# Asymptotic Expansions under degeneracy

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**SAPS4**, *Université du Maine*, Le Mans, France

December 19-20, 2002

## Stationary Ergodic Diffusion Model

- **Observations** : $X = (X_t)_{t\in[0,T]}$, $\dim(X_t) = d$, Continuous
- **Parameter** : $\theta \in \Theta \subset \mathbb{R}^p$, $\dim(\theta) = p$, $\Theta$ : open bounded convex
- **Model** : $dX_t = V_0(X_t,\theta)dt + V(X_t)dw_t$, $X_0 \sim \nu_\theta$

  $V_0 : \mathbb{R}^d \times \Theta \to \mathbb{R}^d$, $V : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^r$

  $w = (w_t)$ : $r$-dimensional standard Wiener process

  $\nu_\theta$ : stationary distribution with a positive density $\dfrac{d\nu_\theta}{dx}$
- **Log-Likelihood** :

$$\ell(\theta) = \log \frac{d\nu_\theta}{dx} + \int_0^T V_0'(VV')^{-1}(X_t,\theta)dX_t - \frac{1}{2}\int_0^T V_0'(VV')^{-1}V_0(X_t,\theta)dt$$

- **Estimator** : $\hat{\theta}_T$ : $(X_t)_{t\in[0,T]} \to \hat{\theta}_T \in \Theta$
- **Asymptotics** : $T \to \infty$

## Estimation for unknown parameter $\theta$ in drift function $V_0$

- **Parameter of interest** : $\theta = (\theta^1, \dots, \theta^p) \in \Theta \subset \mathbb{R}^p$,

$$dX_t = V_0(X_t, \theta)dt + V(X_t)dw_t, \ X_0 \sim \nu_\theta, \ t \in [0, T].$$

- **True value** : $\theta_0 \in \Theta$.

- **Conditional MLE** $\widehat{\theta}_T^{(c)}$ : $\delta_a \ell^{(c)}(\widehat{\theta}_T^{(c)}) = 0, \ a = 1, \dots, p, \ \delta_a = \partial/\partial\theta^a$.

$$\ell^{(c)}(\theta) = \int_0^T V_0'(VV')^{-1}(X_t, \theta)dX_t - \frac{1}{2}\int_0^T V_0'(VV')^{-1}V_0(X_t, \theta)dt.$$

- **(Exact) MLE** $\widehat{\theta}_T$ : $\delta_a \ell(\widehat{\theta}_T) = 0, \ a = 1, \dots, p, \ \delta_a = \partial/\partial\theta^a$.

$$\ell(\theta) = \log \frac{d\nu_\theta}{dx}(X_0) + \ell^{(c)}(\theta).$$

- $M$-**estimator** $\widehat{\theta}_T^{\psi}$ : $\psi_{a;}(\widehat{\theta}_T^{\psi}) = 0, \ a = 1, \dots, p$, for an estimating fuction $\psi = (\psi_{1;}, \dots, \psi_{p;})$.

## Literature on distributional expansion for ergodic diffusion

- Second order
  - Yoshida(1997) : martingale expansion, (global approach) (conditional) MLE, $d = 1$, $p = 1$
  - Sakamoto-Yoshida(1998) : $M$-estimator, $d = 1$, $p = 1$, Numerical studies on (conditional) MLE for OU, etc
  - Uchida-Yoshida(2001), (Mykland(1992, 1993))

- Third or higher order
  - Kusuoka-Yoshida(2000) : $\epsilon$-Markov mixing, (local approach) Diffusion functional having stoch. exp. , arbitrary $d$, $p$
  - S-Y(1998) : third order MLE, arbitrary $d$, $p$
  - S-Y(1999) : representation of expansion, third order $M$-estimator, arbitrary $d$, $p$
  - S-Y(2000) : under degeneracy
  - Sakamoto(2000), Kutoyants-Yoshida(2001)

## Second order expansion of MLE for univariate ergodic diffusion

**Theorem 1 (Yoshida(P.T.R.F.,1997)).** *For any* $\theta \in \Theta \subset \mathbb{R}$, *Let* $X = (X_t)_{t\in[0,T]}$ *be a one-dimensional stationary ergodic diffusion process satisfying* $dX_t = V_0(X_t,\theta)dt + dw_t$ *with a stationary distribution* $\nu_\theta$ *given by*

$$\nu_\theta(dx) = \frac{n_\theta(x)}{\int_{-\infty}^{\infty} n_\theta(u)du}dx, \quad n_\theta(x) = \exp\left(\int_0^x 2V_0(u,\theta)du\right).$$

