of Ibragimov-Khasminskii (Theorem A22, in [13]) that the distribution of W_T^2 converge to the distribution of W^2. The detailed proof is given in [12].

3.2 KL expansion of Gaussian process $\eta_f(t)$

In this section, we establish the KL expansion of the Gaussian process $\eta_f(t)$, for $0 \leq t \leq 1$, the representation for the limit statistic defined by the equation (4) allow to find the threshold c_α.

Under hypothesis $\mathcal{H}_0$ the process $\eta_f(t)$ admits the following representation

$$\eta_f(t) = 2\left(1 - |2t-1|\right)\int_0^1 \frac{1_{\{s>t\}} - s}{1 - |2s-1|}\, dW_s,$$

The last integral is with respect to double-sided Wiener process, i.e,. $W_s = W^+(s)$, $s \geq 1/2$ and $W_s = W^-(-s)$, $s \leq 1/2$, where $W^+(\cdot)$ and $W^-(\cdot)$ are two independent Wiener processes.

By the law of large numbers we have

$$\sqrt{T}\left(f_T^\circ(x) - f_{S_0}(x)\right) = 2f_{S_0}(x)W\left(\frac{1}{T}\int_0^T \left(\frac{1_{\{X_t>x\}} - F_{S_0}(X_t)}{f_{S_0}(X_t)}\right)^2 dt\right)$$

$$\longrightarrow 2f_{S_0}(x)W\left(\int_{-\infty}^{\infty}\left(\frac{1_{\{y>x\}} - F_{S_0}(y)}{f_{S_0}(y)}\right)^2 dF_{S_0}(y)\right).$$

where $W(u)$, $u \geq 0$ is some Wiener process. Making the transformation $t = F_{S_0}(x)$ we obtain

$$\begin{aligned}\eta_T(t) &\longrightarrow 2\,(1-|2t-1|)\,W\left(\int_0^1 \left(\frac{1_{\{s>t\}}-s}{1-|2s-1|}\right)^2 ds\right)\\ &= 2\,(1-|2t-1|)\int_0^1 \frac{1_{\{s>t\}}-s}{1-|2s-1|}\,dW_s = \eta_f(t)\end{aligned}$$

The covariance function of $\eta_f(\cdot)$, for $0 \leq s,t \leq 1$, can be written as follows:

$$\begin{aligned}R_f(s,t) &= 4(1-|2t-1|)(1-|2s-1|)\int_0^1 \frac{(1_{\{u>t\}}-u)(1_{\{u>s\}}-u)}{(1-|2u-1|)^2}\,du\\ &= 4(1-|2t-1|)(1-|2s-1|)\left\{\int_0^{s\wedge t} \frac{u^2}{(1-|2u-1|)^2}\,du\right.\\ &\quad \left. - \int_{s\wedge t}^{s\vee t} \frac{u(1-u)}{(1-|2u-1|)^2}\,du + \int_{s\vee t}^1 \frac{(1-u)^2}{(1-|2u-1|)^2}\,du\right\}.\end{aligned}$$

Let $s \leq t$, then by a direct calculation we have

$$R_f(s,t) = \begin{cases} 4st\ln(4st) + 4s(1-t), & s,t \leq 1/2,\\ 4s(1-t)\ln(4s(1-t)) + 4s(1-t), & s \leq 1/2,\ t > 1/2,\\ 4(1-s)(1-t)\ln(4(1-s)(1-t)) + 4s(1-t), & s,t > 1/2.\end{cases}$$

Therefore, we can write

$$\begin{aligned}R_f(s,t) &= (1-|2s-1|)(1-|2t-1|)\ln\left[(1-|2s-1|)(1-|2t-1|)\right]\\ &\quad - 4(1_{\{s>t\}}-s)(1_{\{t>s\}}-t), \quad 0 \leq s,t \leq 1.\end{aligned} \tag{5}$$

Of course,

$$R_f(t,t) = 2(1-|2t-1|)^2\ln(1-|2t-1|) + 4t(1-t), \quad 0 \leq t \leq 1. \tag{6}$$

