Optimum Sequential Procedures

for

Detecting Changes in Processes

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Outline

- The change detection (disorder) problem
- Overview of existing results
- Lorden's criterion and the CUSUM test
- A modified Lorden criterion
- Optimality of CUSUM for Itô processes

The Change Detection (Disorder) Problem

We are observing sequentially a process $»_t$ with the following statistics

$$
\begin{aligned}
»_t &\sim \mathbb{P}_\infty && \text{for } 0 \le t \le ¿ \\
&\sim \mathbb{P}_0 && \text{for } ¿ < t
\end{aligned}
$$

– Change time ¿ : deterministic (but unknown) or random.
– Probability measures $\mathbb{P}_\infty; \mathbb{P}_0$: known.

Detect the change "as soon as possible".

Applications include: systems monitoring; quality control; financial decision making; remote sensing (radar, sonar, seismology); occurrence of industrial accidents; speech/image/video segmentation; etc.

The observation process $»_t$ is available sequentially; this can be expressed through the filtration:

$$\mathcal{F}_t = ¾\{»_s : 0 \leq s \leq t\}.$$

For detecting the change we are interested in sequential schemes.

Any sequential detection scheme can be represented by a stopping time T (the time we stop and declare that the change took place).

The stopping time T is adapted to $\mathcal{F}_t$.
In other words, *at every time instant* t we perform a test (whether to stop and declare a change or continue sampling) using only the available information up to time t.

Overview of Existing Results

$\mathbb{P}_\tau$: the probability measure induced, when the change takes place at time ¿ .

$\mathbb{E}_\tau[\cdot]$: the corresponding expectation.

$\mathbb{P}_\infty$: all data under nominal régime.

$\mathbb{P}_0$: all data under alternative régime.

Optimality Criteria

They are basically comprised of two parts:

– The first measures the detection delay

– The second the frequency of false alarms

Possible approaches are Baysian and Min-max.

Bayesian Approach (Shiryayev):

¿ is random and exponentially distributed

$$\inf_{T}\{c\,\mathbb{E}[(T-¿)^{+}]+\mathbb{P}[T<¿]\}$$

The Shiryayev test consists in computing the statistics $¼_t=\mathbb{P}[¿\leq t|\mathcal{F}_t]$; and stop when

$$T_S=\inf_{t}\{t:¼_t\geq º\}:$$

T_S is optimum (Shiryayev 1978):

– In discrete time: when $»_n$ is i.i.d. before and after the change.

– In continuous time: when $»_t$ is a Brownian Motion with constant drift before and after the change.

Min-Max Approach (Shiryayev-Roberts-Pollak):
¿ is deterministic and unknown

$$\inf_T \sup_\tau \mathbb{E}_\tau[(T - ¿)^+ | T > ¿]; \text{ subject } \mathbb{E}_\infty[T] \geq °:$$

Optimality results exists only for discrete time when $»_n$ is i.i.d. before and after the change. Specifically if we define the statistics

$$S_n = (S_{n-1} + 1)\frac{f_0(\xi_n)}{f_\infty(\xi_n)};$$

where $f_\infty(\cdot); f_0(\cdot)$ the common pdf of the data before and after the change then (Yakir 1997) the stopping time

$$T_{SRP} = \inf_n\{n : S_n \geq º\}$$

is optimum.

Lorden's Criterion and the CUSUM Test

An alternative min-max approach consists in defining the following performance measure (Lorden 1971)

$$J(T) = \sup_{\tau} \operatorname{essup} \mathbb{E}_{\tau}[(T - ¿)^{+} | \mathcal{F}_{\tau}]$$

and solve the min-max problem

$$\inf_{T} J(T); \quad \text{subject to } \mathbb{E}_{\infty}[T] \geq °.$$

The test closely related to Lorden's criterion and being to most popular one used in practice is the **Cumulative Sum** (CUSUM) test.

Define the CUSUM statistics y_t as follows:

$$
\begin{aligned}
u_t &= \log\left(\frac{d\mathbb{P}_0}{d\mathbb{P}_\infty}(\mathcal{F}_t)\right); \;\; m_t = \inf_{0\leq s\leq t} u_s \\
y_t &= u_t - m_t.
\end{aligned}
$$

The CUSUM stopping time (Page 1954):

$$T_C = \inf{}_t\{t : y_t \geq º\}.$$

Optimality results:

– Discrete time: when $»_n$ is i.i.d. before and after the change (Moustakides 1986, Ritov 1990).