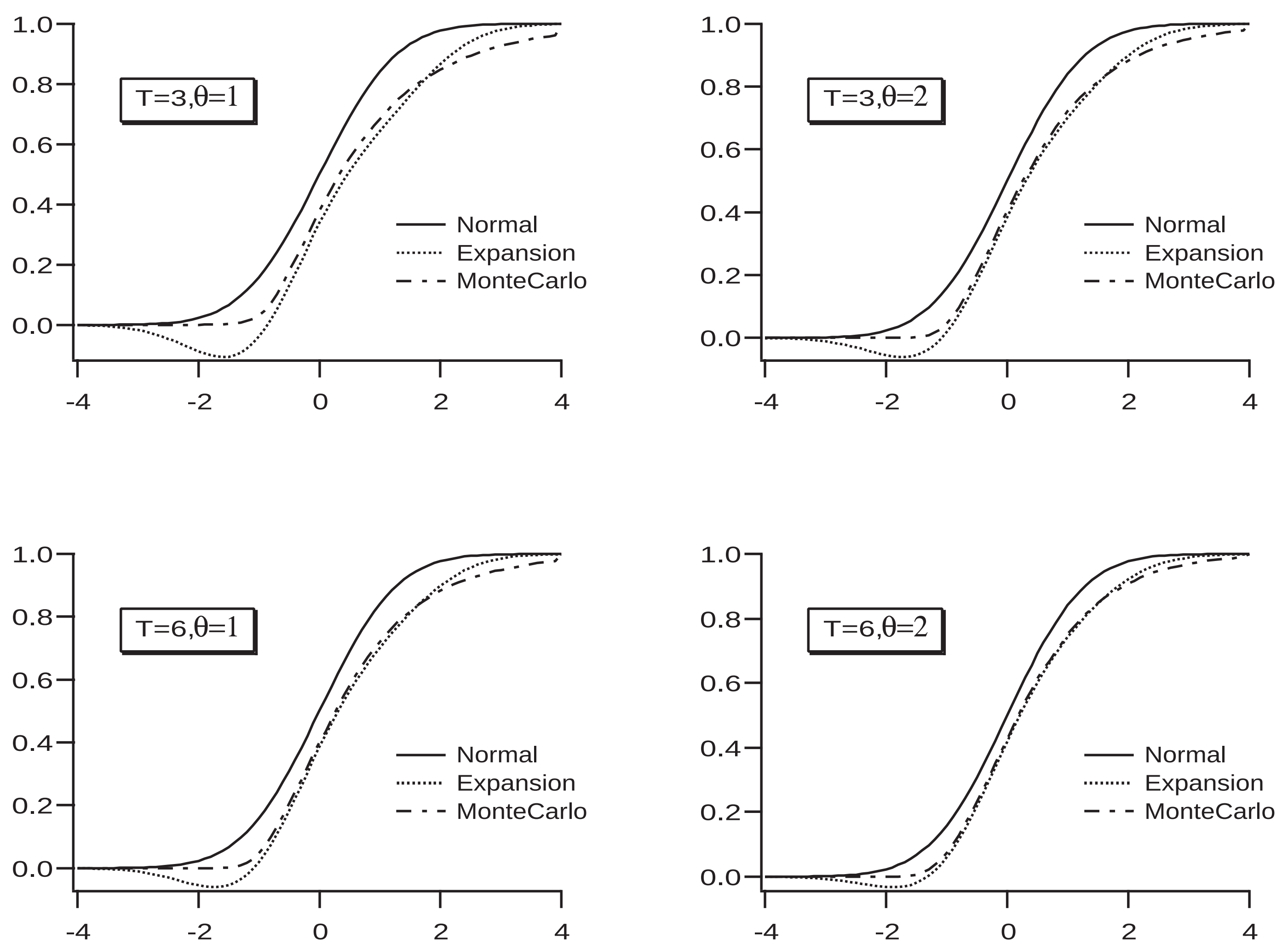
*Suppose that* $\sup_{x\in\mathbb{R}} \partial_x V_0(x,\theta_0) < 0$ *and that* $|\delta^l \partial^j V_0(x,\theta)| \le C_{j,l}(1 + |x|^{C_{j,l}})$, $\forall x,\theta$. *Then it holds that*

$$P(\sqrt{IT}(\hat{\theta}_T^{(c)} - \theta_0) \le x) = \Phi(x) + \frac{1}{\sqrt{T}}(A - Bx^2)\phi(x) + \bar{o}(T^{-1/2}),$$

