On Hypotheses Testing for Ergodic Diffusion Process

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Model

We observe a trajectory $X^T = \{X_t,\ 0 \le t \le T\}$ of diffusion process

$$dX_t = S(X_t)\,\mathrm{d}t + \sigma\left(X_t\right)\,\mathrm{d}W_t, \qquad X_0, \qquad t \ge 0.$$

The functions $S(\cdot)$ and $\sigma(\cdot) > 0$ are localy bounded and such that the weak solution of the equation exists, is unique and the conditions

$$\int_0^x \exp\left\{-2\int_0^z \frac{S(y)}{\sigma(y)^2}\mathrm{d}y\right\}\,\mathrm{d}z \to \pm\infty, \qquad x \to \pm\infty$$

and

$$G\left(S\right) = \int_{-\infty}^{\infty} \frac{1}{\sigma\left(z\right)^2}\exp\left\{2\int_0^z \frac{S(y)}{\sigma(y)^2}\mathrm{d}y\right\}\,\mathrm{d}z < \infty$$

are fulfilled.

The process $\{X_t, t \geq 0\}$ is *recurrent positive* with the invariant distribution

$$F(x) = \int_{-\infty}^{x} \frac{1}{G(S)\,\sigma(z)^2} \exp\left\{2\int_0^z \frac{S(y)}{\sigma(y)^2} \mathrm{d}y\right\} \,\mathrm{d}z.$$

Remind the *empirical distribution function* (EDF)

$$\widehat{F}_T(x) = \frac{1}{T}\int_0^T \chi_{\{X_t < x\}}\,\mathrm{d}t$$

and *local time estimator* (LTE) of the invariant density

$$\begin{aligned} f_T^{\circ}(x) &= \frac{1}{T\sigma(x)^2}\int_0^T \mathrm{sgn}\,(x - X_t)\,\mathrm{d}X_t + \\ &\qquad + \frac{|X_T - x| - |X_0 - x|}{T\sigma(x)^2} \end{aligned}$$

The behavior of the likelihood

$$\ln L(X^T) = f(X_0) \int_0^T \frac{S(X_t)}{\sigma(X_t)^2} \,\mathrm{d}X_t - \\ - \frac{1}{2}\int_0^T \left(\frac{S(X_t)}{\sigma(X_t)}\right)^2 \mathrm{d}t$$

is defined by the properties of the integral

$$\frac{1}{T}\int_0^T \left(\frac{S(X_t)}{\sigma(X_t)}\right)^2 \mathrm{d}t = \\ = \int_{-\infty}^{\infty} \left(\frac{S(x)}{\sigma(x)}\right)^2 \, f_T^{\circ}(x)\,\mathrm{d}x \\ = \int_{-\infty}^{\infty} \left(\frac{S(x)}{\sigma(x)}\right)^2 \, \mathrm{d}\hat{F}_T(x)\,.$$

Moreover, we can replace the stochastic integral by ordinary one

$$\begin{aligned}\ln L(X^T) = & -\ln G(S) + H(X_0, S) + H(X_T, S) \\ & - \frac{T}{2} \int \left[\left(\frac{S(x)}{\sigma(x)} \right)^2 + \left(\frac{S(x)}{\sigma(x)^2} \right)' \sigma(x)^2 \right] \; f_T^\circ(x) \, \mathrm{d}x \\ = & -\ln G(S) + H(X_0, S) + H(X_T, S) \\ & - \frac{T}{2} \int \left[\left(\frac{S(x)}{\sigma(x)} \right)^2 + \left(\frac{S(x)}{\sigma(x)^2} \right)' \sigma(x)^2 \right] \; \mathrm{d}\hat{F}_T(x)\end{aligned}$$

where

$$H(x) = \int_0^x \frac{S(v)}{\sigma(v)^2} \, \mathrm{d}v.$$

Hence

$$\left\{ X_0, X_T, f_T^\circ(x), x \in \mathbb{R} \right\}$$

and

$$\left\{ X_0, X_T, \hat{F}_T(x), x \in \mathbb{R} \right\}$$

are sufficient statistics.

