

ESTIMATION OF SINGULARITY LOCATION  
FOR POISSON PROCESS

**S. Dachian**

*Laboratoire de Mathématiques Appliquées*

*Université Blaise Pascal*

*Clermont-Ferrand, FRANCE*

## The model

- **The process.** — Poisson process of intensity function  $S_\theta(\cdot)$ :

$$X = \{X(t), 0 \leq t \leq T\}.$$

- **The observations.** —  $n$  independent realizations (trajectories) of  $X$ :

$$(X_1, \dots, X_n) = X^n.$$

- **The hypotheses on  $S_\theta(\cdot)$ .** — The intensity function  $S_\theta(\cdot)$  is regular everywhere on  $[0, T]$  except at the point  $\theta$ , where it has a singularity.

- **The unknown parameter.** — The location (the point) of the singularity:

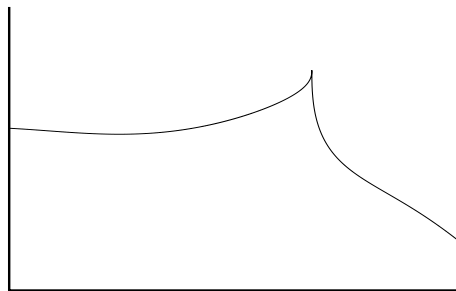
$$\theta \in \Theta = (\alpha, \beta) \subseteq (0, T).$$

- **The types of singularities.** — Three types: “cusp”, singularity of “0”-type and singularity of “ $\infty$ ”-type.

- **The asymptotics.** —  $n \longrightarrow \infty$ .

We consider  $S_\theta(t)$  of the form 
$$S_\theta(t) = \begin{cases} a |t - \theta|^p + \Psi(\theta, t), & \text{if } t \leq \theta \\ b |t - \theta|^p + \Psi(\theta, t), & \text{if } t \geq \theta \end{cases} .$$

We suppose that  $a^2 + b^2 > 0$ ,  $S_\theta(t) > 0$  for all  $t \neq \theta$ , the function  $\Psi(\theta, t)$  is continuous and uniformly in  $t$  Hölder continuous of order  $\mu$  with respect to  $\theta$ .

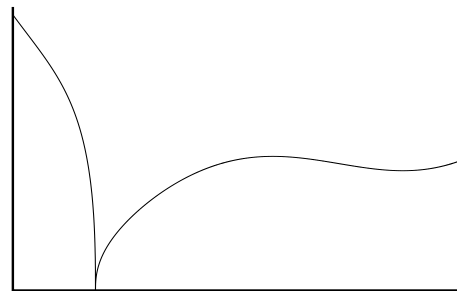


“cusp”

$$0 < p < 1/2$$

$$\mu > p + 1/2$$

$$\Psi(\theta, \theta) > 0$$

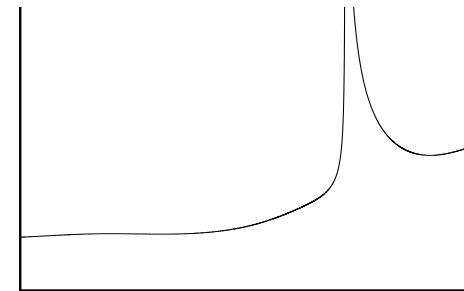


singularity of “0”-type

$$0 < p < 1$$

$$\mu > (p + 1)/2$$

$$\Psi(\theta, \theta) = 0$$



singularity of “∞”-type

$$-1 < p < 0$$

$$\mu > (p + 1)/2$$

## The history of the problem

- PRAKASA RAO, B.L.S., “Estimation of the location of the cusp of a continuous density”, *Annals of Mathematical Statistics*, vol. 20, no. 1, pp. 76–87, 1968.
- IBRAGIMOV, I.A. et KHASHMINSKII, R.Z., “*Statistical Estimation. Asymptotic Theory*”, Springer-Verlag, New York, 1981.
- DACHIAN, S., “Estimation of Cusp Location by Poisson Observations”, *Statistical Inference for Stochastic Processes*, to appear, 2001.
- DACHIAN, S., “Estimation of Singularity Location by Poisson Observations”, in preparation, 2002.
- DACHIAN, S. et KUTOYANTS, YU.A., “On Cusp Estimation of Ergodic Diffusion Process”, *Journal of Statistical Planning and Inference*, to appear, 2001.

The likelihood ratio is:

$$L(\theta, X^n) = \exp \left\{ \sum_{i=1}^n \int_0^T \ln S_\theta(t) dX_i(t) - n \int_0^T [S_\theta(t) - 1] dt \right\}.$$

The maximum likelihood estimator (MLE)  $\hat{\theta}_n$  is defined as one of the solutions of the maximum likelihood equation  $L(\hat{\theta}_n, X^n) = \sup_{\theta \in \Theta} L(\theta, X^n)$ .

The Bayesian estimator (BE) for prior density  $q(\cdot)$  and quadratic loss function is defined by  $\tilde{\theta}_n = \int_{\alpha}^{\beta} \theta q(\theta|X^n) d\theta$ , where the posterior density  $q(\cdot|X^n)$  is given by:

$$q(\theta|X^n) = L(\theta, X^n) q(\theta) \left( \int_{\alpha}^{\beta} L(\theta, X^n) q(\theta) d\theta \right)^{-1}.$$

**The “cusp” case.** — We introduce the stochastic process (on  $\mathbb{R}$ )

$$Z_1(u) = \exp \left\{ \Gamma_\theta W^{p+1/2}(u) - \frac{1}{2} \Gamma_\theta^2 |u|^{2p+1} \right\}$$

where  $\Gamma_\theta^2 = \frac{\text{B}(p+1, p+1)}{\Psi(0, 0)} \left[ \frac{a^2 + b^2}{\cos(\pi p)} - 2ab \right]$ ,  $0 < \Gamma_\theta^2 < +\infty$ ,

and  $W^H(\cdot)$  is a fractional Brownian motion (fBm) of Hurst parameter  $H$ , that is, a centered Gaussian process with covariance

$$\mathbf{E} \left[ W^H(u_1) W^H(u_2) \right] = \frac{1}{2} \left[ |u_1|^{2H} + |u_2|^{2H} - |u_1 - u_2|^{2H} \right].$$

We introduce equally the random variables  $\xi_1$  and  $\zeta_1$  by

$$Z_1(\xi_1) = \sup_{u \in \mathbb{R}} Z_1(u)$$

and  $\zeta_1 = \int_{-\infty}^{+\infty} u Z_1(u) du \left( \int_{-\infty}^{+\infty} Z_1(u) du \right)^{-1}$ .

