# Estimation of Singularity Location for Poisson Process

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## The model

– **The process.** — Poisson process of intensity function $S_\theta(\cdot)$:

$$X = \{X(t),\ 0 \leqslant t \leqslant T\}.$$

– **The observations.** — $n$ independent realizations (trajectories) of $X$:

$$(X_1, \ldots, X_n) = X^n.$$

– **The hypotheses on $S_\theta(\cdot)$.** — The intensity function $S_\theta(\cdot)$ is regular everywhere on $[0,T]$ except at the point $\theta$, where it has a singularity.

– **The unknown parameter.** — The location (the point) of the singularity:

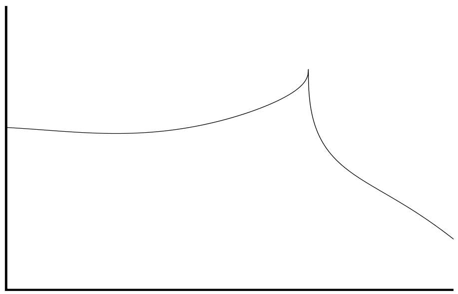
$$\theta \in \Theta = (\alpha,\beta) \subseteq (0,T).$$

– **The types of singularities.** — Three types: "cusp", singularity of "0"-type and singularity of "$\infty$"-type.

– **The asymptotics.** — $n \longrightarrow \infty$.

We consider $S_\theta(t)$ of the form $S_\theta(t) = \begin{cases} a\,|t-\theta|^p + \Psi(\theta,t), & \text{if } t \leqslant \theta \\ b\,|t-\theta|^p + \Psi(\theta,t), & \text{if } t \geqslant \theta \end{cases}$.

We suppose that $a^2+b^2>0$, $S_\theta(t)>0$ for all $t\neq\theta$, the function $\Psi(\theta,t)$ is continuous and uniformly in $t$ Hölder continuous of order $\mu$ with respect to $\theta$.

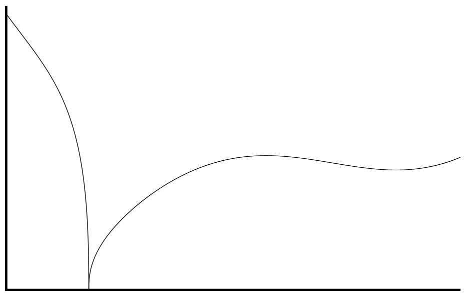


"cusp"

$0<p<1/2$

$\mu>p+1/2$

$\Psi(\theta,\theta)>0$

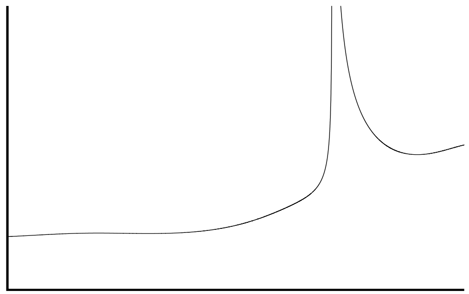


singularity of "0"-type

$0<p<1$

$\mu>(p+1)/2$

$\Psi(\theta,\theta)=0$



singularity of "$\infty$"-type

$-1<p<0$

$\mu>(p+1)/2$

## The history of the problem

– Prakasa Rao, B.L.S., "Estimation of the location of the cusp of a continuous density", *Annals of Mathematical Statistics*, vol. 20, no. 1, pp. 76–87, 1968.

– Ibragimov, I.A. et Khasminskii, R.Z., "*Statistical Estimation. Asymptotic Theory*", Springer-Verlag, New York, 1981.

– Dachian, S., "Estimation of Cusp Location by Poisson Observations", *Statistical Inference for Stochastic Processes*, to appear, 2001.

– Dachian, S., "Estimation of Singularity Location by Poisson Observations", in preparation, 2002.

– Dachian, S. et Kutoyants, Yu.A., "On Cusp Estimation of Ergodic Diffusion Process", *Journal of Statistical Planning and Inference*, to appear, 2001.

The likelihood ratio is:

$$L(\theta, X^n) = \exp\left\{ \sum_{i=1}^{n} \int_0^T \ln S_\theta(t)\, dX_i(t) - n \int_0^T \left[S_\theta(t) - 1\right] dt \right\} .$$

The maximum likelihood estimator (MLE) $\widehat{\theta}_n$ is defined as one of the solutions of the maximum likelihood equation $\quad L(\widehat{\theta}_n, X^n) = \sup_{\theta \in \Theta} L(\theta, X^n)$.

The Bayesian estimator (BE) for prior density $q(\cdot)$ and quadratic loss function is defined by $\widetilde{\theta}_n = \int_\alpha^\beta \theta\, q\big(\theta | X^n\big)\, d\theta$, where the posterior density $q\big(\cdot|X^n\big)$ is given by:

$$q\big(\theta | X^n\big) = L(\theta, X^n)\, q(\theta) \left( \int_\alpha^\beta L(\theta, X^n)\, q(\theta)\, d\theta \right)^{-1} .$$