*where* $I = \nu_{\theta_0}((\delta V_0)^2)$, $A = -\nu_{\theta_0}(\delta V_0 \cdot k)/(2I^{3/2})$, $B = -\{\nu_{\theta_0}(\delta V_0 \cdot \delta^2 V_0) - \nu_{\theta_0}(\delta V_0 \cdot k)\}/(2I^{3/2})$,

$$k = -n_{\theta_0}(x)^{-1}\int_x^\infty 2n_{\theta_0}(u)((\delta V_0(u,\theta_0))^2 - I(\theta_0))du.$$

## Numerical study on expansion of $\widehat{\theta}_T^{(c)}$ for OU in S-Y(1998)



## Third order expansion of MLE for multidimensional diffusion

**Theorem 2 (S-Y(1999)).** *Let* $M$, $\gamma > 0$, *and* $\hat{\rho} > (\rho^{ab})$. *Assume that* [L], [DM1], [DM2], [DM3] *hold true. For any* $\beta \in C_B^2(\Theta)$, *let* $\hat{\theta}_T^* = \hat{\theta}_T - \beta(\hat{\theta}_T)/T$. *Moreover assume that the diffusion process* $X$ *has the geometrically strong mixing property. Then there exist positive constants* $c, \tilde{C}, \tilde{\epsilon}$ *such that for any* $f \in \mathcal{E}(M, \gamma)$

$$\left| E[f(\sqrt{T}(\hat{\theta}_T^* - \theta))] - \int dy^{(0)} f(y^{(0)}) q_{T,2}(y^{(0)}) \right| \leq c\omega(f, \tilde{C}T^{-(\tilde{\epsilon}+2)/2}, \hat{\rho}^{ab}) + o(T^{-1}), \quad (1)$$

*where*

$$\begin{aligned} q_{T,2}(y^{(0)}) =& \phi(y^{(0)}; \rho^{ab}) \Big( 1 + \frac{1}{6\sqrt{T}} c^*_{abc} h^{abc}(y^{(0)}; \rho^{ab}) + \frac{1}{\sqrt{T}} \rho_{aa'}(\mu^{a'} - \tilde{\beta}^{a'}) h^a(y^{(0)}; \rho^{ab}) \\ & + \frac{1}{2T} A^*_{ab} h^{ab}(y^{(0)}; \rho^{ab}) + \frac{1}{24T} c^*_{abcd} h^{abcd}(y^{(0)}; \rho^{ab}) \\ & + \frac{1}{72T} c^*_{abc} c^*_{def} h^{abcdef}(y^{(0)}; \rho^{ab}) \Big). \end{aligned}$$

# Coefficients in asymptotic expansion(1)

$$\rho_{ab} = F_{a,b}, \qquad (\rho^{ab}) = (\rho_{ab})^{-1}, \qquad c^*_{abc} = -3\Gamma^{(-1/3)}_{ab,c}, \qquad \tilde{\beta}^a = \beta^a - \Delta^a, \qquad \mu^a = -\frac{1}{2}\rho^{aa'}\rho^{bc}\Gamma^{(-1)}_{bc,a'},$$

