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Test d'ajustement d'un processus de diffusion ergodique à changement de régime

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Introduction

Ce travail est consacré à la construction et l'étude des tests d'ajustement des processus de diffusion ergodiques. Une attention particulière a été accordée au processus de diffusion à changement de régime dont la dérive est de type signe "*switching diffusion*".

Les tests d'ajustement occupent une place trés importante dans l'inférence statistique, car ils permettent de vérifier l'adéquation des modèles mathématiques par rapport aux données réelles.

Parmi les tests d'ajustement les plus fréquemment utilisés, dans le cadre des observations indépendantes et identiquement distribuées (i.i.d.), nous pouvons citer respectivement le test de Cramér-von Mises (C-vM) et le test bilatéral de Kolmogorov-Smirnov (K-S) :

$$\hat{\Psi}_n = \mathbb{1}_{\{n||\hat{F}_n - F_0||_2 > c_\alpha\}}, \quad \tilde{\Psi}_n = \mathbb{1}_{\{\sqrt{n}||\hat{F}_n - F_0||_\infty > d_\alpha\}}.$$

Ces deux tests sont basés essentiellement sur la comparaison de la fonction de répartition empirique $\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{\{X_i \le x\}}$ avec la fonction de répartition théorique $F_0(x)$ dans les différentes métriques :

$$\hat{\Psi}_n = \mathbb{1}_{\{W_n^2 > c_\alpha\}}, \quad W_n^2 = n\int_{\mathbb{R}} [\hat{F}_n(x) - F_0(x)]^2 \, dF_0(x),$$

et

$$\tilde{\Psi}_n = \mathbb{1}_{\{V_n > d_\alpha\}}, \quad V_n = \sqrt{n}\sup_x |\hat{F}_n(x) - F_0(x)|.$$

Il est souvent important de distinguer les alternatives qui diffèrent de l'hypothèse par le comportement des queues des distributions. Dans ce cas, il est commode d'utiliser le test dit d'Anderson-Darling et basé sur la statistique suivante :

$$A_n^2 = n\int_{\mathbb{R}} \frac{[\hat{F}_n(x) - F_0(x)]^2}{F_0(x)\,(1 - F_0(x))} \, dF_0(x).$$

Une propriété intéressante de ces tests est que les distributions limites des statistiques sous-jacentes, lorsque la taille de l'échantillon n s'accroît à l'infini, et lorsque $X_1, \ldots, X_n$ sont indépendantes de même fonction de répartition continues, sont de simples fonctionnelles d'un pont Brownien. En d'autre termes, si $\{B_t, 0 \le t \le 1\}$ est un pont Brownien, nous avons, quand $n \to \infty$,

$$W_n^2 \Longrightarrow \int_0^1 B_t^2 \, dt, \quad V_n \Longrightarrow \sup_{0 \le t \le 1} |B_t|, \quad A_n^2 \Longrightarrow \int_0^1 \frac{B_t^2}{t(1-t)} \, dt.$$

Par conséquent et comme ces seuils sont les mêmes pour toute hypothèse de base simple avec une fonction de répartition continue, le choix des seuils pour ces tests est simplifié. Ces tests sont dits *distribution-free*. Le choix des seuils pour ces tests nécessite la connaissance de la fonction de répartition limite (et des quantiles appropriés). Pour les statistiques de type C-vM, tels que W_n^2 ou A_n^2, il est généralement réalisé par la combinaison du développement de Karhunen-Loève (K-L) avec la formule de Smirnov. Nous nous référons à Deheuvels et Martynov (2003) pour plus de détails, ainsi qu'une étude des tests basés sur les statistiques :

$$D_n^2 = n \int_{\mathbb{R}} F_0(x)^{2\beta} [\hat{F}_n(x) - F_0(x)]^2 \, dF_0(x) \Longrightarrow \int_0^1 t^{2\beta} B_t^2 \, dt, \quad \beta > -1.$$

La théorie de tests d'ajustement a une longue histoire et elle a été développée pour de nombreux modèles d'observations. Nous pouvons citer ici les travaux de Durbin (1973) ; Shorack et Wellner (1986) ; Greenwood et Nikulin (1996) ; Lehmann et Romano (2005) et les références qui y sont.

Dans le cas où l'hypothèse de base n'est pas simple (par exemple contient une famille paramétrique de fonctions de répartition), les mêmes tests avec la fonction de répartition théorique remplacée par la fonction avec le paramètre estimé, ne sont plus généralement distribution-free et leur distribution limite dépend du vrai modèle sous-jacent.

Néanmoins, il existe un cas particulier d'hypothèse de base paramétriques avec des statistiques distribution-free. Ce cas correspond à des expériences statistiques non régulières, autrement dit les modèles où l'information de Fisher est infinie et le taux de convergence est meilleur que $\sqrt{n}$. Par exemple si nous avons un paramètre de translation et la fonction de densité admet des sauts (discontinuités).

L'objectif principal de ce travail est de construire une théorie analogue dans le cas d'observations de processus de diffusion ergodiques en temps continu.

Le premier chapitre est consacré aux résultats auxiliaires relatifs au processus de diffusion et de test d'ajustement.

Dans le deuxième chapitre, nous commençons avec le processus de diffusion à changement de régime

$$dX_t = -\mathrm{sgn}\,(X_t)\; dt + dW_t, \qquad X_0, \quad 0 \le t \le T,$$

comme modèle de base sous l'hypothèse nulle. Les tests statistiques sont basés sur le temps local $\Lambda_T\,(x)$ de ce processus et sont présentés comme suit :

$$V_T^2 = T \int_{\mathbb{R}} [f_T^\circ(x) - f_{S_0}(x)]^2 \, dx, \quad W_T^2 = T \int_{\mathbb{R}} [f_T^\circ(x) - f_{S_0}(x)]^2 \, dF_{S_0}(x),$$

où l'estimateur de temps local de la densité invariante est défini par l'égalité :

$$f_T^\circ(x) = \frac{\Lambda_T(x)}{T} = \frac{|X_T - x| - |X_0 - x|}{T} - \frac{1}{T} \int_0^T \mathrm{sgn}(X_t - x) \, dX_t.$$

Les tests d'ajustement correspondants sont :

$$\varphi_T(X^T) = 1_{\{V_T^2 > d_\alpha\}}, \quad \phi_T(X^T) = 1_{\{W_T^2 > c_\alpha\}}.$$

Les lois limites de ces statistiques sont représentées par les variables aléatoires suivantes

$$V^2 = \int_{\mathbb{R}} \eta_f^2(x)\, dx, \quad W^2 = \int_{\mathbb{R}} \eta_f^2(x)\, dF_{S_0}(x) = \int_0^1 \tilde{\eta}_f^2(u)\, du.$$

Ici $\eta_f(x)$ and $\tilde{\eta}_f(x)$ sont des processus Gaussiens. Nous proposons de calculer le développement de K-L des processus Gaussiens limites correspondants (voir Théorèmes 2.3-2.4).

Le processus de diffusion à changement de régime appartient à la classe des processus de diffusion à seuil (*threshold diffusion processes*) étudiés par Kutoyants (2004). Un exemple de ces processus est

$$dX_t = -\vartheta_1 \mathrm{sgn}(X_t - \vartheta_2)dt + dW_t, \quad X_0, \quad 0 \leq t \leq T, \tag{1}$$

où le paramètre inconnu ϑ_2 est appelé le seuil. La construction du test d'ajustement pour un tel modèle nécessite la connaissance du comportement des estimateurs de ces paramètres. Par conséquent, le chapitre suivant est consacré à l'étude du comportement de l'estimateur de maximum de vraisemblance et l'estimateur bayésien pour ce modèle. Il est montré que l'estimateur du "paramètre régulier" ϑ_1 est $\sqrt{T}$-normale, que l'estimateur de "seuil" ϑ_2 a une distribution limite avec la normalisation T et sont tous les deux asymptotiquement indépendants (voir Théorèmes 3.1-3.2).

Dans le chapitre quatre, nous étudions les tests d'ajustement de type C-vM pour le modèle (1). Dans le cas où les deux paramètres sont inconnus ces tests ne sont pas asymptotiquement distribution-free. Par contre dans le cas du paramètre de seuil inconnu ces tests sont asymptotiquement distribution-free.

Dans le dernier chapitre, le cas général du processus de diffusion est considéré

$$dX_t = S_0(X_t)\, dt + \sigma(X_t)\, dW_t, \qquad X_0, \quad 0 \leq t \leq T,$$

comme hypothèse de base. Les statistiques de C-vM pour ce modèle sont les suivantes :

$$\Delta_T = T \int_{\mathbb{R}} H(x) \left[\hat{F}_T(x) - F_{S_0}(x)\right]^2 dF_{S_0}(x),$$
$$\delta_T = T \int_{\mu}^{\infty} h(x) \left[f_T^{\circ}(x) - f_{S_0}(x)\right]^2 dF_{S_0}(x),$$

où $\hat{F}_T(x) = \frac{1}{T}\int_0^T \mathbb{1}_{\{X_t < x\}}\, dt$ est la fonction de répartition empirique et μ est la médiane de la loi invariante ($F_0(\mu) = 1/2$), elles ont été étudiées par Kutoyants (2009) qui a montré comment choisir les fonctions de poids $H(\cdot)$ and $h(\cdot)$ pour que ces tests aient la propriété asymptotiquement distribution-free :

$$\Delta_T \Longrightarrow \int_0^{\infty} e^{-y} W(y)^2 dy, \quad \delta_T \Longrightarrow \int_1^{\infty} e^{-y} W(y)^2 dy.$$

Notons ici que la fonction e^{-y} peut être remplacée par toute autre fonction $\psi(y)$ qui rend ces intégrales finies.

Dans ce chapitre, pour certains cas particuliers de la fonction $\psi(y)$, nous établissons les distributions explicites de ces variables aléatoires par le calcul direct de la transformée de Laplace (voir Théorèmes 5.1-5.3). Le choix des seuils est par conséquent réduit à l'application directe de la formule de Smirnov.

Publications

1. Gassem, A., On parameter estimation for switching diffusion process, *Statistics & Probability Letters, Volume 79, Issue 24, 2484-2492, 2009*.
2. Gassem, A., Goodness-of-fit test for switching diffusion, *Statistical Inference for Stochastic processes, Volume 13, Number 2, 97-123, 2010*.
3. Gassem, A., On Cramér-von Mises type test based on local time of switching diffusion process, à paraître dans : *Journal of Statistical Planning and Inference.*
4. Gassem, A., On the goodness-of-fit testing for switching diffusion process, soumis.
5. Gassem, A., On limit distributions of some goodness-of-fit tests statistics for ergodic diffusion processes, soumis.

1 Preliminaries

Sommaire

In this chapter we introduce the main concepts of the stochastic calculus necessary to deal with diffusion processes. We recall some preliminaries on Gaussian process theory and Karhunen-Loève expansions. We recall also some basic results of the large samples theory for the goodness-of-fit tests.

1.1 Stochastic calculus

Throughout this work, we are given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\Omega = \{\omega\}$ is a space of elementary events, $\mathcal{F}$ is a σ-algebra of subsets of Ω and $\mathbf{P}$ is a probability measure defined on the sets from $\mathcal{F}$, *i.e.*, $\mathbf{P}$ is a measurable mapping $\mathbf{P} : \mathcal{F} \to [0,1]$, which is countably additive and $\mathbf{P}\{\Omega\} = 1$. We suppose that this probability space is complete, *i.e.*, it contains all subsets of the sets of measure zero. Random variables $\xi(\omega)$ are defined as measurable mappings $\xi : \Omega \to \mathbb{R}$, *i.e.*, for any set $\mathbb{B} \in \mathcal{B}(\mathbb{R})$, $\mathcal{B}(\mathbb{R})$ is the σ-algebra of Borel subsets of the real line $\mathbb{R}$, the inclusion $\{\omega : \xi(\omega) \in \mathbb{B}\} \in \mathcal{F}$ holds. We say that the stochastic process (collection of random variables) $\{h(t,\omega), 0 \leq t \leq T\}$ is measurable if for any Borel set $\mathbb{B} \in \mathcal{B}(\mathbb{R})$ we have

$$\{(\omega, t) : h(t,\omega) \in \mathbb{B} \in \mathcal{F} \times \mathcal{B}[0,T]\},$$

where $\mathcal{B}[0,T]$ is the Borel σ-algebra of the interval $[0,T]$. Let $\{\mathcal{F}_t, 0 \leq t \leq T\}$ be an increasing family of σ-algebra (filtration), *i.e.*, for any $0 \leq s < t \leq T$ the inclusions $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ hold. The measurable stochastic process $\{h(t,\omega), 0 \leq t \leq T\}$ is said to be $\mathcal{F}_t$-adapted if the random variables $h(t,\omega)$ are $\mathcal{F}_t$-measurable for every $t \in [0,T]$, *i.e.*, for any Borel $\mathbb{B}$, we have

$\{\omega : h(t,\omega) \in \mathbb{B}\} \in \mathcal{F}_t$. An $\mathcal{F}_t$-adapted stochastic process is progressively measurable if for any $t \in [0,T]$ and Borel $\mathbb{B}$ we have

$$\{(t,\omega) : s < t,\, h(s,\omega) \in \mathbb{B}\} \in \mathcal{F}_t \otimes \mathcal{B}[0,t].$$

Let $\mathcal{M}_T$ be the class of progressively measurable random functions $h(\cdot)$ such that

$$\mathbf{P}\left\{\int_0^T h(t,\omega)^2\, dt < \infty\right\} = 1.$$

We say that $h(\cdot) \in \mathcal{M}_T^2$ if $h(\cdot) \in \mathcal{M}_T$ and the expectation :

$$\mathbf{E}\left\{\int_0^T h(t,\omega)^2 dt\right\} < \infty.$$

Definition 1.1 *A random process $W = \{W_t, \mathcal{F}_t, 0 \le t \le T\}$ is called standard Wiener process if it fulfills the following three conditions :*

1. $W_0 = 0$ *a.s.*
2. *The increments of W on disjoint intervals are independent. For example, for any partition $0 < t_1 < \ldots < t_n$ the random variables $W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent.*
3. *The random variable W_t is Gaussian with the following first moments :* $\mathbf{E}W_t = 0, \quad \mathbf{E}W_t\, W_s = t \wedge s, \quad t,s \in [0,T]$.

Stochastic integrals. Itô formula

In this section, the notion of stochastic integration is introduced.

The stochastic Itô integral :

$$\mathcal{I}_T(h) = \int_0^T h(t,\omega)\, dW_t,$$

is defined for the functions $h(\cdot) \in \mathcal{M}_T$ as follows. Let $\{h_n(\cdot,\omega),\, n = 1,2,\ldots\}$ be a sequence of elementary functions, *i.e.*, $h_n(t,\omega) = h_{n,l}(\omega)$ for $t \in [t_l^{(n)}, t_{l+1}^{(n)})$, where the random variables $h_{n,l}(\omega)$ are $\mathcal{F}_{t_l^{(n)}}$ measurable and $\left\{t_l^{(n)}, l = 0,1,\ldots,L_n\right\}$ is some subdivision of the interval $[0,T]$. Then the stochastic integral of $h_n(\cdot,\omega)$ is defined as

$$\mathcal{I}_T(h_n) = \sum_{l=1}^{L_n-1} h_n(\omega)\left(W_{t_{l+1}^{(n)}} - W_{t_l^{(n)}}\right).$$

For the functions $h(\cdot,\omega) \in \mathcal{M}_T$ we can take such a sequence of elementary functions $\{h_n(\cdot,\omega), n = 1,2,\ldots\}$ that

$$\mathbf{P} - \lim_{n\to\infty} \int_0^T [h(t,\omega) - h_n(t,\omega)]^2\, dt = 0$$

and the stochastic integral is defined as the limit

$$\mathcal{I}_T(h) = \mathbf{P} - \lim_{n\to\infty} \int_0^T h_n(t,\omega)\, dW_t.$$

It has the following properties.