**Theorem.** — In the case of “cusp”, we have the following lower bound on the risks of all the estimators of  $\theta$ :

$$\lim_{\delta \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \inf_{\bar{\theta}_n} \sup_{|\theta - \theta_0| < \delta} \mathbf{E}_\theta \left( n^{1/(2p+1)} (\bar{\theta}_n - \theta) \right)^2 \geq \mathbf{E}\zeta_1^2$$

for all  $\theta_0 \in \Theta$ , where inf is taken on the set of all the estimators  $\bar{\theta}_n$  of  $\theta$ .

**Definition.** — We say that an estimator  $\bar{\theta}_n$  is asymptotically efficient if

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \sup_{|\theta - \theta_0| < \delta} \mathbf{E}_\theta \left( n^{1/(2p+1)} (\bar{\theta}_n - \theta) \right)^2 = \mathbf{E}\zeta_1^2$$

for all  $\theta_0 \in \Theta$ .

**Theorem.** — In the case of “cusp”, the BE  $\tilde{\theta}_n$  and the MLE  $\hat{\theta}_n$  have uniformly in  $\theta \in \mathbf{K}$  (for any compact  $\mathbf{K} \subset \Theta$ ) the following properties:

- $\tilde{\theta}_n$  and  $\hat{\theta}_n$  are consistent, that is,

$$\tilde{\theta}_n \xrightarrow{\mathbf{P}_\theta} \theta \quad \text{and} \quad \hat{\theta}_n \xrightarrow{\mathbf{P}_\theta} \theta,$$

- the limit distributions of  $\tilde{\theta}_n$  and  $\hat{\theta}_n$  are given by

$$n^{1/(2p+1)}(\tilde{\theta}_n - \theta) \Longrightarrow \zeta_1 \quad \text{and} \quad n^{1/(2p+1)}(\hat{\theta}_n - \theta) \Longrightarrow \xi_1,$$

- for any  $k > 0$  the convergence of moments equally holds:

$$\lim_{n \rightarrow \infty} \mathbf{E}_\theta \left| n^{1/(2p+1)}(\tilde{\theta}_n - \theta) \right|^k = \mathbf{E} |\zeta_1|^k,$$

$$\lim_{n \rightarrow \infty} \mathbf{E}_\theta \left| n^{1/(2p+1)}(\hat{\theta}_n - \theta) \right|^k = \mathbf{E} |\xi_1|^k.$$

Moreover, the BE  $\tilde{\theta}_n$  are asymptotically efficient.



**The “0”-type and “ $\infty$ ”-type singularity cases.** — We introduce the stochastic process (on  $\mathbb{R}$ )

$$Z_2(u) = \exp \left\{ p \int_{-\infty}^{+\infty} \ln \left| 1 - \frac{u}{z} \right| (\nu(dz) - \mathbf{E}\nu(dz)) - \frac{a-b}{p+1} |u|^{p+1} \text{sign}(u) + \right. \\ \left. + \ln \frac{a}{b} \int_0^u \nu(dz) - \int_{-\infty}^{+\infty} \left[ \left| 1 - \frac{u}{z} \right|^p - 1 - p \ln \left| 1 - \frac{u}{z} \right| \right] d(z) |z|^p dz \right\},$$

where  $d(z) = \begin{cases} a, & \text{if } z \leq 0 \\ b, & \text{if } z \geq 0 \end{cases}$  and  $\nu$  is a Poisson process of intensity  $d(z) |z|^p$ .

We introduce equally the random variables  $\xi_2$  (in the case of “0”-type singularity only) and  $\zeta_2$  (in both cases) by

$$Z_2(\xi_2) = \sup_{u \in \mathbb{R}} Z_2(u)$$

and

$$\zeta_2 = \int_{-\infty}^{+\infty} u Z_2(u) du \left( \int_{-\infty}^{+\infty} Z_2(u) du \right)^{-1}.$$

**Theorem.** — In the case of “0”-type or “ $\infty$ ”-type singularity, we have the following lower bound on the risks of all the estimators of  $\theta$ :

$$\lim_{\delta \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \inf_{\bar{\theta}_n} \sup_{|\theta - \theta_0| < \delta} \mathbf{E}_\theta \left( n^{1/(p+1)} (\bar{\theta}_n - \theta) \right)^2 \geq \mathbf{E}\zeta_2^2$$

for all  $\theta_0 \in \Theta$ , where inf is taken on the set of all the estimators  $\bar{\theta}_n$  of  $\theta$ .

**Definition.** — We say that an estimator  $\bar{\theta}_n$  is asymptotically efficient if

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \sup_{|\theta - \theta_0| < \delta} \mathbf{E}_\theta \left( n^{1/(p+1)} (\bar{\theta}_n - \theta) \right)^2 = \mathbf{E}\zeta_2^2$$

for all  $\theta_0 \in \Theta$ .

