

Filtering with Exponential Criteria

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The model

- nonobservable signal process (X_t) with values in \mathbb{R}^n ;
- observations (Y_t) from \mathbb{R}^d ;
- **exponential** type payoff function L_T

The aim

To find \bar{h} which minimizes the payoff function.

LEG problem in one slide

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The observation model

the precise definition

- **signal** $X = (X_t, t \geq 0)$ — n -dimensional continuous Gaussian process with

- mean function $m = (m_t, t \geq 0)$:

$$\mathbb{E}X_t = m_t$$

- covariance function $\Gamma = (\Gamma(t, s), t \geq 0, s \geq 0)$:

$$\mathbb{E}(X_t - m_t)(X_s - m_s)' = \Gamma(t, s), \quad t \geq 0, s \geq 0.$$

- **observation:**

$$Y_t = \int_0^t A(s)X_s ds + \tilde{B}_t, \quad t \geq 0.$$

- where

- \tilde{B}_t is \mathbb{R}^d valued Wiener process, independent of X ;
- $A := (A(s), s \geq 0)$ is continuous with values in the set of $d \times n$ matrices.

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LEG Filtering Problem

The precise statement

To minimize wrt $h \in \mathcal{H}$ and $g \in \mathcal{G}$ the quantity:

Exponential Criterium,¹

$$L_T(h, g, \mu) = \mathbb{E} \left[\mu \exp \left\{ \frac{\mu}{2} (X_T - g)' M (X_T - g) + \frac{\mu}{2} \int_0^T (X_s - h(s))' Q_s (X_s - h(s)) ds \right\} \right],$$

► A Quiz

where

- h : $(\mathcal{Y}_s, 0 \leq s \leq T)$ adapted process
- g : \mathcal{Y}_T measurable variable
- $M, Q_s, 0 \leq s \leq T$: given nonnegative symmetric deterministic matrices

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Two different cases

There are two different cases for LEG filtering problem:

- $\mu < 0$ - risk-preferring filtering problem,
- $\mu > 0$ - risk-averse filtering problem.

Our approach

- Solve the problem for $\mu < 0$ (it is easier).
- Extend results to the general case using the analytical properties.

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The second problem: Risk Sensitive Filtering

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Recursive equation as a **definition** of the Risk-sensitive Filtering:

Exponential Criterium, II

$$\hat{h}(t) = \arg \min_{g \in \mathcal{Y}_t} \mathbb{E} \left[\mu \exp \left\{ \frac{\mu}{2} (X_t - g)' Q_t (X_t - g) + \frac{\mu}{2} \int_0^t (X_s - \bar{h}(s))' Q_s (X_s - \hat{h}(s)) ds \right\} / \mathcal{Y}_t \right],$$

where g is a \mathcal{Y}_t measurable variable.

▶ RS problem, result

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Quadratic type payoff function

Q: What happens for the quadratic type payoff function?

▸ Exponential Criterion, I

A: Solutions of LEG and RS: $g = \bar{h}_T$, $\bar{h}_t = \hat{h}_t = \pi_t(X)$, where $\pi_t(X) := \mathbb{E}[X_t | \mathcal{Y}_t]$ (can be computed using Kalman filter).

Are they equal?

Q: Can we always take $\bar{h} = \hat{h}$?

A: Sometimes yes, sometimes no ...

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History 1: control & partial observations

A. Bensoussan & J.H. van Schuppen, 1985

Markov observation model

$$\begin{cases} dX_t = (a_t X_t + u_t) dt + dB_t \\ dY_t = A_t X_t dt + d\tilde{B}_t. \end{cases}$$

An important formula :

$$\mathbb{E} \left\{ \exp \left(-\frac{1}{2} \int_0^T X_s^2 ds \right) \middle| F_T^Y \right\} = \exp \left\{ -\frac{1}{2} \int_0^T \bar{\gamma}(s) ds \right\} \times \\ \times \exp \left\{ -\frac{1}{2} \int_0^T V_s^2 ds \right\} \cdot \exp \left\{ \int_0^T A_s Z_s d\nu_s - \frac{1}{2} \int_0^T \|A_s Z_s\|^2 ds \right\}$$

with

$$dV_t = (a_t V_t + u_t) dt - \bar{\gamma}(t) V_t dt + \bar{\gamma}(t) A_t [dY_t - A_t V_t dt]$$