1. If $h(\cdot) \in \mathcal{M}_T^2$, then $\mathbf{E}\mathcal{I}_T(h) = 0$, and $\mathbf{E}\{\mathcal{I}_T(h)|\mathcal{F}_t\} = \mathcal{I}_t(h)$.
2. For any two functions $h(\cdot), g(\cdot) \in \mathcal{M}_T^2$

$$\mathbf{E}\mathcal{I}_T(h)\,\mathcal{I}_T(g) = \mathbf{E}\left\{\int_0^T h(t,\omega)\,g(t,\omega)\,dt\right\}.$$

   In particular

$$\mathbf{E}\mathcal{I}_T(h)^2 = \mathbf{E}\left\{\int_0^T h(t,\omega)^2 dt\right\}.$$

3. If $h(\cdot) \in \mathcal{M}_T$, then

$$\mathbf{E}\left\{\exp\left(\mathcal{I}_t(h) - \frac{1}{2}\int_0^T h(t,\omega)^2 dt\right)\right\} \le 1.$$

Theorem 1.1 *(Burkholder-Davis-Gundy inequality). Let $h(\cdot) \in \mathcal{M}_T^2$ and for some $m \ge 1$*

$$\mathbf{E}\left\{\int_0^T |h(t,\omega)|^{2m} dt\right\} < \infty.$$

Then, there exists a constant C_m such that

$$\mathbf{E}\left|\sup_{0\le t\le T}\mathcal{I}_t(h)\right|^{2m} \le C_m\,\mathbf{E}\left(\int_0^T h(t,\omega)^2 dt\right)^m.$$

In particular,

$$\mathbf{E}\left|\sup_{0\le t\le T}\mathcal{I}_t(h)\right|^2 \le 4\,\mathbf{E}\left\{\int_0^T h(t,\omega)^2 dt\right\}.$$

The proof of these properties can be found in any book on stochastic calculus (see, e.g., Durett 1996, Karatzas and Shreve 1991, Lipster and Shiryayev 2001, Revuz and Yor 1991).

Let $g(t,\omega)$ be $\mathcal{F}_t$-adapted for almost all $t \in [0,T]$,

$$\mathbf{P}\left\{\int_0^T |g(t,\omega)|\,dt < \infty\right\} = 1,$$

and $h(\cdot) \in \mathcal{M}_T$. Then the stochastic process

$$X_t = X_0 + \int_0^t g(s,\omega)\,ds + \int_0^t h(s,\omega)\,dW_s, \quad t \in [0,T], \tag{1.1}$$

is called the Itô process. Here X_0 is a $\mathcal{F}_0$-measurable random variable. In the shortened form it is usually written as

$$dX_t = g(t,\omega)\,dt + h(t,\omega)\,dW_t, \quad X_0, \quad t \in [0,T]. \tag{1.2}$$

This last equality is called the stochastic differential of the Iô process. It can be shown that the trajectory $\{X_t, 0 \le t \le T\}$ is continuous with probability 1 and $\mathcal{F}_t$-adapted.

The class of Itô processes is closed with respect to smooth transformations in the following sense. Let $\{X_t, 0 \le t \le T\}$ be an Itô process with stochastic differential (1.2) and $G(x,t)$ be a differentiable function with the following continuous derivatives : $G'_t(x,t)$, $G'_x(x,t)$, $G''_{xx}(x,t)$ (with obvious notation). Then the stochastic process $Y_t = G(X_t,t), 0 \le t \le T$ is the Itô process too with the stochastic differential

$$dY_t = \left[G'_t(X_t,t) + G'_x(X_t,t)\, g(t,\omega) + \frac{1}{2} G''_{xx}(X_t,t)\, h(t,\omega)^2 \right] dt \\ + G'_x(X_t,t)\, h(t,\omega)\, dW_t, \quad Y_0 = G(X_0,0), \quad t \in [0,T]. \tag{1.3}$$

This equality is called the Itô formula and it can be written as

$$dY_t = \left[G'_t(X_t,t) + \frac{1}{2} G''_{xx}(X_t,t)\, h(t,\omega)^2 \right] dt + G'_x(X_t,t)\, h(t,\omega)\, dX_t, \tag{1.4}$$

with the same initial value.

Diffusion Process

In the framework of this work, we are interested in a particular case of the Itô processes. The so-called homogeneous diffusion processes defined as solutions of the stochastic differential equation (SDE)

$$dX_t = S(X_t)\, dt + \sigma(X_t)\, dW_t, \quad X_0 = x, \quad 0 \le t \le T, \tag{1.5}$$

i.e., it is the Itô process (1.4) with $g(t,\omega) = S(X_t)$ and $h(t,\omega) = \sigma(X_t)$. The function $S(\cdot)$ is called *drift* or *trend coefficient*, $\sigma^2(\cdot)$ is called *diffusion coefficient* and $\sigma(\cdot)$ is called *volatility*, where again (1.5) is a short version of the integral representation :

$$X_t = X_0 + \int_0^t S(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s, \quad t \in [0,T]. \tag{1.6}$$

This equality can be considered as an integral equation with respect to the random function $X^T = \{X_t, 0 \le t \le T\}$ and the question of the existence of the solution of this equation naturally arises. Two types of solutions namely, strong and weak solutions are possible.

Consider a family of σ-algebras

$$\mathcal{F}_t^{X_0,W} = \sigma\{X_0, W_s, 0 \le s \le t\}, \quad 0 \le t \le T,$$

generated by the initial value X_0 and by the given Wiener process up to time t.

Definition 1.2 *We say that equation (1.5) has a strong solution X^T on the given probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with respect to the fixed Wiener process $W^T = \{W_t, 0 \le t \le T\}$ and initial condition X_0 if the random function X^T satisfies the equality (1.5), has continuous sample paths and*

$$X_t \text{ is } \mathcal{F}_t^{X_0,W} \text{ measurable, for all } t \in [0,T]. \tag{1.7}$$

We say that SDE (1.5) has a unique strong solution if for any two solutions $\{X_t^{(1)}, 0 \le t \le T\}$ and $\{X_t^{(2)}, 0 \le t \le T\}$ the equality

$$\mathbf{P}\left\{ \sup_{0 \le t \le T} \left| X_t^{(2)} - X_t^{(1)} \right| > 0 \right\} = 0,$$

holds.

By this definition the solution X_t depends on the initial value X_0 and the trajectory of the Wiener process up to time t.

There are different sufficient conditions on the functions $S(\cdot)$, $\sigma(\cdot)$ ensuring the strong existence and uniqueness of solution of (1.5). The most convenient result for our purposes is the following :

Theorem 1.2 *(cf. Durett 1996, p. 190). If $\mathbf{P}\{|X_0| < \infty\} = 1$ and the functions $S(\cdot)$ and $\sigma(\cdot)$ are locally Lipschitz, i.e., for any positive N, there exists a constant L_N such that*

$$|S(x) - S(y)| + |\sigma(x) - \sigma(x)| < L_N|x - y|, \quad \forall x, y \in [0, N],$$

and for some constant B we have

$$2xS(x) + \sigma^2(x) \le B(1 + |x|^2), \quad \forall x \in \mathbb{R},$$

then the equation (1.5) has a unique strong solution $\{X_t, 0 \le t \le T\}$, continuous with probability 1.

The existence of the strong solution translates into the fact that given the probability space, Wiener process and initial random variable, then for given $S(\cdot)$ and $\sigma(\cdot)$ the process (1.5) with (1.7) can be constructed. It is possible to consider the "inverse" problem, which is described in the definition 1.3.

Definition 1.3 *Suppose that we are given the functions $S(\cdot)$ and $\sigma(\cdot)$ and the distribution function $F(\cdot)$. We say that there exists a weak solution of equation (1.5) if there exists a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a nondecreasing family of sub-sigma algebras $\{\mathcal{F}_t, 0 \le t \le T\}$, a continuous random process $\{X_t, \mathcal{F}_t, 0 \le t \le T\}$ and a Wiener process $\{W_t, \mathcal{F}_t, 0 \le t \le T\}$, such that*

$$\mathbf{P}\left\{\int_0^T \left[|S(X_t)| + \sigma(X_t)^2\right] \, dt < \infty\right\} = 1,$$

and for all $t \in [0, T]$.

$$X_t = X_0 + \int_0^t S(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s, \quad \textit{with} \quad \mathbf{P}\{X_0 < x\} = F(x).$$

Note that existence of a weak solution does not imply the existence of a strong solution but the existence of a strong solution implies the existence of the weak solution (see, e.g., Karatzas and Shreve 1991, Durett 1996).

Throughout this work, we suppose that the SDE (1.5) has a unique weak solution. For our purposes, the following one is sufficient.

Theorem 1.3 *(cf. Durett 1996, p.210). Suppose that the function $S(\cdot)$ is locally bounded, the function $\sigma(\cdot)^2$ is continuous and positive and for some $A > 0$ the condition*

$$xS(x) + \sigma^2(x) \le A(1 + |x|^2), \quad \forall x \in \mathbb{R},$$

holds. Then the SDE (1.5) has a unique weak solution.

Definition 1.4 *A continuous random process X is called diffusion process, if there exist two function S and σ from $\mathbb{R}$ to itself, such that the equality (1.6) is satisfied for any positive T.*

Moreover, any solution of equation (1.6) possess the strong Markov property : *for any stopping time* T *and for any borelian function* $\Phi : C(\mathbb{R}_+) \longrightarrow \mathbb{R}_+$, *we have*

$$\mathbf{E}\left[\Phi(X_{T+t}, t \geq 0 | \mathcal{F}_T\right] = \mathbf{E}\left[\Phi(X_{T+t}, t \geq 0 | X_T\right].$$

Therefore, the solution of (1.6) is a strong Markov process. We say that a Markov process is *ergodic or possess ergodic properties* if there is a random variable X_0, such that, the process X_t having X_0 as initial value is stationary. The law (distribution) of this random variable is called *invariant law (distribution)* of the process X, the density of this distribution with respect to the Lebesgue measure (if it exists) is called *invariant density* of the process X. The next result gives a necessary and sufficient condition on the coefficients of the equation (1.5) ensuring the ergodicity of the solution.

Theorem 1.4 *Let the diffusion coefficient* $\sigma(\cdot)^2$ *be strictly positive and the following conditions be fulfilled*

$$V(x) = \int_0^x \exp\left\{-2\int_0^y \frac{S(v)}{\sigma^2(v)}\,dv\right\} dy \longrightarrow \pm\infty, \quad as \quad x \to \pm\infty,$$

and

$$G(S) = \int_{\mathbb{R}} \frac{1}{\sigma^2(y)} \exp\left\{2\int_0^y \frac{S(v)}{\sigma^2(v)}\,dv\right\} dy < \infty.$$

Then the process X *defined by* (1.6) *process ergodic properties and the invariant density is given by formula*

$$f_S(x) = \frac{1}{G(S)\,\sigma^2(x)} \exp\left\{2\int_0^x \frac{S(v)}{\sigma^2(v)}\,dv\right\}. \tag{1.8}$$

The proof of this theorem can be found in Gikhman and Skorohod (1969). A good exposition of results concerning ergodic properties of the solution of (1.6) can be found in Hasminskii (1980), for more details see Dynkin (1965). It is proved there that under the conditions of Theorem 1.4, the process X is recurrent positive. In fact, the first condition of this theorem guarantees the recurrence of the process X and the second one the positiveness. The recurrence means that for any initial value x and for any open ball $B \in \mathbb{R}$ the process starting at x visits infinitely many times B for almost every ω's. A recurrent process is called positive, if the mathematical expectation of the visiting time of an open ball B is finite for any B. It is well known that for Markov process the positiveness does not depend on the choice of the initial value x. Moreover, if the process X issuing from x_0 is recurrent and positive with respect to an open ball $B_0 \subseteq \mathbb{R} - \{x_0\}$, then it is so for any initial value x and any open ball $B \subseteq \mathbb{R}$. This feature can be interpreted as a kind of stability of the solution.

An important consequence of ergodicity is the so called *law of large numbers*. It can be formulated in the following way.

Theorem 1.5 *(see Hasminskii 1980). Let the conditions of previous theorem be satisfied and let us denote the invariant law by* μ_S. *Then, for any initial value* x *and for any* μ_S*-integrable function* g, *we have*

$$\mathbf{P}\left\{\frac{1}{T}\int_0^T g(X_t)\,dt \overset{T\to\infty}{\longrightarrow} \int_{\mathbb{R}} g(x)\,\mu_S(dx)\right\} = 1,$$

where X is the solution of (1.5).

Theorems 1.3 and 1.4 invoke that under suitable conditions on the trend coefficient $S(\cdot)$ and diffusion coefficient $\sigma(\cdot)^2$, there exists a unique weak solution of equation (1.5) and this solution is an ergodic diffusion process. But the conditions of Theorem 1.4 have a form which is rather complicated. That is why we formulate below a result which gives a very simple condition ensuring the existence, uniqueness and ergodicity of diffusion process.

Proposition 1.1 *Let the diffusion coefficient $\sigma(\cdot)^2$ be bounded away from zero and has at most linear growth (i.e., $|\sigma(x)| \le C(1+|x|)$ for any x) and the condition*

$$\limsup_{|x|\to\infty} \operatorname{sgn}(x) \frac{S(x)}{\sigma^2(x)} < 0. \tag{1.9}$$

is fulfilled. Then the conditions of Theorem 1.3 and 1.4 are satisfied (see, e.g., Kutoyants 2004, p.29).

Likelihood ratio for diffusion processes

Let $X^T = \{X_t, 0 \le t \le T\}$ be a diffusion process given by equation (1.5), with initial value X_0. As mentioned above, this process is a.s. continuous. Consequently, it induces a probability measure $\mathbf{P}_S^{(T)}$ on the measurable space $(\mathcal{C}_T, \mathcal{B}_T) = (\mathcal{C}[0,T], \mathcal{B}(\mathcal{C}[0,T]))$.

Let us consider the stochastic differential equations

$$\begin{aligned} dX_t &= S_1(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0^{(1)}, \quad 0 \le t \le T, \\ dX_t &= S_2(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0^{(2)}, \quad 0 \le t \le T, \end{aligned}$$

and denote by $\mathbf{P}_{S_1}^{(T)}$, $\mathbf{P}_{S_2}^{(T)}$ the probability measures induced in $(\mathcal{C}_T, \mathcal{B}_T)$ by the solutions of these equations respectively. Let us introduce the following condition :

$\mathcal{EM}$. *The functions $S_1(\cdot)$, $S_2(\cdot)$ and $\sigma(\cdot)$ satisfy the condition of Theorem 1.3 and the densities $f_1(\cdot)$, $f_2(\cdot)$ (with respect to the Lebesgue measure) of the corresponding initial values have the same support (if the initial value is nonrandom then we suppose that it takes the same value for all processes).*

Theorem 1.6 *Suppose that the condition $\mathcal{EM}$ is fulfilled and*

$$\mathbf{P}_i \left\{ \int_0^T \left(\frac{S_1(X_t) - S_2(X_t)}{\sigma(X_t)} \right)^2 dt < \infty \right\} = 1 \quad \textit{for} \quad i = 1,2. \tag{1.10}$$

Then the measures $\mathbf{P}_{S_1}^{(T)}$, $\mathbf{P}_{S_2}^{(T)}$ are equivalent and the corresponding Radon-Nikodym derivatives is given by the formula

$$\begin{aligned} \frac{d\mathbf{P}_{S_1}^{(T)}}{d\mathbf{P}_{S_2}^{(T)}} \left(X^T\right) = \frac{f_2(X_0)}{f_1(X_0)} \exp\Bigg\{ &\int_0^T \frac{S_2(X_t) - S_1(X_t)}{\sigma(X_t)^2}\, dX_t \\ &- \frac{1}{2} \int_0^T \frac{S_2(X_t)^2 - S_1(X_t)^2}{\sigma(X_t)^2}\, dt \Bigg\}. \end{aligned}$$

Proof. The proof of this theorem can be found in Lipster and Shiryayev (2001). □

We give below the likelihood ratio formula for the parametric family of the diffusion processes which will be used later. We are given a set $\Theta \subset \mathbb{R}^d$ and a parametric family of diffusion processes

$$dX_t = S(\vartheta, X_t)\, dt + \sigma(X_t)\, dW_t, \quad X_0, \quad 0 \le t \le T, \tag{1.11}$$

where $\vartheta \in \Theta$ is an unknown parameter. Therefore, the statistical experiment (see, e.g., Le Cam 1981, Ibragimov and Khasminskii 1986) is given by the triplet

$$\left(\mathcal{C}_T, \mathcal{B}_T, \left\{\mathbf{P}_\vartheta^{(T)}, \vartheta \in \Theta\right\}\right).$$

Here $\mathbf{P}_\vartheta^{(T)}$ is the measure induced in the space $\mathcal{C}_T$ by the process (1.11). This statistical experiment is the starting point for statistical inference. Therefore all we need is the family of measure $\left\{\mathbf{P}_\vartheta^{(T)}, \vartheta \in \Theta\right\}$. The conditions of the existence and uniqueness of a measure corresponding to a stochastic process having the stochastic differential equation (1.11) are weaker and the notion of the weak solution of equation (1.11) fits statistical problems better.