By the Karhunen-Loève Theorem, we have the following

Proposition 3.2. *The Gaussian process $\eta_f(t)$ has a KL expansion given by*

$$\begin{aligned}\eta_f(t) &= \sum_{n=1}^{\infty} \frac{\xi_n^*}{n\,\pi}\sqrt{2}\,\mathrm{sgn}(1/2-t)\sin(n\,\pi(1-|2\,t-1|))\\ &\quad + \sum_{n=1}^{\infty} \frac{\xi_n}{\nu_n}\left\{\frac{\sqrt{2}}{\mathrm{Si}(\nu_n)}\left[\left(\frac{\alpha(\nu_n)}{\nu_n} - \dot{\alpha}(\nu_n)\right)\sin(\nu_n(1-|2\,t-1|))\right.\right.\\ &\quad \left.\left. - \left(\frac{\sin(\nu_n)}{\nu_n} - \cos(\nu_n)\right)\alpha(\nu_n(1-|2\,t-1|))\right]\right\}, \quad 0 \leq t \leq 1,\end{aligned} \tag{7}$$

where $\{\xi_n, n \geq 1\}$, $\{\xi_n^*, n \geq 1\}$ *denote two independent sequences of independent and identically distributed* $\mathcal{N}(0,1)$ *random variables and* $\nu_n, n = 1, 2, \ldots,$ *are the positive zeros of* $f(\cdot)$*, defined by the equation*

$$f(t) = Si(t)\left[\alpha(t) - t\dot{\alpha}(t)\right] - G(t)\left[\sin(t) - t\cos(t)\right],$$

where

$$\alpha(t) = Ci(t)\sin(t) - Si(t)\cos(t), \quad \dot{\alpha}(t) = \frac{d}{dt}\alpha(t), \quad G(t) = \int_0^t \frac{\alpha(s)}{s}\, ds,$$

with

$$Ci(t) = \gamma + \ln(t) + \int_0^t \frac{\cos(s)}{s}\, ds, \qquad Si(t) = \int_0^t \frac{\sin(s)}{s}\, ds,$$

the cosine and sine integral respectively, $\gamma = 0.577215...$ *the Euler's constant.*

Proof. To determine the eigenfunctions, one must solve the integral equation:

$$\lambda_n \psi_n(t) = \int_0^1 R_f(t,s)\, \psi_n(s)\, ds, \quad t \in [0,1]. \tag{8}$$

The fact that we have an absolute value in the covariance function $R_f(s,t)$, we will take $\psi_n(t) = \psi_{1,n}(t)1_{\{t<1/2\}} + \psi_{2,n}(t)1_{\{t\geq 1/2\}}$.

Here, the $\lambda_n\, n = 1 \ldots \infty$ are eigenvalues yet to be determined. We have the boundary conditions

$$\psi_{1,n}(0) = \psi_{2,n}(1) = 0, \qquad \psi_{1,n}(1/2) = \psi_{2,n}(1/2), \tag{9}$$

$$\lambda_n \psi_n(1/2) = \int_0^1 (1 - |2s-1|)\left[1 + \ln(1 - |2s-1|)\right] \psi_n(s)\, ds. \tag{10}$$

Taking derivative of equation (8) with respect to t, we obtain

$$\lambda_n \psi_n'(t) = \int_0^1 \frac{\partial R_f(t,s)}{\partial t}\, \psi_n(s)\, ds, \quad t \in [0,1].$$

Evaluating the derivative of the last equation we obtain

$$\lambda_n \psi_n''(t) + 4\,\psi_n(t) = \frac{4\, C_n}{1 - |2t-1|}, \quad t \in [0,1]. \tag{11}$$

where

$$C_n = \int_0^1 (1 - |2s-1|)\, \psi_n(s)\, ds. \tag{12}$$

Let $\nu_n = 1/\sqrt{\lambda_n}$, the last equation can be written as follow

$$\begin{aligned} \psi''_{1,n}(t) + 4\,\nu_n^2 \psi_{1,n}(t) &= \frac{2\,\nu_n^2 C_n}{t}, \quad t < 1/2, \\ \psi''_{2,n}(t) + 4\,\nu_n^2 \psi_{2,n}(t) &= \frac{2\,\nu_n^2 C_n}{1-t}, \quad t \geq 1/2. \end{aligned}$$

When $C_n = 0$ the equation (11) has solution

$$\psi_n(t) = \begin{cases} \psi_{1,n}(t) = A_{1,n} \sin(2\nu_n t) + B_{1,n} \cos(2\nu_n t), & t < 1/2, \\ \psi_{2,n}(t) = A_{2,n} \sin(2\nu_n t) + B_{2,n} \cos(2\nu_n t), & t \geq 1/2. \end{cases}$$

