Controlled fractional diffusion and partial observations

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Outline

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Observations

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$$dY_t = \mu X_t dt + dW_t^H\ ,\ Y_0 = 0,$$

with

- $W^H = (W_t^H, t \geq 0)$ —— fBm with Hurst parameter H in $(\frac{1}{2}, 1)$.
- $X = (X_t,\ t \geq 0)$ —- unobservable signal
  - —- depends on unknown parameter ϑ
  - —- possibly controlled

Problems for discussion

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- To find a control to maximize the Fisher information
  Equivalent formulation: to compute explicitly the first eigenvalue of a covariance operator
- To study the long time asymptotic properties of the **implicit** MLE

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Notations

For a fixed value of the parameter ϑ let

- $\mathbf{P}_\vartheta^T$—-the probability measure of (X^T, Y^T) on $\mathcal{C}_{[0,T]} \times \mathcal{C}_{[0,T]}$
- $\mathcal{F}_t^Y$—the natural filtration of Y
- $\mathcal{L}(\vartheta, Y^T)$ —- the likelihood function
- MLE —- $\hat{\vartheta}_T = \arg\max_{\vartheta>0} \mathcal{L}(\vartheta, Y^T)$.
- Fisher information——
  $\mathcal{I}_T(\vartheta, u) = -\mathbf{E}_\vartheta \frac{\partial^2}{\partial \vartheta^2} \ln \mathcal{L}_T(\vartheta, Y^T)$.
- $\mathcal{U}_T$ —some functional space of controls

Two observation models

Controlled fractional O-U

$$\begin{cases} dX_t &= -\vartheta X_t dt + \varepsilon u_t dt + \sigma dV_t^H \,, \quad X_0 = 0\,, \\ dY_t &= \mu X_t dt + dW_t^H \,, \quad Y_0 = 0, \end{cases}$$

with independents fBm's $V^H = (V_t^H, t \geq 0)$ and $W^H = (W_t^H, t \geq 0)$, and $u \in \mathcal{U}_T$

Three cases

- $\varepsilon = 0$, $\sigma = 1$—- estimation problem
- $\varepsilon = 1$, $\sigma = 0$,— optimal design problem
- $\varepsilon = \sigma = 1$—"separation principle"

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Dumped vibration of a pendulum

Design for parameter estimation

Brouste, Cai, Kleptsyna

Introduction
Problems statement
Historical Survey

Main Results

Sketch of the Proof
Change of the observation model
Likelihood function and the Fisher information
Laplace Transform computations

Work In Progress

Observation model,II

$$\begin{cases} \ddot{X}_t & = & -k\dot{X}_t - \vartheta X_t dt + u_t\,, & X_0 = 0 \\ dY_t & = & \mu X_t dt + dW_t^H\,, & Y_0 = 0, \end{cases}$$

$\vartheta >$ is unknown

$k > 0, \quad u \in U_T$

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The goal

$$\mathcal{J}_T(\vartheta) = \sup_{u \in \mathcal{U}_T} \mathcal{I}_T(\vartheta, u) \ .$$

Control in statistics, main goal

To find estimator $\overline{\vartheta}_T$ such that, for any compact $\mathbb{K} \subset \mathbb{R}^+$,

$$\sup_{\vartheta \in \mathbb{K}} \mathcal{J}_T(\vartheta) \mathbf{E}_\vartheta \left(\overline{\vartheta}_T - \vartheta \right)^2 = 1 + o(1), \ T \to \infty.$$

Estimation problem, the goal

To study the long time asymptotic properties of the **implicit** MLE $\hat{\vartheta}_T$

Outline

A couple of references, continuous time statement

- I.Ibragimov, R.Khasminskii (1981) — general approach
- A.Le Breton (1998), Yu.Koutoyants (2002) — partial observations, classical case ($H = 1/2$)
- M.Kleptsyna, A.Le Breton (2002) ——complete observations, fractional case
- P.Chiganskii (2009) —— partial observations, classical nonlinear case, finite space Markov signal
- (2000) A.I. Ovseevich and R.Z. Khasminskii and P.L. Chow — classical case, adaptive design problem
- (2010) A.Brouste, M.Kleptsyna— partial observations, fractional case
- (2011) A.Brouste, M.Kleptsyna and A.Popier——adaptive design problem, fractional type observations

Fractional OU signal, results, I

Optimal Input

The asymptotical optimal input in the class of controls $\mathcal{U}_T$ is

$$u_{opt}(t) = \frac{\kappa_H}{\sqrt{2\lambda}} t^{H-\frac{1}{2}},$$

where the constants λ and κ_H:

$$\kappa_H = 2H\Gamma\left(\frac{3}{2} - H\right)\Gamma\left(\frac{1}{2} + H\right), \lambda = \frac{H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{2(1-H)\Gamma(\frac{3}{2}-H)}$$

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Longtime Properties of MLE

Longtime Properties of MLE

- MLE $\hat{\vartheta}_T$ is uniformly on compacts $\mathbb{K} \subset \mathbb{R}_*^+$ consistent.
- Uniformly asymptotically normal:

$$\sqrt{T}\left(\hat{\vartheta}_T - \vartheta\right) \overset{law}{\Longrightarrow} \mathcal{N}\left(0, \mathcal{I}(\vartheta)^{-1}\right)$$