Theorem 1.7 *Suppose that the functions $S(\vartheta, \cdot)$, for all $\vartheta \in \Theta$ and $\sigma(\cdot)$ satisfy the condition $\mathcal{EM}$ and the following conditions be fulfilled*

$$V(\vartheta, x) = \int_0^x \exp\left\{-2\int_0^y \frac{S(\vartheta, v)}{\sigma^2(v)}\, dv\right\} dy \longrightarrow \pm\infty, \quad \text{as} \quad x \to \pm\infty,$$

and

$$G(\vartheta) = \int_{\mathbb{R}} \frac{1}{\sigma^2(y)} \exp\left\{2\int_0^y \frac{S(\vartheta, v)}{\sigma^2(v)}\, dv\right\} dy < \infty.$$

Then there exists an invariant distribution with density function

$$f(\vartheta, x) = \frac{1}{G(\vartheta)\,\sigma^2(x)} \exp\left\{2\int_0^x \frac{S(\vartheta, v)}{\sigma^2(v)}\, dv\right\}, \tag{1.12}$$

and if we suppose that the initial value X_0 has the density $f_\vartheta(\cdot) = f(\vartheta, \cdot)$, then the process $\{X_t, t \ge 0\}$ is stationary. The measures $\left\{\mathbf{P}_\vartheta^{(T)}, \vartheta \in \Theta\right\}$ are equivalent. The likelihood ratio

$$L\left(\vartheta, \vartheta_1; X^T\right) = \frac{d\mathbf{P}_\vartheta^{(T)}}{d\mathbf{P}_{\vartheta_1}^{(T)}}\left(X^T\right),$$

is given by the formula

$$\begin{aligned} L\left(\vartheta, \vartheta_1; X^T\right) = \frac{G(\vartheta_1)}{G(\vartheta)} &\exp\left\{2\int_0^{X_0} \frac{S(\vartheta, v) - S(\vartheta_1, v)}{\sigma(v)^2}\, dv\right\} \\ &\times \exp\left\{\int_0^T \frac{S(\vartheta, X_t) - S(\vartheta_1, X_t)}{\sigma(X_t)^2}\, dX_t \right. \\ &\qquad \left. - \frac{1}{2}\int_0^T \frac{S(\vartheta, X_t)^2 - S(\vartheta_1, X_t)^2}{\sigma(X_t)^2}\, dt\right\}. \end{aligned}$$

If ϑ_1 is the true value, then with $\mathbf{P}_{\vartheta_1}^{(T)}$ probability 1 (see Kutoyants 2004, p.37)

$$\begin{aligned} L\left(\vartheta, \vartheta_1; X^T\right) = \frac{G(\vartheta_1)}{G(\vartheta)} \exp\left\{2\int_0^{X_0} \frac{S(\vartheta,v)-S(\vartheta_1,v)}{\sigma(v)^2}\,dv\right\} \\ \times \exp\left\{\int_0^T \frac{S(\vartheta,X_t)-S(\vartheta_1,X_t)}{\sigma(X_t)}\,dW_t \right. \\ \left. -\frac{1}{2}\int_0^T \left(\frac{S(\vartheta,X_t)-S(\vartheta_1,X_t)}{\sigma(X_t)}\right)^2 dt\right\}. \end{aligned}$$

1.2 Local time

From now on, we fix a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\{\mathcal{F}_t, t \geq 0\}$ and a random process X strong solution of SDE whose coefficients satisfy the conditions of Theorem 1.2. A very important feature of the diffusion processes from statistical point of view is the fact that the empirical function $\hat{F}_T(x) = \frac{1}{T}\int_0^T \mathbb{1}_{\{X_t \leq x\}}\,dt$ has a Radon-Nikodym derivative with respect to the Lebesgue measure for any positive T. To formulate the corresponding result we need the following:

TANAKA FORMULA. *For any real number x, there exists an increasing continuous process $\Lambda_T(x)$ called the local time of X at the point x such that,*

$$|X_T - x| = |X_0 - x| + \int_0^T \operatorname{sgn}(X_t - x)\,dX_t + \Lambda_T(x), \quad T \geq 0,\ x \in \mathbb{R}.$$

Note that this formula is something like Itô formula for the function $F(y) = |y - x|$, which is not two times continuous differentiable.

In the particular case of a diffusion process given by equation (1.6), the process $\Lambda_T(x)$ can be rewritten as

$$\begin{aligned} \Lambda_T(x) = |X_T - x| - |X_0 - x| - \int_0^T \operatorname{sgn}(X_t - x)\,S(X_t)\,dt \\ - \int_0^T \operatorname{sgn}(X_t - x)\,\sigma(X_t)\,dW_t, \quad T \geq 0,\ x \in \mathbb{R}. \end{aligned}$$

Let $h(\cdot)$ be a measurable function. Then with probability 1

$$\int_0^T h(X_t)\sigma^2(X_t)\,dt = \int_{\mathbb{R}} h(x)\Lambda_T(x)\,dx.$$

We will use this equality (Occupation times formula) in a different form. Thus, if we denote

$$f_T^\circ(x) = \frac{\Lambda_T(x)}{T\,\sigma^2(x)}, \tag{1.13}$$

then the empirical distribution function is

$$\hat{F}_T(y) = \frac{1}{T}\int_0^T \mathbb{1}_{\{X_t \leq y\}}\,dt = \int_{\mathbb{R}} \frac{\mathbb{1}_{\{x \leq y\}}}{T\,\sigma^2(x)}\,\Lambda_T(x)\,dx = \int_{-\infty}^y f_T^\circ(x)\,dx.$$

These and many other properties of the local time can be found in Revuz and Yor (1991); Karatzas and Shreve (1991).

The function $f_T^\circ(\cdot)$ is called *local time estimator of invariant density*. It has been introduced and studied by Kutoyants (1997, 1998, 2004). Who gives, the following representation of the local time estimator

$$\begin{aligned} f_T^\circ(x) - f(x) &= \frac{2f(x)}{T}\int_{X_0}^{X_T} \frac{1_{\{v>x\}} - F(v)}{f(v)}\,dv \\ &\quad - \frac{2f(x)}{T}\int_0^T \frac{1_{\{X_t>x\}} - F(X_t)}{f(X_t)}\,dW_t, \end{aligned} \tag{1.14}$$

where $F(\cdot)$ and $f(\cdot)$ are respectively the invariant distribution function and the invariant density of an ergodic diffusion process X.

Using this representation and the occupation times formula. We obtain for any integrable function $h(\cdot)$ such that $\mathbf{E}h(\xi) = \int_{\mathbb{R}} h(x)f(x)\,dx = 0$,

$$\begin{aligned} \frac{1}{T}\int_0^T h(X_t)\,dt &= \int_{\mathbb{R}} h(x)f_T^\circ(x)\,dx = \int_{\mathbb{R}} h(x)\left[f_T^\circ(x) - f(x)\right]\,dx \\ &= \frac{M(X_T) - M(X_0)}{T} - \frac{1}{T}\int_0^T m(X_t)\,dW_t, \end{aligned} \tag{1.15}$$

where we used the following auxiliary notations:

$$\begin{aligned} M(y) &= \int_0^y \int_{\mathbb{R}} 2\,h(x)f(x)\left(\frac{\mathbb{1}_{\{v>x\}} - F(v)}{\sigma^2(v)f(v)}\right)dx\,dv, \\ &= \int_0^y \frac{2}{\sigma^2(v)f(v)}\int_{-\infty}^v h(x)f(x)\,dx\,dv \end{aligned} \tag{1.16}$$

$$m(y) = \frac{2}{\sigma^2(y)f(y)}\int_{-\infty}^y h(x)f(x)\,dx. \tag{1.17}$$

The equality (1.15) plays a very important role in this work and is one of the main technical tools to prove the statistical properties for the model of ergodic diffusion.

Note that the local time estimators of the density of wide class of stationary processes were studied by Davydov and Bosq (1999).

1.3 Gaussian processes and Karhunen-Loève expansion

Let $Z = \{Z_t,\ t \in \mathbb{R}\}$ be a Gaussian process with mean value zero and continuous covariance function

$$R(s,t) = \mathbf{E}(Z_t Z_s), \quad s,t \in \mathbb{R}.$$

Below, we will be mainly concerned with the study of the quadratic functional

$$\int_{\mathbb{R}} Z_t^2\,dt, \tag{1.18}$$

with dt denoting the Lebesgue measure on $\mathbb{R}$. To render (1.18) meaningful, we consider the minimal assumption that

$$\mathbf{E}\left(\int_{\mathbb{R}} Z_t^2\,dt\right) = \int_{\mathbb{R}} R(t,t)\,dt < \infty. \tag{1.19}$$

Below, we briefly discuss the meaning and implications of this assumption. The condition (1.18) entails that $Z \in L^2(\mathbb{R})$ a.s. Moreover, making use of the Cauchy-Schwarz inequality, it can be seen that, for each $s, t \in L^2(\mathbb{R})$,

$$R(s,t)^2 = \mathbf{E}(Z_t Z_s)^2 \le \mathbf{E}(Z_t^2)\mathbf{E}(Z_s^2) = R(s,s)R(t,t).$$

When combining this last inequality with (1.19), we obtain the following inequality:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} R(s,t)^2 ds\, dt \le \left(\int_{\mathbb{R}} R(t,t)\, dt \right)^2 < \infty. \tag{1.20}$$

Routine analytical arguments show that, under (1.20) only, the Fredholm transformation $f(\cdot) \in L^2(\mathbb{R}) \longrightarrow \mathcal{I}_Z f \in L^2(\mathbb{R})$, defined by

$$\mathcal{I}_Z f(t) = \int_{\mathbb{R}} R(t,s)\, f(s)\, ds \quad \text{for} \quad t \in \mathbb{R}, \tag{1.21}$$

is a continuous linear mapping of $L^2(\mathbb{R})$ onto itself. The condition (1.20) implies the existence of $\{\lambda_n, e_n(\cdot) : n \ge 1\}$ with the following properties. The $\{\lambda_n,\, n \ge 1\}$ are nonnegative constants such that $\lambda_1 > \lambda_2 > \cdots > 0$, with $\{e_n(\cdot) : n \ge 1\}$ form an orthonormal sequence of functions in $L^2(\mathbb{R})$, fulfilling

$$\int_{\mathbb{R}} e_n(t)\, e_m(t)\, dt = \begin{cases} 1 & \text{if} \quad n = m, \\ 0 & \text{if} \quad n \ne m. \end{cases}$$

By Mercer's theorem (see, e.g., Anderson and Darling 1952, p.198; Riesz and Nagy 1955; Neveu 1968, p. 49; Loève 1978, p. 143-144; Novitskii 1984; Buescu 2004), it is well known that $R(s,t)$ can be represented as

$$R(s,t) = \sum_{n=1}^{\infty} \lambda_n\, e_n(s)\, e_n(t), \tag{1.22}$$

where the series in (1.22) is absolutely and uniformly convergent. This last property entails that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} R(s,t)^2 ds\, dt = \sum_{n=1}^{\infty} \lambda_n^2 < \infty.$$

The λ_n (resp. $e_n(\cdot)$) are the eigenvalues (resp. eigenfunctions) of the Fredholm operator (1.21), since they fulfill the relations,

$$\lambda_n\, e_n(\cdot) = \int_{\mathbb{R}} R(t,\cdot)\, e_n(t)\, dt, \quad \int_{\mathbb{R}} e_n^2(t)\, dt = 1, \tag{1.23}$$

The following result is widely known as Karhunen-Loève representation of Z (see, e.g., Kac and Siegert 1947; Kac 1951; Ash and Gardner 1975, p. 39-40). This representation decomposes Z into the sum of the series

$$Z_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \xi_n\, e_n(t), \quad t \in \mathbb{R}, \tag{1.24}$$

where the random variables $\{\xi_n : n \ge 1\}$ are i.i.d. $\mathcal{N}(0,1)$. In general, the series in (1.24) is convergent in mean square. This follows from the observation that, in terms of $\{\lambda_n : n \ge 1\}$, the condition (1.19) is equivalent to

$$\mathbf{E}\left(\int_{\mathbb{R}} Z_t^2\, dt \right) = \sum_{n=1}^{\infty} \lambda_n < \infty. \tag{1.25}$$

Obviously, the condition (1.19) (or equivalently (1.25)) is strictly stronger than (1.20). Moreover, is readily checked that, under (1.19) (or equivalently (1.25)), the quadratic functional (1.18) can be rewritten as the sum of the series

$$Q^2 = \int_{\mathbb{R}} Z_t^2 \, dt = \sum_{n=1}^{\infty} \lambda_n \, \xi_n^2. \tag{1.26}$$

An easy argument, which we omit, shows that the series in (1.26) is a.s. convergent if and only if (1.19) (or equivalently (1.25)) holds.

The relation (1.26) readily implies (see, e.g., Kac 1951; Rosenblatt 1952), that the characteristic function of the distribution of Q^2 is given by,

$$\begin{aligned} \varphi(v) = \mathbf{E}\left[e^{ivQ^2}\right] &= \mathbf{E}\left[\exp\left(iv \sum_{n=1}^{\infty} \lambda_n \, \xi_n^2\right)\right] = \prod_{n=1}^{\infty} \mathbf{E}\left[\exp\left(iv\lambda_n \, \xi_n^2\right)\right] \\ &= \prod_{n=1}^{\infty} \left(1 - 2iv\lambda_n\right)^{-1/2}, \quad v \in \mathbb{R}. \end{aligned} \tag{1.27}$$

By the method of Smirnov (1937) (see also the correction by Darling 1957; Martynov (1975, 1977)), the distribution function $F(x)$ of the variable Q^2, admits the representation

$$F(x) = 1 + \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \int_{\delta_{2n-1}}^{\delta_{2n}} \frac{e^{-xu/2}}{u\sqrt{|D(u)|}} \, du \quad \text{for} \quad x \geq 0, \tag{1.28}$$

where $D(u)$ is the Fredholm determinant defined under (1.25), for $u \geq 0$, by

$$D(u) = \prod_{n=1}^{\infty} (1 - u\lambda_n) = \prod_{n=1}^{\infty} \left(1 - \frac{u}{\delta_n}\right).$$

Here $\delta_n = 1/\lambda_n$, for $n \geq 1$.

1.4 Goodness-of-fit tests

The problem of goodness-of-fit testing is one of the central themes of statistical theory and practice. It is important to verify the degree of correspondence between observed outcomes and expected outcomes that is the foundation of classical statistics. The classical nonparametric approaches to this hypothesis testing problem can be found in Durbin (1973); Greenwood and Nikulin (1996); Lehmann and Romano (2005)). The most known tests are: Pearson's goodness-of-fit Chi-Squared test, Kolmogorov-Smirnov and Cramér-von Mises tests.