Now, using the Lagrange method we have

$$\begin{cases} \dot{A}_{1,n} \sin(2\nu_n t) + \dot{B}_{1,n} \cos(2\nu_n t) = 0 \\ \dot{A}_{1,n} \cos(2\nu_n t) - \dot{B}_{1,n} \sin(2\nu_n t) = \frac{\nu_n C_n}{t} \end{cases} \Longrightarrow \begin{cases} \dot{A}_{1,n} = \nu_n C_n \frac{\cos(2\nu_n t)}{t}, \\ \dot{B}_{1,n} = -\nu_n C_n \frac{\sin(2\nu_n t)}{t}. \end{cases}$$

Then, the first equation has a solution of the following form

$$\psi_{1,n}(t) = A_{1,n} \sin(2\nu_n t) + B_{1,n} \cos(2\nu_n t) + \nu_n C_n\, \alpha(2\nu_n t), \quad t < 1/2,$$

where

$$\alpha(t) = \mathrm{Ci}(t) \sin(t) - \mathrm{Si}(t) \cos(t)$$

with

$$\mathrm{Ci}(t) = \gamma + \ln(t) + \int_0^t \frac{\cos(s)}{s}\, ds, \qquad \mathrm{Si}(t) = \int_0^t \frac{\sin(s)}{s}\, ds.$$

In the same manner for the second equation, we have

$$\begin{cases} \dot{A}_{2,n} \sin(2\nu_n t) + \dot{B}_{2,n} \cos(2\nu_n t) = 0 \\ \dot{A}_{2,n} \cos(2\nu_n t) - \dot{B}_{2,n} \sin(2\nu_n t) = \frac{\nu_n C_n}{1-t} \end{cases} \Longrightarrow \begin{cases} \dot{A}_{2,n} = \nu_n C_n \frac{\cos(2\nu_n t)}{1-t}, \\ \dot{B}_{2,n} = -\nu_n C_n \frac{\sin(2\nu_n t)}{1-t}. \end{cases}$$

Using the following equalitie

$$\begin{aligned} \cos(2\nu_n t) &= \cos(2\nu_n (1-t)) \cos(2\nu_n) + \sin(2\nu_n (1-t)) \sin(2\nu_n), \\ \sin(2\nu_n t) &= \cos(2\nu_n (1-t)) \sin(2\nu_n) - \sin(2\nu_n (1-t)) \cos(2\nu_n). \end{aligned}$$

Then, the second equation has a solution of the following form

$$\psi_{1,n}(t) = A_{2,n}\sin(2\nu_n t) + B_{2,n}\cos(2\nu_n t) + \nu_n C_n\,\alpha(2\nu_n(1-t)), \quad t \geq 1/2.$$

Therefore, using the boundary conditions (9), the general solution is of the following form

$$\begin{aligned}\psi_n(t) = (A_{1,n}1_{\{t<1/2\}} - A_{2,n}1_{\{t\geq 1/2\}})\sin(\nu_n(1-|2t-1|)) \\ + \nu_n C_n\,\alpha(\nu_n(1-|2t-1|)), \quad t\in[0,1].\end{aligned} \tag{13}$$

Now, returning to the equation (10), replacing $\psi_n(s)$ by this general form and making the change of variable $u = \nu_n s$, we have

$$\begin{aligned}\psi_n(1/2) &= \frac{\nu_n^2}{2}\int_0^1 s\,(1+\ln(s))\,[(A_{1,n}-A_{2,n})\sin(\nu_n s) + 2\,C_n\nu_n\alpha(\nu_n s)]\,ds \\ &= \frac{1}{2}(A_{1,n}-A_{2,n})\int_0^{\nu_n} u\,[1-\ln(\nu_n)+\ln(u)]\sin(u)\,du \\ &\quad + C_n\nu_n\int_0^{\nu_n} u\,[1-\ln(\nu_n)+\ln(u)]\,\alpha(u)\,du.\end{aligned}$$

By a simple integration by part we have

$$\int_0^{\nu_n} u\,[1-\ln(\nu_n)+\ln(u)]\sin(u)\,du = 2\sin(\nu_n) - \nu_n\cos(\nu_n) - \mathrm{Si}(\nu_n).$$
$$\int_0^{\nu_n} u\,[1-\ln(\nu_n)+\ln(u)]\,\alpha(u)\,du = 2\,\alpha(\nu_n) - \nu_n\dot\alpha(\nu_n) - G(\nu_n).$$

where

$$\dot\alpha(t) = \frac{d}{dt}\alpha(t) = \mathrm{Si}(t)\sin(t) + \mathrm{Ci}(t)\cos(t), \qquad \mathrm{G}(t) = \int_0^t \frac{\alpha(s)}{s}\,ds.$$