$$\begin{aligned}
A^*_{ab} = & -\tau_{ab} - \rho^{cd}\Big(F_{bcd,a} + F_{ab,cd} - F_{ac,bd} - F_{[a,c],[b,d]} + 2F_{[ab,c],d} + 2F_{[ac,b],d} + 4F_{[b,d],ac} \\
& + F_{[cd,b],a} + 2F_{[[b,c],a],d} + 2F_{[[b,c],d],a}\Big) \\
& + \rho^{cd}\rho^{ef}\Big(\frac{1}{2}\Gamma^{(-1)}_{ce,b}\Gamma^{(-1)}_{df,a} - \Gamma^{(1)}_{ac,e}\Gamma^{(1)}_{bd,f} + \Gamma^{(-1)}_{cd,e}(\Gamma^{(1)}_{ab,f} + \Gamma^{(-1)}_{fb,a}) + \Gamma^{(-1)}_{ce,a}(\Gamma^{(1)}_{bd,f} + \Gamma^{(-1)}_{bd,f})\Big) \\
& + \rho_{aa'}\rho_{bb'}(\mu^{a'} - \tilde{\beta}^{a'})(\mu^{b'} - \tilde{\beta}^{b'}) + 2\rho_{aa'}(\Delta^c \eta^{*a'}{}_{c,b} - \delta_b\beta^{a'}),
\end{aligned}$$

$$\begin{aligned}
c^*_{abcd} = & -12(F_{[[a,b],c],d} + F_{[a,b],cd} + F_{[ab,c],d}) + 3F_{[a,b],[c,d]} - 4F_{abc,d} \\
& + 12\Gamma^{(-1/3)}_{ab,c}\rho_{dd'}(\tilde{\beta}^{d'} - \mu^{d'}) + 12\rho^{ef}(\Gamma^{(-1)}_{ab,e} + \Gamma^{(1)}_{ae,b})\Gamma^{(-1)}_{cf,d},
\end{aligned}$$

$$\Gamma^{(\alpha)}_{ab,c} = F_{ab,c} - F_{[a,b],c} + \frac{1-\alpha}{2}\sum_{(ab,c)}^{[3]} F_{[a,b],c}, \qquad \eta^{*a}{}_{b,c} = -\rho^{aa'}(\Gamma^{(1)}_{a'c,b} + \Gamma^{(-1)}_{bc,a'}).$$

The coefficients $c^*_{abc}$, $A^*_{ab}$, $c^*_{abcd}$, and $\rho_{aa'}(\mu^{a'} - \tilde{\beta}^{a'})$ are functions of $F$, $\tau$, and $\Delta$.

## Coefficients in asymptotic expansion(2)

$$\Delta^a = \rho^{aa'}\nu_{\theta_0}(\delta_{a'}\frac{d\nu_{\theta_0}}{dx}), \quad \tau_{ab} = \mathrm{Cov}[\delta_a\frac{d\nu_{\theta_0}}{dx}, \delta_b\frac{d\nu_{\theta_0}}{dx}],$$

$$F_{A_1,A_2} = \nu_{\theta_0}(B_{A_1}\cdot B_{A_2}), \qquad F_{A_1,[A_2,A_3]} = \nu_{\theta_0}(B_{A_1}\cdot[B_{A_2}\cdot B_{A_3}]),$$
$$F_{[A_1,A_2],[A_3,A_4]} = \nu_{\theta_0}([B_{A_1}\cdot B_{A_2}]\cdot[B_{A_3}\cdot B_{A_4}]),$$
$$F_{[[A_1,A_2],A_3],A_4]} = \nu_{\theta_0}([[B_{A_1}\cdot B_{A_2}]\cdot B_{A_3}]\cdot B_{A_4}]),$$

where $B(x,\theta) = V_0'(VV')^{-1}V_0(x,\theta)$, $B_A(x,\theta) = \delta_{a_1}\cdots\delta_{a_k}B(x,\theta)$, $\delta_{a_j} = \partial/\partial\theta^{a_j}$, $A = a_1\cdots a_k$,

$$[f] = -V'\nabla G_{f-\nu(f)}, \quad \mathcal{A}G_{f-\nu(f)} = f - \nu(f),$$
$$\mathcal{A} = \sum_{i=1}^{d} V_0^i(x,\theta_0)\frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{r} V_k^i(x)V_k^j(x)\frac{\partial^2}{\partial x^i\partial x^j}$$