The advantage of classical tests is that they are distribution-free, *i.e.*, the distribution of the underlying statistics do not depend on the basic model and this property allows to choose the universal thresholds, which can be used for all models.

Remind the construction of the C-vM test. Suppose that we observe n i.i.d. random variables $(X_1, \ldots, X_n) = X^n$ with continuous distribution function $F(x)$ and the basic hypothesis is simple: $\mathcal{H}_0 : \; F(x) = F_0(x)$, then we can introduce the well-known C-vM family statistics,

$$W_n^2 = n \int_{\mathbb{R}} \psi\left(F_0(x)\right) \left[\hat{F}_n(x) - F_0(x)\right]^2 dF_0(x), \tag{1.29}$$

where
$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i \le x\}},$$
is the empirical distribution function and $\psi(\cdot)$ is some nonnegative weight function. After the transformation $t = F_0(x)$ we get the standard form of the statistic:
$$W_n^2 = n\int_0^1 \psi(t)(\tilde{F}_n(t) - t)^2 dt,$$
where $\tilde{F}_n(t)$ is the empirical distribution function of a random sample of size n from the uniform distribution on $(0,1)$ (under hypothesis $\mathcal{H}_0$) and the function $\psi(t)$ may be continuous on $[0,1]$, or it may in certain circumstances tend to infinity at $t = 0$ or $t = 1$ (or both).

Recall that the standard Brownian bridge is a centered Gaussian process with continuous paths p.s., such that $B = \{B_t : t \in [0,1]\}$ with $\mathbf{E}B_t = 0$. The covariance function is given by
$$K(t,s) = \mathbf{E}B_t B_s = t \wedge s - t\, s, \quad \forall s,t \in [0,1].$$
Then the limit distribution of W_n^2 under hypothesis $\mathcal{H}_0$ is the distribution of the functional
$$W^2 = \int_0^1 \psi(t) B_t^2 dt.$$
The hypothesis $\mathcal{H}_0$ is accepted, if $W_n^2 < c_{n,\alpha}$ and rejected, if $W_n^2 \ge c_{n,\alpha}$, where the constant $c_{n,\alpha}$ is solution of the equation $\mathbf{P}(W_n^2 \ge c_{n,\alpha}) = \alpha$. The constant $c_{n,\alpha}$ can be approximated by its limit ($n \to \infty$) value c_α, solution of the equation $\mathbf{P}(W^2 \ge c_\alpha) = \alpha$.

The classical solution of this problem requires a preliminary computation of the sequence of eigenvalues $\lambda_1 > \lambda_2 > \cdots > 0$ and eigenfunctions $e_1(t), e_2(t), \ldots$ of the Fredholm operator (1.21). In particular, if the weight function $\psi(t) = 1$, we obtain the ordinary C-vM statistic, its limit distribution coincides with the distribution of the quadratic form
$$\mathcal{C}_n^2 = \int_0^1 (\tilde{F}_n(t) - t)^2 dt \Longrightarrow \mathcal{C}^2 = \int_0^1 B_t^2\, dt = \sum_{n=1}^{\infty} \frac{\xi_n^2}{n^2\pi^2},$$
where the K-L expansion of the Brownian bridge process B_t is given by
$$B_t = \sum_{n=1}^{\infty} \frac{1}{\pi n} \sqrt{2}\,\xi_n \sin(\pi n t), \quad \text{for} \quad t \in [0,1].$$
The function $\psi(t) = [t(1-t)]^{-1}$ gives the Anderson-Darling statistic, the limit distribution of this statistic coincides with the distribution of the quadratic form
$$\mathcal{A}_n^2 = \int_0^1 \frac{(\tilde{F}_n(t) - t)^2}{t(1-t)} dt \Longrightarrow \mathcal{A}^2 = \int_0^1 \frac{B_t^2}{t(1-t)} dt = \sum_{n=1}^{\infty} \frac{\xi_n^2}{n(n+1)},$$
where, we have on the interval $(0,1)$
$$\frac{B_t}{\sqrt{t(1-t)}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} \xi_n 2\sqrt{\frac{2n+1}{n(n+1)}} \sqrt{t(t+1)} P_n'(2t-1),$$

with P_n' denoting the derivative of the nth Legendre polynomial (see, e.g., Robin 1959, formula (35), p. 13 or Magnus et al 1966, p.232).

$$P_n(t) = \frac{1}{n!2^n}\frac{d^n}{dt^n}(t^2-1)^n.$$

Thus, using the equality (1.27) we obtain,

$$\lim_{n\to\infty} \mathbf{E}\left[e^{iv\mathcal{C}_n^2}\right] = \mathbf{E}\left[e^{iv\mathcal{C}^2}\right] = \prod_{n=1}^{\infty}\left(1-\frac{2iv}{n^2\pi^2}\right)^{-1/2} = \{D(2iv)\}^{-1/2},$$

with

$$D(u) = \frac{\sin(\sqrt{u})}{\sqrt{u}}.$$

For the Anderson-Darling statistic

$$\lim_{n\to\infty} \mathbf{E}\left[e^{iv\mathcal{A}_n^2}\right] = \mathbf{E}\left[e^{iv\mathcal{A}^2}\right] = \prod_{n=1}^{\infty}\left(1-\frac{2iv}{n(n+1)}\right)^{-1/2} = \{D(2iv)\}^{-1/2},$$

with

$$D(u) = -\frac{1}{\pi\, u}\cos\left(\frac{\pi}{2}\sqrt{1+4u}\right).$$

These relations provides, for large n, a reasonable approximation to the null distribution of the test statistic $\mathcal{C}_n^2$, (resp. $\mathcal{A}_n^2$), whose distribution becomes close to that of $\mathcal{C}^2$, (resp. $\mathcal{A}^2$) for large n.

The distribution function $F_1(x)$, (resp. $F_2(x)$) of the limiting variable $\mathcal{C}^2$, (resp. $\mathcal{A}^2$) is given by the formula

$$F_1(x) = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\int_{(2n-1)\pi}^{2n\pi}\frac{e^{-xu^2/2}}{\sqrt{-u\sin(u)}}\,du \quad \text{for} \quad x \geq 0,$$

and

$$F_2(x) = 1 + \frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}(-1)^n\int_{4n-1}^{4n+1}\frac{u\, e^{-x(u^2-1)/8}}{\sqrt{(u^2-1)\cos(u\pi/2)}}\,du \quad \text{for} \quad x \geq 0.$$

The percentage points of $\mathcal{C}^2$ and $\mathcal{A}^2$ were calculated by Anderson and Darling (1952, 1954) using different formulas and these values are reproduced by Shorack and Wellner (1986).

It is important to extend the test $\mathcal{H}_0$ to the composite null hypothesis (see, e.g., Cramer 1946):

$$\mathcal{H}_0^* : F(x) = F_*(x,\vartheta), \quad \vartheta \in \Theta,$$

where the parameter ϑ ranges over a set Θ. For the case when Θ consists of an interval of the reals, $a \leq \vartheta \leq b$, a test of $\mathcal{H}_0^*$ analogous to W_n^2 (with $\psi \equiv 1$ in (1.29)) was introduced by Darling (1955):

$$C_n^2 = n\int_{\mathbb{R}}\left[\hat{F}_n(x) - F_*(x,\hat{\vartheta}_n)\right]^2 dF_*(x,\hat{\vartheta}_n),$$

where $\hat{\vartheta}_n = \hat{\vartheta}_n(X^n)$ is an estimate of the parameter ϑ. $\mathcal{H}_0^*$ is to be rejected if C_n^2 is sufficiently large. The main result obtained by Darling (1955) is that, under suitable regularity conditions, if we have the convergence:

$$\sqrt{n}\,(\hat{\vartheta}_n - \vartheta) \Longrightarrow \xi \sim \mathcal{N}(0, \sigma^2(\vartheta)),$$

then the limiting distributions of C_n^2 is the distribution of the functional

$$C^2 = \int_{\mathbb{R}} \left[B(F_*(x,\vartheta)) - \xi \frac{\partial}{\partial \vartheta} F_*(x,\vartheta) \right]^2 dF_*(x,\vartheta).$$

Note that the limit distribution C^2 depends on $F_*(x,\vartheta)$ and the test is no more distribution-free.

In the *superfficient case, i.e.*, if $\mathbf{Var}(\hat{\vartheta}_n)$ goes to zero sufficiently rapidly (see Darling 1955, Theorem 2.1), the limiting distributions of C_n^2 and W_n^2 are the same, and the test is distribution-free.

Goodness-of-fit test with simple basic hypothesis

2

Sommaire

In this chapter we consider the Cramér-von Mises goodness-of-fit tests for hypotheses that the observed diffusion process has sign-type trend coefficient based on empirical density function. It is shown that the limit distributions of the proposed tests statistics are defined by the integral type functionals of continuous Gaussian processes. We provide the Karhunen-Loéve expansion of the corresponding limiting processes. Approximations of the thresholds are given through the representation for the limit statistics. The behavior of these statistics under the alternative hypothesis is also studied and the consistency of the tests proven.

2.1 Introduction

Let $X^T = \{X_t, 0 \leq t \leq T\}$ be an observed trajectory of ergodic diffusion process

$$dX_t = S(X_t)\,dt + dW_t, \quad X_0, \quad 0 \leq t \leq T, \tag{2.1}$$

where X_0 is its initial value, independent of the Wiener process $\{W_t, t \geq 0\}$ and the trend coefficient $S(\cdot)$ is supposed to be unknown to the observer. Diffusion processes of this type are widely used as models in many different fields such as biology, physics, economics and finance. Nevertheless, few works are devoted to the goodness-of-fit testing for diffusions.

All statistical inference concerns the trend coefficient $S(\cdot)$ only. We have two hypotheses: the basic hypothesis in our consideration is always simple $\mathcal{H}_0 : S(\cdot) = S_0(\cdot)$ and the alternative corresponds to the process

(2.1) with a different trend coefficient $S(\cdot) \neq S_0(\cdot)$. As usual in goodness-of-fit testing there are two problems. The first one is to find the threshold which provides the asymptotic size α of the test and the second is to describe the behavior of the power function for some classes of alternatives.

The Cramér-von Mises statistics, based on the observation X^T solution of (2.1) can be:

$$V_T^2 = T \int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_0}(x)\right]^2 dx, \quad W_T^2 = T \int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_0}(x)\right]^2 dF_{S_0}(x),$$

where $f_T^\circ(x)$ is the estimate of the invariant density function $f_{S_0}(x)$ (see, Section 1.2). The tests $\varphi_T = 1_{\left\{V_T^2 > d_\alpha\right\}}$ and $\phi_T = 1_{\left\{W_T^2 > c_\alpha\right\}}$ were proposed by Dachian and Kutoyants (2007), but they are not distribution-free.

These statistics converge in distribution under the null hypothesis to quadratic functionals of Gaussian processes (see Kutoyants 2004). However, due to the structure of the covariance of these processes, the choice of the thresholds d_α, c_α for these tests are much more complicated (see Dachian and Kutoyants, 2007). In practice it is very difficult to find explicit expressions of these limits processes even for relatively simple diffusion process. To circumvent this difficulty, weighting functions of these statistics were introduced to make these tests asymptotically distribution-free (see Kutoyants 2009). Note that some other asymptotically distribution-free tests were proposed by Dachian and Kutoyants (2007), Negri (2008).

In the present work, we are interested in the case where the trend coefficient $S_0(\cdot)$ of the process (2.1) under hypothesis takes just two values $+1$ and -1 (simple switching $S_0(x) = -\operatorname{sgn}(x)$), *i.e.*, is a discontinuous function (see Kutoyants 2000, for the further details). We study the limit distribution of the corresponding Cramér-von Mises type statistics based on the empirical density estimator (local time estimator) of the invariant density. The behavior of these statistics under the alternative hypothesis is also studied and the consistency of the tests proven. The main tool is the K-L expansions of the limit Gaussian processes. Finally, using these expansion and numerical simulation we compute the thresholds of these tests.

2.2 Auxiliary results

Let us consider a one dimensional diffusion process which is a solution of the following stochastic differential equation

$$dX_t = S(X_t)\,dt + dW_t, \quad X_0, \quad 0 \leq t \leq T, \tag{2.2}$$

where $\{W_t, t \geq 0\}$ is a Wiener process, and the initial random value X_0 is independent of W_t. We suppose that the trend coefficient $S(\cdot)$ is unknown to the observer and belongs to the set $\mathcal{C}_0$ of all real functions satisfying the conditions of Theorem 1.3 and Theorem 1.4 with $\sigma(\cdot) = 1$. By these conditions the process $\{X_t, t \geq 0\}$ is ergodic, *i.e.*, there exists an invariant distribution $F_S(\cdot)$ with density defined for every $x \in \mathbb{R}$ by the equality

$$f_S(x) = \frac{1}{G(S)} \exp\left\{2\int_0^x S(v)\,dv\right\},$$

where $G(S)$ is the normalizing constant:

$$G(S) = \int_{\mathbb{R}} \exp\left\{2\int_0^y S(v)\,dv\right\}\,dy < \infty.$$

Moreover, for any measurable function $h(\cdot)$ with $\mathbf{E}_S|h(\xi)| < \infty$ (here ξ has distribution function $F_S(\cdot)$), the law of large numbers holds with probability 1

$$\frac{1}{T}\int_0^T h(X_t)\,dt \longrightarrow \mathbf{E}_S h(\xi).$$

The invariant density function $f_S(x)$ can be estimated by the local time type estimator $f_T^\circ(x)$ defined by the equality

$$f_T^\circ(x) = \frac{\Lambda_T(x)}{T} = \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon\,T}\int_0^T 1_{\{\,|X_t-x\,|<\varepsilon\,\}}\,dt \quad \text{for} \quad x\in\mathbb{R}.$$

Theorem 2.1 *Let $S(\cdot)\in\mathcal{C}_0$, then this estimator is unbiased*

$$\boldsymbol{E}_S f_T^\circ(x) = f_S(x),$$

and is $\sqrt{T}$ asymptotically normal (as $T\to\infty$)

$$\eta_T(x) = \sqrt{T}\left(f_T^\circ(x) - f_S(x)\right) \Longrightarrow \mathcal{N}\left(0, R_f(x,x)\right).$$

Moreover, the process $(\eta_T(x), x\in\mathbb{R})$ converges weakly to the zero mean Gaussian process $(\eta_f(x), x\in\mathbb{R})$ with the covariance function

$$\begin{aligned} R_f(x,y) &= \boldsymbol{E}_S\left[\eta_f(x)\eta_f(y)\right] \\ &= 4f_S(x)f_S(y)\,\boldsymbol{E}_S\left(\frac{[1_{\{\xi>x\}}-F_S(\xi)][1_{\{\xi>y\}}-F_S(\xi)]}{f_S(\xi)^2}\right), \end{aligned}$$

Of course,

$$R_f(x,x) = \boldsymbol{E}_S\left[\eta_f(x)^2\right] = 4f_S(x)^2\boldsymbol{E}_S\left(\frac{1_{\{\xi>x\}}-F_S(\xi)}{f_S(\xi)}\right)^2.$$

Proof. For the proof of this theorem, see Kutoyants (2004), Proposition 1.57 and Theorem 4.13. □

2.3 Cramér-von Mises tests

Let us introduce the simple switching diffusion process

$$dX_t = -\mathrm{sgn}(X_t)\,dt + dW_t, \quad X_0, \quad 0\le t\le T, \tag{2.3}$$

where the trend coefficient $S_0(x) = -\mathrm{sgn}(x)$ is a discontinuous function and taking just two values +1 and -1.

It is easy to see that $S_0(\cdot)\in\mathcal{C}_0$ and (2.3) is an ergodic diffusion process. Its stationary density $f_{S_0}(x)$ and invariant distribution function $F_{S_0}(x)$ are

$$f_{S_0}(x) = e^{-2|x|}, \quad F_{S_0}(x) = 1_{\{x>0\}} - \frac{1}{2}\,\mathrm{sgn}(x)\,e^{-2|x|}, \quad x\in\mathbb{R}.$$

Here, we suppose that the initial random variable X_0 has $f_{S_0}(\cdot)$ as a density function, so the process (2.3) is stationary.

