Reflected BSDEs and Applications

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Outlines

1. Reflected BSDEs
2. Systems of reflected BSDEs
3. Application in switching problems and systems of variational inequalities.

1. Reflected BSDEs. First results

- $B := (B_t)_{t \leq T}$ a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$;
- $(F_t)_{t \leq T}$ the completed natural filtration of B.

Let:

(i) $\xi \in L^2(F_T, \mathbf{R}, P)$;

(ii) $f(t, \omega, y, z)$ a function with values in $\mathbf{R}$;

(iii) $S := (S_t)_{t \leq T}$ an $\mathbf{R}$-valued uniformly square integrable s.t. $S_T \leq \xi$.

(iv) Assume adaptation w.r.t. $(F_t)_{t \leq T}$.

Then:

Theorem (El-Karoui et al., [97]): There exists a unique triple $(Y_t, Z_t, K_t)_{t\leq T}$, valued in $\mathbf{R}^{1+d+1}$ and F_t-adapted (K continuous and increasing) such that :

$$\begin{cases} E[\sup_{t\leq T}|Y_t|^2 + \int_0^T |Z_s|^2 ds + K_T^2] < \infty; \\ \\ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \\ \qquad\qquad K_T - K_t - \int_t^T Z_s dB_s, \forall t \leq T; \\ \\ Y \geq S \text{ and } \int_0^T (Y_t - S_t)dK_t = 0. \end{cases} \quad (1)$$

In addition, Y satisfies : $\forall t \leq T$,

$$Y_t = \operatorname{esssup}_{\tau \geq t} E[\int_t^\tau f(s, \omega, Y_s, Z_s) ds + S_\tau 1\!\!1_{[\tau < T]} + \xi 1\!\!1_{[\tau = T]} | \mathcal{F}_t]. \quad \blacksquare$$

Comparison : If $f \leq f'$, $S \leq S'$ and $\xi \leq \xi'$, then:

$$Y \leq Y'. \blacksquare$$

Motivation: Pricing of American options in constrained markets.

Generalization (H.-Popier, '10): The data in L^p with $p \in (1, 2)$. ■

Connection with PDEs

For $(t, x) \in [0, T] \times R^k$, let $X^{t,x}$ be s.t.:

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s, \ s \in [t, T]; \\ X_s^{t,x} = x \text{ for } s \leq t; \end{cases}$$

Let $(Y_t, Z_t, K_t)_{t \leq T}$ be the solution of the reflected BSDE associated with $(\varphi(s, X_s^{t,x}, y, z), \ell(s, X_s^{t,x}), \pi(X_T^{t,x}))$: $\forall s \leq T$,

$$\begin{cases} Y_s = \pi(X_T^{t,x}) + \int_s^T \varphi(r, X_r^{t,x}, Y_r, Z_r)dr + K_T - K_s - \int_s^T Z_r dB_r, \\ Y_s \geq \ell(s, X_s^{t,x}), \\ \int_0^T (Y_s - \ell(s, X_s^{t,x}))dK_s = 0. \end{cases}$$

Then there exists a deterministic continuous function $\varpi(t,x)$ such that:

(i) For any (t,x), $s \in [t,T]$, $Y_s = \varpi(s, X_s^{t,x})$.

(ii) ϖ is a unique solution in viscosity sense of the following variational inequality (or PDE with obstacle):

$$\begin{cases} \min\{\varpi(t,x) - \ell(t,x); \\ \quad -\partial_t \varpi(t,x) - \mathcal{L}\varpi(t,x) - \ell(t,x,\varpi(t,x),\sigma^\top(t,x)D_x\varpi(t,x))\} = 0; \\ v(T,x) = \pi(x). \end{cases}$$

■

2. Systems of Reflected BSDEs with Oblique Reflection

Let $m \geq 2$ and $\mathcal{J} = \{1, \dots, m\}$ and $\mathcal{J}^{-i} := \mathcal{J} - \{i\}$. A system of reflected BSDEs with inter-connected obstacles or oblique reflection is: $\forall i \in \mathcal{J}$,

$$\begin{cases} Y_t^i = \xi_i + \int_t^T f_i(s, \omega, Y_s^1, \dots, Y_s^m, Z_s^i)ds + K_T^i - K_t^i - \int_t^T Z_s^i dB_s; \\ Y_t^i \geq \max_{j \in \mathcal{J}^{-i}}\{Y_t^j - g_{ij}(t)\}, \ \forall t \leq T \\ \int_0^T (Y_s^i - \max_{j \in \mathcal{J}^{-i}}\{Y_s^j - g_{ij}(s)\})dK_s^i = 0. \end{cases} \tag{2}$$