**The "cusp" case.** — We introduce the stochastic process (on $\mathbb{R}$)

$$Z_1(u) = \exp\left\{\Gamma_\theta\, W^{p+1/2}(u) - \frac{1}{2}\,\Gamma_\theta^2\,\left|u\right|^{2p+1}\right\}$$

where $$\Gamma_\theta^2 = \frac{\mathrm{B}\bigl(p+1\,,\,p+1\bigr)}{\Psi(0,0)}\left[\frac{a^2+b^2}{\cos(\pi p)} - 2ab\right], \quad 0 < \Gamma_\theta^2 < +\infty,$$

and $W^H(\cdot)$ is a fractional Brownian motion (fBm) of Hurst parameter $H$, that is, a centered Gaussian process with covariance

$$\mathbf{E}\Big[W^H\,(u_1)\,W^H\,(u_2)\Big] = \frac{1}{2}\Big[\left|u_1\right|^{2H} + \left|u_2\right|^{2H} - \left|u_1-u_2\right|^{2H}\Big].$$

We introduce equally the random variables $\xi_1$ and $\zeta_1$ by

$$Z_1(\xi_1) = \sup_{u\in\mathbb{R}} Z_1(u)$$

and $$\zeta_1 = \int_{-\infty}^{+\infty} u\,Z_1(u)\,du\,\left(\int_{-\infty}^{+\infty} Z_1(u)\,du\right)^{-1}.$$

**Theorem.** — In the case of "cusp", we have the following lower bound on the risks of all the estimators of $\theta$:

$$\lim_{\delta\to 0} \varliminf_{n\to\infty} \inf_{\overline{\theta}_n} \sup_{|\theta-\theta_0|<\delta} \mathbf{E}_\theta \left(n^{1/(2p+1)} \left(\overline{\theta}_n - \theta\right)\right)^2 \geqslant \mathbf{E}\zeta_1^2$$

for all $\theta_0 \in \Theta$, where inf is taken on the set of all the estimators $\overline{\theta}_n$ of $\theta$.

**Definition.** — We say that an estimator $\overline{\theta}_n$ is asymptotically efficient if

$$\lim_{\delta\to 0} \lim_{n\to\infty} \sup_{|\theta-\theta_0|<\delta} \mathbf{E}_\theta \left(n^{1/(2p+1)} \left(\overline{\theta}_n - \theta\right)\right)^2 = \mathbf{E}\zeta_1^2$$

for all $\theta_0 \in \Theta$.

**Theorem.** — In the case of "cusp", the BE $\widetilde{\theta}_n$ and the MLE $\widehat{\theta}_n$ have uniformly in $\theta \in \mathbf{K}$ (for any compact $\mathbf{K} \subset \Theta$) the following properties:

- $\widetilde{\theta}_n$ and $\widehat{\theta}_n$ are consistent, that is,

$$\widetilde{\theta}_n \xrightarrow{\mathbf{P}_\theta} \theta \quad \text{and} \quad \widehat{\theta}_n \xrightarrow{\mathbf{P}_\theta} \theta,$$

- the limit distributions of $\widetilde{\theta}_n$ and $\widehat{\theta}_n$ are given by

$$n^{1/(2p+1)}\left(\widetilde{\theta}_n - \theta\right) \Longrightarrow \zeta_1 \quad \text{and} \quad n^{1/(2p+1)}\left(\widehat{\theta}_n - \theta\right) \Longrightarrow \xi_1,$$

- for any $k > 0$ the convergence of moments equally holds:

$$\lim_{n\to\infty} \mathbf{E}_\theta \left| n^{1/(2p+1)}\left(\widetilde{\theta}_n - \theta\right)\right|^k = \mathbf{E}\left|\zeta_1\right|^k,$$
$$\lim_{n\to\infty} \mathbf{E}_\theta \left| n^{1/(2p+1)}\left(\widehat{\theta}_n - \theta\right)\right|^k = \mathbf{E}\left|\xi_1\right|^k.$$

Moreover, the BE $\widetilde{\theta}_n$ are asymptotically efficient.

**The "0"-type and "$\infty$"-type singularity cases.** — We introduce the stochastic process (on $\mathbb{R}$)

$$Z_2(u) = \exp\Biggl\{ p \int_{-\infty}^{+\infty} \ln\left|1 - \frac{u}{z}\right| \bigl(\nu(dz) - \mathbf{E}\nu(dz)\bigr) - \frac{a-b}{p+1}\,|u|^{p+1}\,\mathrm{sign}(u) + \\ + \ln\frac{a}{b}\int_0^u \nu(dz) - \int_{-\infty}^{+\infty} \left[\left|1 - \frac{u}{z}\right|^p - 1 - p\,\ln\left|1 - \frac{u}{z}\right|\right] d(z)\ |z|^p\ dz\Biggr\},$$

where $\quad d(z) = \begin{cases} a, & \text{if } z \leqslant 0 \\ b, & \text{if } z \geqslant 0 \end{cases}$ and $\nu$ is a Poisson process of intensity $d(z)\,|z|^p$.

We introduce equally the random variables $\xi_2$ (in the case of "0"-type singularity only) and $\zeta_2$ (in both cases) by

$$Z_2(\xi_2) = \sup_{u\in\mathbb{R}} Z_2(u)$$

and

$$\zeta_2 = \int_{-\infty}^{+\infty} u\,Z_2(u)\,du \left(\int_{-\infty}^{+\infty} Z_2(u)\,du\right)^{-1}.$$

**Theorem.** — In the case of "0"-type or "$\infty$"-type singularity, we have the following lower bound on the risks of all the estimators of $\theta$:

$$\lim_{\delta\to 0}\ \varliminf_{n\to\infty}\ \inf_{\overline{\theta}_n}\ \sup_{|\theta-\theta_0|<\delta} \mathbf{E}_\theta\left(n^{1/(p+1)}\left(\overline{\theta}_n-\theta\right)\right)^2 \geqslant \mathbf{E}\zeta_2^2$$

for all $\theta_0 \in \Theta$, where inf is taken on the set of all the estimators $\overline{\theta}_n$ of $\theta$.