# Third order expansion of MLE for CIR model

$$dX_t = (p - qX_t)dt + r\sqrt{X_t}dw_t, \quad \theta = (p,q), \; \Theta = \{(p,q) \,:\, 2p > r > 0, q > 0\}$$

$$F_{1,1} = \frac{2q}{r^2(2p-r^2)}, \quad F_{1,2} = F_{2,1} = -\frac{1}{r^2}, \quad F_{2,2} = \frac{p}{qr^2},$$

$$F_{[1,1],1} = -\frac{4q}{r^2(2p-r^2)^2}, \quad F_{[1,1],2} = \frac{2}{(2p-r^2)}, \quad F_{[1,2],1} = F_{[1,2],2} = 0$$

$$F_{[2,2],1} = \frac{1}{r^2 q}, \quad F_{[2,2],2} = -\frac{p}{r^2 q^2} \quad F_{[1,1],[1,1]} = \frac{8q}{r^2(2p-r^2)^3}, \quad F_{[1,1],[1,2]} = 0$$

$$F_{[1,1],[2,2]} = -\frac{2}{r^2(2p-r^2)q}, \quad F_{[1,2],[1,2]} = F_{[1,2],[2,2]} = 0$$

$$F_{[2,2],[2,2]} = \frac{p}{r^2 q^3}, \quad F_{[[1,1],1],1} = \frac{8q}{r^2(2p-r^2)^3}, \quad F_{[[1,1],1],2} = -\frac{4}{r^2(2p-r^2)^2}$$

$$F_{[[1,1],2],1} = F_{[[1,1],2],2} = F_{[[1,2],1],1} = F_{[[1,2],1],2} = 0$$

$$F_{[[1,2],2],1} = F_{[[1,2],2],2} = F_{[[2,2],1],1} = F_{[[2,2],1],2} = 0$$

$$F_{[[2,2],2],1} = -\frac{1}{r^2 q^2}, \quad F_{[[2,2],2],2} = \frac{p}{r^2 q^3},$$

$$\Delta^1 = \Delta^2 = 0, \quad \tau_{11} = 4\text{PolyGamma}[1,1], \quad \tau_{12} = \tau_{21} = \frac{2}{r^2 q}, \quad \tau_{22} = \frac{2p}{q^2 r^2}.$$

## Stochastic Expansion of MLE $\widehat{\theta}_T$ ($p = 1$)

- **MLE $\widehat{\theta}_T$** $\quad \delta\ell_T(\widehat{\theta}_T) = 0, \quad \delta = \partial/\partial\theta,$ : Likelihood eq.

- **Third order stochastic expansion**

$$\sqrt{T}(\widehat{\theta}_T - \theta_0) = S_T(Z_1, Z_2, Z_3) + \frac{1}{T\sqrt{T}} R_3,$$

$$S_T(z_1, z_2, z_3) = z_1 + \frac{1}{\sqrt{T}}\left(z_1 z_2 + \frac{1}{2}\bar{\upsilon}_2^{-1}\bar{\nu}_3 z_1^2\right) + \frac{1}{T}\left(\frac{1}{6}(\bar{\nu}_4 + 3\bar{\nu}_3^2) z_1^3 + \frac{3}{2}\bar{\nu}_3 z_1^2 z_2 + z_1 z_2^2 + \frac{1}{2} z_1^2 z_3\right),$$

where $\bar{\upsilon}_2 = -\frac{1}{T} E_{\theta_0}[\delta^2 \ell_T(\theta_0)]$, $\bar{\nu}_3 = \frac{1}{T}\bar{\upsilon}_2^{-1} E_{\theta_0}[\delta^3 \ell_T(\theta_0)]$, $\bar{\nu}_4 = \frac{1}{T}\bar{\upsilon}_2^{-1} E_{\theta_0}[\delta^4 \ell_T(\theta_0)]$,

$$Z_1 = \frac{1}{\sqrt{T}}\bar{\upsilon}_2^{-1}\delta\ell_T(\theta_0),$$

$$Z_2 = \frac{1}{\sqrt{T}}\bar{\upsilon}_2^{-1}\left(\delta^2\ell_T(\theta_0) - E_{\theta_0}[\delta^2\ell_T(\theta_0)]\right),$$

$$Z_3 = \frac{1}{\sqrt{T}}\bar{\upsilon}_2^{-1}\left(\delta^3\ell_T(\theta_0) - E_{\theta_0}[\delta^3\ell_T(\theta_0)]\right).$$