We make the following

Assumptions:

For $i, j \in \mathcal{J}$,

- $f_i : (t, \omega, \overrightarrow{y}, z) \mapsto f_i(t, \omega, \overrightarrow{y}, z)$ $(\overrightarrow{y} = (y^1, ..., y^m))$ a function

s.t.:

(i) f_i is Lipschitz in (y_i, z) uniformly in the other components ;

(ii) $(\sup_{\overrightarrow{y}, y_i=0} |f_i(t, \overrightarrow{y}, 0)|)_{t \le T} \in L^2(dP)$;

(iii) f_i satisfies: $\forall i \in \mathcal{J}$, for any $k \in \mathcal{J}^{-i}$, the mapping:

$$y_k \in R \mapsto f_i(t, y_1, ..., y_{k-1}, y_k, y_{k+1}, ..., y_m, z)$$

is non-decreasing when the other components are fixed.

• (i) $g_{ij}(t,\omega) \geq 0$ ($i \neq j$) and unif. in L^2 ;

(ii) for any $t \leq T$, there is no sequence of indices $i_1, ..., i_k$ such that $i_1 = i_k$, $card\{i_1, ..., i_k\} = k - 1$ and

$$\mathbf{P}[g_{i_1 i_2}(t) + g_{i_2 i_3}(t) + \cdots + g_{i_{k-1} i_k}(t) + g_{i_k i_1}(t) = 0] > 0.$$

This condition is called the non-free loop property.

• ξ_i is and F_T-r.v., $E[\xi_i^2] < \infty$ and

$$\forall i \in \mathcal{J},\ \xi_i \geq \max_{j \in \mathcal{J}^{-i}} (\xi_j - g_{ij}(T)).$$

Theorem: (Ham.-Zhang '09)

Assume that for any $i, j \in \mathcal{J}$, f_i, g_{ij} and ξ_i verify the above Assumptions. Then there exist an F_t-adapted processes $((Y^i, Z^i, K^i))_{i=1,m}$ that satisfy system (2).

The main steps of the proof are:

Step 1. Bounds: Let us introduce

$$\underline{f}_i(t,y,z) = \inf_{\overrightarrow{y}\in\mathbf{R}^m, y^i=y} f_i(t,\overrightarrow{y},z)$$

and

$$\bar{f}_i(t,y,z) = \sup_{\overrightarrow{y}\in\mathbf{R}^m, y^i=y} f_i(t,\overrightarrow{y},z).$$

Next let consider the two following BSDEs:

$$\begin{cases} \bar{Y}\in\mathcal{S}^2, \bar{Z}\in\mathcal{H}^{2,d} \\ \bar{Y}_t = \sum_{i=1,m}|\xi_i| + \int_t^T \sum_{i=1,m}|\bar{f}_i(s,\bar{Y}_s,\bar{Z}_s)|ds - \int_t^T \bar{Z}_s dB_s \end{cases}$$

and

$$\begin{cases} \underline{Y}^i\in\mathcal{S}^2, \underline{Z}^i\in\mathcal{H}^{2,d} \\ \underline{Y}^i_t = -|\xi_i| + \int_t^T \underline{f}_i(s,\underline{Y}^i_s,\underline{Z}^i_s)ds - \int_t^T \underline{Z}^i_s dB_s \end{cases}$$

Step 2: Construction of the approximating scheme

For $i \in \mathcal{J}$,

$$Y^{i,0} = \underline{Y}^i$$

and for $n \geq 1$,

$$\begin{cases} Y_t^{i,n} = \xi^i - \int_t^T Z_s^{i,n} dB_s + K_T^{i,n} - K_t^{i,n} + \\ \int_t^T f_i(s, Y_s^{1,n-1}, \ldots, Y_s^{i-1,n-1}, Y_s^{i,n}, Y_s^{i,n-1}, \ldots, Y_s^{m,n-1}, Z_s^{i,n}) ds \\ \\ Y_t^{i,n} \geq \max_{j \in \mathcal{J}^{-i}} \{ Y_t^{j,n-1} - g_{ij}(t) \}, \ \forall t \leq T \\ \\ \int_0^T (Y_s^{i,n} - \max_{j \in \mathcal{J}^{-i}} \{ Y_s^{j,n-1} - g_{ij}(s) \}) dK_s^{i,n} = 0. \end{cases} \quad (3)$$