**Definition.** — We say that an estimator $\overline{\theta}_n$ is asymptotically efficient if

$$\lim_{\delta\to 0}\ \lim_{n\to\infty}\ \sup_{|\theta-\theta_0|<\delta} \mathbf{E}_\theta\left(n^{1/(p+1)}\left(\overline{\theta}_n-\theta\right)\right)^2 = \mathbf{E}\zeta_2^2$$

for all $\theta_0 \in \Theta$.

**Theorem.** — The BE $\widetilde{\theta}_n$ (in both cases) and the MLE $\widehat{\theta}_n$ (in the case of "0"-type singularity only) have uniformly in $\theta \in \mathbf{K}$ (for any compact $\mathbf{K} \subset \Theta$) the following properties:

- $\widetilde{\theta}_n$ and $\widehat{\theta}_n$ are consistent, that is,

$$\widetilde{\theta}_n \xrightarrow{\mathbf{P}_\theta} \theta \quad \text{and} \quad \widehat{\theta}_n \xrightarrow{\mathbf{P}_\theta} \theta,$$

- the limit distributions of $\widetilde{\theta}_n$ and $\widehat{\theta}_n$ are given by

$$n^{1/(p+1)}\big(\widetilde{\theta}_n - \theta\big) \Longrightarrow \zeta_2 \quad \text{and} \quad n^{1/(p+1)}\big(\widehat{\theta}_n - \theta\big) \Longrightarrow \xi_2,$$

- for any $k > 0$ the convergence of moments equally holds:

$$\lim_{n\to\infty} \mathbf{E}_\theta \left| n^{1/(p+1)}\big(\widetilde{\theta}_n - \theta\big)\right|^k = \mathbf{E}\left|\zeta_2\right|^k,$$
$$\lim_{n\to\infty} \mathbf{E}_\theta \left| n^{1/(p+1)}\big(\widehat{\theta}_n - \theta\big)\right|^k = \mathbf{E}\left|\xi_2\right|^k.$$

Moreover, the BE $\widetilde{\theta}_n$ are asymptotically efficient.

# Ideas of the proof

We use the Ibragimov and Khasminskii method which consist in studying the normalized likelihood ratio process

$$Z_n(u) = \frac{L(\theta_u, X^n)}{L(\theta, X^n)}\ , \quad u \in U_n,$$

where we denote $\theta_u = \theta + u\, n^{-1/\nu}$ (with $\nu = 2p+1$ or $\nu = p+1$) and the set $U_n = \big(n^{1/\nu}(\alpha - \theta)\,, n^{1/\nu}(\beta - \theta)\big)$, and establishing the three following properties:

- The finite-dimensional distributions of $Z_n(u)$ converge to those of $Z(u)$ (with $Z = Z_1$ or $Z = Z_2$) uniformly in $\theta \in \mathbf{K}$.
- $\mathbf{E}_\theta \left| Z_n^{1/2}(u_1) - Z_n^{1/2}(u_2) \right|^2 \leqslant C\ |u_1 - u_2|^\nu$ uniformly in $\theta \in \mathbf{K}$.
- $\mathbf{E}_\theta\, Z_n^{1/2}(u) \leqslant \exp\big\{-c\ |u|^\nu\big\}$ uniformly in $\theta \in \mathbf{K}$.

## The simulations

- **The model.** — $S_\theta(t) = 2 - \left|t-\theta\right|^{1/10}$
- **The true value of the parameter.** — $\theta = 1{,}5$
- **The rate of convergence of the BE and the MLE.** — $n^{1/(2p+1)} = n^{5/6}$
- **The limit distributions of the BE and the MLE.** —