One-sided parametric alternative

A. *Contiguous parametric alternatives (Regular case)*

$$\mathrm{d}X_t = S\left(\vartheta, X_t\right)\,\mathrm{d}t + \sigma\left(X_t\right)\,\mathrm{d}W_t, \qquad X_0.$$

Hypotheses testing problem

$$\mathscr{H}_0 : \qquad \vartheta = \vartheta_0,$$
$$\mathscr{H}_1 : \qquad \vartheta > \vartheta_0,$$

or if we put $\vartheta = \vartheta_0 + \frac{u}{\sqrt{T}}$, then

$$\mathscr{H}_0 : \qquad u = 0,$$
$$\mathscr{H}_1 : \qquad u > 0,$$

$\mathcal{P}$. **Regularity condition**: *The process is ergodic under* $\mathscr{H}_0$ *and*

$$\left\| \frac{S(\vartheta_0 + h, \cdot) - S(\vartheta_0, \cdot) - h\,\dot{S}\left(\vartheta_0, \cdot\right)}{\sigma\left(\cdot\right)} \right\| = o\left(h\right)$$

where

$$\left\|g\left(\cdot\right)\right\|^2 = \mathbf{E}_{\vartheta_0}\, g\left(\xi\right)^2.$$

We say that a test $\phi_T^*(X) \in \mathcal{K}_\varepsilon$ is locally asymptotically uniformly most powerful in the class $\mathcal{K}_\varepsilon$ if for any other test $\phi_T(X) \in \mathcal{K}_\varepsilon$ we have: for any $K > 0$

$$\lim_{T\to\infty} \inf_{0\le u\le K} \left[\beta_T\left(u, \phi_T^*\right) - \beta_T\left(u, \phi_T\right)\right] \ge 0.$$

Fisher information

$$\mathrm{I}\left(\vartheta_0\right) = \int_{-\infty}^{\infty} \left(\frac{\dot S\left(\vartheta_0, x\right)}{\sigma\left(x\right)}\right)^2 f\left(\vartheta_0, x\right)\, \mathrm{d}x,$$

and for $t \in (T, T+1]$ we put

$$\sigma(X_t) = 1, \qquad \dot S\left(\vartheta_0, X_t\right) = \sqrt{T\,\mathrm{I}\left(\vartheta_0\right)},$$
$$\mathrm{d}X_t - S\left(\vartheta_0, X_t\right)\, \mathrm{d}t = \mathrm{d}\tilde W_t$$

Then we introduce the stopping time

$$\tau_T = \inf\left\{\tau : \frac{1}{T}\int_0^{\tau}\left(\frac{\dot S\left(\vartheta_0, X_t\right)}{\sigma\left(X_t\right)}\right)^2 \,\mathrm{d}t \geq \mathrm{I}\left(\vartheta_0\right)\right\}$$

$$\Delta_\tau\left(\vartheta_0, X^T\right) = \\ = \frac{1}{\sqrt{T}}\int_0^{\tau_T}\frac{\dot S\left(\vartheta_0, X_t\right)}{\sigma\left(X_t\right)^2}\,\left[\mathrm{d}X_t - S\left(\vartheta_0, X_t\right)\,\mathrm{d}t\right].$$

Theorem. *Let the condition $\mathcal{P}$ be fulfilled and* $\mathrm{I}\left(\vartheta_0\right) > 0$ *then the test*

$$\phi_T^*\left(X\right) = \chi_{\left\{\Delta_\tau\left(\vartheta_0, X^T\right) \geq z_\varepsilon\, \mathrm{I}\left(\vartheta_0\right)^{1/2}\right\}}$$

is locally asymptotically uniformly most powerful in the class $\mathcal{K}_\varepsilon$ and

$$\beta_T\left(u, \phi_T^*\right) = \mathbf{P}\left\{\zeta > z_\varepsilon - u\,\mathrm{I}\left(\vartheta_0\right)^{1/2}\right\} + o\left(1\right),$$

where $\zeta \sim \mathcal{N}\left(0, 1\right)$.

B. *Contiguous parametric alternatives (Non-regular case)*

We observe the ergodic diffusion process

$$\mathrm{d}X_t = S(\vartheta, X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \quad X_0, \quad 0 \le t \le T$$

and we would like to test hypotheses

$$\begin{aligned} \mathscr{H}_0 : &\qquad \vartheta = \vartheta_0, \\ \mathscr{H}_1 : &\qquad \vartheta > \vartheta_0 \end{aligned}$$

in the situations when the regularity condition $\mathcal{P}$ is not fulfilled. Particularly we study three models. The first one corresponds to *change-point testing*, the second to *delay testing* and the third to *cusp testing*.

