Mixed fractional Brownian motion: the filtering perspective

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7 October 2013 / Le Mans

Outline

Objects of studies

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- $$X_t = B_t + B_t^H$$

  where
  - B_t —- is the standard Brownian motion
  - B_t^H — is an independent fractional Brownian motion
  - $H \in (0, 1]$—- is known Hurst parameter

- $$Y_t = \int_0^t f(s)\,ds + X_t, 0 \le t \le T.$$

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B^H_t is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$, i.e. zero mean Gaussian process with the correlation function

$$K(s,t) = \mathbb{E}B^H(t)B^H(s) = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big),$$

$$s,t \in [0,T].$$

B^H is not a semimartingale on its own filtration, unless $H = \frac{1}{2}$ or $H = 1$.

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B_t^H is a fractional Brownian motion with Hurst parameter $H \in (0,1)$, i.e. zero mean Gaussian process with the correlation function

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Questions for discussions

- Stochastic analysis of X:
  - structure of the fundamental martingale
  - the semimartingale representation
  - the density with respect to the standard and fractional Wiener measures
- Stochastic analysis of Y: the density with respect to measure μ^X
- Potential applications; MLE asymptotic properties

Important references

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- 2000 Patric Cheridito

$$\sup_{\tau} \sum_{j=0}^{n-1} \mathbb{E}\left|\mathbb{E}\left(X_{t_{j+1}} - X_{t_j} \middle| \mathbb{F}_{t_j}^X\right)\right| < \infty$$

- 2003 Fabrice Boduin and David Nualart

$$X := B + V, \; \partial^2 K/\partial s \partial t \in L^2([0, T]^2)$$

Hida-Hitsuda criterion.

- 2007 Harry van Zanten equivalence of $\xi = \sum_{k=1}^{n} \alpha_k B^{H_k}$ of n independent fBm's to a single fBm. (Spectral techniques for processes with stationary increments).

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Main Tools

$\mathbb{F}^X = (\mathbb{F}_t^X)$ and $\mathbb{F} = (\mathbb{F}_t)$, $t \in [0, T]$ —the natural filtrations of X and (B, B^H) respectively.

Fundamental Martingale

$$M_t = \mathbb{E}\big(B_t | \mathbb{F}_t^X\big), \quad t \in [0, T].$$

M encodes many of the essential features of the process X, making its structure particularly transparent.

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Outline

Fundamental Martingale Representation via X

Fundamental Martingale Representation

$$M_t = \int_0^t g(s,t)dX_s, \quad \langle M \rangle_t = \int_0^t g(s,t)ds, \quad t \geq 0.$$

The kernel $g(s,t)$ solves integro-differential equation:

Equation for the Kernel

$$g(s,t) + H\frac{d}{ds}\int_0^t g(r,t)|s-r|^{2H-1} sign(s-r)dr = 1, 0 < s < t \leq T$$

The family of functions $\{g(s,t), 0 \leq s \leq t \leq T\}$ plays the key role in our approach to analysis of the mixed fBm.

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X is a stochastic integral w.r.t M

Representation of X via M

The following representation holds:

$$X_t = \int_0^t G(s,t)dM_s, \quad t \in [0,T],$$

where

$$G(s,t) := 1 - \frac{d}{d\langle M\rangle_s}\int_0^t g(\tau,s)d\tau, \quad 0 \le s \le t \le T.$$

and, in particular, $\mathbb{F}_t^X = \mathbb{F}_t^M$, P-a.s. for all $t \in [0,T]$.

The fundamental Semimartingale

Let $Y = (Y_t)$ defined by

$$Y_t = \int_0^t f(s)ds + X_t, \quad t \in [0, T],$$

Then Y admits the representation

$$Y_t = \int_0^t G(s, t)dZ_s$$

The fundamental semimartingale $Z = (Z_t)$

$$Z_t = \int_0^t g(s, t)dY_s = M_t + \int_0^t \Phi(s)d\langle M\rangle_s,$$

and

$$\Phi(t) = \frac{d}{d\langle M\rangle_t} \int_0^t g(s, t)f(s)ds.$$

The measures μ^X and μ^Y

In particular, $\mathbb{F}_t^Y = \mathbb{F}_t^Z$, P-a.s. for all $t \in [0, T]$ and, if

$$E \exp\left\{ -\int_0^T \Phi(t) dM_t - \frac{1}{2}\int_0^T \Phi^2(t) d\langle M\rangle_t \right\} = 1,$$

then the measures μ^X and μ^Y are equivalent and the corresponding Radon-Nikodym density is given by

$$\frac{d\mu^Y}{d\mu^X}(Y) = \exp\left\{ \int_0^T \hat{\Phi}(t) dZ_t - \frac{1}{2}\int_0^T \hat{\Phi}^2(t) d\langle M\rangle_t \right\},$$

where $\hat{\Phi}(t) = E\big(\Phi(t) | \mathbb{F}_t^Y\big)$.