The goodness-of-fit testing problem is introduced as follows:

Suppose that we observe the process $\{X_t, 0 \le t \le T\}$, solution of the stochastic differential equation (2.2) and we wish to test the following two hypotheses

$$\mathcal{H}_0 : \qquad S(\cdot) = S_0(\cdot), \qquad \text{against alternative} \qquad \mathcal{H}_1 : \qquad S(\cdot) \neq S_0(\cdot).$$

In order to test these hypotheses, the local time estimator (LTE) is used for construction of goodness-of-fit tests based on the Cramér-von Mises statistics:

$$V_T^2 = T \int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_0}(x)\right]^2 dx, \quad W_T^2 = T \int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_0}(x)\right]^2 dF_{S_0}(x).$$

The test procedure starts by fixing the number $\alpha \in (0,1)$. Let us consider the class of asymptotic test of level $1-\alpha$ or size α. Given any statistical decision function $\psi_T = \psi_T(X^T)$, the value of $\psi_T(X^T)$ is the probability to reject hypothesis $\mathcal{H}_0$ having the observation $X^T = \{X_t, 0 \le t \le T\}$. Let $\mathbf{E}_S$ denote the mathematical expectation with respect to the measures $\mathbf{P}_S$ induced by the process $X^T = \{X_t, 0 \le t \le T\}$ in the space $C[0,T]$ (the space of all the continuous functions on $[0,T]$).

Furthermore, define the class of tests of asymptotic level $1-\alpha$ as

$$\mathcal{K}_\alpha = \{\psi_T : \limsup_{T\to\infty} \mathbf{E}_{S_0} \psi_T(X^T) \le \alpha\}.$$

The power of the test based on ψ_T is the probability of the true decision under $\mathcal{H}_1$, and is given by

$$\beta_T\left(S, \psi_T\right) = \mathbf{E}_S \psi_T(X^T).$$

A test procedure is consistent if

$$\lim_{T\to\infty} \mathbf{E}_S \psi_T(X^T) = 1.$$

Let $(\eta_f(x),\, x \in \mathbb{R})$ denote a Gaussian process with zero mean and covariance function

$$R_f(x,y) = \mathbf{E}_{S_0}\left[\eta_f(x)\eta_f(y)\right],$$

and consider the following statistical decision functions

$$\varphi_T(X^T) = 1_{\{V_T^2 > d_\alpha\}}, \quad \phi_T(X^T) = 1_{\{W_T^2 > c_\alpha\}},$$

where the critical values d_α and c_α are defined by the equations

$$\mathbf{P}\left\{\int_{\mathbb{R}} \eta_f^2(x)\, dx > d_\alpha\right\} = \alpha, \quad \mathbf{P}\left\{\int_{\mathbb{R}} \eta_f^2(x)\, dF_{S_0}(x) > c_\alpha\right\} = \alpha. \tag{2.4}$$

Let $\mathbf{P}$ be the probability distribution corresponding to the null hypothesis $\mathcal{H}_0$.

The objective is to find a test ψ_T which rejects $\mathcal{H}_0$ when it is true with a probability asymptotically equal to α and is consistent. Introduce the conditions (under alternative)

$$\int_{\mathbb{R}} \mathbf{E}_S \eta_f^2(x)\, dx < \infty, \quad \int_{\mathbb{R}} \mathbf{E}_S \eta_f^2(x)\, dF_{S_0}(x) < \infty. \tag{2.5}$$

Proposition 2.1 *Let conditions* (2.5) *be fulfilled. Then the C-vM type tests* $\varphi_T(X^T)$ *and* $\phi_T(X^T)$, *belong to* $\mathcal{K}_\alpha$ *and are consistent.*

Proof. The proof of $\varphi_T \in \mathcal{K}_\alpha$ and $\phi_T \in \mathcal{K}_\alpha$ is based on the general result by Gikhman and Skorohod (1969), Theorem 9.7.1. Hence, we check the conditions of this theorem and show that the distribution of V_T^2 and W_T^2 converge (under $\mathcal{H}_0$) to the distribution of $\int_{\mathbb{R}} \eta_f^2(x)\,dx$ and $\int_{\mathbb{R}} \eta_f^2(x)\,dF_{S_0}(x)$ respectively. The proof is detailed in the Theorem 2.2 below.

To prove the consistency it is enough to verify that, under $\mathcal{H}_1$,

$$\mathbf{P}\left(\lim_{T\to\infty} V_T^2 = \infty\right) = 1, \quad \mathbf{P}\left(\lim_{T\to\infty} W_T^2 = \infty\right) = 1.$$

Let $L^2(\mathbb{R},\nu)$ be the basic Hilbert space of our framework. We employ the usual notation for its norm

$$\|u(x)\|_2 = \|f_T^\circ(x) - f_{S_0}(x)\|_2 = \left(\int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_0}(x)\right]^2 \nu(dx)\right)^{\frac{1}{2}},$$

where $\nu(dx)$ is either dx or $dF_{S_0}(x)$. Thus,

$$\sqrt{T}\,\|u(x)\|_2 \geq \sqrt{T}\,\|f_S(x) - f_{S_0}(x)\|_2 - \sqrt{T}\,\|f_T^\circ(x) - f_S(x)\|_2 \longrightarrow \infty,$$

as from Theorem 2.1, the normed difference $\sqrt{T}\,(f_T^\circ(x) - f_S(x))$ converges weakly to the corresponding limit Gaussian process. On the other hand, $S \neq S_0$ implies that $\|f_S(x) - f_{S_0}(x)\|_2 > 0$. Therefore,

$$\mathbf{P}\left(V_T^2 > d_\alpha\right) = \mathbf{P}\left(V_T > \sqrt{d_\alpha}\right) \longrightarrow 1,$$
$$\mathbf{P}\left(W_T^2 > c_\alpha\right) = \mathbf{P}\left(W_T > \sqrt{c_\alpha}\right) \longrightarrow 1.$$

□

The Theorem 2.2 is used to study the limit distribution of these statistics under hypothesis for the switching diffusion process model (2.3).

Theorem 2.2 *Under hypothesis* $\mathcal{H}_0$, *we have, the convergence in distribution*

$$T\int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_0}(x)\right]^2 \nu(dx) \Longrightarrow \int_{\mathbb{R}} \eta_f^2(x)\,\nu(dx).$$

Proof. First, we prove that

$$\sup_T T\,\mathbf{E}_{S_0}\int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_0}(x)\right]^2 \nu(dx) < \infty.$$

In fact, the following inequalities

$$\int_{\mathbb{R}} \mathbf{E}_{S_0}\, g(\xi,x)^2 \nu(dx) < \infty, \quad \int_{\mathbb{R}} \mathbf{E}_{S_0}\, G(\xi,x)^2 \nu(dx) < \infty, \tag{2.6}$$

are verified via the use of Lemmas A.3 and A.4. By the Itô formula, the process $\{\eta_T(x), x\in\mathbb{R}\}$ admits the representation (1.14):

$$\eta_T(x) = \sqrt{T}\,(f_T^\circ(x) - f_{S_0}(x)) = \frac{G(X_T,x) - G(X_0,x)}{\sqrt{T}} - \int_0^T \frac{g(X_t,x)}{\sqrt{T}}\,dW_t,$$

where

$$G(y,x) = \int_0^y g(v,x)\,dv = 2\,f_{S_0}(x)\,\int_0^y \left(\frac{1_{\{v>x\}} - F_{S_0}(v)}{f_{S_0}(v)}\right)\,dv.$$

Therefore

$$\begin{aligned}
\int_{\mathbb{R}} \eta_T^2(x)\,\nu(dx) = &\frac{1}{T}\int_{\mathbb{R}} \left(G(X_T,x) - G(X_0,x)\right)^2\,\nu(dx)\\
&- \frac{2}{T}\int_{\mathbb{R}} \left(G(X_T,x) - G(X_0,x)\right)\int_0^T g(X_t,x)\,dW_t\,\nu(dx)\\
&+ \frac{1}{T}\int_{\mathbb{R}} \left(\int_0^T g(X_t,x)\,dW_t\right)^2\,\nu(dx).
\end{aligned}$$

Note that by the stationarity, the following inequality holds

$$\begin{aligned}
&\mathbf{E}_{S_0}\int_{\mathbb{R}} \left(G(X_T,x) - G(X_0,x)\right)^2\,\nu(dx)\\
&\qquad \le 2\,\mathbf{E}_{S_0}\int_{\mathbb{R}} \left(G(X_T,x)^2 + G(X_0,x)^2\right)\,\nu(dx)\\
&\qquad = 4\,\mathbf{E}_{S_0}\int_{\mathbb{R}} G(\xi,x)^2\,\nu(dx).
\end{aligned}$$

Applying the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
&\mathbf{E}_{S_0}\int_{\mathbb{R}} \left(\left(G(X_T,x) - G(X_0,x)\right)\int_0^T g(X_t,x)\,dW_t\right)\nu(dx)\\
&\le 2\sqrt{T}\,\left(\int_{\mathbb{R}} \mathbf{E}_{S_0}\,G(\xi,x)^2\,\nu(dx)\right)^{1/2} \times \left(\int_{\mathbb{R}} \mathbf{E}_{S_0}\,g(\xi,x)^2\nu(dx)\right)^{1/2}.
\end{aligned}$$

Due to inequalities (2.6) we have

$$\begin{aligned}
&\lim_{T\to\infty}\sup_T \frac{1}{T}\,\mathbf{E}_{S_0}\int_{\mathbb{R}} \left(G(X_T,x) - G(X_0,x)\right)^2\,\nu(dx) = 0,\\
&\lim_{T\to\infty}\sup_T \frac{1}{T}\,\mathbf{E}_{S_0}\int_{\mathbb{R}} \left(G(X_T,x) - G(X_0,x)\right)\int_0^T g(X_t,x)\,dW_t\nu(dx) = 0,
\end{aligned}$$

which achieves the proof of the first step.

The second step is to prove that

$$\mathbf{E}_{S_0}\left|\eta_T(x)^2 - \eta_T(y)^2\right| \le 2\sqrt{22}\,|x-y|^{1/2}.$$

Indeed, starting from the following inequality

$$\begin{aligned}
&\mathbf{E}_{S_0}\left|\eta_T(x)^2 - \eta_T(y)^2\right|\\
&\qquad \le \left(\mathbf{E}_{S_0}\left|\eta_T(x) - \eta_T(y)\right|^2\right)^{1/2} \times \left(\mathbf{E}_{S_0}\left|\eta_T(x) + \eta_T(y)\right|^2\right)^{1/2}\\
&\qquad \le \sqrt{2}\left(\mathbf{E}_{S_0}\left|\eta_T(x) - \eta_T(y)\right|^2\right)^{1/2} \times \left(\mathbf{E}_{S_0}\eta_T(x)^2 + \mathbf{E}_{S_0}\eta_T(y)^2\right)^{1/2}\\
&\qquad \le 2\,\left(\mathbf{E}_{S_0}\left|\eta_T(x) - \eta_T(y)\right|^2\right)^{1/2} \times \left(\sup_{x\in\mathbb{R}} R_f(x,x)\right)^{1/2},
\end{aligned}$$

we first consider the case where $y < x$ and let

$$\psi(x,z) = f_{S_0}(x)\left(1_{\{z>x\}} - F_{S_0}(z)\right) f_{S_0}(z)^{-1}.$$

Then,

$$\begin{aligned}
\mathbf{E}_{S_0} |\eta_T(x) - \eta_T(y)|^2 &= \mathbf{E}_{S_0}\left[\frac{2}{\sqrt{T}}\int_0^T (\psi(x,X_t) - \psi(y,X_t))\, dW_t\right]^2 \\
&= 4\,\mathbf{E}_{S_0}\left(\psi(x,\xi) - \psi(y,\xi)\right)^2 = 4(I^y + I_y^x + I_x),
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
I^y &= (f_{S_0}(x) - f_{S_0}(y))^2 \int_{-\infty}^{y} \left(\frac{F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv, \\
I_y^x &\le 2\, f_{S_0}(x)^2 \int_y^x \left(\frac{F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv \\
&\quad + 2\, f_{S_0}(y)^2 \int_y^x \left(\frac{1 - F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv, \\
I_x &= (f_{S_0}(x) - f_{S_0}(y))^2 \int_x^{\infty} \left(\frac{1 - F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv.
\end{aligned}$$

In the case $x < 0$, we have

$$\begin{aligned}
(f_{S_0}(x) - f_{S_0}(y))^2 &\le f_{S_0}(x)\,(f_{S_0}(x) - f_{S_0}(y)) \le f_{S_0}(x) - f_{S_0}(y) \\
&= 2\int_y^x f_{S_0}(u)\, du \le 2(x-y).
\end{aligned}$$

Hence,

$$\mathbf{E}_{S_0}\left(\psi(\xi,x) - \psi(\xi,y)\right)^2 \le \frac{17}{4}(x-y).$$

In fact

$$\begin{aligned}
I^y &\le 2(x-y) \int_{-\infty}^{y} \left(\frac{F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv = \frac{1}{4}\,(x-y)\, e^{2y} \le \frac{1}{4}\,(x-y), \\
I_x &\le 2(x-y)\, f_{S_0}(x) \int_x^{\infty} \left(\frac{1 - F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv \\
&= \frac{1}{4}\,(x-y)\,\left(4 + 8\,x\,e^{2x} - e^{4x} - 2\,e^{2x}\right) \le (x-y), \\
I_y^x &\le \frac{f_{S_0}(x)^2}{2} \int_y^x f_{S_0}(v)\, dv + 2\, f_{S_0}(y)^2 \left(e^{-2y} - \frac{1}{2}\right)^2 \int_y^x f_{S_0}(v)\, dv \\
&\le \frac{1}{2}\,(x-y) + \frac{5}{2}(x-y) = 3\,(x-y).
\end{aligned}$$

In the case $y > 0$, by symmetry and similarly to the case $x < 0$, we obtain the same result.

In the case $x > 0$ and $y < 0$, we have

$$(f_{S_0}(x) - f_{S_0}(y))^2 \le |f_{S_0}(y) - f_{S_0}(x)| \le 2\int_y^x f_{S_0}(u)\, du \le 2(x-y).$$

Hence

$$\mathbf{E}_{S_0}\left(\psi(\xi,x) - \psi(\xi,y)\right)^2 \le \frac{11}{2}(x-y),$$

resulting from the following analyze:

$$
\begin{aligned}
I^y &\leq 2(x-y)\int_{-\infty}^{y}\left(\frac{F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\,dv = \frac{1}{4}\,(x-y)\,e^{2y} \leq \frac{1}{4}\,(x-y),\\
I_x &\leq 2(x-y)\int_{x}^{\infty}\left(\frac{1-F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\,dv = \frac{1}{4}\,(x-y)\,e^{-2x} \leq \frac{1}{4}\,(x-y),\\
I_y^x &\leq \frac{f_{S_0}(x)^2}{2}\int_y^0 f_{S_0}(v)\,dv + 2\,f_{S_0}(x)^2\left(e^{2x}-\frac{1}{2}\right)^2\int_0^x f_{S_0}(v)\,dv\\
&\quad + \frac{f_{S_0}(y)^2}{2}\int_0^x f_{S_0}(v)\,dv + 2\,f_{S_0}(y)^2\left(e^{-2y}-\frac{1}{2}\right)^2\int_y^0 f_{S_0}(v)\,dv\\
&\leq \frac{5}{2}\,(x-y) + \frac{5}{2}(x-y) = 5\,(x-y).
\end{aligned}
$$

Returning to (2.7), we obtain

$$\mathbf{E}_{S_0}\,|\eta_T(x)-\eta_T(y)|^2 \leq 22\,(x-y).$$

So we have the result for all $x, y \in \mathbb{R}$.