Change-point testing

Example: $\mathsf{d}X_t = -\,\mathsf{sgn}\left(X_t - \vartheta\right)\ \mathsf{d}t + \mathsf{d}W_t,$

Suppose the trend coefficient $S\left(\vartheta, x\right)$ is a discontinuous along the curve

$$\left\{x_*\left(\theta\right),\ \theta \in [\alpha, \beta]\right\}$$

function, i.e.

$$r\left(\vartheta\right) = S\left(\vartheta, x_*\left(\vartheta\right)+\right) - S\left(\vartheta, x_*\left(\vartheta\right)-\right) \neq 0$$

Let us put

$$\Gamma_\vartheta = \left|\dot{x}_*\left(\vartheta\right)\right| \left(\frac{r\left(\vartheta\right)}{\sigma\left(x_*\left(\vartheta\right)\right)}\right)^2 f\left(\vartheta, x_*\left(\theta\right)\right)$$

We test the hypotheses

$$\mathscr{H}_0: \qquad \vartheta = \vartheta_0,$$
$$\mathscr{H}_1: \qquad \vartheta = \vartheta_0 + \frac{u}{T}, \qquad u > 0,$$

Introduce three independent random variables: $\zeta(u) \sim \mathcal{N}\left(\frac{u}{2}\Gamma^2_{\vartheta_0}, u\Gamma^2_{\vartheta_0}\right)$, $\eta \sim \mathrm{Exp}\,(1)$ and $\eta(u)$ with distribution function $\left(v = u\Gamma^2_{\vartheta_0}\right)$

$$F_{\eta(u)}\,(x) = \Phi\left(\frac{x}{\sqrt{v}} + \frac{\sqrt{v}}{2}\right) + \\ + e^{-x}\left[1 - \Phi\left(\frac{x}{\sqrt{v}} - \frac{\sqrt{v}}{2}\right)\right], \qquad x \geq 0,$$

We study the likelihood ratio test (LRT) based on the statistic

$$\hat{L}\left(X^T\right) = \sup_{\vartheta > \vartheta_0} L\left(\vartheta, \vartheta_0, X^T\right)$$

Proposition *The LRT*

$$\hat{\phi}_T\left(X^T\right) = \chi_{\left\{\hat{L}_T\left(X^T\right) > \frac{1}{\varepsilon}\right\}}$$

belongs to $\mathscr{K}_\varepsilon'$, *is consistent and for any local alternative* $\vartheta = \vartheta_0 + T^{-1}u,\ u > 0$, *the power function*

$$\beta_T\left(u, \hat{\phi}\right) = \mathbf{P}\left\{\zeta(u) + \max\left[\eta(u), \eta\right] > \ln\frac{1}{\varepsilon}\right\} + o\left(1\right).$$

Windows

Suppose that we know that under alternative $\vartheta \in (\vartheta_0, \beta]$, then we can introduce the window

$$\mathbb{A} = [x_* (\vartheta_0), x_* (\beta)]$$

and modify the likelihood ratio as follows

$$\ln L_T^* \left(\vartheta, \vartheta_0, X^T\right) = -\int_0^T \frac{h(\vartheta, X_t)^2}{2\sigma(X_t)^2} \chi_{\{X_t \in \mathbb{A}\}} \mathrm{d}t$$

$$+ \int_0^T \frac{h(\vartheta, X_t)}{\sigma(X_t)^2} \chi_{\{X_t \in \mathbb{A}\}} [\mathrm{d}X_t - S(\vartheta_0, X_t)\, \mathrm{d}t]$$

where $h(\vartheta, x) = S(\vartheta, x) - S(\vartheta_0, x)$. Then we put

$$\hat{L}_T^* \left(X^T\right) = \sup_{\vartheta > \vartheta_0} L_T^* \left(\vartheta, \vartheta_0, X^T\right)$$

Proposition. *The LRT*

$$\widehat{\phi}_T^*\left(X^T\right) = \chi_{\left\{\widehat{L}_T^*\left(X^T\right) > \frac{1}{\varepsilon}\right\}}$$

belongs to $\mathscr{K}_\varepsilon'$, is consistent and for any local alternative $\vartheta = \vartheta_0 + T^{-1}u,\ u > 0$, the power function

$$\beta_T\left(u, \widehat{\phi}^*\right) = \mathbf{P}\left\{\zeta(u) + \max\left[\eta(u), \eta\right] > \ln\frac{1}{\varepsilon}\right\} + o\left(1\right).$$

This means that having observations in $\mathbb{A}$ only we have no loss of information.