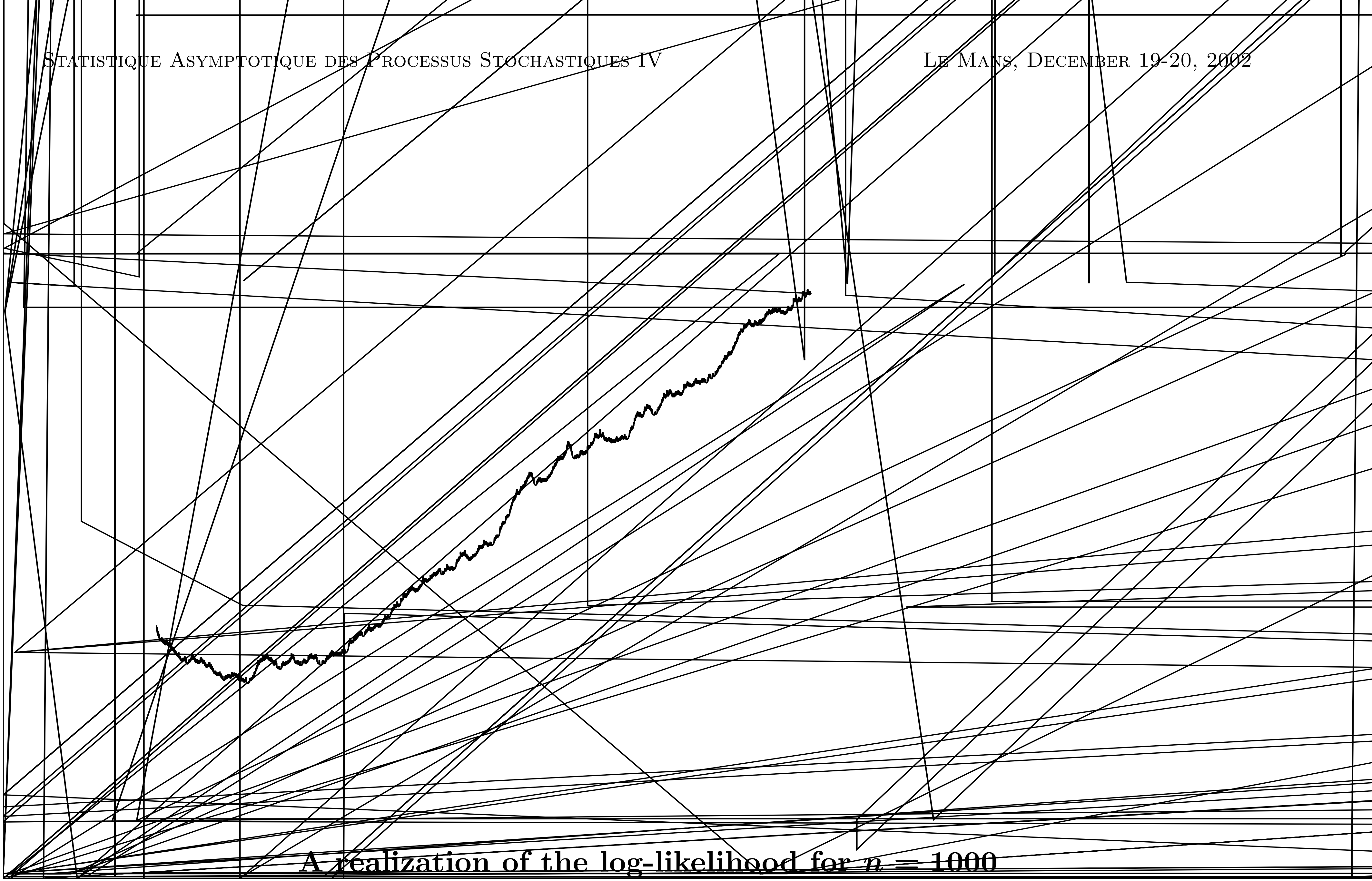
$$n^{5/6}\big(\widetilde{\theta}_n - \theta\big) \Longrightarrow \zeta_1 = \frac{1}{\gamma}\,\zeta \approx 14\zeta \quad \text{and} \quad n^{5/6}\big(\widehat{\theta}_n - \theta\big) \Longrightarrow \xi_1 = \frac{1}{\gamma}\,\xi \approx 14\xi,$$

where $\gamma = \Gamma_\theta^{2/(2p+1)} \approx 0{,}07$ and the random variables $\xi$ and $\zeta$ are defined by

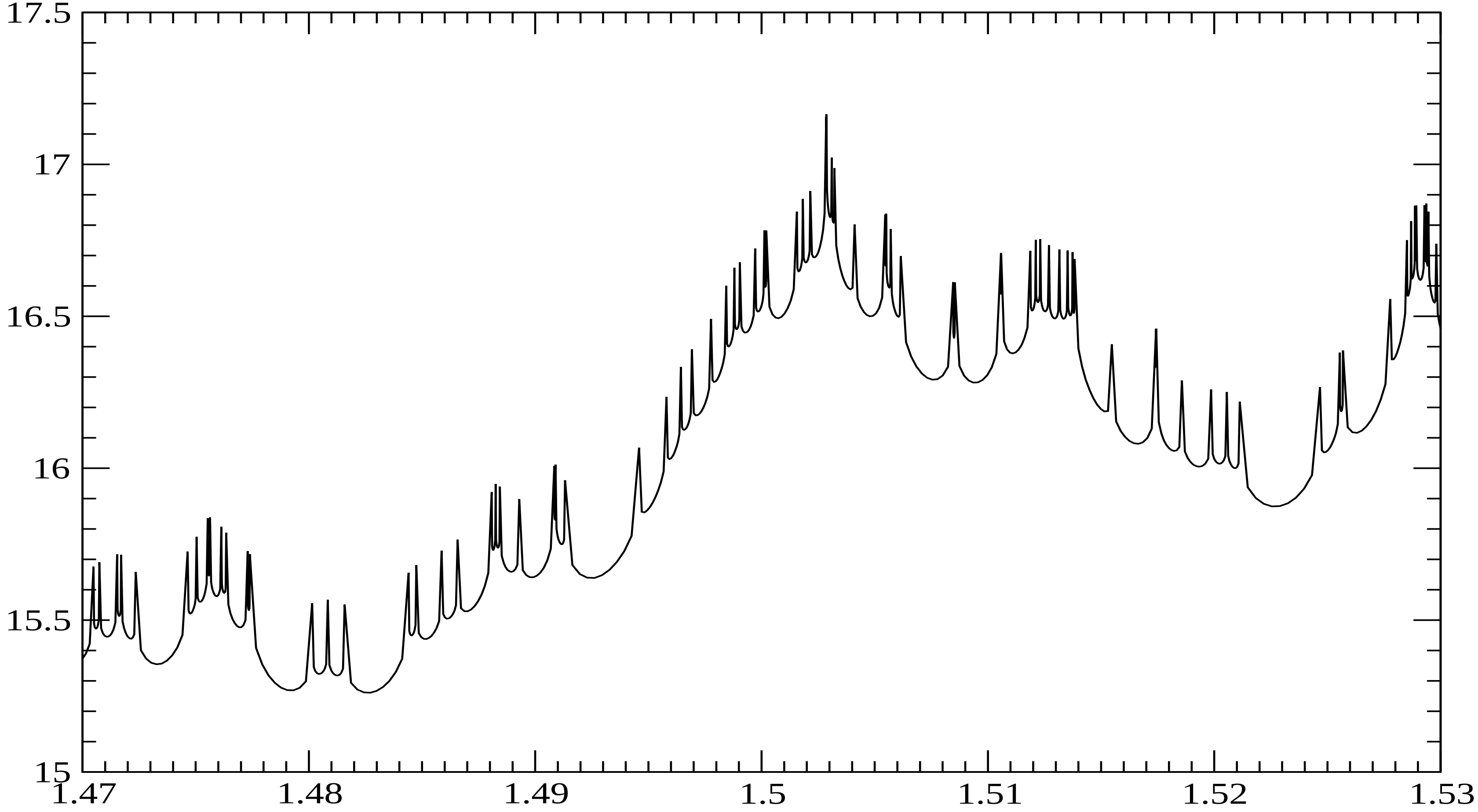
$$Z(\xi) = \sup_{u\in\mathbb{R}} Z(u) \quad \text{and} \quad \zeta = \int_{-\infty}^{+\infty} u\,Z(u)\,du\,\left(\int_{-\infty}^{+\infty} Z(u)\,du\right)^{-1},$$