Finally we have

$$\begin{aligned}(A_{1,n}-A_{2,n})\,[\sin(\nu_n) - \nu_n\cos(\nu_n) - \mathrm{Si}(\nu_n)] \\ + 2\,C_n\nu_n\,[\alpha(\nu_n) - \nu_n\dot\alpha(\nu_n) - \mathrm{G}(\nu_n)] = 0.\end{aligned} \tag{14}$$

In the same manner for equality (12), we obtain

$$\begin{aligned}C_n &= \frac{1}{2\,\nu_n^2}\int_0^{\nu_n} u\,[(A_{1,n}-A_{2,n})\sin(u) + 2\,C_n\nu_n\alpha(u)]\,du \\ &= \frac{1}{2\,\nu_n^2}(A_{1,n}-A_{2,n})\,[\sin(\nu_n) - \nu_n\cos(\nu_n)] \\ &\quad + \frac{1}{\nu_n^2}\,C_n\nu_n\,[\alpha(\nu_n) - \nu_n\dot\alpha(\nu_n) + \nu_n]\,,\end{aligned}$$

and we have

$$(A_{1,n} - A_{2,n})\,(\sin(\nu_n) - \nu_n \cos(\nu_n)) + 2\,C_n \nu_n\,(\alpha(\nu_n) - \nu_n \dot{\alpha}(\nu_n)) = 0. \qquad (15)$$

Using the boundary conditions (9) and equalities (14), (15), we get the linear system

$$\begin{bmatrix} \sin(\nu_n) & \sin(\nu_n) & 0 \\ \mathrm{Si}(\nu_n) & -\mathrm{Si}(\nu_n) & 2\,\nu_n \mathrm{G}(\nu_n) \\ \sin(\nu_n) - \nu_n \cos(\nu_n) & -(\sin(\nu_n) - \nu_n \cos(\nu_n)) & 2\,\nu_n\,(\alpha(\nu_n) - \nu_n \dot{\alpha}(\nu_n)) \end{bmatrix} \times$$
$$\times \begin{bmatrix} A_{1,n} \\ A_{2,n} \\ C_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (16)$$

which has non-zero solutions in $(A_{1,n}, A_{2,n}, C_n)$, if and only if

$$\det \begin{bmatrix} \sin(\nu_n) & \sin(\nu_n) & 0 \\ \mathrm{Si}(\nu_n) & -\mathrm{Si}(\nu_n) & 2\,\nu_n \mathrm{G}(\nu_n) \\ \sin(\nu_n) - \nu_n \cos(\nu_n) & -(\sin(\nu_n) - \nu_n \cos(\nu_n)) & 2\,\nu_n\,(\alpha(\nu_n) - \nu_n \dot{\alpha}(\nu_n)) \end{bmatrix}$$
$$= 4\,\nu_n \sin(\nu_n)\, f\,(\nu_n) = 0, \qquad (17)$$

where

$$f\,(\nu_n) = \mathrm{Si}(\nu_n)\,(\alpha(\nu_n) - \nu_n \dot{\alpha}(\nu_n)) - \mathrm{G}(\nu_n)\,(\sin(\nu_n) - \nu_n \cos(\nu_n))\,. \qquad (18)$$

Next we consider the function $\psi_n(t)$ of the form (13) and $A_{1,n}, A_{2,n}, C_n$ fulfilling (16), and by $\nu_{1,n}, \nu_{2,n}, n = 1, 2, \ldots$, respectively solutions of $\sin(\nu_n) = 0$ and $f(\nu_n) = 0$. We have the following two cases:

Case 1: When $\nu_n = \nu_{1,n} = n\,\pi$, for some $n = 1, 2, \ldots$, $\sin(\nu_{1,n}) = 0$, so that (16) implies that $A_{1,n} = A_{2,n}$ and $C_n = 0$. The requirement that $\int_0^1 \psi_n^2(t)\,dt = 1$ yields, therefore

$$\psi_n(t) = \sqrt{2}\,\mathrm{sgn}(1/2 - t)\,\sin(n\,\pi(1 - |2\,t - 1|))\,, \quad t \in [0, 1].$$

