Mixed Gaussian processes: a filtering approach

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9 April 2015/ Kiev

Outline

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Motivation and the Challenge of this talk

Challenge

To present a new approach to analysis of mixed Gaussian processes based on the linear filtering theory.

Motivation

To construct the likelihood type estimates for mixed Gaussian noises systems:

$$Y_t = \int_0^t f(\theta, s)\, ds + X_t, 0 \le t \le T,$$

Object of studies

Mixed Gaussian process

where

$$X_t = B_t + G_t, \quad t \in [0, T], \quad T > 0,$$

with

- B_t — the standard Brownian motion
- G_t — an independent centered Gaussian process

with the covariance function (frequently) in the form:

$$\Gamma(s, t) = \mathbb{E} G_t G_s = \int_0^t \int_0^s K(u, v) du dv$$

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Two Partial Cases

Fractional type Mixed Noises

- $G = B^H$, $H \in (0,1)$ — a fractional Brownian motion

$$\Gamma(s,t) = \mathbb{E}B^H_t B^H_s = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big)$$

- $G = B^{L,H}$ $H \in (0,1)$ — a Riemann-Liouville process

$$B^{L,H}_t = \int_0^t (t-s)^{H-1/2} dB_s$$

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Questions around

- Stochastic analysis of X:
  - **canonical representations**: multiplicity of the innovations, structure of the fundamental martingale:

    $$X_t = \int_0^t G(s,t)dM_s \quad M_t = \int_0^t g(s,t)dX_s$$

    with an innovation martingale M; equivalence of the filtrations $F_t^X = F_t^M$
  - the semimartingale representation
  - the density with respect to the standard and fractional Wiener measures
- Stochastic analysis of Y: the density with respect to measure μ^X

Outline

1. Introduction
   - Problems statement and questions around
   - Historical survey
2. Results
   - From L^2 to L^1
   - Even from L^1, but in a partial case
   - Semimartingale Structure of X
3. The Proofs
   - Integro-Differential Equation
   - Diffusion type representation, Equivalence of measures
4. Concluding Remarks

50 years ago; independently; at the same time

L. Shepp

- $\mu^X \sim \mu^B$ if and only if $K \in L^2\big([0, T]^2\big)$
- the density $d\mu^X/d\mu^B$ involves Carleman-Fredholm determinant and resolvent kernel of the covariance operator associated with K
- it doesn't immediately reveal the innovation structure

50 years ago; independently; at the same time

I. Gohberg and M. Krein

- Factorization theory of Fredholm operators in Hilbert spaces:

$$(Id + K)^{-1} = (Id + L_+)(Id + L_-)$$

  with right and left Volterra kernels L.

- Resolvent identity for a continuous kernel K

$$L(s,t) + \int_0^t L(r,t)K(r,s)dr = -K(s,t), \quad 0 \le s \le t \le T.$$

- The crusial role of the equation

$$g(s,t)\int_0^t g(r,t)K(r,s)dr = 1, \quad g(s,s) \neq 0.$$

50 years ago; independently; at the same time

T.Kailath

- the relevance of I. Gohberg and M. Krein theory:
- Shepp's density formula is rewritten in the form

$$\frac{d\mu^X}{d\mu^B}(X) = \exp\left(-\int_0^T \varphi_t(X)dX_t - \frac{1}{2}\int_0^T \varphi_t^2(X)dt\right),$$

where

- $\varphi_t(X) = \int_0^t L(s,t)dX_s$ with $L \in L^2([0,T]^2)$ being the unique solution of the Wiener-Hopf integral equation
-

$$L(s,t) + \int_0^t L(r,t)K(r,s)dr = -K(s,t), \quad 0 \le s \le t \le T.$$

50 years ago; independently; at the same time

M.Hitsuda

- $\mu^X \sim \mu^B$ if and only if X can be represented as

$$X_t = \overline{B}_t - \int_0^t \int_0^s \ell(r,s) d\overline{B}_r ds,$$

  with *some* Brownian motion $\overline{B}$ and *some* Volterra kernel $\ell \in L^2([0,T]^2)$.
- Actually the kernel ℓ solves the Riccati-Volterra equation:

$$\ell(s,t) = K(s,t) - \int_0^{t \wedge s} \ell(s,r)\ell(t,r)dr.$$

Recovered 10 years ago

Important references

- 2000 P.Cheridito

$$\sup_{\tau} \sum_{j=0}^{n-1} \mathbb{E}\left|\mathbb{E}\left(X_{t_{j+1}} - X_{t_j} \middle| \mathbb{F}^X_{t_j}\right)\right| < \infty$$

- 2003 F.Boduin and D.Nualart

$$X := B + V,\ \partial^2 K/\partial s \partial t \in L^2([0,T]^2)$$

Hida-Hitsuda criterion.