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History 2: LEG filtering, discrete time setting

J.L. Speyer, 1992

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Discrete time observation model

$$\begin{cases} X_{t+1} &= a_t X_t + \varepsilon_t \\ Y_t &= A_t X_t + \tilde{\varepsilon}_t. \end{cases}$$

Exponential payoff function

$$L_T(h, \mu) = \mathbb{E} \left[\mu \exp \left\{ \frac{\mu}{2} \sum_0^T (X_s - h(s))' Q_s (X_s - h(s)) \right\} \right]$$

Answer - using dynamical programming

$$\begin{aligned} \hat{h}(T) &= \arg \min_{g \in \mathcal{Y}_T} \mathbb{E} \left[\mu \exp \left\{ \frac{\mu}{2} (X_T - g)' Q_T (X_T - g) \right. \right. \\ &\quad \left. \left. + \frac{\mu}{2} \sum_0^{T-1} (X_s - \bar{h}(s))' Q_s (X_s - \hat{h}(s)) \right\} \right] / \mathcal{Y}_T, \end{aligned}$$

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History 3: Risk-Sensitive setting

R.J. Elliott, S. Dey, J.B. Moore, 1994

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Risk-Sensitive Filtering, definition by recursive equation

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proposed by R.J. Elliott, S. Dey, J.B. Moore and .. for RS filtering problem

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A non normalized random measure

$$\nu_t(dx) = \mathbb{E} \left[\mathbb{I}(X_t \in dx) \exp \left\{ \frac{\mu}{2} \int_0^t (X_s - h(s))' Q_s (X_s - h(s)) ds \right\} / \mathcal{Y}_t \right]$$

Information variable: definition

Information variable $\lambda_t(x)$ is the density of the measure $\nu_t(dx)$.

"Information State" approach, first definitions

proposed by R.J. Elliott, S. Dey, J.B. Moore and .. for RS filtering problem

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A non normalized random measure

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Information variable: definition

Information variable $\lambda_t(x)$ is the density of the measure $\nu_t(dx)$.

Information variable, equation & solution

Markov case

Equation

$$d\lambda_t(x) = \left[\frac{1}{2} \lambda_{xx} - a_t(x\lambda)_x + \frac{\mu}{2} Q(x - h())^2 \lambda \right] dt + A_t \lambda d\nu_t$$

Solution

$$\begin{aligned} \lambda_t(x) = & C \exp \left\{ -\frac{1}{2} (x - z_t^h)' \bar{\gamma}^{-1}(t, t) (x - z_t^h) \right. \\ & \left. + \frac{\mu}{2} \int_0^t (z_s^h - h(s))' Q_s (z_s^h - h(s)) ds + \frac{\mu}{2} \int_0^t \text{tr}(\bar{\gamma}(s) Q_s) ds \right\} \\ & \times \exp \left\{ \int_0^t (z_s^h - \pi_s(X))' A_s' d\nu_s - \frac{1}{2} \int_0^t \|A_s(z_s^h - \pi_s(X))\|^2 ds \right\}, \end{aligned}$$

with $C = [(2\pi)^n \det(\bar{\gamma}(t, t))]^{-\frac{1}{2}}$

The equations for z^h and $\bar{\gamma}$ can be written also.

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Three steps towards the solution

- Integrate the measure ν_t .
- Minimize the quadratic form
- Obtain the answer
 - $\hat{h}_t = z^{\hat{h}}$
 - $\hat{h} = \bar{h}$

Remaining questions

- Probabilistic sense of involved functions z^h and $\bar{\gamma}$?
- What to do if the observation model is not Markovian?