- the sequence $(Y^{i,n}, K^{i,n}, K^{i,n})_n$ is defined for any $i \in \mathcal{J}$;
- by induction and comparison we have

$$\underline{Y}^i \leq Y^{i,n} \leq Y^{i,n+1} \leq \bar{Y}.$$

Therefore there exist:

(i) an RCLL process Y^i such that $Y^{i,n} \nearrow Y^i$

(ii) a process Z^i of $\mathcal{H}^{2,d}$ such that, at least for a subsequence, $(Z^{i,n})_n$ converges weakly to Z^i in $\mathcal{H}^{2,d}$ and strongly in $L^p(dt \otimes dP)$ for $p < 2$ (by a Peng's result)

(iii) an RCLL non decreasing process K^i such that for any stopping time τ, $(K^{i,n}_\tau)_n$ converges to K^i_τ in $L^p(dP)$.

Step 3: The triplets $((Y^i, Z^i, K^i))_{i=1,m}$ is a solution of system (2). We mainly show that Y^i is continuous in using the non-free loop property satisfied by g_{ij}. ■

Remark: The solution is unique if f_i, $i = 1, m$, is uniformly Lipschitz in $(y^1, ..., y^m, z)$. ■

Application in Switching Problems

(a) Introduction

(i) $X := (X_t)_{t \le T}$ a stochastic process which stands for a performance of m economies ;

(ii) An agent invests his money in the economy which provides him/her the highest performance ;

(iii) if the money is invested in the $i-th$ economy, the yield per dt is $\psi_i(t, X_t)dt$;

(iv) switching from an economy i to another one j costs $g_{ij}(t, \omega)$.

A management strategy of the agent has two components:

(i) $\delta = (\tau_n)_{n \geq 0}$ the sequence of stopping times such that $\tau_n \leq \tau_{n+1}$ and $\tau_n \to T$ P-$a.s.$. At τ_n the agent switches his money from the current economy to another one ;

(ii) $\xi = (\xi_n)_{n \geq 0}$ a sequence of $r.v.'s$ such that:

$$\xi_0 = 1,\ \xi_n(\omega) \in \mathcal{J} \text{ and } \xi_n \ is\ F_{\tau_n} - meas..$$

The pair (δ, ξ) is called a strategy of management of the agent. ■

(b) The payoff

Let $(u_t)_{t\leq T}$ be the indicator process of the economy where money is invested, i.e.,

$$u_0 = 1 \text{ and } u_t = \xi_n \text{ if } t \in]\tau_n, \tau_{n+1}] \ (n \geq 0).$$

When a strategy (δ, ξ) is implemented the yield is given by:

$$\Gamma(\delta, \xi) := \mathbf{E}[\int_0^T \psi_{u_s}(s, X_s)ds - \sum_{n\geq 1} g_{u_{\tau_{n-1}}, u_{\tau_n}}(\tau_n)1\!\!1_{[\tau_n < T]}].$$

(c) Problems

(i) Existence of an optimal strategy (δ^*, ξ^*), i.e.,

$$\Gamma(\delta^*, \xi^*) = \sup_{(\delta,\xi)} \Gamma(\delta, \xi).$$

(ii) What can be said about

$$\sup_{(\delta,\xi)} \Gamma(\delta, \xi)$$

in terms of characterization, properties, simulation,... ■

(d) Connection with Systems of RBSDEs ($m = 3$)

Let $(Y^i, Z^i, K^i)_{i=1,3}$ be the solution of the following system of reflected BSDEs: For $i = 1, ..., 3$,

$$\begin{cases} Y_t^i = \int_t^T \psi_i(s, X_s)ds + K_T^i - K_t^i - \int_t^T Z_s^i dB_s; \\ \\ Y_t^i \geq \max_{j \in \mathcal{J}^{-i}} \{Y_t^j - g_{ij}(t)\}, \ \forall t \leq T \\ \\ \int_0^T (Y_s^i - \max_{j \in \mathcal{J}^{-i}} \{Y_s^j - g_{ij}(s)\}) dK_s^i = 0. \end{cases} \quad (4)$$

Then:

Theorem: (Djehiche-H.-Popier, '07)

(i)

$$Y_0^1 = \sup_{(\delta,\xi)} \Gamma(\delta, \xi).$$

(ii) The strategy $(\delta^*, \xi^*) = ((\tau_n^*)_{n\geq 1}, (\xi_n^*)_{n\geq 1})$ defined by:

$$\tau_1^* = \inf\{s \geq 0, Y_s^1 = \max_{j\in\mathcal{J}^{-1}} (-g_{1j}(s) + Y_s^j)\},$$

$$\xi_1^* = \mathrm{argmax}_{j\in\mathcal{J}^{-1}}(-g_{1j}(\tau_1^*) + Y_{\tau_1^*}^j),$$

$$\tau_2^* = \inf\{s \geq \tau_1^*, Y_s^{\xi_1^*} = \max_{j\in\mathcal{J}^{-\xi_1^*}} (g_{\xi_1^* j}(s) + Y_s^j)\},$$

$$\xi_2^* = \mathrm{argmax}_{j\in\mathcal{J}^{-\xi_1^*}}(-g_{\xi_1^* j}(\tau_2^*) + Y_{\tau_2^*}^j), \text{ etc},$$

4. Application in systems of PDEs

Let $T > 0$, $m \geq 2 \in \mathbf{N}$.

We are concerned with the problem of existence and uniqueness in viscosity sense for the following system of variational inequalities with inter-connected obstacles: $\forall\ i \in \mathcal{J}$, $(t,x) \in [0,T] \times \mathbf{R}^k$,

$$\begin{cases} \min\Big\{ v_i(t,x) - \max_{j \in \mathcal{J}^{-i}} (-g_{ij}(t,x) + v_j(t,x)); \\ -\partial_t v_i(t,x) - \mathcal{L}v_i(t,x) - f_i(t,x,v^1(t,x),\ldots, \\ \qquad v^m(t,x), \sigma^\top(t,x) D_x v^i(t,x)) \Big\} = 0\ ; \\ \\ v_i(T,x) = h_i(x) \end{cases} \quad (5)$$

where the unknowns are the functions $(v^i)_{i \in \mathcal{J}}$.

The functions $(h_i)_{i\in\mathcal{J}}$, $(g_{ij})_{i,j\in\mathcal{J}}$ and $(f_i)_{i\in\mathcal{J}}$ are given. The generator $\mathcal{L}$ is given by:

$$\mathcal{L}\varphi(t,x) := \frac{1}{2}Tr[(\sigma.\sigma^{\top})(t,x)D^2_{xx}\varphi(t,x)] + b(t,x)^{\top}.D_x\varphi(t,x).$$

Remark: In the case when $f_i(t, x, y^1, ..., y^m, z) \equiv \psi_i(t, x)$, system (5) is just the HJB system of equations associated with the switching problem.

(i) Definition of a solution of system (5)

• Let $u : (t,x) \in [0,T] \times \mathbf{R}^k \mapsto u(t,x) \in \mathbf{R}$ be a locally bounded function. We define its *lsc* (resp. *usc*) envelope u_* (resp. u^*) as follows:

$$u_*(t,x) = \underline{\lim}_{(t',x')\to(t,x),\, t'<T} u(t',x')$$

and

$$u^*(t,x) = \overline{\lim}_{(t',x')\to(t,x),\, t'<T} u(t',x').$$

• For a function $u(t,x)$ lsc (resp. usc), we denote $J^{-}u(t,x)$ (resp. $J^{+}u(t,x)$) the parabolic subjet (resp. superjet) of u at (t,x), as the triples $(p,q,M) \in R^{1+k} \times \mathbf{S}^k$ satisfying:

$$\begin{aligned} u(t',x') \quad &\geq \quad (\text{resp. } \leq) \\ &u(t,x) + p(t'-t) + < q, x'-x > \\ &\quad + \frac{1}{2} < x'-x, M(x'-x) > + o(|t'-t| + |x'-x|^2) \end{aligned}$$

where $\mathbf{S}^k$ is the set of symmetric real matrices of dimension k.