**Theorem.** — The BE  $\tilde{\theta}_n$  (in both cases) and the MLE  $\hat{\theta}_n$  (in the case of “0”-type singularity only) have uniformly in  $\theta \in \mathbf{K}$  (for any compact  $\mathbf{K} \subset \Theta$ ) the following properties:

- $\tilde{\theta}_n$  and  $\hat{\theta}_n$  are consistent, that is,

$$\tilde{\theta}_n \xrightarrow{\mathbf{P}_\theta} \theta \quad \text{and} \quad \hat{\theta}_n \xrightarrow{\mathbf{P}_\theta} \theta,$$

- the limit distributions of  $\tilde{\theta}_n$  and  $\hat{\theta}_n$  are given by

$$n^{1/(p+1)}(\tilde{\theta}_n - \theta) \Longrightarrow \zeta_2 \quad \text{and} \quad n^{1/(p+1)}(\hat{\theta}_n - \theta) \Longrightarrow \xi_2,$$

- for any  $k > 0$  the convergence of moments equally holds:

$$\lim_{n \rightarrow \infty} \mathbf{E}_\theta \left| n^{1/(p+1)}(\tilde{\theta}_n - \theta) \right|^k = \mathbf{E} |\zeta_2|^k,$$

$$\lim_{n \rightarrow \infty} \mathbf{E}_\theta \left| n^{1/(p+1)}(\hat{\theta}_n - \theta) \right|^k = \mathbf{E} |\xi_2|^k.$$

Moreover, the BE  $\tilde{\theta}_n$  are asymptotically efficient.

## Ideas of the proof

We use the Ibragimov and Khasminskii method which consist in studying the normalized likelihood ratio process

$$Z_n(u) = \frac{L(\theta_u, X^n)}{L(\theta, X^n)}, \quad u \in U_n,$$

where we denote  $\theta_u = \theta + u n^{-1/\nu}$  (with  $\nu = 2p + 1$  or  $\nu = p + 1$ ) and the set  $U_n = (n^{1/\nu}(\alpha - \theta), n^{1/\nu}(\beta - \theta))$ , and establishing the three following properties:

- The finite-dimensional distributions of  $Z_n(u)$  converge to those of  $Z(u)$  (with  $Z = Z_1$  or  $Z = Z_2$ ) uniformly in  $\theta \in \mathbf{K}$ .
- $\mathbf{E}_\theta \left| Z_n^{1/2}(u_1) - Z_n^{1/2}(u_2) \right|^2 \leq C |u_1 - u_2|^\nu$  uniformly in  $\theta \in \mathbf{K}$ .
- $\mathbf{E}_\theta Z_n^{1/2}(u) \leq \exp\{-c |u|^\nu\}$  uniformly in  $\theta \in \mathbf{K}$ .

## The simulations

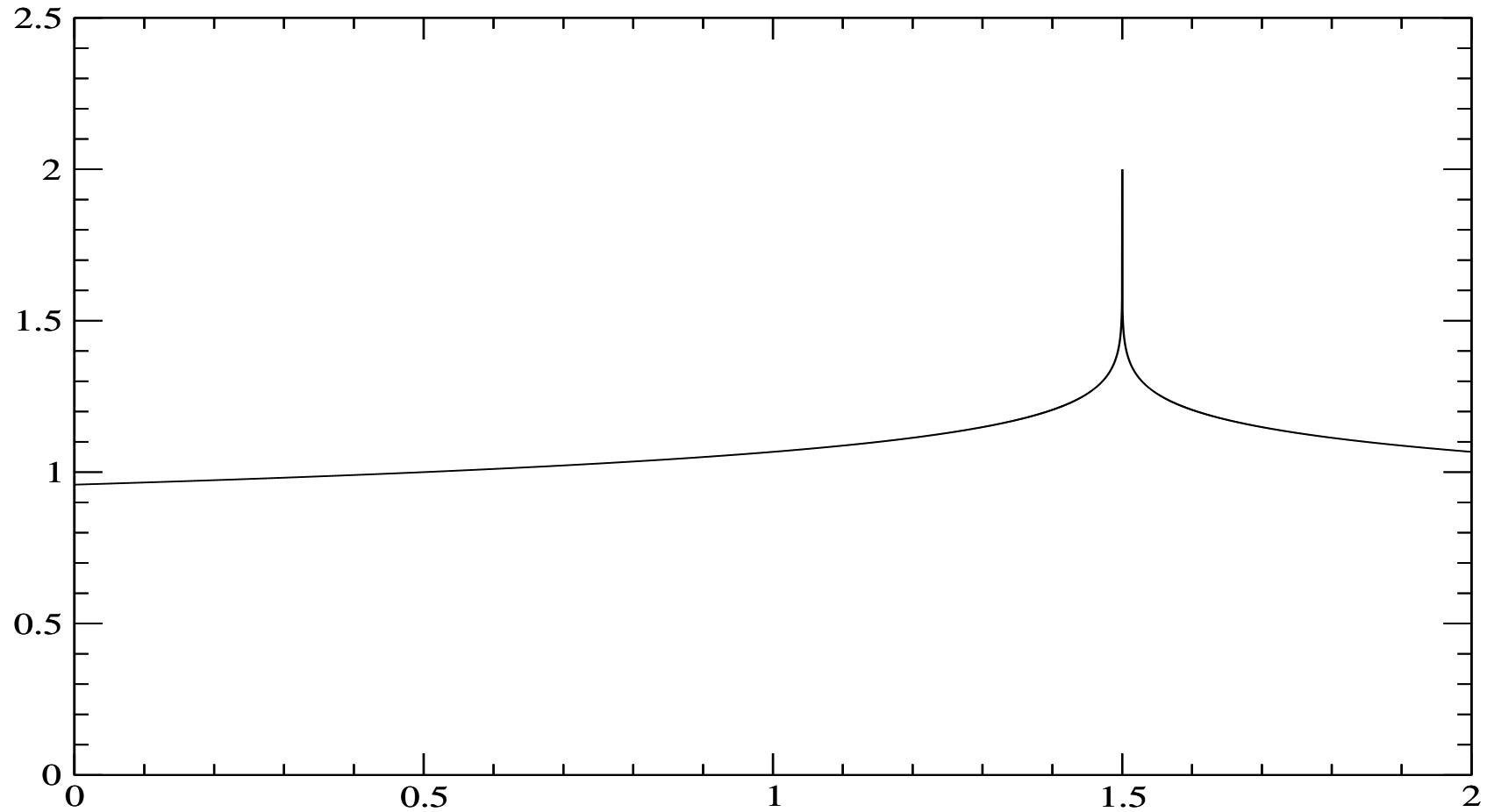
- **The model.** —  $S_\theta(t) = 2 - |t - \theta|^{1/10}$
- **The true value of the parameter.** —  $\theta = 1,5$
- **The rate of convergence of the BE and the MLE.** —  $n^{1/(2p+1)} = n^{5/6}$
- **The limit distributions of the BE and the MLE.** —

$$n^{5/6}(\tilde{\theta}_n - \theta) \implies \zeta_1 = \frac{1}{\gamma} \zeta \approx 14\zeta \quad \text{and} \quad n^{5/6}(\hat{\theta}_n - \theta) \implies \xi_1 = \frac{1}{\gamma} \xi \approx 14\xi,$$