where $Z(u) = \exp\left\{W^{p+1/2}(u) - \frac{1}{2}\,\left|u\right|^{2p+1}\right\} = \exp\left\{W^{3/5}(u) - \frac{1}{2}\,\left|u\right|^{6/5}\right\}.$

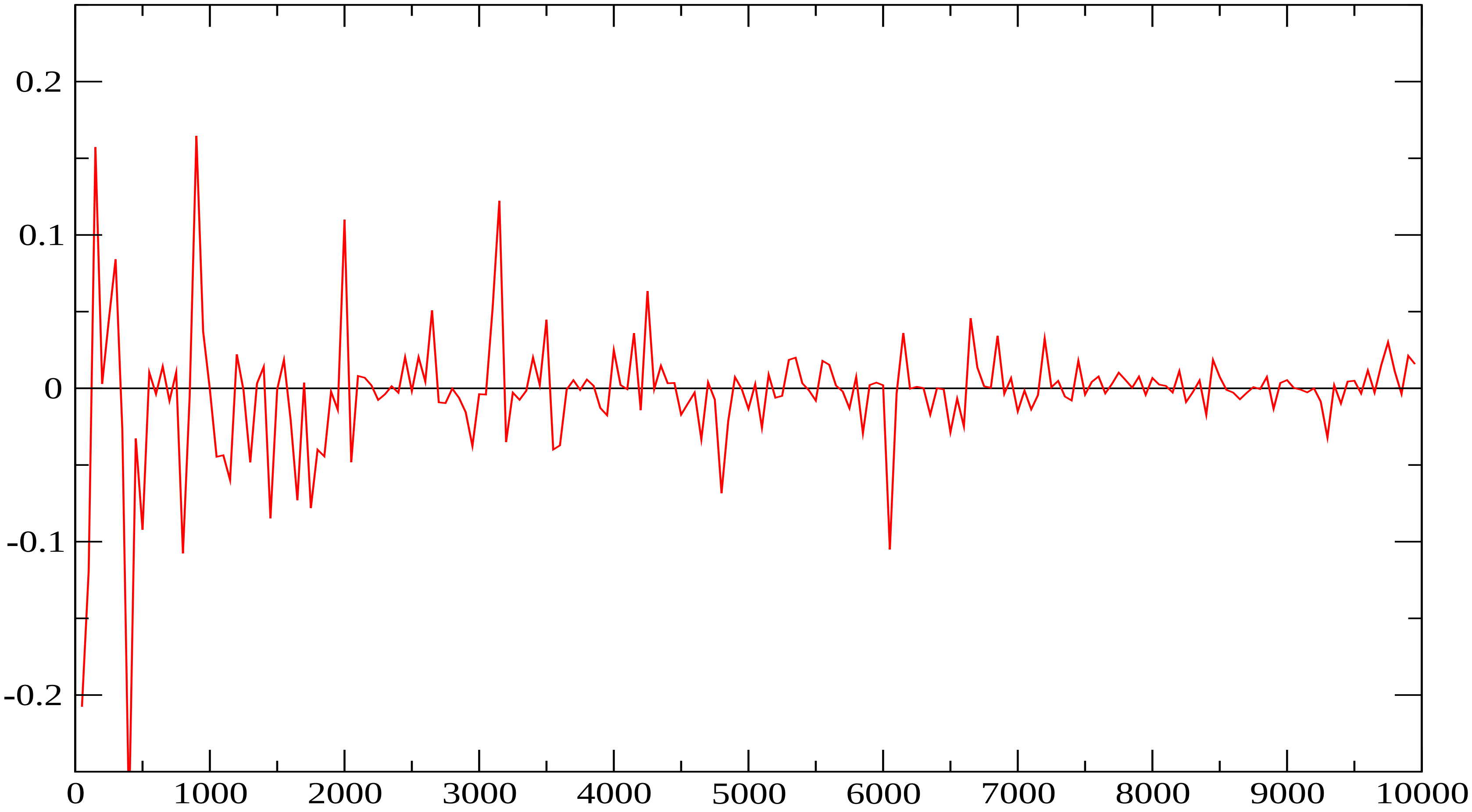
**The intensity function $S_\theta(t) = 2 - |t-\theta|^{1/10}$ for $\theta = 1{,}5$**



