On compound Poisson type limiting likelihood ratio processes.

(joint work with I. Negri)

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Parameter estimation
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Asymptotics: \( n \to \infty \) (or \( T \to \infty \), or \( \varepsilon \to 0 \), \ldots).
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Likelihood: density $L^{(n)}_\theta$ of $P^{(n)}_\theta$ (its Radon-Nikodym derivative w.r.t. some reference measure) evaluated in $X^{(n)}$, that is, $L_n(\theta) = L^{(n)}_\theta (X^{(n)})$. 
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Maximum likelihood estimator: \( \hat{\theta}_n = \arg\sup_{\theta \in \Theta} L_n(\theta) \).
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Maximum likelihood estimator: $\hat{\theta}_{n} = \operatorname{arg\,sup}_{\theta \in \Theta} L_{n}(\theta)$.

Bayesian estimators (for quadratic loss function):

$$\tilde{\theta}_{n} = \frac{\int_{\Theta} \theta p(\theta) L_{n}(\theta) \, d\theta}{\int_{\Theta} p(\theta) L_{n}(\theta) \, d\theta} \quad \text{where } p \text{ is some prior density.}$$
Likelihood ratio analysis


First find a renormalization rate

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such that the renormalized likelihood ratio (process)

\[ Z_n(u) = \frac{L_n(\theta + u \varphi_n)}{L_n(\theta)} = \frac{dP_{\theta+u\varphi_n}^{(n)}}{dP_{\theta}^{(n)}}(X^{(n)}), \quad u \in \mathbb{R}, \]
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converges weekly (in a suitable functional space) to some non-degenerate limiting likelihood ratio (process)

\[ Z(u), \quad u \in \mathbb{R}. \]
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- are **consistent**;
- converge at rate $\varphi_n$ and have **limiting distributions** given by

$$
\xi = \arg\sup_{u \in \mathbb{R}} Z(u) \quad \text{and} \quad \zeta = \frac{\int_{\mathbb{R}} u Z(u) \, du}{\int_{\mathbb{R}} Z(u) \, du}
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that is,

\[
\varphi_n^{-1}(\hat{\theta}_n - \theta) \Rightarrow \xi \quad \text{and} \quad \varphi_n^{-1}(\tilde{\theta}_n - \theta) \Rightarrow \zeta;
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- the Bayesian estimators are **asymptotically efficient**.
Likelihood ratio analysis

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- Limiting likelihood ratio depends on the model.
- We have, in general, $\xi \neq \zeta$ and $M \neq B$.
- $E = B/M$ is the \textit{asymptotic (relative) efficiency} of the MLE.
Limiting likelihood ratio process $Z_0$
The process $Z_0$ on $\mathbb{R}$ defined by

$$\ln Z_0(x) = W(x) - \frac{1}{2} |x|$$

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The random variables

$$\xi_0 = \arg\sup_{x \in \mathbb{R}} Z_0(x) \quad \text{and} \quad \zeta_0 = \frac{\int_{\mathbb{R}} x Z_0(x) \, dx}{\int_{\mathbb{R}} Z_0(x) \, dx}.$$
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The quantities $M_0 = \mathbb{E}\xi_0^2$, $B_0 = \mathbb{E}\zeta_0^2$ and $E_0 = B_0 / M_0$. 
Models with limiting likelihood ratio $Z_0$
Discontinuous signal in a white Gaussian noise:
Ibragimov and Khasminskii (1975) and (1981, Chapter 7.2);
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Discontinuous periodic signal in a time inhomogeneous diffusion: Höpfner and Kutoyants (2009); ...
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Asymptotics: $\varepsilon \to 0.$
The normalized likelihood ratio process:

\[ Z_\varepsilon(u) = \frac{dP^{(\varepsilon)}_{\theta+\varepsilon^2u}(X)}{dP^{(\varepsilon)}_\theta(X)} \]

\[ = \exp \left\{ \frac{1}{\varepsilon} \int_0^1 \left[ S(t - \theta - \varepsilon^2u) - S(t - \theta) \right] dW(t) \right. \]

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It converges weakly in the space \( C_0(-\infty, +\infty) \) to \( Z_0(r^2u) \).
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LMSEs and AE: \( r^{-4} M_0 \), \( r^{-4} B_0 \) and \( E_0 \).
About $\xi_0$, $\zeta_0$, $M_0$, $B_0$ and $E_0$
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Rubin and Song (1995):
- exact values $B_0 = 16 \zeta(3)$ and $E_0 = 8 \zeta(3)/13$, where $\zeta$ is Riemann’s zeta function.
Limiting likelihood ratio processes $Z_{\gamma,f}$
The process $Z_{\gamma,f}$ on $\mathbb{R}$ defined by

$$
\ln Z_{\gamma,f}(x) = \begin{cases} 
\sum_{k=1}^{\Pi_+(x)} \ln \frac{f(\epsilon^+_k + \gamma)}{f(\epsilon^+_k)}, & \text{if } x \geq 0, \\
\sum_{k=1}^{\Pi_-(x)} \ln \frac{f(\epsilon^-_k - \gamma)}{f(\epsilon^-_k)}, & \text{if } x \leq 0,
\end{cases}
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where $\gamma > 0$, $f$ is a strictly positive density of some random variable $\epsilon$ with mean 0 and variance 1, $\Pi_+$ and $\Pi_-$ are two independent Poisson processes of intensity 1 on $\mathbb{R}_+$, and $\epsilon^\pm_k$ are i.i.d. random variables with density $f$ which are also independent of $\Pi_\pm$. 