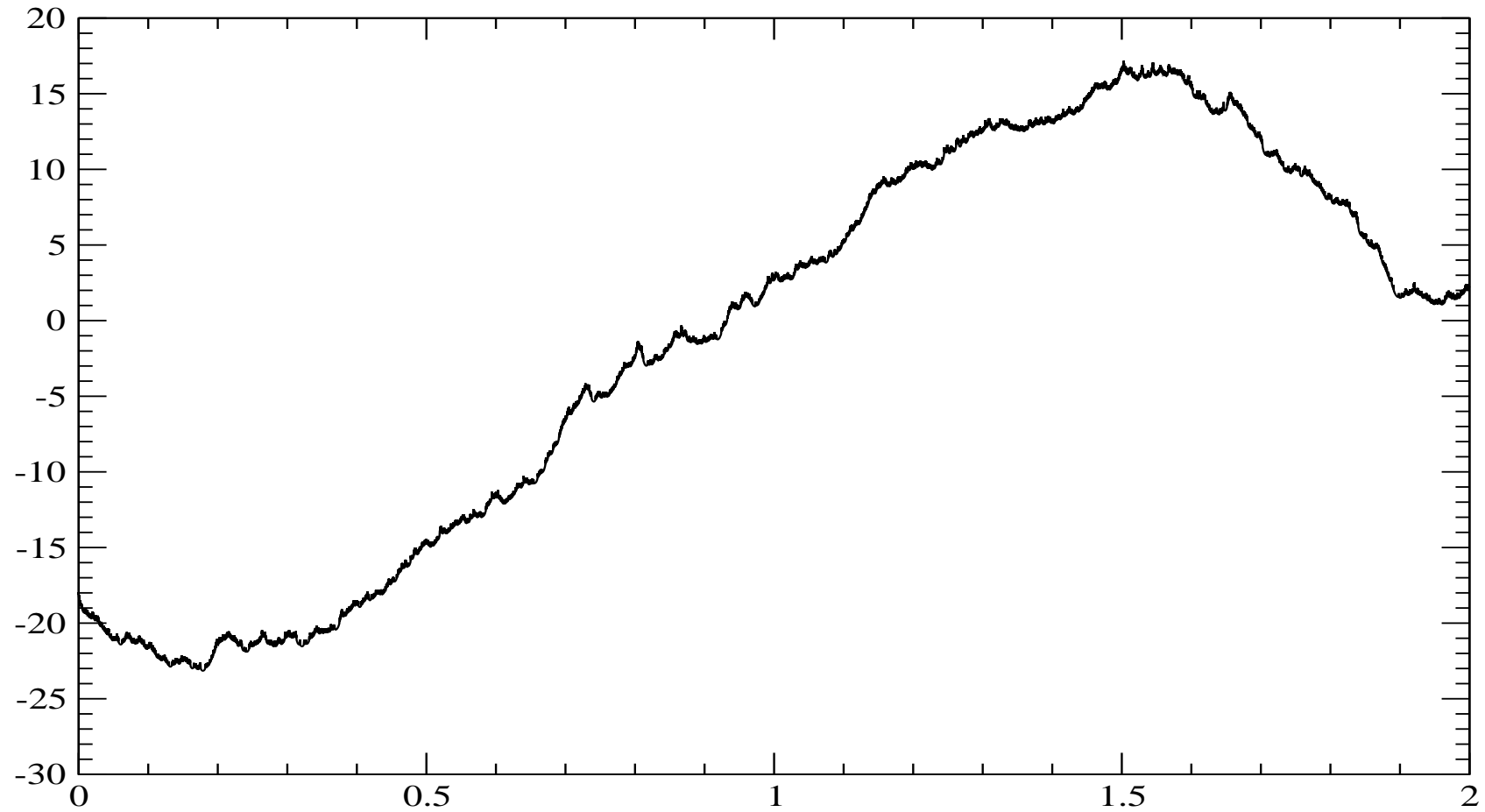
where  $\gamma = \Gamma_\theta^{2/(2p+1)} \approx 0,07$  and the random variables  $\xi$  and  $\zeta$  are defined by

$$Z(\xi) = \sup_{u \in \mathbb{R}} Z(u) \quad \text{and} \quad \zeta = \int_{-\infty}^{+\infty} u Z(u) du \left( \int_{-\infty}^{+\infty} Z(u) du \right)^{-1},$$