Outline

X est **diffusion** type process

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Let $H \in (\frac{3}{4}, 1]$.Then X is a **diffusion** type process:

$$X_t = W_t - \int_0^t \varphi_s(X)ds, \; W_t = \int_0^t \frac{dM_s}{g(s,s)},$$

W is an F^X-Brownian motion; $\varphi_t(X) = \displaystyle\int_0^t \frac{\dot{g}(s,t)}{g(t,t)} dX_s$

The density w.r.t μ^W

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Moreover, the measures μ^X and μ^W are equivalent and

$$\frac{d\mu^X}{d\mu^W}(X) = \exp\left\{-\int_0^T \varphi_t(X)dX_t - \frac{1}{2}\int_0^T \varphi_t^2(X)dt\right\}.$$

Outline

Drift estimation in mixed fractional noise

Let

$$Y_t = \theta t + B_t + B_t^H, \quad t \in [0, T]$$

MLE of θ

The MLE of θ is given by

$$\hat{\theta}_T(Y) = \frac{\int_0^T g(s, T) dY_s}{\int_0^T g(s, T) ds},$$

For $H \in (0, 1)$ this estimator is strongly consistent and the corresponding estimation error is normal

$$\hat{\theta}_T - \theta \sim N\left(0, \frac{1}{\int_0^T g(s, T) ds}\right),$$

Drift estimation in mixed fractional noise

Asymptotic variance

With the following asymptotic variance:

- for $H > \frac{1}{2}$,

$$\lim_{T\to\infty} T^{2-2H}\mathbb{E}(\hat{\theta}_T - \theta)^2 = \frac{2H\Gamma(H+\frac{1}{2})\Gamma(3-2H)}{\Gamma(\frac{3}{2}-H)},$$

  where $\Gamma(\cdot)$ is the standard Gamma function.

- for $H < \frac{1}{2}$,

$$\lim_{T\to\infty} T\mathbb{E}\left(\hat{\theta}_T - \theta\right)^2 = 1.$$

Outline

Indirect approach

Reduction to the case $H > 1/2$

The trick is to transform X into

$$\widetilde{X}_t = \int_0^t \widetilde{\rho}(s,t)dX_s, \quad t \in [0,T],$$

where the kernel $\widetilde{\rho}(s,t)$ is such that the process

$$\widetilde{B}_t = \int_0^t \widetilde{\rho}(s,t)dB_s^H,$$

is a standard Brownian motion.

Indirect approach, II

Reduction to the case $H > 1/2$

Then the Gaussian process

$$\widetilde{U}_t = \int_0^t \widetilde{\rho}(s,t)dB_s$$

has covariance function with integrable partial derivative

$$\widetilde{\kappa}(s,t) := \frac{\partial^2}{\partial s \partial t}\mathbb{E}\widetilde{U}_s\widetilde{U}_t = |t-s|^{-2H}\chi\left(\frac{s \wedge t}{s \vee t}\right), \quad s \neq t,$$

where $\chi(\cdot)$ is a continuous function, and the process $\widetilde{X} = \widetilde{B} + \widetilde{U}$ with $H < \frac{1}{2}$ has the structure, similar to the original process X with $H > \frac{1}{2}$.

Indirect approach, II

Reduction to the case $H > 1/2$

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Outline

Integro-Differential Equation and its alternative forms,I

Alternative Forms,I

The kernel $g(s,t)$ is the unique continuous solution of the following equations:

- for $H \in (0,1]$, the integro-differential equation:

$$g(s,t) + H\frac{d}{ds}\int_0^t g(r,t)|s-r|^{2H-1} sign(s-r)dr = 1.$$

- for $H \in (\frac{1}{2}, 1]$, the **weakly singular integral** equation:

$$g(s,t) + H(2H-1)\int_0^t g(r,t)|s-r|^{2H-2}dr = 1.$$

Integro-Differential Equation and its alternative forms,II

Alternative Forms,II

- for $H \in (0,1)$, the **fractional** integro-differential equation

$$c_H \frac{d}{ds}\int_0^s g(r,t) r^{1/2-H}(s-r)^{1/2-H}\,dr -$$

$$-\alpha_H s^{1-2H}\frac{d}{ds}\int_s^t g(r,t) r^{H-1/2}(r-s)^{H-1/2}\,dr$$

$$= c_H s^{1/2-H}(t-s)^{1/2-H}.$$

Integro-Differential Equation and its alternative forms,III

Alternative Forms,III

- for $H \in (0, \frac{1}{2})$, the **weakly singular integral** equation

$$g(s,t) + \beta_H t^{-2H} \int_0^t g(r,t) \bar{\kappa}\left(\frac{r}{t}, \frac{s}{t}\right) dr =$$

$$c_H s^{1/2-H} (t-s)^{1/2-H},$$

with the kernel

$$\bar{\kappa}(u,v) = (uv)^{1/2-H} \int_{u \vee v}^1 r^{2H-1} (r-u)^{-1/2-H} (r-v)^{-1/2-H} dr$$