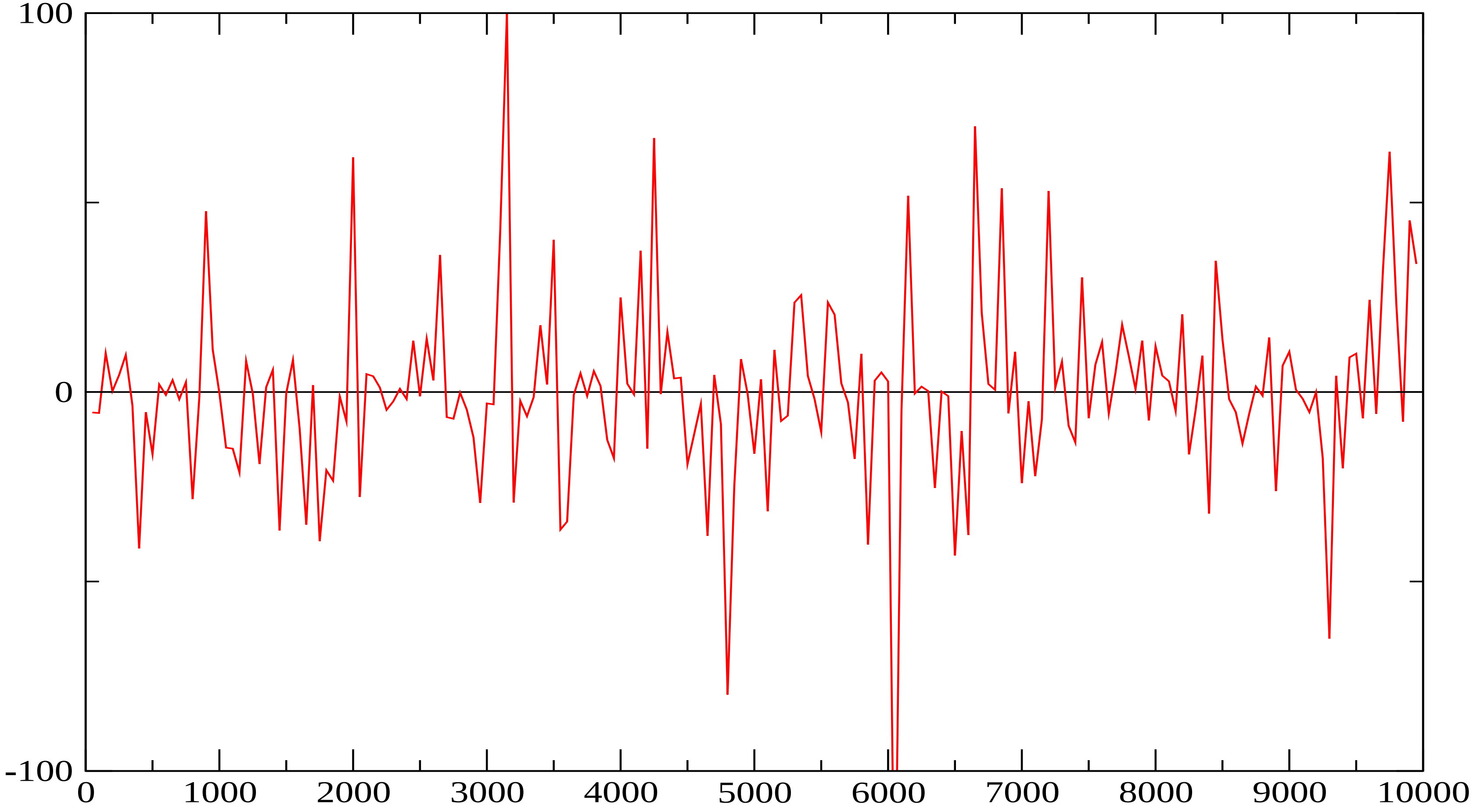
**A realization of the log-likelihood for $n = 1000$**