– Continuous time: when $»_t$ is a Brownian Motion with constant drift before and after the change (Shiryayev 1996, Beibel 1996).

A modified Lorden criterion

Our goal is to extend the optimality of CUSUM to Itô processes. For this it will be necessary to modify Lorden's criterion using the Kullback-Leibler Divergence (KLD).

Similar extension was proposed for the SPRT by Liptser and Shiryayev (1978).

Consider the process $»_t$

$$d»_t = \begin{cases} dw_t; & 0 \leq t \leq ¿ \\ ®_t dt + dw_t; & ¿ < t \end{cases}$$

where w_t is a standard Brownian motion with respect to $\mathcal{F}_t = ¾(»_s; 0 \leq s \leq t)$; $®_t$ is adapted to $\mathcal{F}_t$ and $¿$ denotes the time of change.

To ξ_t we correspond the process u_t defined by

$$du_t = \alpha_t d\xi_t - 0.5\alpha_t^2 dt$$

which we like to play the role of the log-likelihood ratio $u_t = \log(d\mathbb{P}_0/d\mathbb{P}_\infty(\mathcal{F}_t))$. We therefore need to impose the following conditions:

1. $\mathbb{P}_0\left[\int_0^t \alpha_s^2 ds < \infty\right] = \mathbb{P}_\infty\left[\int_0^t \alpha_s^2 ds < \infty\right] = 1$
2. A “Novikov” condition, i.e. $\mathbb{E}_\infty[\exp(\int_{t_{n-1}}^{t_n} \alpha_s^2 ds)] < \infty$ where t_n strictly increasing with $t_n \to \infty$.
3. $\mathbb{P}_0\left[\int_0^\infty \alpha_s^2 ds = \infty\right] = \mathbb{P}_\infty\left[\int_0^\infty \alpha_s^2 ds = \infty\right] = 1$

From conditions 1 & 2 we have validity of Girsanov's theorem, therefore

$$\frac{d\mathbb{P}_0}{d\mathbb{P}_\infty}(\mathcal{F}_t) = e^{u_t}; \quad \frac{d\mathbb{P}_\tau}{d\mathbb{P}_\infty}(\mathcal{F}_t) = e^{u_t - u_\tau}:$$

Furthermore for the KLD we can write

$$\mathbb{E}_\tau\left[\log\left(\frac{d\mathbb{P}_\tau}{d\mathbb{P}_\infty}(\mathcal{F}_t)\right)\,\Big|\,\mathcal{F}_\tau\right]$$

$$= \quad \mathbb{E}_\tau\left[\int_\tau^t \text{®}_s dw_s + \int_\tau^t \frac{1}{2}\text{®}_s^2 ds \,\Big|\, \mathcal{F}_\tau\right]$$

$$= \quad \mathbb{E}_\tau\left[\int_\tau^t \frac{1}{2}\text{®}_s^2 ds \,\Big|\, \mathcal{F}_\tau\right]; \ \text{ for } 0 \le \text{¿} \le t < \infty;$$

This suggests the following modification in Lorden's criterion

$$J(T) = \sup_{\tau\in[0,\infty)} \operatorname{essup} \mathbb{E}_\tau\left[\mathbb{1}_{\{T>\tau\}} \int_\tau^T \frac{1}{2} \text{®}_t^2 dt \,\middle|\, \mathcal{F}_\tau\right];$$

and the corresponding min-max optimization

$$\inf_T J(T); \;\; \text{subject} \;\; \mathbb{E}_\infty\left[\int_0^T \frac{1}{2}\text{®}_t^2 dt\right] \geq \text{°}:$$

The original and the modified criterion coincide when »_t is a Brownian motion with constant drift.