It is possible to construct the test with the same asymptotic properties by the observations in the windows of decreasing to zero width (as in estimation theory). Let

$$\mathrm{d}X_t = -\,\mathrm{sgn}\left(X_t - \vartheta\right)\ \mathrm{d}t + \mathrm{d}W_t.$$

Using observations $\left\{X_t, 0 \leq t \leq \sqrt{T}\right\}$ we construct a consistent estimator, say,

$$\bar{\vartheta}_{\sqrt{T}} = \frac{1}{\sqrt{T}} \int_0^{\sqrt{T}} X_t \,\mathrm{d}t \longrightarrow \vartheta$$

and then put

$$\mathbb{A}_T = \left[\bar{\vartheta}_{\sqrt{T}} - T^{-1/8}, \bar{\vartheta}_{\sqrt{T}} + T^{-1/8}\right].$$

The LRT based on the statistic $\widehat{L}_T^*\left(X^T\right)$ where we replace $\mathbb{A}$ by $\mathbb{A}_T$ has the same asymptotic properties as the LRT $\widehat{\phi}_T\left(X^T\right)$, i.e;, it belongs to $\mathscr{K}_\varepsilon'$ and is consistent.

Delay testing

The observed process is

$$\mathrm{d}X_t = -\gamma \, X_{t-\vartheta} \, \mathrm{d}t + \mathrm{d}W_t, \qquad \hat{X}_0, \qquad\qquad 0 \le t \le T$$

and we test the hypotheses

$$\begin{aligned} \mathscr{H}_0 : & \qquad \vartheta = \vartheta_0, \\ \mathscr{H}_1 : & \qquad \vartheta = \vartheta_0 + \frac{u}{T}, \qquad\qquad u > 0, \end{aligned}$$

where $\gamma > 0$ is known, the parameter $\vartheta \in [\vartheta_0, \beta)$, $0 \le \vartheta_0 < \beta < \pi/2\gamma$.

Proposition. *The LRT*

$$\hat{\phi}_T\left(X^T\right) = \chi_{\left\{\hat{L}_T\left(X^T\right) > \frac{1}{\varepsilon}\right\}}$$

belongs to $\mathscr{K}_\varepsilon'$*, is consistent and for any local alternative* $\vartheta = \vartheta_0 + T^{-1}u,\ u > 0$*, the power function*

$$\beta_T\left(u, \hat{\phi}\right) = \mathbf{P}\left\{\zeta(u) + \max\left[\eta(u), \eta\right] > \ln \frac{1}{\varepsilon}\right\} + o\left(1\right).$$

where we have to put $\Gamma_{\vartheta_0} = \gamma$.

Cusp testing

Let the observed process be

$$\mathrm{d}X_t = -\,\mathrm{sgn}\,(X_t - \vartheta)\,|X_t - \vartheta|^{\kappa}\ \mathrm{d}t + \mathrm{d}W_t,$$

where $\kappa \in (0, 1/2)$. The contiguous alternatives are given by the following hypotheses testing problem

$$\begin{aligned} &\mathscr{H}_0:\ \vartheta = \vartheta_0, \\ &\mathscr{H}_1:\ \vartheta = \vartheta_0 + \frac{u}{T^{\frac{1}{2\kappa+1}}}, \qquad u > 0 \end{aligned}$$

Let us denote y_ε the $(1-\varepsilon)$-quantile ($\mathbf{P}\{\eta < y_\varepsilon\} = 1 - \varepsilon$) of the random variable

$$\eta = \sup_{u \geq 0}\left(W^H(u) - \frac{|u|^{2\kappa+1}}{2}\right),$$

where $W^H(\cdot)$ is a fractional Brownian motion with Hurst constant $H = \kappa + 1/2$.