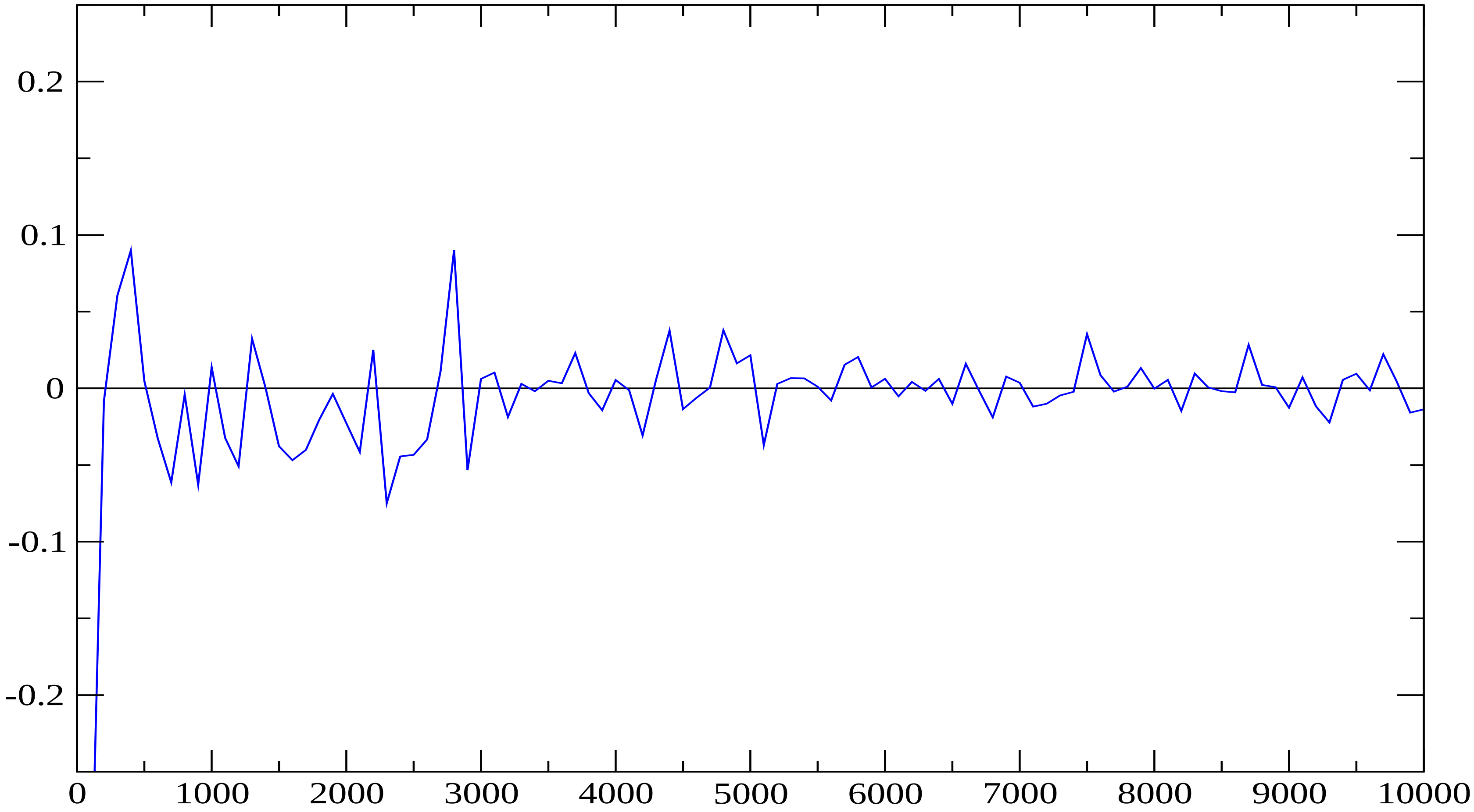

**A "zoom" of the log-likelihood in the vicinity of the true value $\theta = 1{,}5$**



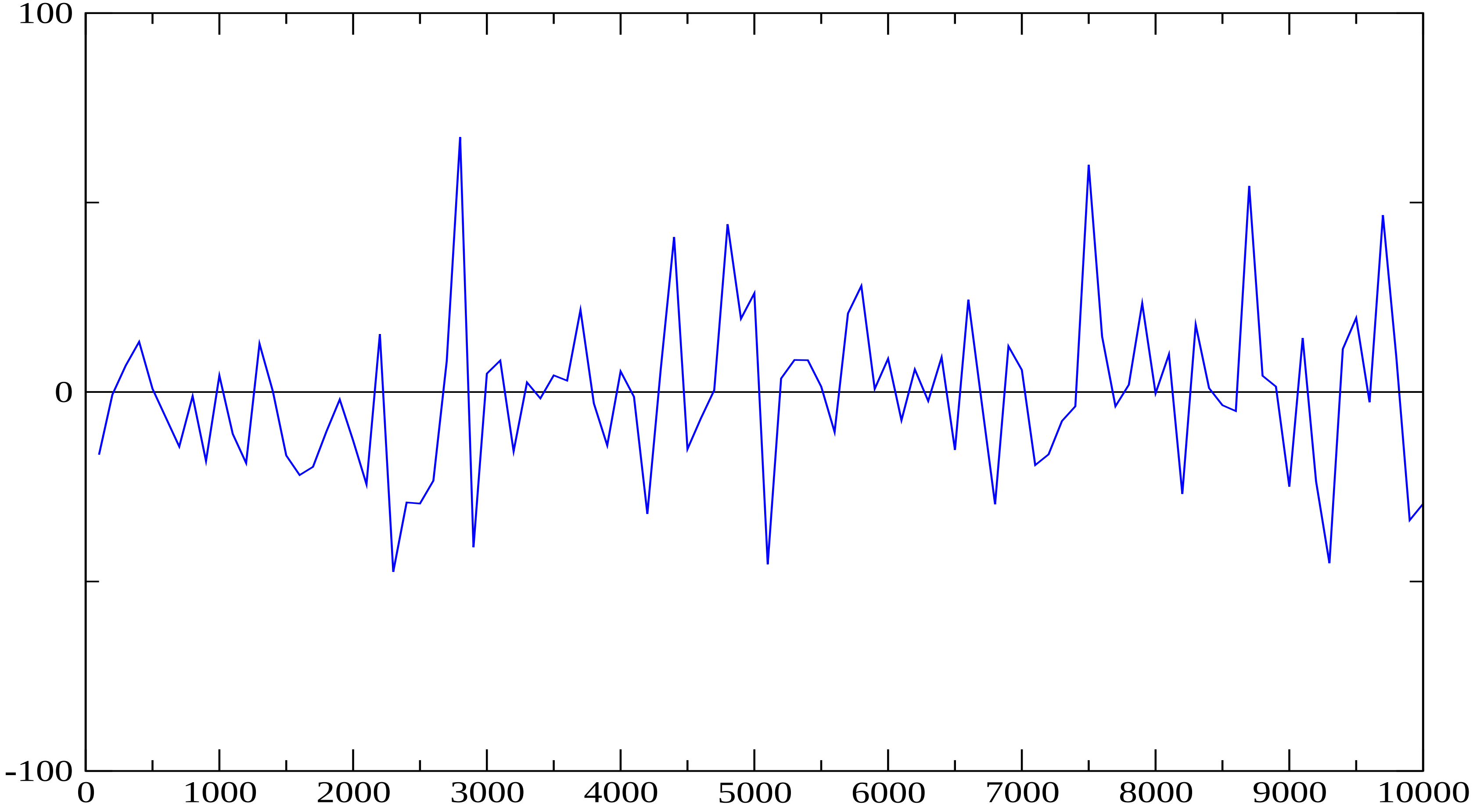

**The error $\widehat{\theta}_n - 1.5$ of the MLE for $n = 50, 100, \ldots, 10000$**



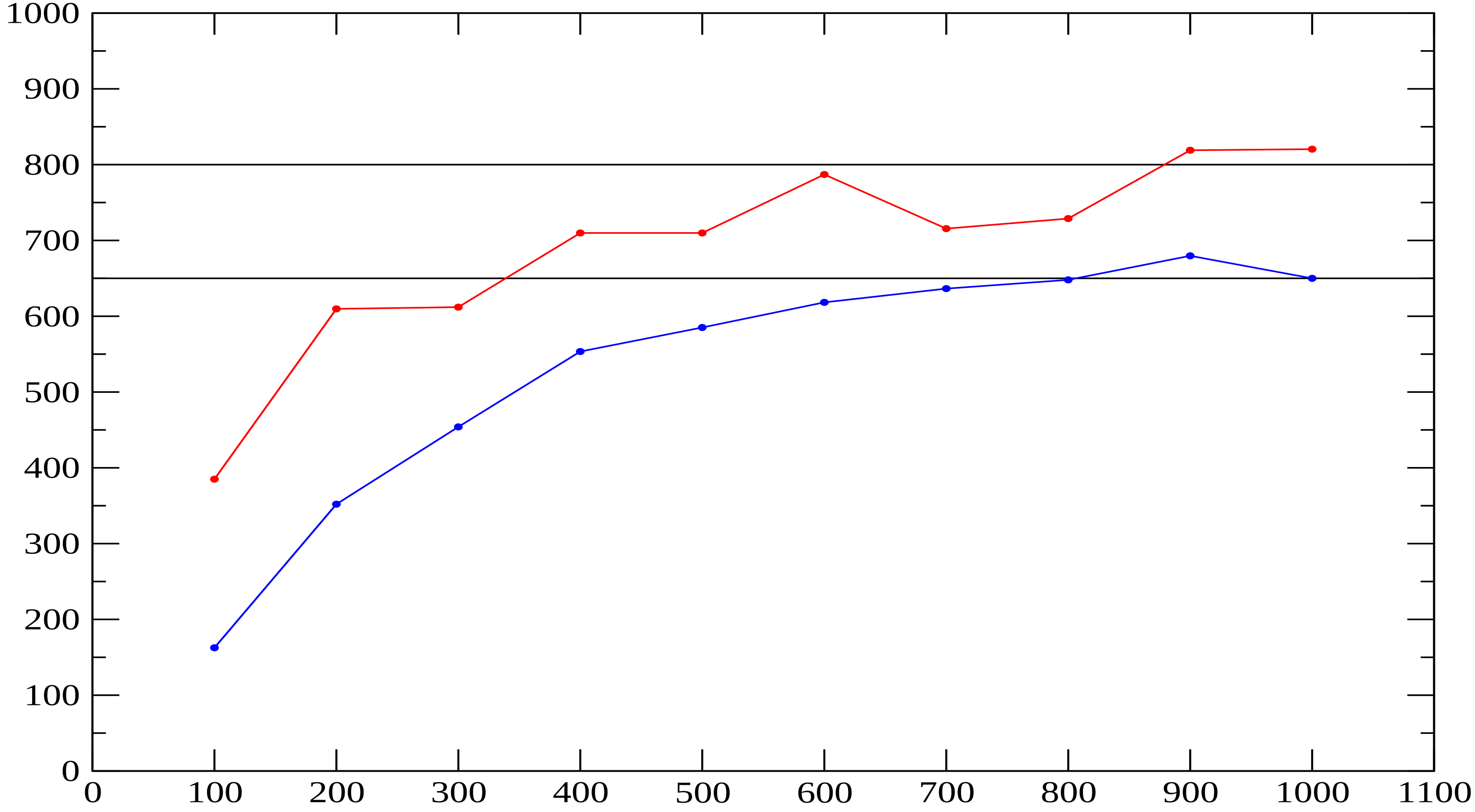

**The normalized error $n^{5/6}(\widehat{\theta}_n - 1.5)$ of the MLE for $n = 50, 100, \ldots, 10000$**




**The error $\widetilde{\theta}_n - 1.5$ of the BE for $n = 100, 200, \ldots, 10000$**




**The normalized error $n^{5/6}(\widetilde{\theta}_n - 1.5)$ of the BE for $n = 100, 200, \ldots, 10000$**




**The empirical variances of $n^{5/6}(\widetilde{\theta}_n - 1.5)$ for BE and $n^{5/6}(\widehat{\theta}_n - 1.5)$ for MLE**