Case 2: When $\nu_n = \nu_{2,n}$, for some $n = 1, 2, \ldots$, $f\,(\nu_n) = 0$, so that (16) implies that $A_{1,n} = -C_n\,(\alpha(\nu_{2,n}) - \nu_{2,n}\dot{\alpha}(\nu_{2,n}))\,/\,(\sin(\nu_{2,n}) - \nu_{2,n}\cos(\nu_{2,n})) = -A_{2,n}$. The requirement that $\int_0^1 \psi_n^2(t)\,dt = 1$ yields, therefore

$$\psi_n(t) = \frac{\sqrt{2}}{\mathrm{Si}(\nu_{2,n})}\left[\left(\frac{\alpha(\nu_{2,n})}{\nu_{2,n}} - \dot{\alpha}(\nu_{2,n})\right)\sin(\nu_{2,n}(1 - |2\,t - 1|)) - \left(\frac{\sin(\nu_{2,n})}{\nu_{2,n}} - \cos(\nu_{2,n})\right)\alpha\,(\nu_{2,n}(1 - |2\,t - 1|))\right], \quad t \in [0, 1].$$

and this ends the proof.

3.3 Simulation

As a direct consequence of (7), the distribution of $\int_0^1 \eta_f^2(t)\,dt$, coincide with the distribution of the quadratic form (under hypothesis $\mathcal{H}_0$), we obtain the identity

$$\int_0^1 \eta_f^2(t)\,dt \stackrel{d}{=} \sum_{n=1}^{\infty} \frac{\xi_n^{*2}}{n^2\pi^2} + \sum_{n=1}^{\infty} \frac{\xi_n^2}{\nu_n^2}, \tag{19}$$

allowing to evaluate the distribution of $\int_0^1 \eta_f^2(t)\,dt$ by a number of methods (see, e.g., Smirnov [23], Martynov [20], and the references therein).

We may check from the formulas (7) and (19) that

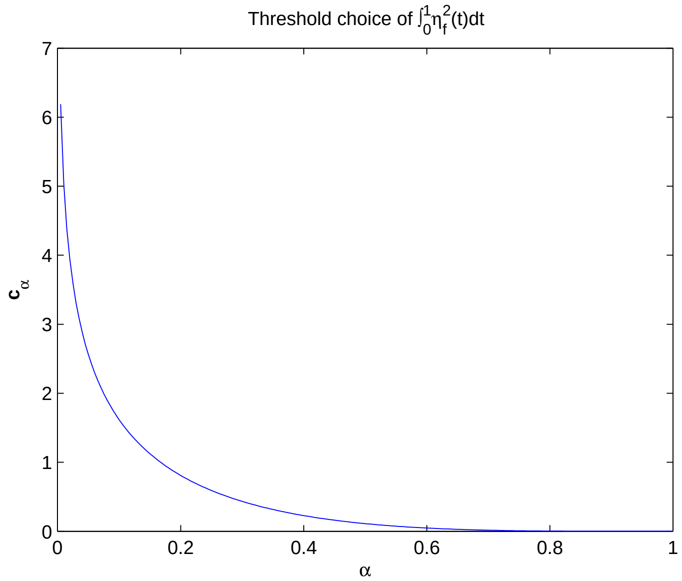
$$\int_0^1 R_f(t,t)\,dt = \mathbf{E}\int_0^1 \eta_f^2(t)\,dt = \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} + \sum_{n=1}^{\infty} \frac{1}{\nu_n^2} = \frac{1}{6} + \frac{5}{18} = \frac{4}{9}, \tag{20}$$

where for the first sum we have used the Euler's formula $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ and numerically (Secant Method) for the second sum.
To check (20), we infer from (6) that

$$\int_0^1 R_f(t,t)\,dt = 2\int_0^1 \left[2t(1-t) + t^2\ln(t)\right]dt = 2\left(\frac{1}{3} - \frac{1}{9}\right) = \frac{4}{9}, \tag{21}$$

We see therefore that (20),(21) are in agreement.

The random variable $\int_0^1 \eta_f^2(t)\,dt$ is a weighted sum of independent χ_1^2 components, Thus, calculating the $1-\alpha$ quantiles of this random variable we obtaining the threshold c_α of C-vM test defined in equation (4).



Particulary, if $\alpha = 0,05$ we obtain $c_{0,05} = 2,4202$.

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