Finally the convergence of marginal distributions of $\eta_T(\cdot)$ was proved by Kutoyants (2004).

It is clear that for any constant $L \geq 0$

$$\int_{\mathbb{R}} \eta_T^2(x)\,\nu(dx) = \int_{|x|\leq L} \eta_T^2(x)\,d\nu(x) + \int_{|x|>L} \eta_T^2(x)\,\nu(dx).$$

According to Theorem 9.7.1 in Gikhman and Skorohod (1969), by these relations, the distribution of the integral $\int_{|x|\leq L}\eta_T^2(x)\,\nu(dx)$ convergence to the distribution of the integral $\int_{|x|\leq L}\eta^2(x)\,\nu(dx)$.

Now we prove that $\int_{|x|>L}\eta_T^2(x)\,\nu(dx)$ tends to zero at infinity i.e. we prove that for any $\varepsilon > 0$

$$\lim_{L\to\infty}\sup_T \mathbf{P}\left(\int_{|x|>L}\eta_T(x)^2\,\nu(dx) > \varepsilon\right) = 0.$$

From Lemma A.3 we have

$$\mathbf{E}_{S_0}\,|\eta_T(x)|^2 \leq 2\,e^{-2\,|x|}.$$

Therefore, by Tchebychev's inequality we have

$$
\begin{aligned}
\sup_T \mathbf{P}\left(\int_{|x|>L}\eta_T(x)^2\,\nu(dx) > \varepsilon\right) &\leq \frac{1}{\varepsilon}\,\sup_T \int_{|x|>L}\mathbf{E}_{S_0}\eta_T(x)^2\,\nu(dx)\\
&\leq \begin{cases} e^{-2L}/\varepsilon, & \text{when} \quad \nu(dx) = dx,\\ e^{-4L}/2\varepsilon, & \text{when} \quad \nu(dx) = dF_{S_0}(x).\end{cases}
\end{aligned}
$$

Now, for $L \geq -\ln(2\,\varepsilon^2)$ and $L \geq -\ln(4\,\varepsilon^2)/2$ respectively we have the conclusion and the ends of the proof. □

Note that the transformation $Y_s = F_{S_0}(X_s)$ simplifies the exposition, because by the Itô formula the diffusion process Y_s satisfies the differential equation

$$dY_s = f_{S_0}(X_s)\left[2S_0(X_s)ds + dW_s\right], \quad Y_0 = F_{S_0}(X_0),$$

with reflecting bounds in 0 and 1 and (under hypothesis $\mathcal{H}_0$) has uniform on $[0,1]$ invariant distribution. Therefore,

$$W_T^2 = T\int_{\mathbb{R}} \left[f_T^{\circ}(x) - f_{S_0}(x)\right]^2 dF_{S_0}(x) = \int_0^1 \tilde{\eta}_T^2(t)\,dt \Longrightarrow \int_0^1 \tilde{\eta}_f^2(t)\,dt.$$

2.4 K-L expansion of Gaussian processes $\eta_f(\cdot)$ and $\tilde{\eta}_f(\cdot)$

In this section, we establish the Karhunen-Loève expansion of the corresponding limiting Gaussian processes $(\eta_f(x), x \in \mathbb{R})$ and $(\tilde{\eta}_f(t), t \in [0,1])$ respectively. The representation for the limit statistics defined by the equation (2.4) allow us to find the thresholds c_α and d_α.

Let us introduce the Gaussian process $(\eta_f(x), x \in \mathbb{R})$, with zero mean and continuous covariance function (see Lemma A.3)

$$\begin{aligned} R_f(x,y) = 2\left(1_{\{x\vee y<0\}}e^{2(x\wedge y)} + 1_{\{x\wedge y\geq 0\}}e^{-2(x\vee y)}\right) \\ - \left(2(|x|+|y|) + \operatorname{sgn}(xy)\right)e^{-2(|x|+|y|)}. \end{aligned} \tag{2.8}$$

Theorem 2.3 *The Gaussian process $(\eta_f(x), x \in \mathbb{R})$ has a K-L expansion given by*

$$\begin{aligned} \eta_f(x) = &-\sum_{n=1}^{\infty}\frac{2}{z_{1,n}}\,\xi_{1,n}\operatorname{sgn}(x)\,e^{-|x|}\,\frac{J_1\left(z_{1,n}\,e^{-|x|}\right)}{J_0\left(z_{1,n}\right)} \\ &+\sum_{n=1}^{\infty}\frac{2}{z_{*,n}}\,\xi_{2,n}\frac{\alpha\left(z_{*,n}\right)\,e^{-|x|}\,J_1\left(z_{*,n}\,e^{-|x|}\right)}{J_1\left(z_{*,n}\right)\,\alpha\left(z_{*,n}\right) - Y_1\left(z_{*,n}\right)\,J_2\left(z_{*,n}\right)} \\ &-\sum_{n=1}^{\infty}\frac{2}{z_{*,n}}\,\xi_{2,n}\,\frac{J_2\left(z_{*,n}\right)\left(e^{-|x|}\,Y_1\left(z_{*,n}\,e^{-|x|}\right) + \frac{2}{\pi z_{*,n}}\right)}{J_1\left(z_{*,n}\right)\,\alpha\left(z_{*,n}\right) - Y_1\left(z_{*,n}\right)\,J_2\left(z_{*,n}\right)}, \end{aligned} \tag{2.9}$$

where $\{\xi_{1,n}, n\geq 1\}$, $\{\xi_{2,n}, n\geq 1\}$ denote two independent sequences of independent and identically distributed $\mathcal{N}(0,1)$ random variables and $z_{1,n}, z_{*,n}, n = 1,2,\ldots,$ are respectively the positive zeros of $J_1(\cdot)$ and $f(\cdot)$, (solutions of $J_1(z_{1,n}) = 0$ and $f(z_{*,n}) = 0$), with,

$$f(z) = J_2\left(z\right)\beta\left(z\right) - \left(J_0\left(z\right) - 1\right)\alpha\left(z\right),$$

where

$$\beta\left(z\right) = Y_0\left(z\right) - \frac{2}{\pi}\left(\ln(z/2) + \gamma\right), \quad \alpha\left(z\right) = Y_2\left(z\right) + \frac{4}{\pi z^2},$$

and $\gamma = 0.577215$...is the Euler constant.

Lemma 2.1 *The Gaussian process $(\tilde{\eta}_f(t),\, 0\leq t\leq 1)$, admits the representation*

$$\tilde{\eta}_f(t) = 2\left(1 - |2t-1|\right)\int_0^1 \frac{1_{\{u\geq t\}} - u}{1 - |2u-1|}\,dW_u \quad \textit{for} \quad 0\leq t\leq 1,$$

and has the covariance function given by

$$\begin{aligned}
K_f(s,t) &= \mathbf{E}\tilde{\eta}_f(t)\tilde{\eta}_f(s) \\
&= (1-|2s-1|)(1-|2t-1|)\ln\left[(1-|2s-1|)(1-|2t-1|)\right] \\
&\quad + 4\, s\wedge t\,(1-s\vee t) \quad \textit{for} \quad 0\le s,t\le 1.
\end{aligned}$$

Of course, for $0 \le t \le 1$*,*

$$K_f(t,t) = \mathbf{E}\tilde{\eta}_f^2(t) = 2(1-|2t-1|)^2\ln(1-|2t-1|) + 4t(1-t) \qquad (2.10)$$

Proof. Note that the following representation is valid (see section 1.2)

$$\begin{aligned}
\eta_T(x) &= \frac{2\, f_{S_0}(x)}{\sqrt{T}}\int_{X_0}^{X_T}\frac{1_{\{v\ge x\}}-F_{S_0}(v)}{f_{S_0}(v)}\,dv \\
&\quad - \frac{2\, f_{S_0}(x)}{\sqrt{T}}\int_0^T\frac{1_{\{X_t\ge x\}}-F_{S_0}(X_t)}{f_{S_0}(X_t)}\,dW_t \quad \text{for} \quad x\in\mathbb{R}.
\end{aligned}$$

Using the local time $\Lambda_T(x)$, the following equality holds, for any integrable function $h(\cdot)$

$$\frac{1}{T}\int_0^T h(X_t)\,dt = \int_{\mathbb{R}} h(x)\frac{\Lambda_T(x)}{T}\,dx = \int_{\mathbb{R}} h(x) f_T^\circ(x)\,dx.$$

According to Theorem 2.1, under hypothesis $\mathcal{H}_0$ we have (equalities in distribution)

$$\begin{aligned}
\eta_T(x) &= 2f_{S_0}(x)W\left(\frac{1}{T}\int_0^T\left(\frac{1_{\{X_t\ge x\}}-F_{S_0}(X_t)}{f_{S_0}(X_t)}\right)^2 dt\right) + o(1) \\
&= 2f_{S_0}(x)W\left(\int_{\mathbb{R}}\left(\frac{1_{\{y\ge x\}}-F_{S_0}(y)}{f_{S_0}(y)}\right)^2 f_T^\circ(y)\,dy\right) + o(1) \\
&\Longrightarrow 2f_{S_0}(x)W\left(\int_{\mathbb{R}}\left(\frac{1_{\{y\ge x\}}-F_{S_0}(y)}{f_{S_0}(y)}\right)^2 dF_{S_0}(y)\right),
\end{aligned}$$

where $W(u)$, $u\ge 0$ is some Wiener process. Making the transformation $t = F_{S_0}(x)$ we obtain

$$\begin{aligned}
\tilde{\eta}_T(t) &\Longrightarrow 2\,(1-|2t-1|)\,W\left(\int_0^1\left(\frac{1_{\{u\ge t\}}-u}{1-|2u-1|}\right)^2 du\right) \\
&= 2\,(1-|2t-1|)\int_0^1\frac{1_{\{u\ge t\}}-u}{1-|2u-1|}\,dW_u \\
&= \tilde{\eta}_f(t).
\end{aligned}$$

The covariance function of $\tilde{\eta}_f(\cdot)$, for $0\le s,t\le 1$, can be written as follows:

$$\begin{aligned}
K_f(s,t) &= \mathbf{E}\tilde{\eta}_f(t)\tilde{\eta}_f(s) \\
&= 4(1-|2t-1|)(1-|2s-1|)\int_0^1\frac{(1_{\{u\ge t\}}-u)(1_{\{u\ge s\}}-u)}{(1-|2u-1|)^2}\,du.
\end{aligned}$$

Therefore, by a direct calculation we obtain the above expression for $K_f(s,t)$. □

Theorem 2.4 *The Gaussian process $\tilde{\eta}_f(t)$, for $0 \le t \le 1$ has a K-L expansion given by*

$$\begin{aligned}\tilde{\eta}_f(t) = &\sum_{n=1}^{\infty} \frac{\xi_{1,n}}{n\,\pi} \sqrt{2}\,\mathrm{sgn}(1/2 - t) \sin\left(n\,\pi(1 - |2\,t - 1|)\right) \\ &+ \sum_{n=1}^{\infty} \frac{\xi_{2,n}}{\nu_n} \left\{ \frac{\sqrt{2}}{\mathrm{Si}(\nu_n)} \left[\left(\frac{\alpha(\nu_n)}{\nu_n} - \dot{\alpha}(\nu_n) \right) \sin\left(\nu_n(1 - |2\,t - 1|)\right) \right.\right. \\ &\left.\left. - \left(\frac{\sin(\nu_n)}{\nu_n} - \cos(\nu_n) \right) \alpha\left(\nu_n(1 - |2\,t - 1|)\right) \right] \right\},\end{aligned} \tag{2.11}$$

where $\{\xi_{1,n}, n \ge 1\}$, $\{\xi_{2,n}, n \ge 1\}$ denote two independent sequences of independent and identically distributed $\mathcal{N}(0,1)$ random variables and $\nu_n, n = 1, 2, \dots$, are the positive zeros of $f(\cdot)$, defined by the equation

$$f\,(t) = G(t)\,[\sin(t) - t\cos(t)] - Si(t)\,[\alpha(t) - t\dot{\alpha}(t)]\,,$$

where

$$\alpha(t) = Ci(t)\sin(t) - Si(t)\cos(t), \quad \dot{\alpha}(t) = \frac{d}{dt}\alpha(t), \quad G(t) = \int_0^t \frac{\alpha(s)}{s}\,ds,$$

with the cosine and sine integral

$$Ci(t) = \gamma + \ln(t) + \int_0^t \frac{\cos(s) - 1}{s}\,ds, \quad Si(t) = \int_0^t \frac{\sin(s)}{s}\,ds,$$

respectively and $\gamma = 0.577215...$ is the Euler constant.

Remark 2.1 *As a direct consequence of equations (2.9) and (2.11), the distributions of $\int_{\mathbb{R}} \eta_f^2(x)\,dx$ and $\int_0^1 \tilde{\eta}_f^2(t)\,dt$, coincide with the distribution of the quadratic forms*

$$V^2 = \int_{\mathbb{R}} \eta_f^2(x)\,dx \overset{law}{=} \sum_{n=1}^{\infty} \frac{4}{z_{1,n}^2}\,\xi_{1,n}^2 + \sum_{n=1}^{\infty} \frac{4}{z_{*,n}^2}\,\xi_{2,n}^2, \tag{2.12}$$

$$W^2 = \int_0^1 \tilde{\eta}_f^2(t)\,dt \overset{law}{=} \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}\,\xi_{1,n}^2 + \sum_{n=1}^{\infty} \frac{1}{\nu_n^2}\,\xi_{2,n}^2. \tag{2.13}$$

These representations allow as to evaluate the distribution of $\int_{\mathbb{R}} \eta_f^2(x)\,dx$ and $\int_0^1 \tilde{\eta}_f^2(t)\,dt$ by a number of methods (see, e.g., Smirnov 1937; Martynov 1975; Deheuvels and Martynov 2003 and the references therein).