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An important particular case is $\varepsilon \sim \mathcal{N}(0,1)$, so

$$
\ln \frac{f(\varepsilon \pm \gamma)}{f(\varepsilon)} = \mp \gamma \varepsilon - \frac{\gamma^2}{2} \sim \mathcal{N}(-\gamma^2/2, \gamma^2).
$$
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$$\zeta_{\gamma,f} = \frac{\int_{\mathbb{R}} x Z_{\gamma,f}(x) \, dx}{\int_{\mathbb{R}} Z_{\gamma,f}(x) \, dx},$$

$$\xi_{\gamma,f}^- = \inf \left\{ z : Z_{\gamma,f}(z) = \sup_{x \in \mathbb{R}} Z_{\gamma,f}(x) \right\},$$

$$\xi_{\gamma,f}^+ = \sup \left\{ z : Z_{\gamma,f}(z) = \sup_{x \in \mathbb{R}} Z_{\gamma,f}(x) \right\},$$

$$\xi_{\gamma,f}^\alpha = \alpha \xi_{\gamma,f}^- + (1 - \alpha) \xi_{\gamma,f}^+, \quad \alpha \in [0,1].$$
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$$\xi_{\gamma,f}^\alpha = \alpha \xi_{\gamma,f}^- + (1 - \alpha) \xi_{\gamma,f}^+, \quad \alpha \in [0, 1].$$

Note that $\xi_{\gamma,f}^\alpha$ is the limiting distribution of the appropriately chosen MLE (which is not unique in underlying models).
Random variables $\xi_{\gamma,f}^\alpha$ and $\zeta_{\gamma,f}$

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The quantities $M_{\gamma,f}^\alpha = \mathbb{E}(\xi_{\gamma,f}^\alpha)^2$, $B_{\gamma,f} = \mathbb{E}(\zeta_{\gamma,f})^2$ and $E_{\gamma,f}^\alpha = B_{\gamma,f}/M_{\gamma,f}^\alpha$. 
Models with limiting likelihood ratio $Z_{\gamma,f}$
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Two-phase regression models.
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Linear case:
Koul and Qian (2002).
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Regularity assumptions on $f$
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We suppose that $\sqrt{f}$ is continuously differentiable in $L^2$, that is, there exists $\psi \in L^2$ verifying

\[ \int_{\mathbb{R}} \left( \sqrt{f(x + h)} - \sqrt{f(x)} - h \psi(x) \right)^2 \, dx = o(h^2) \]

and

\[ \int_{\mathbb{R}} (\psi(x + h) - \psi(x))^2 \, dx = o(1), \text{ and that } \|\psi\| > 0. \]
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Remarks:

- Under this assumptions, the model of i.i.d. observations with density $f(x + \theta)$ is, in particular, LAN at $\theta = 0$ with Fisher information $I = 4 \|\psi\|^2 = 4 \int_{\mathbb{R}} \psi^2(x) \, dx$, and so

$$\lim_{\gamma \to 0} \left( E e^{it \ln \frac{f(\theta + \gamma)}{f(\theta)}} \right)^{1/\gamma^2} = e^{i \left( -\frac{I}{2} \right) t - \frac{1}{2} It^2}.$$
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  \lim_{\gamma \to 0} \left( \mathbb{E} e^{it \ln \frac{f(\epsilon + \gamma)}{f(\epsilon)}} \right)^{1/\gamma^2} = e^{i \left(-\frac{1}{2}I\right)t - \frac{1}{2}It^2}.
  \]

- One can make any other regularity assumptions sufficient for this, e.g., assume $f$ is differentiable and $0 < I = \int_{\mathbb{R}} \frac{(f'(x))^2}{f(x)} \, dx < \infty$. 

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- In Gaussian case the assumptions clearly hold and $I = 1$. 

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- In particular ($\alpha \in [0, 1]$ below is arbitrary),

$$I_{\gamma}^2 \xi_{\gamma,f}^\alpha \implies \xi_0 \text{ and } I_{\gamma}^2 \zeta_{\gamma,f} \implies \zeta_0.$$
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$$I\gamma^2 \xi_{\gamma,f}^\alpha \to \xi_0 \quad \text{and} \quad I\gamma^2 \zeta_{\gamma,f} \to \zeta_0.$$

- Moreover, for any $k > 0$ we have

$$I^k \gamma^{2k} \mathbb{E}(\xi_{\gamma,f}^\alpha)^k \to \mathbb{E}\xi_0^k \quad \text{and} \quad I^k \gamma^{2k} \mathbb{E}\zeta_{\gamma,f}^k \to \mathbb{E}\zeta_0^k.$$
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$$I_\gamma^2 \xi_{\gamma,f} \rightarrow \xi_0 \text{ and } I_\gamma^2 \zeta_{\gamma,f} \rightarrow \zeta_0.$$ 