## Derivation of valid distributional asymptotic expansion

$$E[f(\sqrt{T}(\hat{\theta}_T - \theta_0))] \approx E[f(S_T(Z_1, Z_2, Z_3))], \quad S_T : \text{stoch. exp.}, \quad \Leftarrow \text{(Delta method)}$$

$$\approx \int f(S_T(z_1, z_2, z_3))p_T(z_1, z_2, z_3)dz_1dz_2dz_3,$$

$\underline{p_T : \text{valid asymptotic exp. of } (Z_1, Z_2, Z_3)}$

$$= \int f(z_1 + Q_T(z_1, z_2, z_3))p_T(z_1, z_2, z_3)dz_1dz_2dz_3$$

$S_T = z_1 + Q_T$

$$\approx \int (f(z_1) + \partial f(z_1)Q_T + \frac{1}{2}\partial^2 f(z_1)Q_T^2)p_T(z)dz$$

$\Leftarrow$ (formal) Taylor's expansion

$$= \int f(z_1)\Big((-\partial)\int Q_T p_T dz_2dz_3 + \frac{1}{2}(-\partial)^2 \int Q_T p_T dz_2dz_3\Big)dz_1$$

$\Leftarrow$ IBP over $\mathbb{R}$

For the validity of $p_T$, the regularity of the distribution of $(Z_1, Z_2, Z_3)$

□ non-degeneracy of covariance matrix

□ Cramér type condition

— non-degeneracy of the Malliavin covariance

## Linearly parametrized diffusion model

$$dX_t = -\theta\, m(X_T)dt + dw_t, \quad \theta > 0$$

For this model,

$$\begin{aligned}
\ell_T(\theta) =& \log \frac{d\nu_\theta}{dx} z(X_0) - \theta \int_0^T m(X_t)dX_t - \frac{1}{2}\theta^2 \int_0^T m^2(X_t)dt \\
Z_1 =& \frac{1}{\sqrt{T}} g^{-1}\left(\delta \log \frac{d\nu_\theta}{dx}(X_0) - E[\delta \log \frac{d\nu_\theta}{dx}(X_0)] - \int_0^T m(X_t)dw_t\right) = O_p(1), \\
Z_2 =& \frac{1}{\sqrt{T}} g^{-1}\left(\delta^2 \log \frac{d\nu_\theta}{dx}(X_0) - E[\delta^2 \log \frac{d\nu_\theta}{dx}(X_0)] - \int_0^T \bar{m}^2(X_t)dt\right) = O_p(1), \\
Z_3 =& \frac{1}{\sqrt{T}} g^{-1}\left(\delta^3 \log \frac{d\nu_\theta}{dx}(X_0) - E[\delta^3 \log \frac{d\nu_\theta}{dx}(X_0)]\right) = O_p(\frac{1}{\sqrt{T}}), \\
\sqrt{T}(\widehat{\theta}_T - \theta_0) =& S_T(Z_1, Z_2) + \frac{1}{T\sqrt{T}} R_3', \\
S_T(z_1, z_2) =& z_1 + \frac{1}{\sqrt{T}}\left(z_1 z_2 + \frac{1}{2}\bar{\upsilon}_2^{-1}\bar{\nu}_3 z_1^2\right) + \frac{1}{T}\left(\frac{1}{6}(\bar{\nu}_4 + 3\bar{\nu}_3^2) z_1^3 + \frac{3}{2}\bar{\nu}_3 z_1^2 z_2 + z_1 z_2^2\right).
\end{aligned}$$

Therefore, the previous asymptotic expansion for the general case can be applied if $(Z_1, Z_2)$ admit a third order asymptotic expansion.

## Linear and Linearly parametrized diffusion: OU-process

□ Ornstein-Uhlenbeck process : $dX_t = -\theta X_t dt + dw_t,\ \theta > 0.$

$$\ell_T(\theta) = \frac{1}{2}\log(\theta) - \theta X_0^2 - \theta\int_0^T X_t dX_t - \frac{1}{2}\theta^2\int_0^T X_t^2 dt,$$

$$Z_1 = \frac{1}{\sqrt{T}} g^{-1}\Big(-X_0^2 + \frac{1}{2\theta_0} - \int_0^T X_t dw_t\Big), \quad Z_2 = -\frac{1}{\sqrt{T}} g^{-1}\int_0^T (X_t^2 - \frac{1}{2\theta_0})dt,$$