Let us form the CUSUM statistics y_t for the Itô process

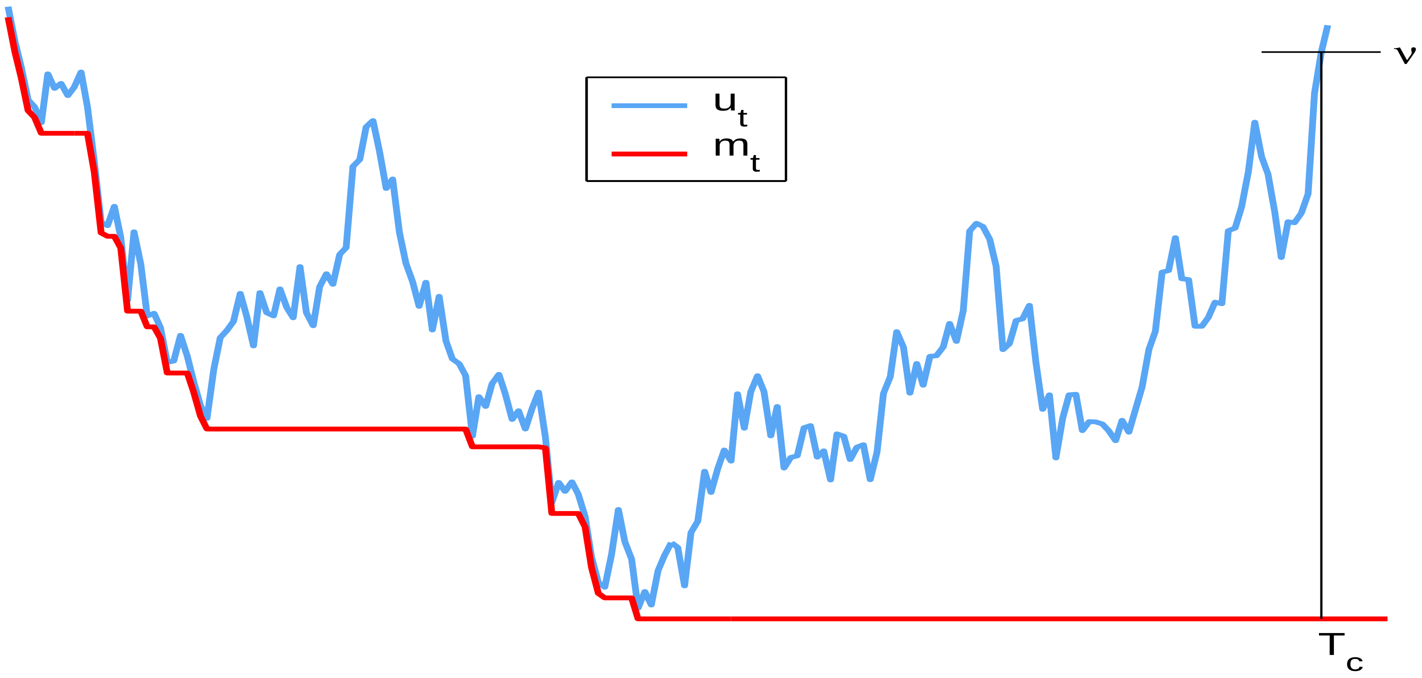
$$
\begin{aligned}
du_t &= ®_t d»_t - 0{:}5®_t^2 dt \\
m_t &= \inf_{0 \le s \le t} u_s \\
y_t &= u_t - m_t
\end{aligned}
$$

and the optimum CUSUM test is

$$T_C = \inf_t \{t : y_t \ge º\}; \text{ where } \mathbb{E}_\infty \left[\int_0^{T_C} \frac{1}{2} ®_t^2 dt \right] = °{:}$$

Since y_t has continuous paths we conclude that when the CUSUM test stops we will have: $y_{T_C} = º{:}$

Optimality of CUSUM for Itô processes



$u_t \geq m_t$ therefore $y_t = u_t - m_t \geq 0$.

m_t is nonincreasing and $dm_t \neq 0$ only when $u_t = m_t$ or $y_t = 0$.

If $f(y)$ continuous; $f(0) = 0$, then $\int_0^\infty f(y_t)dm_t = 0$.