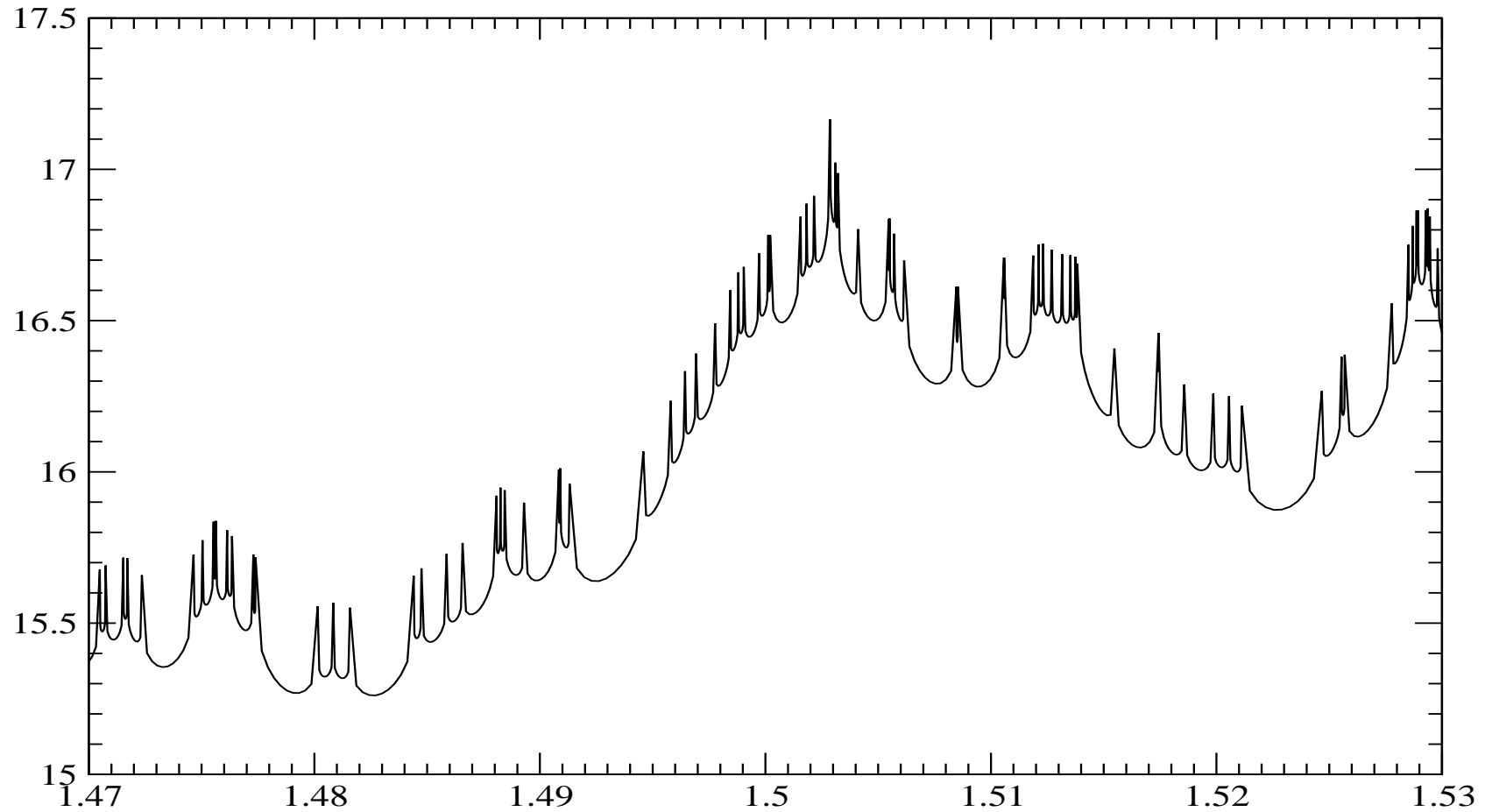
$$\text{where } Z(u) = \exp \left\{ W^{p+1/2}(u) - \frac{1}{2} |u|^{2p+1} \right\} = \exp \left\{ W^{3/5}(u) - \frac{1}{2} |u|^{6/5} \right\}.$$



The intensity function  $S_\theta(t) = 2 - |t - \theta|^{1/10}$  for  $\theta = 1,5$