- 2007 H.van Zanten equivalence of $\xi = \sum_{k=1}^n \alpha_k B^{H_k}$ of n independent fBm's to a single fBm. (Spectral techniques for processes with stationary increments).

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Where we are

What is missed?

- Different objects : general point of view
- Probabilistic interpretation
- What can we do for non L^2 kernels?

The week point: to consider only lower/upper triangular kernels.
What should we do: To forget the factorisation theory and try to find an other point of view

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Outline

Haw far can we go: not too close to L_1

Example

Multiplicity Greater than One

$$X_t = B_t + \xi \int_0^t \frac{1}{\sqrt{|1-s|}} ds,$$

where $\xi \sim N(0,1)$ is independent of B.

- ξ can be recovered precisely from $\mathbb{F}_t^X$ for all $t \geq 1$.
- $\mathbb{F}_t^X$ is discontinuous at $t = 1$,
- with $\mathbb{F}_{t-}^X \subsetneq \mathbb{F}_t^X = \mathbb{F}_t^B \vee \sigma\{\xi\}$ for all $t \geq 1$.

From L^2 to L^1

Assumptions

- $$X_t = B_t + G_t, \quad t \in [0, T], \quad T > 0,$$
- $$\Gamma(s,t) = \mathbb{E} G_t G_s = \int_0^t \int_0^s K(u,v)dudv$$
- $$K(s,t) = |s-t|^{-\alpha} M(s,t), \quad 0 \le \alpha < 1,$$

where $M \in C([0,T]^2)$.

Equations and interpretations

Equations

- $$L(s,t) + \int_0^t L(r,t)K(r,s)dr = -K(s,t), \quad 0 \le s,t \le T.$$

- $$q(s,t) + \int_0^t q(r,t)K(r,s)dr = \phi(s), \quad 0 \le s,t \le T.$$

with $\phi_s = 1 - \int_0^s L(r,s)dr.$

Canonical Representations

Theorem

The process

$$\overline{B}_t = \mathbb{E}\left(\int_0^t \phi_s dB_s \middle| \mathbb{F}_t^X\right)$$

is a Brownian motion, satisfying

$$\overline{B}_t = \int_0^t q(s,t) dX_s,$$

The representation

$$X_t = \int_0^t \hat{q}(s,t) d\overline{B}_s$$

with $\hat{q}(s,t) = -\frac{\partial}{\partial s}\int_s^t q(r,s)dr$, *is canonical, i.e.* $\mathbb{F}_t^X = \mathbb{F}_t^{\overline{B}}$.

Outline

Mixed fractional Brownian motion

We are back to the Krein's remark:

$$g(s,t) + \int_0^t g(r,t)K(r,s)\,dr = 1$$

Notations: $\mathbb{F}^X = (\mathbb{F}_t^X)$ and $\mathbb{F} = (\mathbb{F}_t)$, $t \in [0,T]$ —the natural filtrations of X and (B, B^H) respectively.

Fundamental Martingale

$$M_t = \mathbb{E}\left(B_t | \mathbb{F}_t^X\right), \quad t \in [0,T].$$

M encodes many of the essential features of the process X, making its structure particularly transparent.

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Fundamental Martingale Representation via X

Fundamental Martingale Representation

$$M_t = \int_0^t g(s,t)dX_s, \quad \langle M\rangle_t = \int_0^t g(s,t)ds, \quad t \ge 0.$$

The kernel $g(s,t)$ solves integro-differential equation:

Equation for the Kernel

$$g(s,t)+H\frac{d}{ds}\int_0^t g(r,t)|s-r|^{2H-1}sign(s-r)dr = 1, 0 < s < t \leq T$$

The family of functions $\{g(s,t), 0 \le s \le t \le T\}$ plays the key role in our approach to analysis of the mixed fBm.

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X is a stochastic integral w.r.t M

Representation of X via M

The following representation holds:

$$X_t = \int_0^t G(s,t)dM_s, \quad t \in [0,T],$$

where

$$G(s,t) := 1 - \frac{d}{d\langle M\rangle_s}\int_0^t g(\tau,s)d\tau, \quad 0 \le s \le t \le T.$$

and, in particular, $\mathbb{F}_t^X = \mathbb{F}_t^M$, P-a.s. for all $t \in [0,T]$.

Note that $s > \tau$ is possible.