- Uniform on $\vartheta \in \mathbb{K}$ convergence of the moments: for any $p > 0$,

$$\lim_{T\to\infty} \mathbf{E}_\vartheta \left|\sqrt{T}\left(\hat{\vartheta}_T - \vartheta\right)\right|^p = \mathbf{E}\left|\mathcal{I}(\vartheta)^{-\frac{1}{2}}\zeta\right|^p, \ \zeta \sim \mathcal{N}(0,1).$$

- MLE is efficient:
$\sup_{\vartheta\in\mathbb{R}_*^+} \mathcal{I}(\vartheta) T \mathbf{E}_\vartheta \left(\overline{\vartheta}_T - \vartheta\right)^2 = 1 + o(1)\,,$
as $T \to \infty$.

The Fisher information **does not depend on** H

The Fisher information

- $\varepsilon = 0,\ \sigma = 1$—- estimation problem

$$\mathcal{J}(\vartheta) = \frac{1}{2\vartheta} - \frac{2\vartheta}{\alpha(\alpha+\vartheta)} + \frac{\vartheta^2}{2\alpha^3}$$

and $\alpha = \sqrt{\mu^2 + \vartheta^2}$.

- $\varepsilon = 1,\ \sigma = 0$,— optimal design problem

$$\mathcal{J}_T(\vartheta) = \frac{\mu^2}{\vartheta^4}\,.$$

- $\varepsilon = \sigma = 1$—"separation principle"

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Dumped vibration of a pendulum, Result

Optimal Input

- $k^2 \geq 2\vartheta$
  the optimal input——$u_{opt}(t) = \frac{\kappa_H}{\sqrt{2\lambda}} t^{H-\frac{1}{2}}$
- $k^2 < 2\vartheta$
  the optimal input——$u_{opt}(t) = \frac{\kappa_H}{\sqrt{2\lambda}} t^{H-\frac{1}{2}} e^{i\omega t}$,
  $\omega = \pm\sqrt{\vartheta - \frac{k^2}{2}}$

Constants λ and κ_H:

$$\kappa_H = 2H\Gamma\left(\frac{3}{2} - H\right)\Gamma\left(\frac{1}{2} + H\right), \lambda = \frac{H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{2(1-H)\Gamma(\frac{3}{2}-H)}$$

Optimal input depends on ϑ —two step adaptive estimation procedure is applied.

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Dumped vibration of a pendulum, Result

Optimal Fisher Information

$$\mathcal{J}(\vartheta) = \frac{\mu^2}{\vartheta^4}$$

for $k^2 \geq 2\vartheta$ and

$$\mathcal{J}(\vartheta) = \frac{16\mu^2}{(k^4 - 4k^2\vartheta)^2}$$

for $k^2 < 2\vartheta$.

Outline

Equivalent representation of the observations for two problems

- *Fundamental semimartingale* associated to Y—-

$$Z_t^O = \int_0^t k_H(t,s)dY_s; \quad k_H(t,s) = (t-s)^{(\frac{1}{2}-H)} s^{\frac{1}{2}-H}.$$

- Equivalent representation of the observations:

$$dZ_t^O = \mu\lambda l(t)^* \zeta_t d\langle N\rangle_t + dN_t\,, \quad Z_0^O = 0\,.$$

Estimation Problem

Nonhomogeneous Markov "signal process"

$$\zeta = (\zeta_t, t \geq 0)$$

$$d\zeta_t = -\vartheta\lambda \mathbf{A}(t)\zeta_t d\langle M\rangle_t + b(t)dM_t, \quad \zeta_0 = 0\,,$$

with

$$l(t) = \begin{pmatrix} t^{2H-1} \\ 1 \end{pmatrix} \ \mathbf{A}(t) = \begin{pmatrix} t^{2H-1} & 1 \\ t^{4H-2} & t^{2H-1} \end{pmatrix} \ b(t) = \begin{pmatrix} 1 \\ t^{2H-1} \end{pmatrix}$$

"Control in Statistics" problem

Nonhomogeneous Markov "signal process"

$$\frac{d\zeta_t}{d\langle N\rangle_t} = -\vartheta\lambda \mathbf{A}(t)\zeta_t + b(t)v(t), \quad \zeta_0 = 0,$$

where $v(t) = \frac{d}{d\langle N\rangle_t}\int_0^t k_H(t,s)u(s)ds$

and

Class of Admissible Controls

$$\mathcal{V}_T = \left\{ v \quad | \quad \frac{1}{T}\int_0^T |v(t)|^2 d\langle N\rangle_t \leq 1 \right\}.$$

Outline

Likelihood function for the Estimation Problem,I

Likelihood function

$$\mathcal{L}_T(\vartheta) = \exp\left\{\mu\lambda \int_0^T l^* \pi_t(\zeta) dZ_t^0 - \frac{\mu^2\lambda^2}{2} \int_0^T \pi_t(\zeta) ll^* \pi_t(\zeta)^* d\langle N\rangle_t \right.$$

where

Conditional Expectation of "Signal"

$$d\pi_t(\zeta) = \left(-\vartheta\lambda \mathbf{A} - \mu^2\lambda^2\gamma_{\zeta,\zeta} ll^*\right) \pi_t(\zeta) d\langle N\rangle_t + \mu\lambda\gamma_{\zeta\zeta} l dZ_t^O,$$

$\delta_{\vartheta_1,\vartheta_2} = \pi_t^{\vartheta_1}(\zeta) - \pi_t^{\vartheta_2}(\zeta)$——the difference of two solutions.