– Due to the Fubini theorem we have:

$$1 = \int_{\mathbb{R}} R_f(x,x)\,dx = \sum_{n=1}^{\infty} \frac{4}{z_{1,n}^2} + \sum_{n=1}^{\infty} \frac{4}{z_n'^2} \overset{(a)}{=} \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{z_n'^2},$$

$$\frac{4}{9} = \int_0^1 K_f(t,t)\,dt = \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} + \sum_{n=1}^{\infty} \frac{1}{\nu_n^2} \overset{(b)}{=} \frac{1}{6} + \sum_{n=1}^{\infty} \frac{1}{\nu_n^2},$$

where (a) is Rayleigh's formula :

$$\sum_{n=1}^{\infty} \frac{1}{z_{\nu,n}^2} = \frac{1}{4(\nu+1)},$$

and (b) is Euler's formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

– *Now, the numerical simulation (for $N = 10^5$) gives*

$$|\sum_{n=1}^{N}\frac{4}{z_n'^2}-\frac{1}{2}|\leq 10^{-3},\quad |\sum_{n=1}^{N}\frac{1}{\nu_n^2}-\frac{5}{18}|\leq 1.2\times 10^{-3}.$$

Corollary 2.1 *Under hypothesis $\mathcal{H}_0$, we have, the convergence in distribution*

$$V_T^2\Rightarrow\sum_{n=1}^{\infty}\frac{4}{z_{1,n}^2}\,\zeta_{1,n}^2+\sum_{n=1}^{\infty}\frac{4}{z_{*,n}^2}\,\zeta_{2,n}^2,\quad W_T^2\Rightarrow\sum_{n=1}^{\infty}\frac{\zeta_{1,n}^2}{n^2\pi^2}+\sum_{n=1}^{\infty}\frac{\zeta_{2,n}^2}{\nu_n^2}. \tag{2.14}$$

Proof. It follows from Theorem 2.2 that

$$V_T^2\Longrightarrow\int_{\mathbb{R}}\eta_f^2(x)\,dx,\quad W_T^2\Longrightarrow\int_0^1\tilde{\eta}_f^2(t)\,dt,$$

which, in turn, reduce (2.14) to a direct consequence of Theorem 2.2. □

The random variables $\int_{\mathbb{R}}\eta_f^2(x)\,dx$ and $\int_0^1\tilde{\eta}_f^2(t)\,dt$ are weighted sums of independent χ_1^2 components. We make the truncation of these sum keeping 10^5 terms and using the numerical simulation of the corresponding Gaussian random variables and we obtain the following values for the thresholds d_α and c_α of C-vM type tests

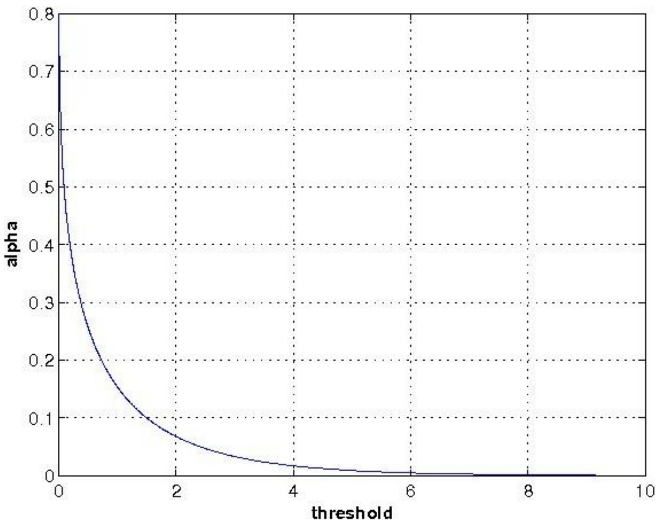

Figure 2.1 – *Thresholds choice of the random variable $\int_{\mathbb{R}}\eta_f^2(x)\,dx$.*

Size of the test α	0.10	0.05	0.025	0.010	0.005
Critical value d_α	1.619	2.563	3.596	5.868	6.197

Table 2.1 – *Values of some quantiles of the random variable $\int_{\mathbb{R}}\eta_f^2(x)\,dx$.*

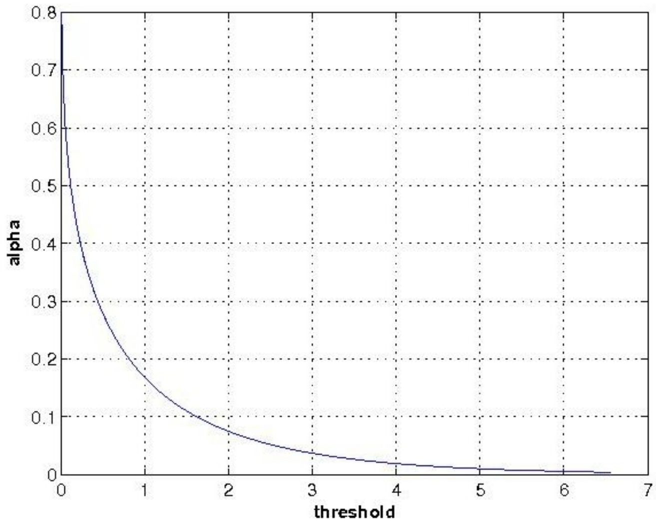

Figure 2.2 – *Thresholds choice of the random variable* $\int_0^1 \tilde{\eta}_f^2(t)\,dt$.

Size of the test α	0.10	0.05	0.025	0.010	0.005
Critical value c_α	1.501	2.420	3.433	5.050	6.004

Table 2.2 – *Values of some quantiles of the random variable* $\int_0^1 \tilde{\eta}_f^2(t)\,dt$.

The approach that can be used here to calculate the quantiles for the distribution of the random variables $\int_{\mathbb{R}} \eta_f^2(x)\,dx$ and $\int_0^1 \tilde{\eta}_f^2(t)\,dt$ in Tables 2.1 and 2.2 is based on the Smirnov formula (1.28)

$$F(x) = 1 + \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \int_{\delta_{2n-1}}^{\delta_{2n}} \frac{e^{-xu/2}}{u\sqrt{|D(u)|}}\,du \quad \text{for} \quad x \geq 0, \tag{2.15}$$

where

$$D(u) = \prod_{n=1}^{\infty} \left(1 - \frac{u}{\delta_n}\right) \quad \text{for} \quad u \geq 0.$$

Let the random variables $\int_{\mathbb{R}} \eta_f^2(x)\,dx$ and $\int_0^1 \tilde{\eta}_f^2(t)\,dt$ be defined respectively, as in (2.12) and (2.13), by

$$\int_{\mathbb{R}} \eta_f^2(x)\,dx = \sum_{n=1}^{\infty} \frac{\xi_n^2}{\delta_n} = \sum_{n=1}^{\infty} \frac{4}{z_{1,n}^2}\,\xi_{1,n}^2 + \sum_{n=1}^{\infty} \frac{4}{z_{*,n}^2}\,\xi_{2,n}^2, \tag{2.16}$$

$$\int_0^1 \tilde{\eta}_f^2(t)\,dt = \sum_{n=1}^{\infty} \frac{\xi_n^2}{\delta_n} = \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}\,\xi_{1,n}^2 + \sum_{n=1}^{\infty} \frac{1}{\nu_n^2}\,\xi_{2,n}^2. \tag{2.17}$$

We deduce from (2.16) that the Fredholm determinant $D(u)$ for $\eta_f(x)$ is such that, with $1/\delta_n = \left\{4/z_{1,n}^2, 4/z_{*,n}^2\right\}$, for $n \geq 1$,

$$D(u) = \prod_{n=1}^{\infty} \left(1 - \frac{(2\sqrt{u})^2}{z_{1,n}^2}\right) \prod_{n=1}^{\infty} \left(1 - \frac{4\,u}{z_{*,n}^2}\right).$$

We use the Euler's formula (A.17), with $\nu = 1$, we obtain

$$D(u) = \frac{J_1(2\sqrt{u})}{\sqrt{u}} \prod_{n=1}^{\infty} \left(1 - \frac{4\,u}{z_{*,n}^2}\right).$$

It is well known that

$$\prod_{n=1}^{\infty} \left(1 - \frac{u}{\pi^2 n^2}\right) = \frac{\sin(\sqrt{u})}{\sqrt{u}}.$$

Therefore, we infer from (2.17) that the Fredholm determinant $D(u)$ for $\tilde{\eta}_f(t)$ is such that, with $1/\delta_n = \left\{1/\pi^2 n^2, 1/\nu_n^2\right\}$, for $n \geq 1$,

$$D(u) = \frac{\sin(\sqrt{u})}{\sqrt{u}} \prod_{n=1}^{\infty} \left(1 - \frac{u}{\nu_n^2}\right).$$

For the numerical computation of the nth integral in (2.15), it is convenient to make a change of variable by writing (see Grad and Solomon 1955)

$$(-1)^n \int_{-1}^{1} \frac{\left[(p_n(z) - \delta_{2n-1})(\delta_{2n} - p_n(z))\right]^{1/2} e^{-x\,p_n(z)/2}}{p_n(z)\,\sqrt{|D(p_n(z))|}} \frac{dz}{\sqrt{1-z^2}},$$

where

$$p_n(z) = \left[(\delta_{2n} - \delta_{2n-1})z + \delta_{2n} + \delta_{2n-1}\right]/2.$$

The integrand in this integral contains singularities concentrated in the factor $1/\sqrt{1-z^2}$ alone. Therefore numerical integration can be carried out by applying a quadrature formula of the form

$$\int_{-1}^{1} \frac{f(z)\,dz}{\sqrt{1-z^2}} \approx \frac{\pi}{m} \sum_{k=1}^{m} f\left(\cos\left\{\frac{(2k-1)\pi}{2m}\right\}\right),$$

which is accurate for sufficiently large values of m. This numerical method was used by Martynov (1975, 1992).

2.5 Proof of Theorem 2.3

Recalling the definition (1.21) of the Fredholm transformation $\mathcal{I}_X$, we set below $X(\cdot) = \eta_f(\cdot)$.

Lemma 2.2 *Let $\{\lambda_n, \psi_n(y) : n \geq 1\}$, for $y \in \mathbb{R}$, denote the eigenvalues and eigenfunctions of the Fredholm tranformation $\mathcal{I}_{\eta_f}$. Then $\psi_n(y)$ is a solution on $\mathbb{R}$ of the differential equation*

$$\lambda_n \left(\psi_n''(y) + 2\,sgn(y)\,\psi_n'(y)\right) e^{2|y|} + 4\left(\psi_n(y) - C_n\right) = 0 \qquad (2.18)$$

where

$$C_n = \int_{\mathbb{R}} e^{-2|x|}\, \psi_n(x)\, dx,$$

with the boundary conditions

$$\psi_n(-\infty) = \psi_n(\infty) = 0, \tag{2.19}$$

$$\lambda_n \psi_n(0^+) = \int_{\mathbb{R}} (-2\,|x| + 1)\, e^{-2\,|x|}\, \psi_n(x)\, dx, \tag{2.20}$$

$$\psi_n(0^+) - \psi_n(0^-) = 0, \tag{2.21}$$

$$\psi_n'(0^-) - \psi_n'(0^+) = 2\left(\psi_n(0^-) + \psi_n(0^+)\right). \tag{2.22}$$

Proof. To determine the eigenfunctions, one must solve the integral equation:

$$\lambda_n \psi_n(y) = \int_{\mathbb{R}} R_f(x,y)\psi_n(x)\, dx, \quad y \in \mathbb{R}.$$

Since we have an absolute value in the covariance function $R_f(x,y)$, we will take $\psi_n(y) = \psi_{2,n}(y)1_{\{y\geq 0\}} + \psi_{1,n}(y)1_{\{y<0\}}$,

$$\begin{aligned}
\lambda_n \psi_{2,n}(y) &= \int_{-\infty}^{0} (2x - 2y + 1)e^{2(x-y)}\, \psi_{1,n}(x)\, dx + 2e^{-2y} \int_0^y \psi_{2,n}(x)\, dx \\
&\quad + 2\int_y^{\infty} e^{-2x}\psi_{2,n}(x)\, dx - \int_0^{\infty} (2y + 2x + 1)e^{-2(x+y)}\psi_{2,n}(x)\, dx, \quad (2.23) \\
\lambda_n \psi_{1,n}(y) &= \int_{-\infty}^{0} (2y + 2x - 1)\, e^{2(x+y)}\psi_{1,n}(x)\, dx + 2e^{2y} \int_y^0 \psi_{1,n}(x)\, dx \\
&\quad + 2\int_{-\infty}^{y} e^{2x}\psi_{1,n}(x)\, dx - \int_0^{\infty} (2x - 2y - 1)\, e^{2(y-x)}\, \psi_{2,n}(x)\, dx. \quad (2.24)
\end{aligned}$$

By setting, $y = 0$ in (2.23) and (2.24), we get

$$\begin{aligned}
\lambda_n \psi_{1,n}(0) &= \int_{-\infty}^{0} (2\,x + 1)\, e^{2\,x}\, \psi_{1,n}(x)\, dx - \int_0^{\infty} (2\,x - 1)\, e^{-2\,x}\, \psi_{2,n}(x)\, dx, \\
\lambda_n \psi_{2,n}(0) &= \int_{-\infty}^{0} (2\,x + 1)\, e^{2\,x}\, \psi_{1,n}(x)\, dx - \int_0^{\infty} (2\,x - 1)\, e^{-2\,x}\, \psi_{2,n}(x)\, dx.
\end{aligned}$$

Thus, when $y = 0$ and $y \to \pm\infty$ we obtain the boundary conditions

$$\psi_{1,n}(-\infty) = \psi_{2,n}(\infty) = 0, \qquad \psi_{2,n}(0) - \psi_{1,n}(0) = 0.$$

Denote by $\tilde{\psi}_{2,n}(y) = e^{2y}\, \psi_{2,n}(y)$ and $\tilde{\psi}_{1,n}(y) = e^{-2y}\, \psi_{1,n}(y)$ and take derivatives of equations (2.23) and (2.24) with respect to y, then we obtain

$$\begin{aligned}
\lambda_n \tilde{\psi}'_{2,n}(y) &= -2\int_{-\infty}^{0} e^{4x}\, \tilde{\psi}_{1,n}(x)\, dx - 2\int_0^{\infty} e^{-4\,x}\, \tilde{\psi}_{2,n}(x)\, dx \\
&\quad + 4\, e^{2y} \int_y^{\infty} e^{-4\,x}\, \tilde{\psi}_{2,n}(x)\, dx \quad \text{for} \quad y \geq 0, \\
\lambda_n \tilde{\psi}'_{1,n}(y) &= 2\int_{-\infty}^{0} e^{4\,x}\, \tilde{\psi}_{1,n}(x)\, dx + 2\int_0^{\infty} e^{-4\,x}\, \tilde{\psi}_{2,n}(x)\, dx \\
&\quad - 4\, e^{-2\,y} \int_{-\infty}^{y} e^{4x}\, \tilde{\psi}_{1,n}(x)\, dx \quad \text{for} \quad y < 0.
\end{aligned}$$

When $y = 0$ we have $\tilde{\psi}'_{2,n}(0) = \tilde{\psi}'_{1,n}(0)$. Thus, we obtain the boundary condition

$$\psi'_{1,n}(0) - \psi'_{2,n}(0) = 2\left(\psi_{1,n}(0) + \psi_{2,n}(0)\right).$$

Evaluating the derivative of the last equations we obtain

$$\begin{aligned}
\lambda_n \tilde{\psi}''_{2,n}(y) - 2\,\lambda_n \tilde{\psi}'_{2,n}(y) + 4\,e^{-2y}\,\tilde{\psi}_{2,n}(y) &= 4\,C_n \quad \text{for} \quad y \geq 0, \\
\lambda_n \tilde{\psi}''_{1,n}(y) + 2\,\lambda_n \tilde{\psi}'_{1,n}(y) + 4\,e^{2y}\,\tilde{\psi}_{1,n}(y) &= 4\,C_n \quad \text{for} \quad y < 0,
\end{aligned}$$

and we have

$$\begin{aligned}
\lambda_n\left(\psi''_{2,n}(y) + 2\,\psi'_{2,n}(y)\right)e^{2y} + 4\left(\psi_{2,n}(y) - C_n\right) &= 0 \quad \text{for} \quad y \geq 0, \\
\lambda_n\left(\psi''_{1,n}(y) - 2\,\psi'_{1,n}(y)\right)e^{-2y} + 4\left(\psi_{1,n}(y) - C_n\right) &= 0 \quad \text{for} \quad y < 0,
\end{aligned}$$

where

$$C_n = \int_{-\infty}^{0} e^{2x}\,\psi_{1,n}(x)\,dx + \int_{0}^{\infty} e^{-2x}\,\psi_{2,n}(x)\,dx.$$

□

Lemma 2.3 *The only solutions on $\mathbb{R}$ of the differential equation* (2.18) *fulfilling* (2.19) *are of the form*

$$\begin{aligned}
\psi_n(y) = {} & (A_{1,n}1_{\{y<0\}} + A_{2,n}1_{\{y\geq 0\}})\nu_n\,e^{-|y|}\,J_1\left(\nu_n\,e^{-|y|}\right) \\
& + B_{1,n}\nu_n\left(e^{-|y|}Y_1\left(\nu_n\,e^{-|y|}\right) + \frac{2}{\nu_n\pi}\right) \quad \textit{for} \quad y \in \mathbb{R},
\end{aligned} \tag{2.25}$$

where $\nu_n = 2/\sqrt{\lambda_n}$ and $A_{1,n}$, $A_{2,n}$, $B_{1,n}$ are arbitrary constants.