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Conditional expectation of the exponent: definition

What we want to find (notation)

$$\mathcal{I}_T = \mathbb{E} \left[\exp \left\{ \frac{\mu}{2} (X_T - g)' M (X_T - g) + \frac{\mu}{2} \int_0^T (X_s - h(s))' Q_s (X_s - h(s)) ds \right\} / \bar{\mathcal{Y}}_T \right],$$

where

- h : (\mathcal{Y}_s) adapted process
- g : \mathcal{Y}_T measurable variable
- $M, Q_s, 0 \leq s \leq T$: nonnegative symmetric deterministic matrices

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Conditional expectation of the exponent: definition

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Conditional expectation of the exponent: the answer

The Cameron-Martin type formula

For any $T > 0$

$$\begin{aligned} \mathcal{I}_T = C \exp & \left\{ \frac{1}{2} (z_T^h - g)' [Id - \mu M \bar{\gamma}(T, T)]^{-1} \mu M (z_T^h - g) \right. \\ & \left. + \frac{\mu}{2} \int_0^T (z_s^h - h(s))' Q_s (z_s^h - h(s)) ds + \frac{\mu}{2} \int_0^T \text{tr}(\bar{\gamma}(s, s) Q_s) ds \right\} \\ & \times \exp \left\{ \int_0^T (z_s^h - \pi_s(X))' A_s' d\nu_s - \frac{1}{2} \int_0^T \|A_s(z_s^h - \pi_s(X))\|^2 ds \right\}. \end{aligned}$$

with $C = [\det(Id - \mu M \bar{\gamma}(T, T))]^{-\frac{1}{2}}$.

► Proof of LEG

Greek and Latin letters in the previous formula

- Innovation $d\nu_t = dY_t - A_t\pi_t(X)dt$, $\nu_0 = 0$,
- $\bar{\gamma} = (\bar{\gamma}(t, s), 0 \leq s \leq t)$: unique solution of Riccati-Volterra equation

$$\bar{\gamma}(t, s) = \Gamma(t, s) - \int_0^s \bar{\gamma}(t, r)[A'_r A_r - \mu Q_r] \bar{\gamma}'(s, r) dr,$$

such that $\bar{\gamma}(t, t) \geq 0$.

- $z^h = (z_s^h, s \geq 0)$ unique solution of Ito-Volterra equation

$$z_t^h = m_t + \int_0^t \bar{\gamma}(t, s) \mu Q_s [z_s^h - h(s)] ds + \int_0^t \bar{\gamma}(t, s) A'_s [dY_s - A_s z_s^h ds].$$

Greek and Latin letters in the previous formula

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Probabilistic sense of Greek & Latin letters

Auxiliary filtering problem

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Auxiliary observations, I

$\bar{Y}_t = (Y_t^1, Y_t^2)$, such that

$$\begin{cases} dY_t^1 = dY_t, Y_0^1 = 0 \\ dY_t^2 = Q_t^{\frac{1}{2}} X_t dt + d\bar{B}_t, Y_0^2 = 0, \end{cases}$$

where \bar{B}_t is a standard Brownian motion, independent of (X, \tilde{B}) .

Auxiliary observations, II

$$d\xi_t = (X_t - h(t))' Q_t^{\frac{1}{2}} dY_t^2, \xi_0 = 0.$$

► Bayes formula and Girsanov theorem

Probabilistic sense of Greek & Latin letters

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Auxiliary observations, II

$$d\xi_t = (X_t - h(t))' Q_t^{\frac{1}{2}} dY_t^2, \xi_0 = 0.$$

► Bayes formula and Girsanov theorem

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Probabilistic sense

- $z_t^h = \bar{\pi}_t(X) - \bar{\gamma}_{X\xi}(t)$

with

- $\bar{\pi}_t(X) = \mathbb{E}(X/\bar{\mathcal{Y}}_t)$ – the conditional expectation of X
- $\bar{\gamma}_{X\xi}(t) = \mathbb{E}[(X_t - \bar{\pi}_t(X))(\xi_t - \bar{\pi}_t(\xi))/\bar{\mathcal{Y}}_t]$ – the conditional covariance
- $\bar{\gamma}(t, t) = \mathbb{E}[(X_t - \bar{\pi}_t(X))(X_t - \bar{\pi}_t(X))'/\bar{\mathcal{Y}}_t]$ – the variance of the filtering error