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The fundamental Semimartingale

Let $Y = (Y_t)$ defined by

$$Y_t = \int_0^t f(s)ds + X_t, \quad t \in [0, T],$$

Then Y admits the representation

$$Y_t = \int_0^t G(s,t)dZ_s$$

The fundamental semimartingale $Z = (Z_t)$

$$Z_t = \int_0^t g(s,t)dY_s = M_t + \int_0^t \Phi(s)d\langle M\rangle_s,$$

and

$$\Phi(t) = \frac{d}{d\langle M\rangle_t} \int_0^t g(s,t)f(s)ds.$$

The measures μ^X and μ^Y

In particular, $\mathbb{F}_t^Y = \mathbb{F}_t^Z$, P-a.s. for all $t \in [0, T]$ and, if

$$E \exp\left\{ -\int_0^T \Phi(t) dM_t - \frac{1}{2}\int_0^T \Phi^2(t) d\langle M\rangle_t \right\} = 1,$$

then the measures μ^X and μ^Y are equivalent and the corresponding Radon-Nikodym density is given by

$$\frac{d\mu^Y}{d\mu^X}(Y) = \exp\left\{ \int_0^T \hat{\Phi}(t) dZ_t - \frac{1}{2}\int_0^T \hat{\Phi}^2(t) d\langle M\rangle_t \right\},$$

where $\hat{\Phi}(t) = E\left(\Phi(t)|\mathbb{F}_t^Y\right)$.

Outline

X est **diffusion** type process

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Let $H \in (\frac{3}{4}, 1]$.Then X is a **diffusion** type process:

$$X_t = W_t - \int_0^t \varphi_s(X)ds, \; W_t = \int_0^t \frac{dM_s}{g(s,s)},$$

W is an F^X-Brownian motion; $\varphi_t(X) = \int_0^t \frac{\dot{g}(s,t)}{g(t,t)} dX_s$

The density w.r.t μ^W

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Moreover, the measures μ^X and μ^W are equivalent and

$$\frac{d\mu^X}{d\mu^W}(X) = \exp\left\{-\int_0^T \varphi_t(X)dX_t - \frac{1}{2}\int_0^T \varphi_t^2(X)dt\right\}.$$

X est **fractional-diffusion** type process

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For $H \in (0, \frac{1}{4})$, X is a **fractional diffusion** type process

$$X_t = \overline{B}_t^H - \int_0^t \rho(s,t)\varphi_s(X)ds,$$

where $\overline{B}^H$ is fBm with $\mathbb{F}_t^{\overline{B}^H} = \mathbb{F}_t^X$, $\varphi_t(X) = \int_0^t L(s,t)dX_s$ and

$$L(s,t) := \frac{\partial}{\partial t} g(s,t) \Big/ \sqrt{\frac{d}{dt}\langle M\rangle_t} - \frac{\partial}{\partial t}\widetilde{\rho}(s,t),$$

The density w.r.t μ^{B^H}

The density w.r.t μ^{B^H}

The measures μ^X and μ^{B^H} are equivalent if and only if $H \in \left(0, \frac{1}{4}\right)$ and

$$\frac{d\mu^X}{d\mu^{B^H}}(X) = \exp\left\{-\int_0^T \varphi_t(X)d\widetilde{X}_t - \frac{1}{2}\int_0^T \varphi_t^2(X)dt\right\},$$

where $\widetilde{X}_t = \int_0^t \widetilde{\rho}(s,t)dX_s$.

Mixed Riemann–Liouville process

$$X_t = B_t + B_t^{L,H}$$

Everything remains valid with $g(s,t)$ solving the equation

$$g(s,t) - \frac{\partial}{\partial s}\int_0^t \Gamma(r,s)\frac{\partial}{\partial r}g(r,t)dr + g(t,t)\frac{\partial}{\partial s}\Gamma(s,t) = 1,$$

$$0 < s,t \leq T,$$

where $\Gamma(s,t)$ is the covariance function of $B^{L,H}$.

Outline

Integro-Differential Equation and its alternative forms,I

Alternative Forms,I

The kernel $g(s,t)$ is the unique continuous solution of the following equations:

- for $H \in (0,1]$, the integro-differential equation:

$$g(s,t) + H\frac{d}{ds}\int_0^t g(r,t)|s-r|^{2H-1} sign(s-r)dr = 1.$$

- for $H \in (\frac{1}{2}, 1]$, the **weakly singular integral** equation:

$$g(s,t) + H(2H-1)\int_0^t g(r,t)|s-r|^{2H-2}dr = 1.$$

Integro-Differential Equation and its alternative forms,II

Alternative Forms,II

- for $H \in (0, \frac{1}{2})$, the **weakly singular integral** equation

$$g(s,t) + \beta_H t^{-2H} \int_0^t g(r,t)\bar{\kappa}\left(\frac{r}{t}, \frac{s}{t}\right) dr =$$

$$c_H s^{1/2-H}(t-s)^{1/2-H},$$

with the kernel

$$\bar{\kappa}(u,v) = |u-v|^{-2H} N(u,v),$$

where $N \in C([0,1]^2)$.