• For a function $u(t,x) \in \mathbf{R}$, lsc (resp. usc), we denote by $\bar{J}^- u(t,x)$ (resp. $\bar{J}^+ u(t,x)$) the parabolic limiting subjet (resp. superjet) of u at (t,x), as the set of triples $(p,q,M) \in \mathbf{R}^{1+k} \times \mathbf{S}^k$ such that:

$$(p,q,M) = \lim_n (p_n, q_n, M_n), (t,x) = \lim_n (t_n, x_n)$$
$$\text{with } (p_n, q_n, M_n) \in J^- u(t_n, x_n)$$

$$\text{(resp. } J^+ u(t_n, x_n)) \text{ and } u(t,x) = \lim_n u(t_n, x_n).$$

• (i) A function $(v_1, ..., v_m) : [0, T] \times \mathbf{R}^k \to \mathbf{R}^m$ s.t. $\forall\, i \in \mathcal{J}$, v_i is lsc (resp. usc), is called a viscosity supersolution (resp. subsolution) of (5) if :

$\forall\, i \in \mathcal{J}$, $(t, x) \in [0, T) \times \mathbf{R}^k$ and $(p, q, M) \in \bar{J}^- v_i(t, x)$ (resp. $\bar{J}^+ v^i(t, x)$) we have:

$$\begin{cases} \min\Big\{ v_i(t,x) - \max_{j \in \mathcal{J}^{-i}} (-g_{ij}(t,x) + v_j(t,x)); \\ -p - b(t,x)^\top.q - \frac{1}{2}\mathrm{Trace}((\sigma.\sigma^\top)(t,x).M) \\ \qquad\qquad -f_i(t,x,v^1(t,x),...,v^m(t,x),\sigma^\top(t,x).q)\Big\} \\ \geq \text{(resp. } \leq \text{)}\ 0\,; \\ \\ v_i(T,x) \geq \text{(resp. } \leq \text{)}\ h_i(x). \end{cases}$$

■

• (ii) A locally bounded function $(v_1, ..., v_m) : [0, T] \times \mathbf{R}^k \to \mathbf{R}^m$ is called a viscosity solution to (5) if $(v_{1*}, ..., v_{m*})$ (resp. $(v_1^*, ..., v_m^*)$) is a viscosity supersolution (resp. subsolution) of (5). ■

System (5) is connected with systems of reflected BSDEs with inter-connected obstacles of the following type: for any $i \in \mathcal{J}$ and $(t,x) \in [0,T] \times \mathbf{R}^k$,

$$\begin{cases} Y_s^{i;t,x} = h^i(X_T^{t,x}) + \\ \qquad \int_s^T f_i(r, X_r^{t,x}, Y_r^{1;t,x}, \dots, Y_r^{m;t,x}, Z_r^{i;t,x})dr \\ \qquad\qquad + K_T^{i;t,x} - K_s^{i;t,x} - \int_s^T Z_r^{i;t,x} dB_r; \\ \\ Y_s^{i;t,x} \geq \max_{j \in \mathcal{J}^{-i}} \{ Y_s^{j;t,x} - g_{ij}(s, X_s^{t,x}) \}; \\ \\ (Y_s^{i;t,x} - \max_{j \in \mathcal{J}^{-i}} \{ Y_s^{j;t,x} - g_{ij}(s, X_s^{t,x}) \}) dK_s^{i;t,x} = 0. \blacksquare \end{cases} \tag{6}$$

Assumptions: Mainly the same as the ones of Theorem on existence of a solution for the system (2) + $f_i(t, x, y^1, ..., y^m, z)$ uniformly Lipschitz in $(y^1, ..., y^m, z, z)$, for any $i \in \mathcal{J}$.

Theorem: (H.-Morlais, '12) There exist deterministic continuous functions with polynomial growth $(v^i(t, x))_{i \in \mathcal{J}}$ such that

$$\forall s \in [t, T], \; Y_s^{i;t,x} = v^i(s, X_s^{t,x}).$$

Moreover $(v^i)_{i \in \mathcal{J}}$ is the unique viscosity solution of system (2). ■

Sketch of the proof:

Step 1: Show a comparison principle:

$$\text{Subsolutions} \leq \text{Supersolutions}.$$

Step 2: Let $v^{i,n}$, $i \in \mathcal{J}$, be the deterministic continuous functions such that:

$$Y_s^{i;t,x;n} = v^{i,n}(s, X_s^{t,x}), s \in [t, T]$$

which exist by El-Karoui et al.'s result. Then:

$$(v^{i,n})_n \nearrow v^i \text{ and } (v^i)_{i=1,m}$$

is a solution for (5). ∎