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Consider $\bar{h} = (\bar{h}(t), t \geq 0)$: solution of LEG filtering problem

LEG, definition

$$\bar{h} = \arg \min_{h \in \mathcal{H}} \mathbb{E} \left[\mu \exp \left\{ \frac{\mu}{2} \int_0^T (X_s - h(s))' Q_s (X_s - h(s)) ds \right\} \right].$$

LEG, solution

$$\bar{h}(t) = m_t + \int_0^t \bar{\gamma}(t, s) A'_s [dY_s - A_s \bar{h}(s) ds],$$

finding γ : Riccati-Volterra equation

$$\bar{\gamma}(t, s) = \Gamma(t, s) - \int_0^s \bar{\gamma}(t, r) [A'_r A_r - \mu Q_r] \bar{\gamma}'(s, r) dr,$$

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Consider $\bar{h} = (\bar{h}(t), t \geq 0)$: solution of LEG filtering problem

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$$\bar{h}(t) = m_t + \int_0^t \bar{\gamma}(t, s) A'_s [dY_s - A_s \bar{h}(s) ds],$$

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$$\bar{\gamma}(t, s) = \Gamma(t, s) - \int_0^s \bar{\gamma}(t, r) [A'_r A_r - \mu Q_r] \bar{\gamma}'(s, r) dr,$$

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Consider $\bar{h} = (\bar{h}(t), t \geq 0)$: solution of LEG filtering problem

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LEG answer: optimal risk

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$$\begin{aligned} \mathbb{E} \left[\mu \exp \left\{ \frac{\mu}{2} \int_0^T (X_s - \bar{h}(s))' Q_s (X_s - \bar{h}(s)) ds \right\} \right] \\ = \mu \exp \left\{ \frac{\mu}{2} \int_0^T \text{tr}(\bar{\gamma}(s, s) Q_s) ds \right\}. \end{aligned}$$

RS problem, result

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Consider $\hat{h} = (\hat{h}(t), t \geq 0)$: solution of RS filtering problem

▶ The second problem: Risk Sensitive Filtering

RS, solution

$$\hat{h}(t) = m_t + \int_0^t \bar{\gamma}(t, s) A'_s [dY_s - A_s \hat{h}(s) ds],$$

we have also the equality of two solutions $\hat{h} = \bar{h}$.

RS problem, result

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Consider $\hat{h} = (\hat{h}(t), t \geq 0)$: solution of RS filtering problem

► The second problem: Risk Sensitive Filtering

RS, solution

$$\hat{h}(t) = m_t + \int_0^t \bar{\gamma}(t, s) A'_s [dY_s - A_s \hat{h}(s) ds],$$

we have also the equality of two solutions $\hat{h} = \bar{h}$.

Proof of LEG

The following uniform lower bound holds:

$$\begin{aligned} & E \left[\mu \exp \left\{ \frac{\mu}{2} \int_0^T (X_s - h(s))' Q_s (X_s - h(s)) ds \right\} \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\exp \left\{ \frac{\mu}{2} \int_0^T (X_s - h(s))' Q_s (X_s - h(s)) ds \right\} / \mathcal{Y}_T \right] \right] \end{aligned}$$

► Conditional expectation of the exponent: the answer

$$\begin{aligned} & \geq \mu \exp \left\{ \frac{\mu}{2} \int_0^T \text{tr}(\bar{\gamma}(s, s) Q_s) ds \right\} \\ & \times \mathbb{E} \left[\exp \left\{ \int_0^T (z_s^h - \pi_s(X))' A_s' d\nu_s - \frac{1}{2} \int_0^T \|A_s(z_s^h - \pi_s(X))\|^2 ds \right\} \right] \\ & = \mu \exp \left\{ \frac{\mu}{2} \int_0^T \text{tr}(\bar{\gamma}(s, s) Q_s) ds \right\}. \end{aligned}$$