Integral Equation with $H > 1/2$, I

Properties of $g(s,t)$ on the diagonal

The function $g(t,t)$, $t \in [0,T]$ satisfies the properties:

- $g(t,t)$ is continuous on $[0,T]$ with $g(0,0) := \lim_{t\to 0} g(t,t) = 1$
- $g(t,t) > 0$ for all $t \in [0,T]$
- $$\int_0^t g(s,t)ds = \int_0^t g^2(s,s)ds.$$

Integral Equation with $H > 1/2$, II

Properties of $\dot{g}(s,t) = \frac{\partial}{\partial t} g(s,t)$

The kernel $g(s,t)$ satisfies the following properties

- $g(s,t)$ is continuously differentiable at $t \in (0,T]$ for any $s > 0$, $s \neq t$;
- the derivative $\dot{g}(s,t) := \frac{\partial}{\partial t} g(s,t)$ satisfies the equation

$$\dot{g}(s,t) + H(2H-1)\int_0^t \dot{g}(r,t)|r-s|^{2H-2}dr =$$

$$-H(2H-1)g(t,t)|r-s|^{2H-2},\ s \in (0,t).$$

- $\dot{g}(\cdot,t) \in L^2([0,t])$ for $H > 3/4$

Outline

$H > 3/4$

X is a diffusion type process

$$M_t = \int_0^t g(s,t)dX_s = \int_0^t g(s,s)dX_s + \int_0^t \Big(g(r,t) - g(r,r)\Big)dX_r$$

$$\int_0^t g(s,s)dX_s + \int_0^t \int_r^t \dot{g}(r,s)ds dX_r =$$

$$\int_0^t g(s,s)dX_s + \int_0^t \int_0^s \dot{g}(r,s)dX_r ds,$$

where the last equality holds since $\dot{g}(\cdot, s) \in L^2([0,s])$

$H > 3/4$,II

X is a diffusion type process, II

Hence

$$W_t = \int_0^t \frac{1}{g(s,s)} dM_s = X_t + \int_0^t \int_0^s \frac{\dot{g}(r,s)}{g(s,s)} dX_r ds =: $$

$$X_t + \int_0^t \varphi_s(X) ds.$$

An interesting toy

Singular perturbations

Fix $\varepsilon > 0$ and let g_ε be the solution of the equation:

$$\varepsilon g_\varepsilon(u) + \frac{d}{du}\int_0^1 g_\varepsilon(v)|u-v|^{2H-1} sign(u-v)dv = 1,\ u \in [0,1],$$

A simple question

What can we say about g_ε when $\varepsilon \longrightarrow 0$?

An interesting toy

Singular perturbations

Fix $\varepsilon > 0$ and let g_ε be the solution of the equation:

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Some answers

The main tool

An asymptotic formula for the eigenfunctions of the corresponding integro-differential operator

About convergence

- Weak convergence with rate ε
- L^2 convergence with depending on H rate
- Boundary layer construction with $\frac{1}{\sqrt{\varepsilon}}$ rate

Spectrum of the integro-differential operator

Integro-differential operator

$$(Kf)(u) = \frac{d}{du}\int_{-1}^{1} f(v)|u-v|^{1-\alpha} sign(u-v)dv, \ \alpha = 2-2H$$

Eigenvalues

$$\lambda_n := \frac{\pi(1-\alpha)}{\Gamma(\alpha)\sin\frac{1}{2}(1-\alpha)\pi}\nu_n^{\alpha-1},$$

where

$$\nu_n = \frac{1}{2}(n-1)\pi + \frac{1}{8}(1+\alpha)\pi + O(n^{-1}), \qquad n\to\infty.$$

Spectrum of the integro-differential operator

Eigenfunctions, $\alpha \in (0,1)$

The normalized eigenfunctions of K with $\alpha \in (0,1)$ satisfy

$$\phi_n(x) = \cos\left(\nu_n(x+1) - \frac{1+\alpha}{8}\pi\right) + \frac{1}{c}\frac{1}{\pi}\int_0^\infty D(\tau)\left(e^{(x-1)\nu_n\tau} - (-1)^n e^{-(x+1)\nu_n\tau}\right)d\tau + r_n(x),$$

where the residual term satisfies $|r_n(x)| \le Cn^{-1}$ with a constant C, depending only on α.

Spectrum of the integro-differential operator

Eigenfunctions, $\alpha \in (1,2)$

For $\alpha \in (1,2)$ the normalized eigenfunctions satisfy

$$\varphi_n(x) = -\cos\left(\nu_n(x+1) - \frac{1+\alpha}{8}\pi\right) + \frac{\alpha-1}{2c}\frac{1}{\pi}\int_0^\infty D(\tau)\left(e^{-(x+1)\tau\nu_n} - (-1)^n e^{(x-1)\tau\nu_n}\right)d\tau + r_n(x),$$

where the residual term satisfies $|r_n(x)| \le Cn^{-1}$ with a constant C, depending only on α.