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$\delta_{\vartheta_1,\vartheta_2} = \pi_t^{\vartheta_1}(\zeta) - \pi_t^{\vartheta_2}(\zeta)$——the difference of two solutions.

One condition to check

Laplace transform method

Suppose that exists $a_0 < 0$ such that for all $a > a_0$, $\forall u_1, u_2 \in \mathbb{R}$,

$$\lim_{T\to\infty} L_T(a, \vartheta + \frac{u_1}{\sqrt{T}}, \vartheta + \frac{u_2}{\sqrt{T}}) = \exp\left(-a\frac{(u_2 - u_1)^2}{2}\mathcal{I}(\vartheta)\right),$$

where

$$L_T(a, \vartheta_1, \vartheta_2) = \mathbf{E}_{\vartheta_1} \exp\left\{-a\frac{\mu^2\lambda^2}{2}\int_0^T \delta^*_{\vartheta_1,\vartheta_2} ll^* \delta_{\vartheta_1,\vartheta_2} d\langle N\rangle_t\right\}.$$

Then we have the desired properties of MLE.

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Then we have the desired properties of MLE.

An important question

Question for discussion

How to compute the Laplace transforms of a quadratic functional of a Gaussian process?

Fischer information,"Control in Statistics" problem

Likelihood function

$$\mathcal{L}_T(\vartheta) = \exp\left\{ \mu\lambda \int_0^T \ell(t)^* \zeta_t dZ_t - \frac{\mu^2\lambda^2}{2} \int_0^T \zeta_t^* \ell(t)\ell(t)^* \zeta_t d\langle N\rangle_t \right\}$$

Fisher information

$$\mathcal{I}_T(\vartheta, v) = \int_0^T \left(\frac{\partial \zeta_t}{\partial \vartheta}\right)^* \mu^2\lambda^2 \ell(t)\ell(t)^* \frac{\partial \zeta_t}{\partial \vartheta} d\langle N\rangle_t .$$

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"Control in Statistics" problem, reformulation

Reformulation

$$\mathcal{J}_T(\vartheta) = \sup_{v \in \mathcal{V}_T} \mathcal{I}_T(\vartheta, v)$$

$$= T \sup_{v \in L^2[0,T]} \int_0^T \int_0^T K_T(s,\sigma) v(s) v(\sigma) ds d\sigma$$

$$= T \sup_{v \in L^2[0,T], \, ||v||=1} (\mathcal{K}_T v, v) \, .$$

with some deterministic positive defined function $K_T(t,s)$.

$\mathcal{J}_T(\vartheta)$——first eigenvalue of $\mathcal{K}_T$

"Control in Statistics" problem, reformulation

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Two bounds

Lower bound

$$\frac{\mathcal{J}_T(\vartheta)}{T} \geq \frac{\mu^2}{\vartheta^4}$$

Laplace transform method for the upper bound

Suppose that for $-\frac{\vartheta^4}{\mu^2} < a < 0$ $L_T(a) < \infty$, where

$$L_T(a) = \mathbf{E}_\vartheta \exp\left\{-a \int_0^T \left[\frac{\mu}{2}\left(\frac{\partial}{\partial\vartheta}\xi_t^1\right) b(t) t^{\frac{1}{2}-H}\right]^2 dt\right\}$$

Then $\frac{\mathcal{J}_T(\vartheta)}{T} \leq \frac{\mu^2}{\vartheta^4}$.

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Outline

Laplace Transform computations

Let $X = (X_t, t \geq 0)$ be a $\mathbb{R}^p$-valued continuous centered Gaussian process with covariance function $K = (K(t,s), t \geq 0, s \geq 0)$. Then

Theorem

For any $t \geq 0$

$$\mathbb{E}\exp\{-\frac{1}{2}\int_0^t X_s' Q_s X_s ds\} = \exp(-\frac{1}{2}\int_0^t [\mathrm{tr}(\gamma(s,s)Q(s))]ds)\,,$$

where $\gamma = (\gamma(t,s), 0 \leq s \leq t)$ *is the unique solution of the Riccati-Volterra equation:*

$$\gamma(t,s) = K(t,s) - \int_0^s \gamma(t,u)Q_u\gamma'(s,u)du\,, \quad 0 \leq s \leq t\,.$$

Work in progress

- Estimation problem with different Hurst indexes in signal and observation
- Discrete time setting
- Mixed fbm noises for different $H's$