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Proof of LEG

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- Fixe $\mu = -1$, $M = 0$.
- Denote by $G_t = \exp \left\{ -\frac{1}{2} \int_0^t [(X - h)' Q (X - h)] ds \right\}$.
- We want to find $\pi_T(G)$.
- General filtering theorem:

$$\pi_t(G) = \exp \left\{ -\frac{1}{2} \int_0^t \alpha_2(s) ds + \int_0^t [\alpha_1(s) - \pi_s(X)]' A_s' d\nu_s - \frac{1}{2} \int_0^t \|A_s(\alpha_1(s) - \pi_s(X))\|^2 ds \right\},$$

- with

$$\alpha_1(t) = \frac{\pi_t(XG)}{\pi_t(G)},$$

and

$$\alpha_2(t) = \frac{\pi_t(G(X - h)' Q (X - h))}{\pi_t(G)}.$$

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Bayes formula and Girsanov theorem

► Auxiliary filtering problem

- Introduce the new measure \tilde{P}_t :

$$\frac{d\tilde{P}_t}{dP} = \exp \left\{ - \int_0^t (X_s - h(s))' Q_s^{\frac{1}{2}} d\bar{B}_s - \frac{1}{2} \int_0^t (X_s - h(s))' Q_s (X_s - h(s)) ds \right\}.$$

- Thanks to our construction:

- $e^{-\xi_t} = G_t \rho_t$, with $\rho_t = \frac{d\tilde{P}_t}{dP}$
- X does not depend on Y^2 .

- Bayes formula and Girsanov theorem:

$$\frac{\mathbb{E}(X_t e^{-\xi_t}) / \bar{\mathcal{Y}}_t}{\mathbb{E}(e^{-\xi_t}) / \bar{\mathcal{Y}}_t} = \frac{\tilde{\mathbb{E}}(X_t G) / \bar{\mathcal{Y}}_t}{\tilde{\mathbb{E}}(G) / \bar{\mathcal{Y}}_t} = \frac{\pi_t(XG)}{\pi_t(G)} = \alpha_1(t),$$

- $$\frac{\mathbb{E}(e^{-\xi_t} (X_t - h_t)' Q_t (X_t - h_t)) / \bar{\mathcal{Y}}_t}{\mathbb{E}(e^{-\xi_t}) / \bar{\mathcal{Y}}_t} = \alpha_2(t).$$

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Bayes formula and Girsanov theorem

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Using a conditionally Gaussian properties of (X, ξ) wrt $\bar{\mathcal{Y}}_t$

$$\frac{\mathbb{E}(X_t e^{-\xi_t / \bar{\mathcal{Y}}_t})}{\mathbb{E}(e^{-\xi_t / \bar{\mathcal{Y}}_t})} = \bar{\pi}_t(X) - \bar{\gamma}_{X\xi}(t),$$

and

$$\frac{\mathbb{E}((X_t - h(t))' Q_t (X_t - h(t)) e^{-\xi_t / \bar{\mathcal{Y}}_t})}{\mathbb{E}(e^{-\xi_t / \bar{\mathcal{Y}}_t})}$$
$$= [\bar{\pi}_t(X) - \bar{\gamma}_{X\xi}(t) - h(t)]' Q_t [\bar{\pi}_t(X) - \bar{\gamma}_{X\xi}(t) - h(t)] + \text{tr}(\bar{\gamma}_{XX}(t) Q_t).$$

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Methodology

- Write equations for
 - the variance of the filtering error $\bar{\gamma}_{xx}(t)$;
 - for the difference $z_t^h = \bar{\pi}_t(X) - \bar{\gamma}_{X\xi}(t)$.
- Solve the LEG filtering problem: take $\bar{h} = z_t^{\bar{h}}$.

Notation

$(B, \tilde{B}) = ((B_t, \tilde{B}_t), t \geq 0)$ is a $(n + d)$ -dimensional standard Brownian motion.