Outline

Integral Equation with $H > 1/2$, I

Properties of $g(s,t)$ on the diagonal

The function $g(t,t)$, $t \in [0,T]$ satisfies the properties:

- $g(t,t)$ is continuous on $[0,T]$ with $g(0,0) := \lim_{t\to 0} g(t,t) = 1$
- $g(t,t) > 0$ for all $t \in [0,T]$
- $$\int_0^t g(s,t)ds = \int_0^t g^2(s,s)ds.$$

Integral Equation with $H > 1/2$, II

Properties of $\dot{g}(s,t) = \frac{\partial}{\partial t} g(s,t)$

The kernel $g(s,t)$ satisfies the following properties

- $g(s,t)$ is continuously differentiable at $t \in (0,T]$ for any $s > 0$, $s \neq t$;
- the derivative $\dot{g}(s,t) := \frac{\partial}{\partial t} g(s,t)$ satisfies the equation

$$\dot{g}(s,t) + H(2H-1)\int_0^t \dot{g}(r,t)|r-s|^{2H-2}dr =$$

$$-H(2H-1)g(t,t)|r-s|^{2H-2}, \; s \in (0,t).$$

- $\dot{g}(\cdot,t) \in L^2([0,t])$ for $H > 3/4$

Outline

$H > 3/4$

X is a diffusion type process

$$M_t = \int_0^t g(s,t)dX_s = \int_0^t g(s,s)dX_s + \int_0^t \Big(g(r,t) - g(r,r)\Big)dX_r$$

$$\int_0^t g(s,s)dX_s + \int_0^t \int_r^t \dot{g}(r,s)ds dX_r =$$

$$\int_0^t g(s,s)dX_s + \int_0^t \int_0^s \dot{g}(r,s)dX_r ds,$$

where the last equality holds since $\dot{g}(\cdot,s) \in L^2([0,s])$

$H > 3/4$,II

X is a diffusion type process, II

Hence

$$W_t = \int_0^t \frac{1}{g(s,s)} dM_s = X_t + \int_0^t \int_0^s \frac{\dot{g}(r,s)}{g(s,s)} dX_r ds =:$$

$$X_t + \int_0^t \varphi_s(X) ds.$$

$H \leq 3/4$

Singularity of Measures

$$L_t := \int_0^t g(s,t)dW_s,$$

must be a semimartingale under P. Define

$$\psi(s,t) = -\int_s^t g(r,r)\sum_{m=1}^{n_0-1}(-1)^m \kappa^{(m)}(r,s)dr, \quad 0 < s < t \leq T,$$

where n_0 is the least integer greater than $\frac{1}{4H-2}$. Than $\psi(\cdot,t) \in L^2([0,t])$.

$H \le 3/4$,II

Singularity of Measures,II

Define

$$U_t := \int_0^t \psi(s,t) dW_s$$

$$V_t := \int_0^t \big(g(s,t) - g(s,s) + \psi(s,t)\big) dW_s.$$

Then

$$\int_0^t g(s,t) dW_s = V_t + \int_0^t g(s,s) dW_s - U_t.$$

- U has zero quadratic variation, but unbounded first variation
- V has bounded first variation.

Singular perturbations

Singular perturbations

Fix $\varepsilon > 0$ and let g_ε be the solution of the equation:

$$\varepsilon g_\varepsilon^{(\varphi)}(u) + \int_0^1 g_\varepsilon^{(\varphi)}(v)\kappa(u,v)dv = \varphi(u), \quad u \in [0,1],$$

where φ is a sufficiently smooth function Let $g^{(\varphi)}$ be the solution of auxiliary integral equation of the first kind

$$\int_0^1 g^{(\varphi)}(v)\kappa(u,v)\,dv = \varphi(u).$$

Singular perturbations

Singular perturbations

Let $\psi(u)$ be a function, such that $g^{(\psi)}$ exists, then

$$\left|\int_0^1 (g_\varepsilon^{(\varphi)}(s) - g^{(\varphi)}(s))\psi(s)\,ds\right| \le$$

$$2\varepsilon\left(\int_0^1 (g^{(\psi)}(u))^2 du\right)^{1/2}\left(\int_0^1 \left(g^{(\varphi)}(u)\right)^2 du\right)^{1/2}.$$

Concluding Remarks

Open Questions

- Pathways convergence for a singulary perturbed equations
- Boundary layer construction