$$Z_3 = 0, \quad g = \frac{1}{2\theta_0} + \frac{1}{T}\frac{1}{2\theta_0^2}.$$

Third order stochastic expansion becomes

$$\sqrt{T}(\hat{\theta}_T - \theta_0) = S_T(Z_1, Z_2) + \frac{1}{T\sqrt{T}} R_3,$$

$$S_T(z_1, z_2) = z_1 + \frac{1}{\sqrt{T}}\Big(z_1 z_2 + \frac{1}{2}\bar{\upsilon}_2^{-1}\bar{\nu}_3 z_1^2\Big) + \frac{1}{T}\Big(\frac{1}{6}(\bar{\nu}_4 + 3\bar{\nu}_3^2)z_1^3 + \frac{3}{2}\bar{\nu}_3 z_1^2 z_2 + z_1 z_2^2\Big),$$

Can we obtain the valid expansion of $(Z_1, Z_2)$ ?

## Degeneracy of stochastic expansion

Itô's formula says that

$$Z_1 - \theta Z_2 = \frac{1}{\sqrt{T}} g^{-1}\left(-X_0^2 + \frac{1}{2\theta_0} - \frac{1}{2}X_T^2 + \frac{1}{2}X_0^2\right) = o_p(1) \to 0 \quad \text{as } T \to \infty$$

$Z_1$ and $Z_2$ are asymptotically linearly dependent :

$$\underline{\mathrm{Cov}[Z_1, Z_2] \to 0} \quad \text{as } T \to \infty$$

### □ Decomposition

$$S_T(Z_1, Z_2) = S_T^{(0)}(Z_1) + \frac{1}{T} S_T^{(1)} + \frac{1}{T\sqrt{T}} \tilde{R}_3,$$

where

$$S_T^{(0)}(z_1) = z_1 + \frac{1}{\sqrt{T}}\left(\frac{1}{\theta_0} + \frac{1}{2}\bar{\nu}_3\right) z_1^2 + \frac{1}{T}\left(\frac{1}{6}(\bar{\nu}_4 + 3\bar{\nu}_3^2) + \frac{3}{2\theta_0}\bar{\nu}_3 + \frac{1}{\theta_0^2}\right) z_1^3,$$

$$S_T^{(1)} = -\frac{1}{\theta_0} g^{-1}\left(-X_0^2 + \frac{1}{2\theta_0} - \frac{1}{2}X_T^2 + \frac{1}{2}X_0^2\right) Z_1.$$

## Asymptotic expansion having degenerate component

$S_T = S_T^{(0)} + u_T S_T^{(1)}$, $E[f(S_T^{(0)})] = \Psi_T[f] + \bar{o}(u_T)$, $u_T \to 0$ as $T \to \infty$

**Theorem 3.** *Let* $m \in \mathbb{N}$, $M > 0$, $\gamma > 0$, $\ell_1 > 0$ *and* $\ell_2 > 0$ *s.t.* $m > \gamma/2 + 1$, $\ell_1 \geq \ell_2 - 1$, $\ell_2 \geq (2m+1) \vee (d+3)$.

*Assume (1)* $\forall p > 1$, $\sup_T ||S_T^{(1)}||_{p,\ell_2} + \sup_T ||S_T^{(0)}||_{p,\ell_2} < \infty$,

*(2)* $\exists S_\infty^{(0)}, S_\infty^{(1)}$, $(S_T^{(0)}, S_T^{(1)}) \xrightarrow{d} (S_\infty^{(0)}, S_\infty^{(1)})$.

*there exits a functional* $\xi_T$ *s.t.* *(3)* $\forall p > 1$, $\sup_T ||\xi_T||_{p,\ell_1} < \infty$,

*(4)* $\exists \alpha > 0$, $P[|\xi_T| > 1/2] = O(u_T^\alpha)$,

*(5)* $\forall p > 1$, $\sup_T E[1_{|\xi_T|<1} \sigma_{S_T^{(0)}}^{-p}] < \infty$.