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Markov observation model, independent noises

Observation model

$$\begin{cases} dX_t = a_t X_t dt + \sigma_1(t) dB_t, \\ dY_t = A_t X_t dt + d\tilde{B}_t. \end{cases}$$

The solution of LEG filtering problem

$$\begin{cases} d\bar{h}_t = a_t \bar{h}_t dt + \bar{\gamma}_{xx}(t) A'_s [dY_t - A_t \bar{h}_t dt], \\ \frac{d\bar{\gamma}_{xx}(t)}{dt} = a_t \bar{\gamma}_{xx}(t) + \bar{\gamma}_{xx}(t) a'_t - \bar{\gamma}_{xx}(t) [-\mu Q_t + A'_t A_t] \bar{\gamma}_{xx}(t) \\ + \sigma_1(t) \sigma'_1(t). \end{cases}$$

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Correlated signal - observation noises

Observation model

$$\begin{cases} dX_t = a_t X_t dt + \sigma_1(t) dB_t + \sigma_2(t) d\tilde{B}_t \\ dY_t = A_t X_t dt + d\tilde{B}_t. \end{cases}$$

The solution of LEG filtering problem

$$\begin{cases} d\bar{h}_t = a_t \bar{h}_t dt + [\sigma_2(t) + \bar{\gamma}_{XX}(t) A'_s] [dY_t - A_t \bar{h}_t dt] \\ \frac{d\bar{\gamma}_{XX}(t)}{dt} = [a_t - \sigma_2(t) A_t] \bar{\gamma}_{XX}(t) + \bar{\gamma}_{XX}(t) [a_t - \sigma_2(t) A_t]' \\ \quad - \bar{\gamma}_{XX}(t) [-\mu Q_t + A'_t A_t] \bar{\gamma}_{XX}(t) + \sigma_1(t) \sigma_1'(t) \end{cases}$$

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The solution of LEG filtering problem

$$\begin{cases} d\bar{h}_t = a_t \bar{h}_t dt + [\sigma_2(t) + \bar{\gamma}_{XX}(t) A'_s][dY_t - A_t \bar{h}_t dt] \\ \frac{d\bar{\gamma}_{XX}(t)}{dt} = [a_t - \sigma_2(t) A_t] \bar{\gamma}_{XX}(t) + \bar{\gamma}_{XX}(t) [a_t - \sigma_2(t) A_t]' \\ \quad - \bar{\gamma}_{XX}(t) [-\mu Q_t + A'_t A_t] \bar{\gamma}_{XX}(t) + \sigma_1(t) \sigma_1'(t) \end{cases}$$

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Observation model

$$\begin{cases} X_t = \int_0^t a(t, s) X_s ds + \int_0^t K(t, s) dB_s \\ dY_t = A_t X_t dt + d\tilde{B}_t, \end{cases}$$

with a nice kernel K .

Solution of the LEG filtering problem

$$\begin{cases} \bar{h}_t = \int_0^t a(t, s) \bar{h}_s ds + \int_0^t \bar{\gamma}(t, s) A'_s [dY_s - A_s \bar{h}_s ds] \\ \bar{\gamma}(t, s) = \int_0^s [a(t, r) \bar{\gamma}(s, r) + \bar{\gamma}(t, r) a(s, r)'] - \\ - \int_0^s \bar{\gamma}(t, r) [-\mu Q_r + A'_r A_r] \bar{\gamma}(s, r) dr + \int_0^s K(t, r) K(s, r) dr \end{cases}$$

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Observation model

$$\begin{cases} dX_t = a_t X_t dt + dG_t \\ dY_t = A_t X_t dt + d\tilde{G}_t, \end{cases}$$

where \tilde{G}_t and G_t two independent Ornstein-Uhlenbeck processes:

noises

$$\begin{cases} dG_t = \beta_t G_t dt + dB_t \\ d\tilde{G}_t = \tilde{\beta}_t \tilde{G}_t dt + d\tilde{B}_t \end{cases}$$