- Moreover, for any $k > 0$ we have

$$I_k^k \gamma^{2k} \mathbb{E}(\xi_{\gamma,f}^\alpha)^k \rightarrow \mathbb{E}\xi_0^k \text{ and } I_k^k \gamma^{2k} \mathbb{E}\zeta_{\gamma,f}^k \rightarrow \mathbb{E}\zeta_0^k.$$ 

- In particular, $I^2 \gamma^4 M_{\gamma,f}^\alpha \rightarrow 26$, $I^2 \gamma^4 B_{\gamma,f} \rightarrow 16 \zeta(3) \approx 19.23$ and $E_{\gamma,f}^\alpha \rightarrow 8 \zeta(3)/13 \approx 0.74$. 

Asymptotical Statistics of Stochastic Processes (S.A.P.S.) VIII – p.15/21
Second possible asymptotics: $\gamma \rightarrow \infty$ (1/3)
The process $Z_{\gamma,f}$ converges weakly in the space $\mathcal{D}_0(-\infty, +\infty)$ to the process $Z_{\infty}(x) = 1\{ -\eta < x < \tau \}$, $x \in \mathbb{R}$, where $\eta$ and $\tau$ are two independent exponential random variables with parameter 1.
The process $Z_{\gamma,f}$ converges weakly in the space $\mathcal{D}_0(-\infty, +\infty)$ to the process $Z_\infty(x) = 1_{\{-\eta < x < \tau\}}$, $x \in \mathbb{R}$, where $\eta$ and $\tau$ are two independent exponential random variables with parameter 1.

In particular,

$$
\zeta_{\gamma,f} \Rightarrow \zeta_\infty = \frac{\int_{\mathbb{R}} x Z_\infty(x) \, dx}{\int_{\mathbb{R}} Z_\infty(x) \, dx} = \frac{\tau - \eta}{2},
$$

$$
\xi^-_{\gamma,f} \Rightarrow \xi^-_\infty = \inf\left\{ z : Z_\infty(z) = \sup_{x \in \mathbb{R}} Z_\infty(x) \right\} = -\eta,
$$

$$
\xi^+_{\gamma,f} \Rightarrow \xi^+_\infty = \sup\left\{ z : Z_\infty(z) = \sup_{x \in \mathbb{R}} Z_\infty(x) \right\} = \tau,
$$

$$
\xi^\alpha_{\gamma,f} \Rightarrow \xi^\alpha_\infty = \alpha \xi^-_\infty + (1 - \alpha) \xi^+_\infty = (1 - \alpha) \tau - \alpha \eta.
$$
Second possible asymptotics: $\gamma \to \infty$ (2/3)
Moreover, for any $k > 0$ we have

$$\mathbb{E}^{\zeta}_{\gamma,f} \rightarrow \mathbb{E}^{\zeta}_{\infty} \quad \text{and} \quad \mathbb{E}(\xi_{\gamma,f}^{\alpha})^k \rightarrow \mathbb{E}(\xi_{\infty}^{\alpha})^k.$$
Moreover, for any \( k > 0 \) we have

\[
E\zeta_{\gamma,f}^k \to E\zeta_{\infty}^k \quad \text{and} \quad E(\xi_{\gamma,f}^\alpha)^k \to E(\xi_{\infty}^\alpha)^k.
\]

In particular,

\[
B_{\gamma,f} \to B_{\infty} = E\left(\frac{\tau - \eta}{2}\right)^2 = \frac{1}{2},
\]

\[
M_{\gamma,f}^\alpha \to M_{\infty}^\alpha = E\left((1 - \alpha)\tau - \alpha\eta\right)^2 = 6\left(\alpha - \frac{1}{2}\right)^2 + \frac{1}{2},
\]

\[
E_{\gamma,f}^\alpha \to E_{\infty}^\alpha = \frac{1}{12\left(\alpha - \frac{1}{2}\right)^2 + 1}.
\]
Second possible asymptotics: $\gamma \rightarrow \infty$ (3/3)
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- In the latter problem, the best choice of $\alpha$ is $\alpha = 1/2$ (which makes the MLE asymptotically efficient). This choice was also suggested in TAR models (with limiting likelihood ratio $Z_{\gamma,f}$). We see that for large values of $\gamma$ this choice is confirmed, but for small values of $\gamma$ the choice of $\alpha$ seems less important.
Numerical simulations (Gaussian $f$)
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Obtained by simulating $10^7$ trajectories of $Z_{\gamma,f}$
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asymptotical Statistics of Stochastic Processes (S.A.P.S.) VIII – p.20/21
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Obtained by simulating $10^7$ trajectories of $Z_{\gamma,f}$
\(\alpha_\circ = 1/2 \pm \sqrt{13/(96 \zeta(3))} - 1/12 \approx 0.5 \pm 0.17\), so \(E_{\alpha_\circ} = E_0 = 8 \zeta(3)/13 \approx 0.74\).