Let us introduce the process ($v \geq 0$)

$$Y(v, \tilde{u}) = W^H(v) - \frac{|v - \tilde{u}|^{2\kappa+1} - |\tilde{u}|^{2\kappa+1}}{2}$$

where $\tilde{u} = u\,\Gamma_{\vartheta_0}^{\kappa+1/2} u$ and

$$\Gamma_\vartheta^2 = \frac{1}{G(\vartheta)}\ \frac{\Gamma(1+\kappa)\Gamma(\frac{1}{2}-\kappa)}{2^{2\kappa-1}\sqrt{\pi}(2\kappa+1)}\left[1 - \cos(\pi\kappa)\right],$$

Proposition. *The LRT*

$$\hat{\phi}_T\left(X^T\right) = \chi_{\left\{\ln \hat{L}_T\left(X^T\right) \geq y_\varepsilon\right\}} \in \mathscr{K}_\varepsilon'$$

is consistent and its power function for the local alternatives $\vartheta = \vartheta_0 + T^{-1/(2\kappa+1)}u$ *is*

$$\beta_T\left(u, \hat{\phi}_T\right) = \mathbf{P}\left\{\sup_{v\geq 0} Y(v, \tilde{u}) \geq y_\varepsilon\right\} + o(1).$$

As in the change-point testing we need not to have all observations and can use only that in the window

$$\mathbb{A} = [\vartheta_0, \beta]\,.$$

The LRT based on the modified LR statistic $\hat{L}^*_T\left(X^T\right)$ will have the same asymptotic properties as if $\mathbb{A} = \mathbb{R}$.

Goodness-of-fit tests

Let the observed process be

$$\mathrm{d}X_t = S\left(X_t\right)\,\mathrm{d}t + \mathrm{d}W_t, \qquad 0 \le t \le T,$$

Fix some function $S_0\left(\cdot\right)$ and consider the problem

$$\begin{aligned} \mathscr{H}_0: &\qquad S\left(\cdot\right) = S_0(\cdot) \\ \mathscr{H}_1: &\qquad S\left(\cdot\right) \neq S_0(\cdot) \end{aligned}$$

Then it is easy to see that if

$$\sup_x \left|S\left(x\right) - S_0\left(x\right)\right| > 0,$$

then

$$\sup_x \left|f_S\left(x\right) - f_{S_0}\left(x\right)\right| > 0$$

and

$$\sup_x \left|F_S\left(x\right) - F_{S_0}\left(x\right)\right| > 0.$$

Therefore, it is possible to construct the goodness-of-fit tests based on the statistics

$$\delta_T\left(X^T\right) = \sup_x \sqrt{T}\ \left|f_T^\circ(x) - f_{S_0}(x)\right|$$

and

$$\gamma_T\left(X^T\right) = \sup_x \sqrt{T}\ \left|\hat{F}_T(x) - F_{S_0}(x)\right|.$$

Condition

$\boldsymbol{A}_0$. *The function* $S\left(\cdot\right)$ *satisfies the condition*

$$\overline{\lim_{|x|\to\infty}}\ \operatorname{sgn}\left(x\right)\ S\left(x\right) < 0$$

Let us introduce a Gaussian process $\eta(x)$ with $\mathbf{E}\,\eta(x) = 0$ and the covariance function

$$R_{S_0}(x,y) = 4\, f_{S_0}(x)\, f_{S_0}(y)$$
$$\mathbf{E}_{S_0}\left(\frac{\left[\chi_{\{\xi>x\}} - F_{S_0}(\xi)\right]\left[\chi_{\{\xi>y\}} - F_{S_0}(\xi)\right]}{f_{S_0}(\xi)^2}\right).$$

Denote by y_ε the value defined by the equation

$$\mathbf{P}\left\{\sup_x |\eta(x)| > y_\varepsilon\right\} = \varepsilon$$

Proposition. *Let the condition $\mathcal{A}_0$ be fulfilled, then the test*

$$\phi_T^*\left(X^T\right) = \chi_{\left\{\delta_T(X^T) > y_\varepsilon\right\}}$$

belongs to $\mathscr{K}_\varepsilon'$ and is consistent.