If $f(y)$ is a twice continuously differentiable function with $f'(0) = 0$, using standard Itô calculus we have

$$\begin{aligned} df(y_t) &= f'(y_t)(du_t - dm_t) + 0{:}5®_t^2 f''(y_t)dt \\ &= f'(y_t)du_t + 0{:}5®_t^2 f''(y_t)dt \end{aligned}$$

Theorem 1: T_C is a.s. finite and

$$\mathbb{E}_\tau\left[\mathbb{1}_{\{T_C>\tau\}}\int_\tau^{T_C}\frac{1}{2}®_t^2 dt\,\middle|\,\mathcal{F}_\tau\right] = [g(^o) - g(y_\tau)]\mathbb{1}_{\{T_C>\tau\}}$$

$$\mathbb{E}_\infty\left[\mathbb{1}_{\{T_C>\tau\}}\int_\tau^{T_C}\frac{1}{2}®_t^2 dt\,\middle|\,\mathcal{F}_\tau\right] = [h(^o) - h(y_\tau)]\mathbb{1}_{\{T_C>\tau\}}.$$

where

$$g(y) = y + e^{-y} - 1; \qquad h(y) = e^y - y - 1{:}$$

Since $g(y); h(y)$ are increasing and strictly convex with $g(0) = h(0) = 0$, we now conclude

$$
\begin{aligned}
J(T_C) &= \sup_{\tau} \operatorname{essup} \mathbb{E}_\tau \left[\int_\tau^{T_C} \text{®}_s^2 ds \middle| \mathcal{F}_\tau \right] \\
&= \sup_{\tau} \operatorname{essup} [g({}^{o}) - g(y_\tau)] \mathbb{1}_{\{T_C > \tau\}} \\
&= g({}^{o}) - g(0) = g({}^{o})
\end{aligned}
$$

Similarly

$$\mathbb{E}_\infty \left[\int_0^{T_C} \text{®}_s^2 ds \right] = h({}^{o}) - h(0) = h({}^{o}) = {}^{\circ}:$$

The threshold can thus be computed: $e^{\nu} - {}^{o} - 1 = {}^{\circ}$.

Using again standard Itô calculus we have the following generalization of Theorem 1.

Corollary:

$$\mathbb{E}_\tau \left[\int_\tau^T \tfrac{1}{2} \text{®}_t^2 dt \,\middle|\, \mathcal{F}_\tau \right] = \mathbb{E}_\tau \left[g(y_T) - g(y_\tau) | \mathcal{F}_\tau \right] \mathbb{1}_{\{T>\tau\}}$$

$$\mathbb{E}_\infty \left[\int_\tau^T \tfrac{1}{2} \text{®}_t^2 dt \,\middle|\, \mathcal{F}_\tau \right] = \mathbb{E}_\infty \left[h(y_T) - h(y_\tau) | \mathcal{F}_\tau \right] \mathbb{1}_{\{T>\tau\}}$$

where T stopping time.

Remark 1: The false alarm constraint can be written as

$$\mathbb{E}_\infty \left[\int_0^T \tfrac{1}{2} \text{®}_t^2 dt \right] = \mathbb{E}_\infty [h(y_T)] \geq \text{°}$$

Remark 2: We can limit ourselves to stopping times that satisfy the false alarm constraint with equality, that is,

$$\mathbb{E}_\infty\left[\int_0^T \tfrac{1}{2}®_t^2 dt\right] = \mathbb{E}_\infty[h(y_T)] = ° = h(º):$$

Remark 3: The modified performance measure $J(T)$ can be suitably lower bounded as follows

$$\begin{aligned} J(T) &= \sup_\tau \operatorname{essup} \mathbb{E}_\tau\left[\mathbb{1}_{\{T>\tau\}} \int_\tau^T \frac{1}{2}®_t^2 dt \,\middle|\, \mathcal{F}_\tau\right] \\ &\geq \frac{\mathbb{E}_\infty\left[e^{y_T} g(y_T)\right]}{\mathbb{E}_\infty\left[e^{y_T}\right]}: \end{aligned}$$

Theorem 2: Any stopping time T that satisfies the false alarm constraint with equality has a performance measure $J(T)$ that is no less than $J(T_C) = g(o)$.