Change of the observation model

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Another form of the same observation model:

$$\begin{cases} d\tilde{X}_t = \tilde{a}_t \tilde{X}_t dt + \tilde{\sigma}_1(t) dB_t + \tilde{\sigma}_2(t) d\tilde{B}_t \\ dY_t = \tilde{A}_t \tilde{X}_t dt + d\tilde{B}_t, \end{cases}$$

with

$$\tilde{X}'_t = \begin{pmatrix} X_t & \tilde{G}_t & G_t \end{pmatrix}, \tilde{\sigma}'_1 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}, \tilde{\sigma}'_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix},$$

$$\tilde{a}_t = \begin{pmatrix} a(t) & 0 & \beta_t \\ 0 & \tilde{\beta}_t & 0 \\ 0 & 0 & \beta_t \end{pmatrix} \text{ and } \tilde{A}_t = \begin{pmatrix} A_t & \tilde{\beta}_t & 0 \end{pmatrix}.$$

Change of criterium

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Another form of the same criterium

$$L_T(h, \mu) = \mathbb{E} \left[\mu \exp \left\{ \frac{\mu}{2} \int_0^T (\tilde{X}_s - h(s))' \tilde{Q}_s (\tilde{X}_s - h(s)) ds \right\} \right]$$

$$\text{with } \tilde{Q}_t = \begin{pmatrix} Q_t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The solution of the LEG filtering problem: \bar{h}^1 is the first component of \bar{h}_t from "Correlated noises" model.

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For a given **positive** symmetric deterministic matrices

$\Omega_s, 0 \leq s \leq T$ let us denote by $\Phi_t(h) = (X_t', h_t')\Omega_t \begin{pmatrix} X_t \\ h_t \end{pmatrix}$.

“LEG setting”

$$\bar{h} = \arg \min_{h \in \mathcal{H}} \mathbb{E} \left[\mu \exp \left\{ \frac{\mu}{2} \int_0^T \Phi_s(h) ds \right\} \right],$$

where h is a (\mathcal{Y}_s) adapted process.

“RS setting”

$$\hat{h}(t) = \arg \min_{g \in \mathcal{Y}_t} \mathbb{E} \left[\mu \exp \left\{ \frac{\mu}{2} \Phi_t(g) + \frac{\mu}{2} \int_0^t \Phi_s(\hat{h}) ds \right\} / \mathcal{Y}_t \right],$$

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Equality of two solutions, yes

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The question

Does the equality $\bar{h} = \hat{h}$ hold ?

One possible answer

Yes for **degenerated** matrices Ω :

$$\Omega_{1,1} = \Omega_{2,2} = -\Omega_{1,2} = -\Omega'_{2,1} = Q.$$

Equality of two solutions, yes

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Equality of two solutions, no

LEG problem, solution

$$\Omega = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, A = 1, \mu = -1 \text{ and } X_t = B_t.$$

$$\bar{h}(t) = \int_0^t \bar{H}(T, t, s) dY_s,$$

where

$$\bar{H}(T, t, s) = \frac{\sinh(\sqrt{3}s) \cosh(T-t)}{\sqrt{\alpha_t \alpha_s}},$$

$$\alpha_t = \frac{\sqrt{3}+1}{2} \cosh(T+(\sqrt{3}-1)t) + \frac{\sqrt{3}-1}{2} \cosh(T-(\sqrt{3}+1)t).$$

$\bar{h}(t)$ depends of T .

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RS problem, solution

$$\hat{h}(t) = \int_0^t \hat{H}(t, s) dY_s,$$

where

$$\hat{H}(t, s) = \frac{\sinh(\sqrt{3}s) \sqrt[3]{\cosh(\sqrt{3}t)}}{\sqrt{3}(\cosh(\sqrt{3}t))^{2/3}},$$

and so \hat{h} and \bar{h} are different.

Open questions

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A couple of unexplored cases

- 1 The discrete time setting
- 2 Equality of the solutions of the two problems for non-Gaussian models