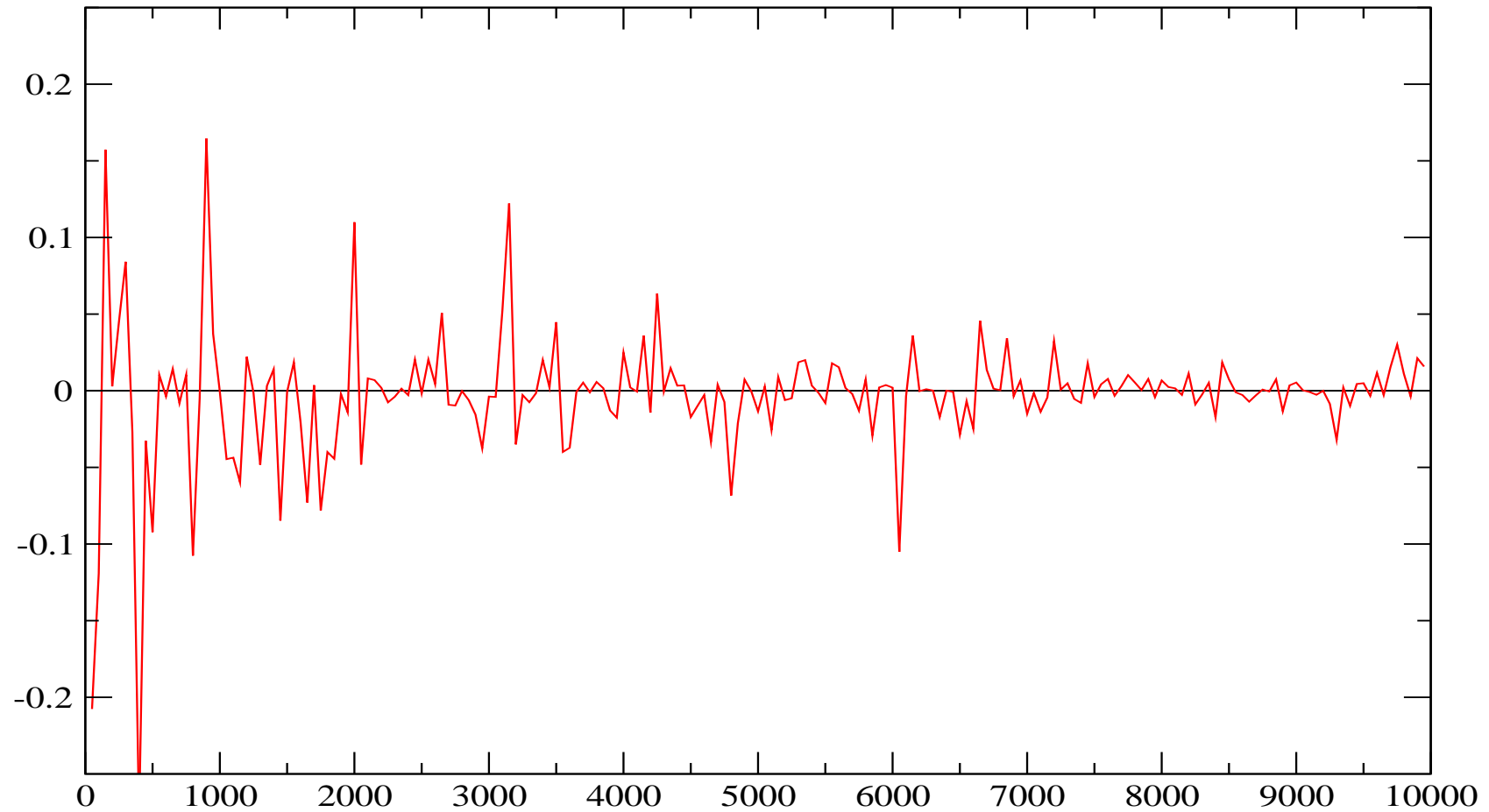


**A realization of the log-likelihood for  $n = 1000$**

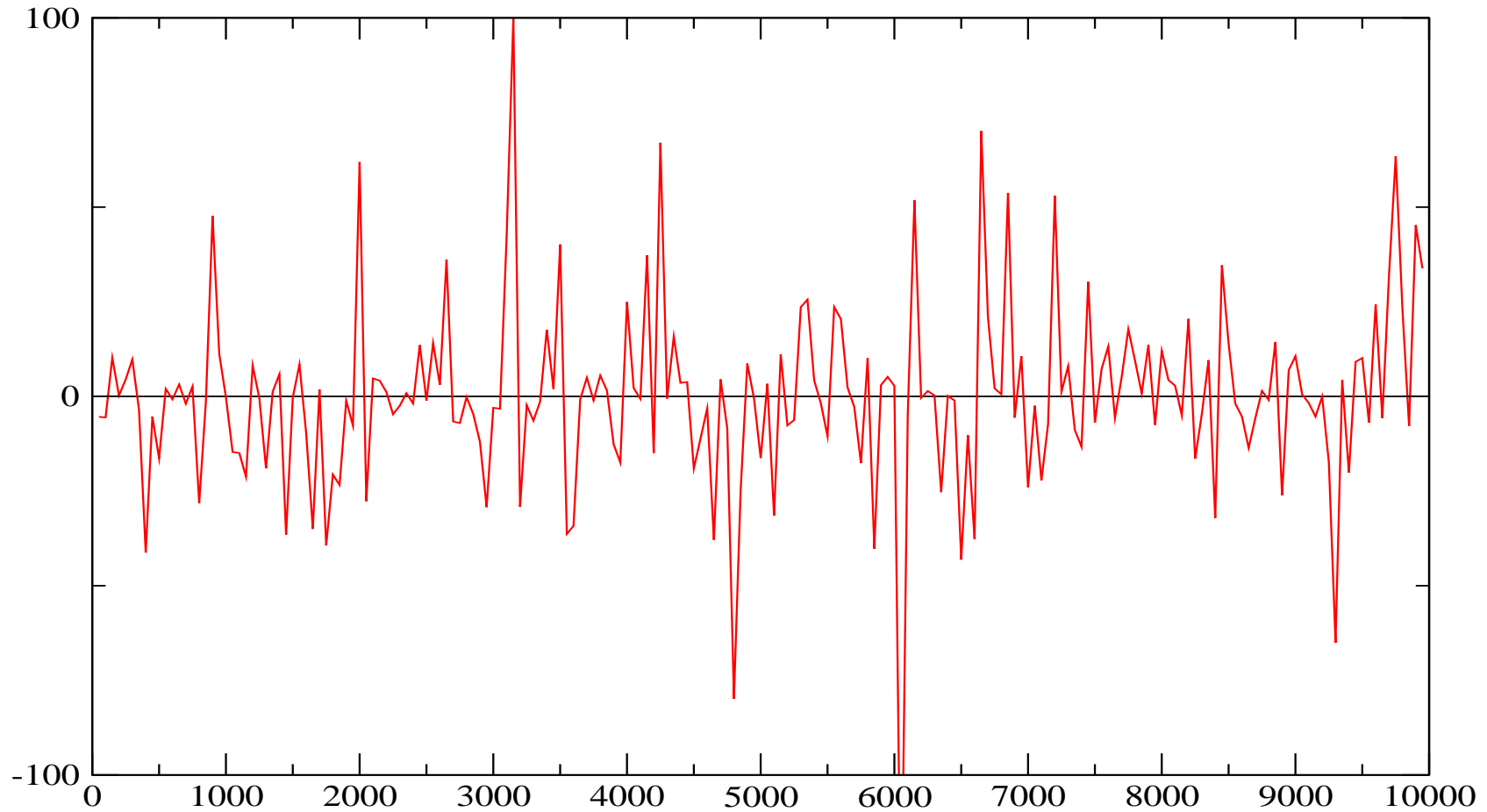


A “zoom” of the log-likelihood in the vicinity of the true value  $\theta = 1,5$