Proof: To show $J(T) \geq g(o)$, since

$$J(T) \geq \frac{\mathbb{E}_\infty\left[e^{y_T} g(y_T)\right]}{\mathbb{E}_\infty\left[e^{y_T}\right]};$$

it is sufficient to show that

$$\frac{\mathbb{E}_\infty\left[e^{y_T} g(y_T)\right]}{\mathbb{E}_\infty\left[e^{y_T}\right]} \geq g(o)$$

or equivalently: $\mathbb{E}_\infty\left[e^{y_T}\{g(y_T) - g(o)\}\right] \geq 0$

We recall that we consider stopping times with

$$\mathbb{E}_\infty\left[\int_0^T \frac{1}{2}\text{®}_t^2 dt\right] = \mathbb{E}_\infty[h(y_T)] = \text{°} = h({}^{o});$$

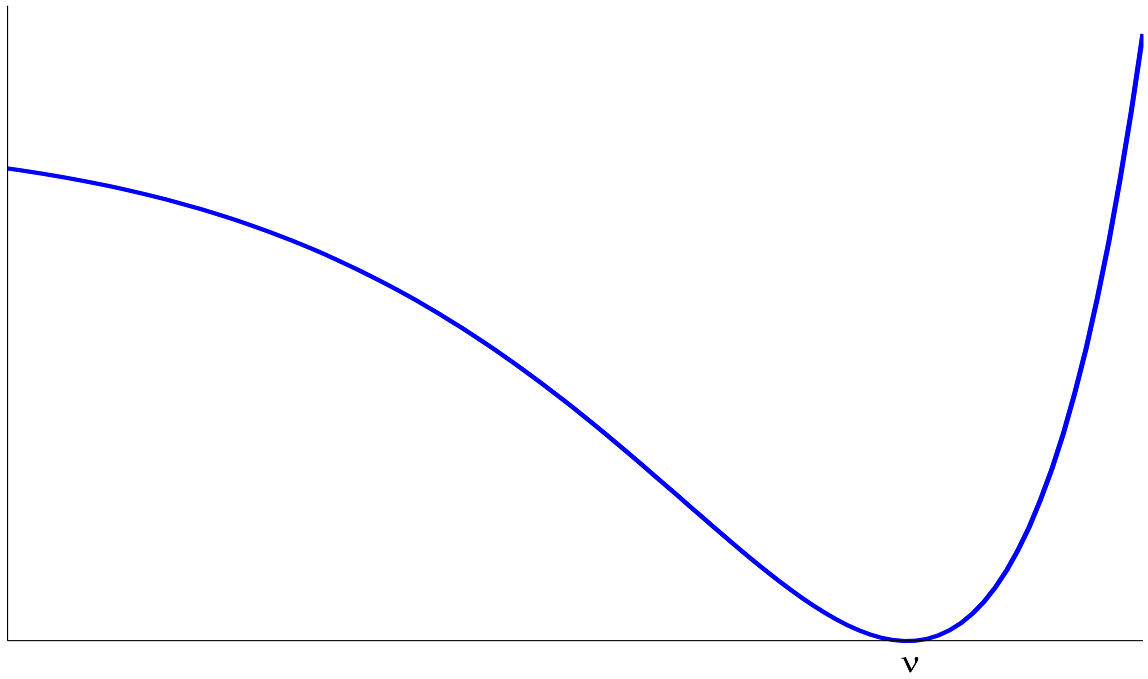
therefore the inequality we like to prove is equivalent to

$$\mathbb{E}_\infty\left[e^{y_T}\{g(y_T) - g({}^{o})\} + h({}^{o}) - h(y_T)\right] \geq 0:$$

The function

$$p(y) = e^{y}\{g(y) - g({}^{o})\} + h({}^{o}) - h(y)$$

for $y \geq 0$, can be shown to exhibit a global minimum at $y = {}^{o}$



Because $p({}^{o}) = 0$, we conclude that $p(y) \geq 0$, thus

$$\mathbb{E}_\infty[p(y_T)] \geq 0$$

with equality iff $y_T = {}^{o}$ (i.e. the CUSUM stopping time).

Conclusion

- We introduced a modification of Lorden's criterion based on the Kullback-Leibler Divergence for the problem of detecting changes in Itô processes.
- With the help of the new criterion we introduced a constrained min-max optimization problem that defines the optimum sequential scheme for the change detection problem.
- We demonstrated that the CUSUM test is the solution to the above optimization problem.