*Then for any* $f$ *satisfying* $|f(x)| \leq M(1 + |x|^\gamma)$,

$$E[f(S_T)] = \Psi_T[f] + u_T \int_{\mathbb{R}^d} f(x) g_\infty(x) dx + \bar{o}(u_T),$$

*where*

$$g_\infty(x) = -\partial_x \left( \mathbb{E}[S_\infty^{(1)} \mid S_\infty^{(0)} = x] p^{S_\infty^{(0)}}(x) \right).$$

## Rough sketck of the proof

$$E[f(S_T)] = E[f(S_T^{(0)})] + u_T E[\partial f(S_T^{(0)}) S_T^{(1)}] + u_T^2 \int_0^1 dv(1-v) E[\partial^2 f(S_T^{(0)} + v u_T S_T^{(1)})[(S_T^{(1)})^{\otimes 2}]]$$

$$= \int f(x)\Psi_T(dx) + u_T E[f(S_T^{(0)}) \Phi_1^{S_T^{(0)}}(S_T^{(1)})] + o(u_T)$$

$$= \int f(x)\Psi_T(dx) + u_T \int f(x) E[\Phi_1^{S_T^{(0)}}(S_T^{(1)}) | S_T^{(0)} = x] p^{S_T^{(0)}}(x) dx + o(u_T)$$

$$g_T(x) := E[\Phi_1^{S_T^{(0)}}(S_T^{(1)}) | S_T^{(0)} = x] p^{S_T^{(0)}}(x)$$

$$= \frac{1}{(2\pi)^d} \int e^{-iux} \int e^{iu\tilde{x}} E[\Phi_1^{S_T^{(0)}}(S_T^{(1)})] | S_T^{(0)} = \tilde{x}] p^{S_T^{(0)}}(\tilde{x}) d\tilde{x} du$$

$$= \frac{1}{(2\pi)^d} \int e^{-iux} E[e^{iuS_T^{(0)} X_T} \Phi_1^{S_T^{(0)}}(S_T^{(1)})] du = \frac{1}{(2\pi)^d} \int e^{-iux} E[iue^{iuS_T^{(0)}} S_T^{(1)}] du$$

$$\longmapsto \frac{1}{(2\pi)^d} \int e^{-iux} \mathbb{E}[iue^{iuS_\infty^{(0)}} S_\infty^{(1)}] du$$

$$= \frac{1}{(2\pi)^d} \int e^{-iux} \int \partial_{\tilde{x}} e^{iu\tilde{x}} \mathbb{E}[S_\infty^{(1)} | S_\infty^{(0)} = \tilde{x}] p^{S_\infty^{(0)}}(\tilde{x}) d\tilde{x} du$$

$$= \frac{1}{(2\pi)^d} \int e^{-iux} \int e^{iu\tilde{x}} \Big( - \partial_{\tilde{x}} \mathbb{E}[S_\infty^{(1)} | S_\infty^{(0)} = \tilde{x}] p^{S_\infty^{(0)}}(\tilde{x}) \Big) d\tilde{x} du = g_\infty(x).$$

## Validity of the formal expansion for OU-process

Mixing property of OU implies that $X_0$, $X_T$, $Z_1$ are mutually asymptotically independent, and therefore,

$$\begin{aligned} S_T^{(1)} &= -\frac{1}{\theta_0} g^{-1}\left(-X_0^2 + \frac{1}{2\theta_0} - \frac{1}{2}X_T^2 + \frac{1}{2}X_0^2\right) Z_1 \\ &\overset{d}{\Rightarrow} -\frac{1}{\theta_0} g^{-1}\left(-\xi_1^2 + \frac{1}{2\theta_0} - \frac{1}{2}\xi_2^2 + \frac{1}{2}\xi_1^2\right) S_\infty^{(0)} =: S_\infty^{(1)}, \end{aligned}$$

where $\xi_1$, $\xi_2$, $S_\infty^{(0)}$ are independent and $\xi_1, \xi_2 \sim \nu$(stationary dist.). This yields

$$\mathbb{E}[S_\infty^{(1)} | S_\infty^{(0)}] = -\frac{1}{\theta_0} g^{-1} S_\infty^{(0)} \mathbb{E}\left[-\xi_1^2 + \frac{1}{2\theta_0} - \frac{1}{2}\xi_2^2 + \frac{1}{2}\xi_1^2\right] = 0.$$

In this way, the asymptotic expansion of MLE coincides with that of $S_T^{(0)}$. Consequently, the formal expansion for general diffusion is valid in the exceptional case(OU-process).