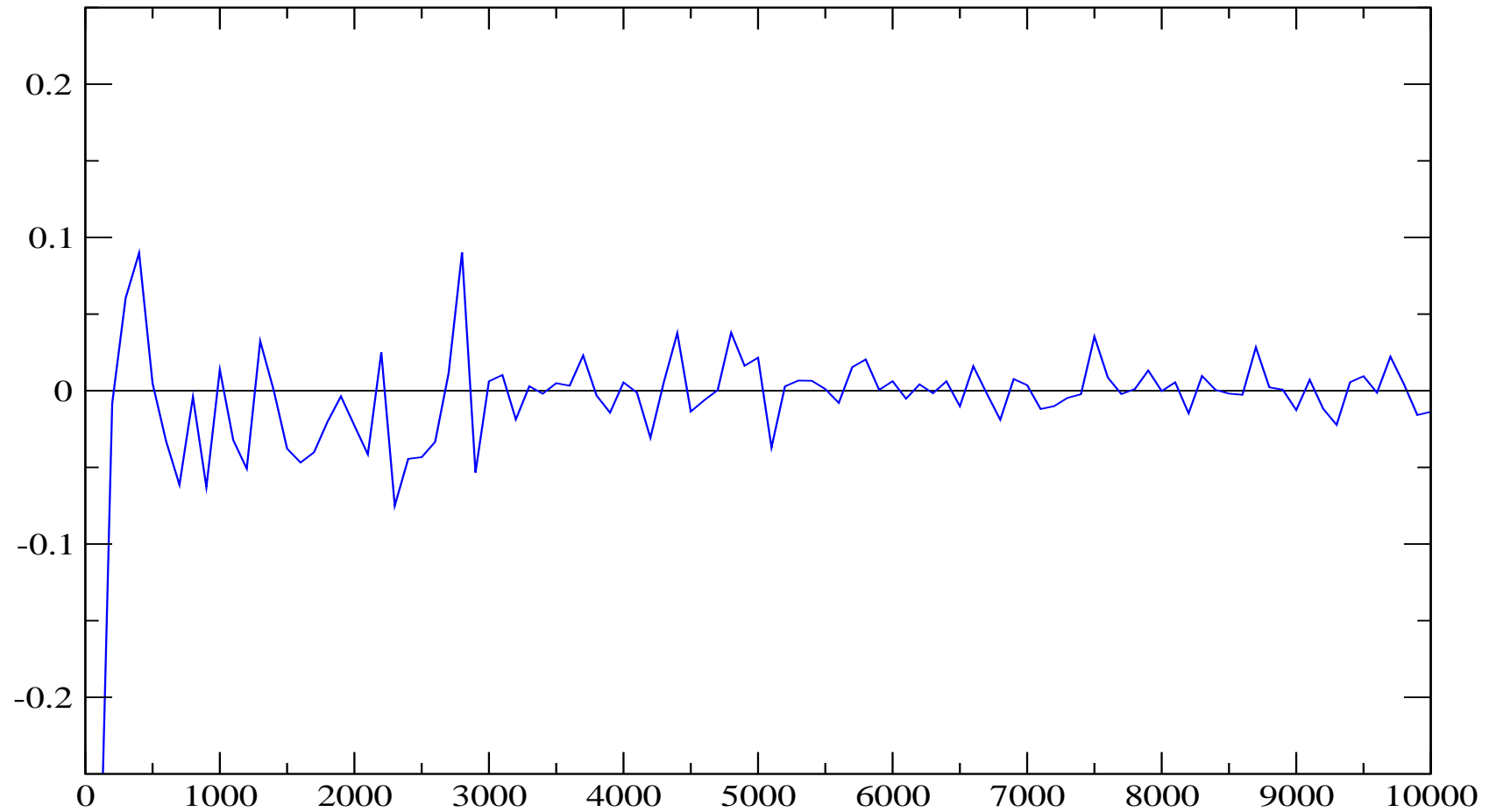




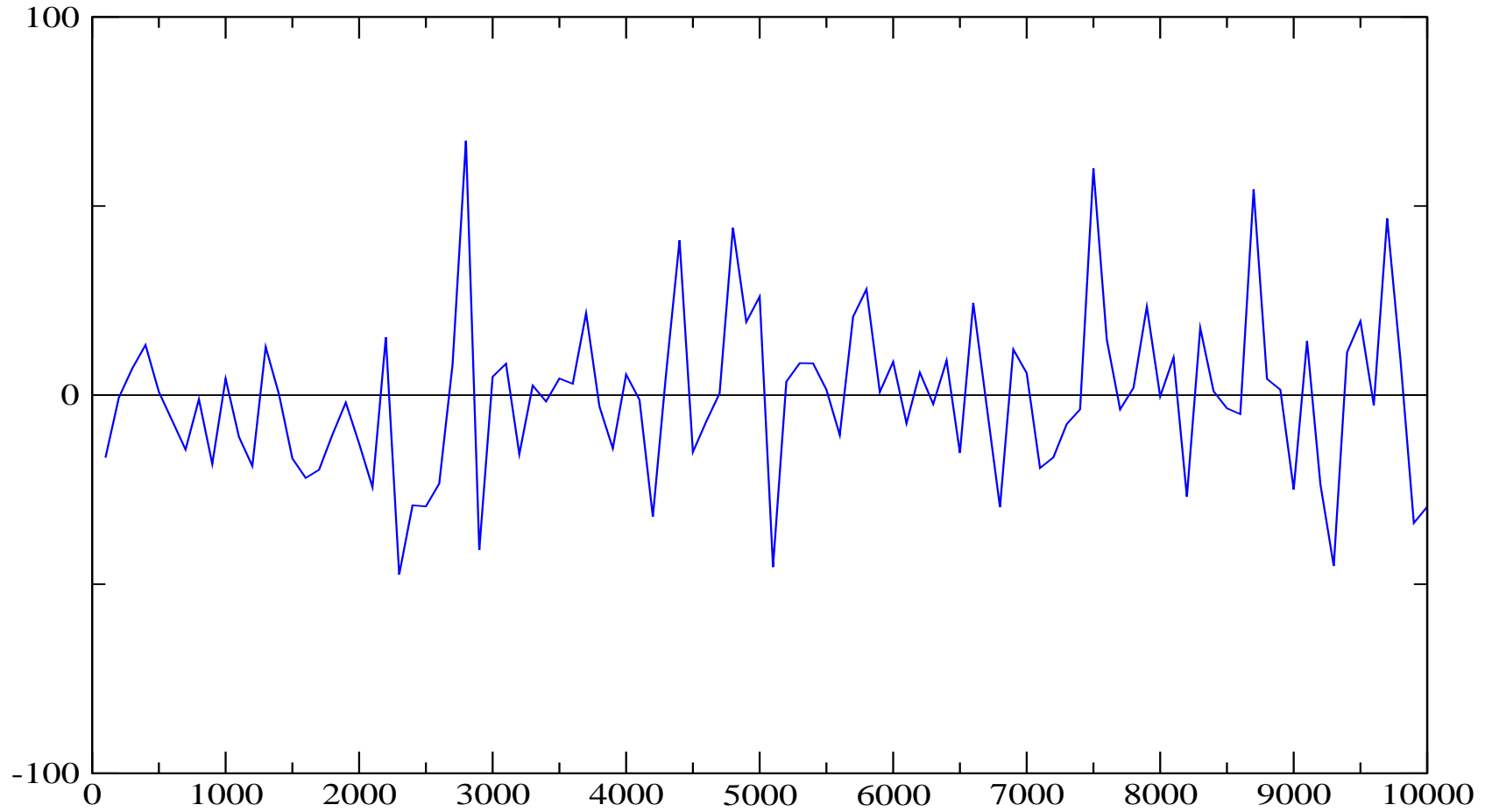
The error  $\hat{\theta}_n - 1.5$  of the MLE for  $n = 50, 100, \dots, 10000$



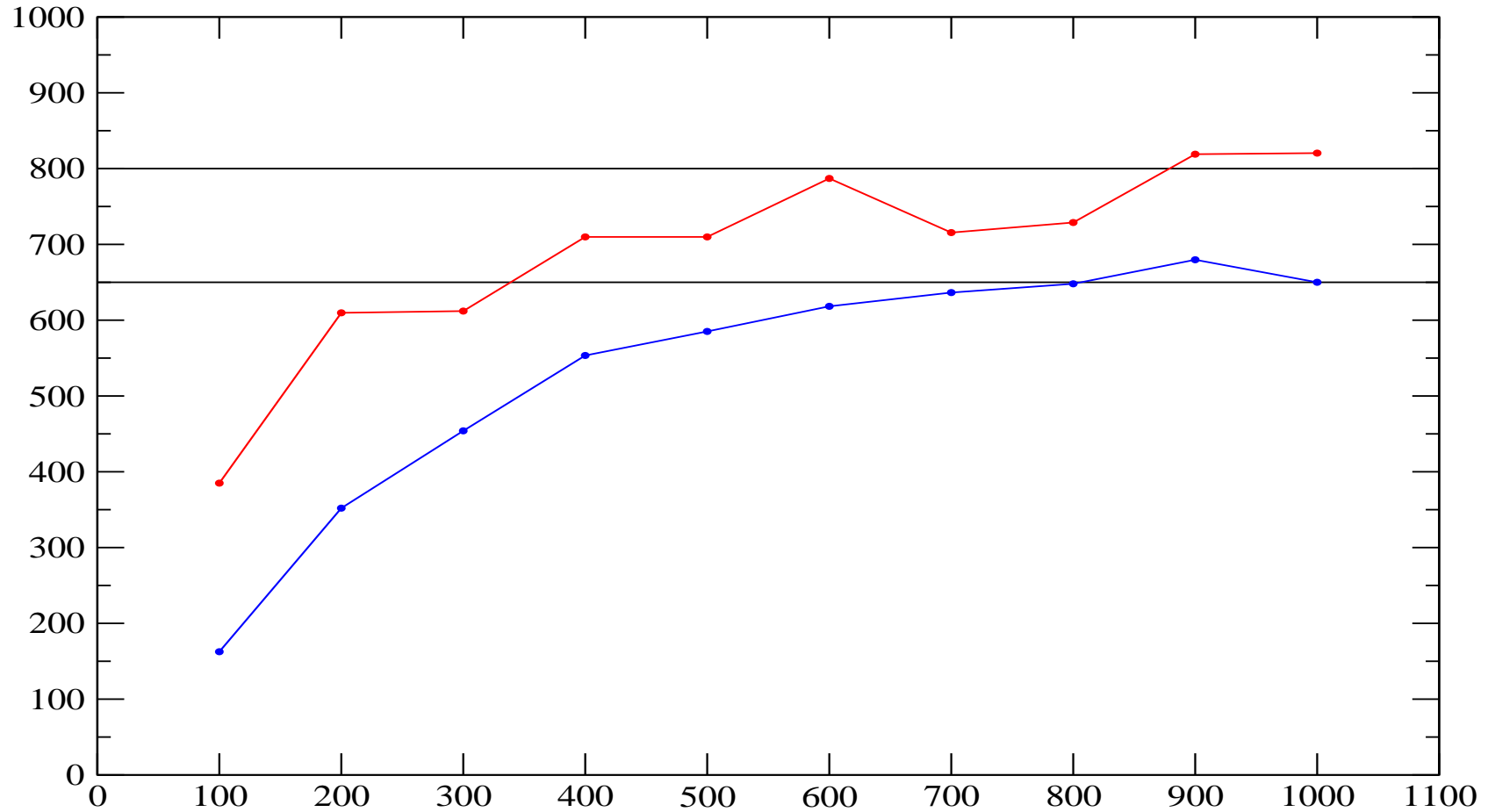
The normalized error  $n^{5/6}(\hat{\theta}_n - 1.5)$  of the MLE for  $n = 50, 100, \dots, 10000$



The error  $\tilde{\theta}_n - 1.5$  of the BE for  $n = 100, 200, \dots, 10000$



The normalized error  $n^{5/6}(\tilde{\theta}_n - 1.5)$  of the BE for  $n = 100, 200, \dots, 10000$



The empirical variances of  $n^{5/6}(\tilde{\theta}_n - 1.5)$  for BE and  $n^{5/6}(\hat{\theta}_n - 1.5)$  for MLE