Proof. Let $\psi_n(y) = \psi_{2,n}(y)1_{\{y\geq 0\}} + \psi_{1,n}(y)1_{\{y<0\}}$ and denote by $\bar{\psi}_{2,n}(y) = \psi_{2,n}(y) - C_n$ and $\bar{\psi}_{1,n}(y) = \psi_{1,n}(y) - C_n$. The equation (2.18) can be written as

$$\bar{\psi}''_{2,n}(y) + 2\,\bar{\psi}'_{2,n}(y) + \nu_n^2\,e^{-2y}\,\bar{\psi}_{2,n}(y) = 0 \quad \text{for} \quad y \geq 0, \tag{2.26}$$

$$\bar{\psi}''_{1,n}(y) - 2\,\bar{\psi}'_{1,n}(y) + \nu_n^2\,e^{2y}\,\bar{\psi}_{1,n}(y) = 0 \quad \text{for} \quad y < 0, \tag{2.27}$$

where $\nu_n = 2/\sqrt{\lambda_n}$. If we make the change of variables $t = \nu_n\,e^{-y}$ for $y \geq 0$, then

$$\begin{aligned}
\frac{d\bar{\psi}_{2,n}}{dy} &= \frac{d\bar{\psi}_{2,n}}{dt} \times \frac{dt}{dy} = -t\,\frac{d\bar{\psi}_{2,n}}{dt}, \\
\frac{d^2\bar{\psi}_{2,n}}{dy^2} &= \frac{d}{dy}\Big(\frac{d\bar{\psi}_{2,n}}{dy}\Big) = \frac{d}{dt}\Big(\frac{d\bar{\psi}_{2,n}}{dy}\Big) \times \frac{dt}{dy} = t\,\frac{d}{dt}\Big(t\,\frac{d\bar{\psi}_{2,n}}{dt}\Big) \\
&= t^2\,\frac{d^2\bar{\psi}_{2,n}}{dt^2} + t\,\frac{d\bar{\psi}_{2,n}}{dt}.
\end{aligned}$$

Using similar arguments in the case $t = \nu_n\,e^{y}$ for $y < 0$, we rewite the equations (2.26) and (2.27) as follows

$$\begin{aligned}
t^2\,\bar{\psi}''_{2,n}(t) - t\,\bar{\psi}'_{2,n}(t) + t^2\,\bar{\psi}_{2,n}(t) &= 0, \\
t^2\,\bar{\psi}''_{1,n}(t) - t\,\bar{\psi}'_{1,n}(t) + t^2\,\bar{\psi}_{1,n}(t) &= 0.
\end{aligned}$$

We set $\bar{\psi}_{2,n}(t) = t\,\hat{\psi}_{2,n}(t)$ and $\bar{\psi}_{1,n}(t) = t\,\hat{\psi}_{1,n}(t)$, then we have

$$\begin{aligned} t^2\,\hat{\psi}''_{2,n}(t) + t\,\hat{\psi}'_{2,n}(t) + (t^2-1)\,\hat{\psi}_{2,n}(t) = 0, \\ t^2\,\hat{\psi}''_{1,n}(t) + t\,\hat{\psi}'_{1,n}(t) + (t^2-1)\,\hat{\psi}_{1,n}(t) = 0. \end{aligned}$$

The last equations are the Bessel differential equations of order 1 ($\nu = 1$) and have solutions

$$\begin{aligned} \hat{\psi}_{2,n}(t) &= A_{2,n}\,J_1(t) + B_{2,n}\,Y_1(t), \\ \hat{\psi}_{1,n}(t) &= A_{1,n}\,J_1(t) + B_{1,n}\,Y_1(t). \end{aligned}$$

Therefore the general solutions are of the following forms

$$\begin{aligned} \psi_{2,n}(y) &= \nu_n\left(A_{2,n}\,e^{-y}\,J_1\left(\nu_n\,e^{-y}\right) + B_{2,n}\,e^{-y}Y_1\left(\nu_n\,e^{-y}\right)\right) + C_n, \quad y \geq 0, \\ \psi_{1,n}(y) &= \nu_n\left(A_{1,n}\,e^{y}\,J_1\left(\nu_n\,e^{y}\right) + B_{1,n}\,e^{y}\,Y_1\left(\nu_n\,e^{y}\right)\right) + C_n, \quad y < 0. \end{aligned}$$

By (A.3), we get, as $y \to \pm\infty$,

$$\begin{aligned} \psi_{2,n}(y) &\simeq \nu_n\left(\frac{1}{2}\,A_{2,n}\,\nu_n e^{-2y} - B_{2,n}\frac{2}{\nu_n\pi}\right) + C_n \to C_n - B_{2,n}\frac{2}{\pi}, \\ \psi_{1,n}(y) &\simeq \nu_n\left(\frac{1}{2}\,A_{1,n}\,\nu_n e^{2y} - B_{1,n}\frac{2}{\nu_n\pi}\right) + C_n \to C_n - B_{1,n}\frac{2}{\pi}. \end{aligned}$$

Thus, $\psi_{1,n}(-\infty) = \psi_{2,n}(\infty) = 0$ holds if and only if $C_n = B_{1,n}\frac{2}{\pi} = B_{2,n}\frac{2}{\pi}$. Therefore, we obtain (2.25). □

Lemma 2.4 *Let $\psi_n(y)$ be as in* (2.25)*. Then its derivative can be written as*

$$\begin{aligned} \psi'_n(y) = {} & (A_{1,n}1_{\{y<0\}} - A_{2,n}1_{\{y\geq 0\}})\nu_n^2\,e^{-2|y|}\,J_0\left(\nu_n\,e^{-|y|}\right) \\ & - sgn(y)B_{1,n}\nu_n^2\,e^{-2|y|}\,Y_0\left(\nu_n\,e^{-|y|}\right). \end{aligned} \tag{2.28}$$

Proof. By (A.5), we have the recurrence relations:

$$J'_1(x) = J_0(x) - \frac{J_1(x)}{x}, \quad Y'_1(x) = Y_0(x) - \frac{Y_1(x)}{x}.$$

Therefore, we get

$$\begin{aligned} \psi'_n(y) = {} & A_{1,n}1_{\{y<0\}}\,\nu_n\,e^{-|y|}\left[J_1\left(\nu_n\,e^{-|y|}\right) + \nu_n\,e^{-|y|}\,J'_1\left(\nu_n\,e^{-|y|}\right)\right] \\ & - A_{2,n}1_{\{y\geq 0\}}\,\nu_n\,e^{-|y|}\left[J_1\left(\nu_n\,e^{-|y|}\right) + \nu_n\,e^{-|y|}\,J'_1\left(\nu_n\,e^{-|y|}\right)\right] \\ & - \mathrm{sgn}(y)B_{1,n}\nu_n^2\,e^{-2|y|}\left[Y_1\left(\nu_n\,e^{-|y|}\right) + \nu_n\,e^{-|y|}\,Y'_1\left(\nu_n\,e^{-|y|}\right)\right] \\ = {} & (A_{1,n}1_{\{y<0\}} - A_{2,n}1_{\{y\geq 0\}})\nu_n^2\,e^{-2|y|}\,J_0\left(\nu_n\,e^{-|y|}\right) \\ & - \mathrm{sgn}(y)B_{1,n}\nu_n^2\,e^{-2|y|}\,Y_0\left(\nu_n\,e^{-|y|}\right). \end{aligned}$$

which yields (2.28). □

Proof of Theorem 2.3. Returning to the equation (2.20) and replacing $\psi_n(y) = \psi_{2,n}(y)1_{\{y\geq 0\}} + \psi_{1,n}(y)1_{\{y<0\}}$ by its general form (2.25), we have

$$\begin{aligned}
\psi_{2,n}(0) &= -\frac{\nu_n^2}{4}\int_0^{\infty}(2\,x-1)\,e^{-2\,x}\,(\psi_{2,n}(x)+\psi_{1,n}(-x))\,dx \\
&= -\frac{\nu_n^3}{2}\bigg[(A_{1,n}+A_{2,n})\int_0^{\infty} x\;e^{-3\,x}\,J_1\left(\nu_n\,e^{-x}\right)dx \\
&\quad + 2\,B_{1,n}\int_0^{\infty} x\;e^{-3\,x}\,Y_1\left(\nu_n\,e^{-x}\right)dx\bigg].
\end{aligned}$$

Below we change the variable $t = \nu_n\,e^{-y}$ in the first integral, use equalities (A.6), (A.8) and integration by parts

$$\begin{aligned}
\int_0^{\infty}\nu_n^3\,x\;e^{-3\,x}\,J_1\left(\nu_n\,e^{-x}\right)dx &= -\int_0^{\nu_n}\ln\left(t/\nu_n\right)\;t^2\,J_1(t)\,dt \\
&= -\ln\left(t/\nu_n\right)\;t^2\,J_2(t)\Big|_0^{\nu_n} + \int_0^{\nu_n} t\,J_2(t)\,dt \\
&= 2\int_0^{\nu_n} J_1(t)\,dt - \int_0^{\nu_n} t\,J_0(t)\,dt \\
&= -2\;J_0\left(\nu_n\right) - \nu_n\,J_1\left(\nu_n\right) + 2.
\end{aligned}$$

The same arguments applied to the second integral together with (A.3), (A.1) provide

$$\begin{aligned}
&\int_0^{\infty}\nu_n^3\,x\;e^{-3\,x}\,Y_1\left(\nu_n\,e^{-x}\right)dx = -\int_0^{\nu_n}\ln\left(t/\nu_n\right)\;t^2\,Y_1(t)\,dt \\
&\quad= -\ln\left(t/\nu_n\right)\;t^2\,Y_2(t)\Big|_0^{\nu_n} + 2\int_0^{\nu_n} Y_1(t)\,dt - \int_0^{\nu_n} t\,Y_0(t)\,dt \\
&\quad= -\left(\ln\left(t/\nu_n\right)\;t^2\,Y_2(t) + 2\;Y_0(t)\right)\Big|_0^{\nu_n} - t\,Y_1(t)\Big|_0^{\nu_n} \\
&\quad= -2\;Y_0\left(\nu_n\right) - \nu_n\,Y_1\left(\nu_n\right) - \frac{2}{\pi}\left(1-2\left(\ln(\nu_n/2)+\gamma\right)\right).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\psi_{2,n}(0) &= (A_{1,n}+A_{2,n})\left(J_0\left(\nu_n\right) + \frac{\nu_n}{2}\,J_1\left(\nu_n\right) - 1\right) \\
&\quad + 2\,B_{1,n}\left(Y_0\left(\nu_n\right) + \frac{\nu_n}{2}\,Y_1\left(\nu_n\right) + \frac{1}{\pi}\left(1-2\left(\ln(\nu_n/2)+\gamma\right)\right)\right).
\end{aligned}$$

Finally, we obtain

$$(A_{1,n}+A_{2,n})\left(J_0\left(\nu_n\right)-1\right) + 2\,B_{1,n}\left(Y_0\left(\nu_n\right) - \frac{2}{\pi}\left(\ln(\nu_n/2)+\gamma\right)\right). \tag{2.29}$$

Now, using the boundary condition (2.21), we have

$$(A_{1,n}-A_{2,n})J_1\left(\nu_n\right) = 0. \tag{2.30}$$

Then, using the boundary condition (2.22),

$$\begin{aligned}
&(A_{1,n}+A_{2,n})\left[J_0\left(\nu_n\right) - \frac{2}{\nu_n}J_1\left(\nu_n\right)\right] \\
&\qquad\qquad + 2\,B_{1,n}\left[Y_0\left(\nu_n\right) - \frac{2}{\nu_n}Y_1\left(\nu_n\right) - \frac{4}{\pi\nu_n^2}\right] = 0.
\end{aligned}$$

By (A.6), we have the recurrence relations:

$$J_2(x) + J_0(x) = \frac{2}{x} J_1(x), \quad Y_2(x) + Y_0(x) = \frac{2}{x} Y_1(x).$$

Therefore, we get

$$(A_{1,n} + A_{2,n}) J_2(\nu_n) + 2 B_{1,n} \left(Y_2(\nu_n) + \frac{4}{\pi} \right) = 0. \tag{2.31}$$

The equations (2.29)-(2.31), can be written as a linear system

$$\begin{bmatrix} J_1(\nu_n) & -J_1(\nu_n) & 0 \\ J_2(\nu_n) & J_2(\nu_n) & 2\alpha(\nu_n) \\ J_0(\nu_n) - 1 & J_0(\nu_n) - 1 & 2\beta(\nu_n) \end{bmatrix} \begin{bmatrix} A_{1,n} \\ A_{2,n} \\ B_{1,n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.32}$$

where

$$\alpha(\nu_n) = Y_2(\nu_n) + \frac{4}{\pi \nu_n^2}, \quad \beta(\nu_n) = Y_0(\nu_n) - \frac{2}{\pi} \left(\ln(\nu_n/2) + \gamma \right).$$

System (2.32) has non-zero solutions in $(A_{1,n}, A_{2,n}, B_{1,n})$, if and only if

$$\det \begin{bmatrix} J_1(\nu_n) & -J_1(\nu_n) & 0 \\ J_2(\nu_n) & J_2(\nu_n) & 2\alpha(\nu_n) \\ J_0(\nu_n) - 1 & J_0(\nu_n) - 1 & 2\beta(\nu_n) \end{bmatrix} = 4 J_1(\nu_n)\, f(\nu_n) = 0, \tag{2.33}$$

where

$$f(\nu_n) = J_2(\nu_n)\, \beta(\nu_n) - (J_0(\nu_n) - 1)\, \alpha(\nu_n). \tag{2.34}$$

Next we consider the function $\psi_n(y)$ of the form (2.25) and $A_{1,n}, A_{2,n}, B_{1,n}$ fulfilling (2.32). Now $\nu_{1,n}, \nu_{2,n}, n = 1, 2, \ldots$, denote respectively the positive zeros of $J_1(\cdot), f(\cdot)$. We have the following two cases:

Case 1: Let $\nu_n = \nu_{1,n}$, for some $n = 1, 2, \ldots$, then $J_1(\nu_{1,n}) = 0$ and (2.32) implies that $A_{1,n} = -A_{2,n}$ and $B_{1,n} = 0$. The requirement that $\int_{-\infty}^{\infty} \psi_n^2(y)\, dy = 1$ yields (see, Lemma A.1),

$$\psi_n(y) = -\mathrm{sgn}(y)\, e^{-|y|} \frac{J_1\left(\nu_{1,n} e^{-|y|} \right)}{J_0(\nu_{1,n})} \quad \text{for} \quad y \in \mathbb{R}. \tag{2.35}$$

Case 2: Let $\nu_n = \nu_{2,n}$, for some $n = 1, 2, \ldots$, then $f(\nu_{2,n}) = 0$ and (2.32) implies that $A_{1,n} = A_{2,n} = -B_{1,n} \alpha(\nu_{2,n}) / J_2(\nu_{2,n})$. The requirement that $\int_{-\infty}^{\infty} \psi_n^2(y)\, dy = 1$ yields (see, Lemma A.2),

$$\begin{aligned} \psi_n(y) = & \frac{\alpha(\nu_{2,n})\, e^{-|y|} J_1\left(\nu_{2,n} e^{-|y|} \right)}{J_1(\nu_{2,n})\, \alpha(\nu_{2,n}) - Y_1(\nu_{2,n})\, J_2(\nu_{2,n})} \\ & - \frac{J_2(\nu_{2,n}) \left(e^{-|y|} Y_1\left(\nu_{2,n} e^{-|y|} \right) + \frac{2}{\pi \nu_{2,n}} \right)}{J_1(\nu_{2,n})\, \alpha(\nu_{2,n}) - Y_1(\nu_{2,n})\, J_2(\nu_{2,n})} \quad \text{for} \quad y \in \mathbb{R}. \end{aligned} \tag{2.36}$$

2.6 Proof of Theorem 2.4

To determine the eigenfunctions, we propose to solve the integral equation:

$$\lambda_n \psi_n(t) = \int_0^1 K_f(t,s)\, \psi_n(s)\, ds \quad \text{for} \quad 0 \le t \le 1. \tag{2.37}$$

Since we have an absolute value in the covariance function $K_f(s,t)$, the function $\psi_n(t)$ is split into:

$$\psi_n(t) = \psi_{1,n}(t) 1_{\{t<1/2\}} + \psi_{2,n}(t) 1_{\{t \ge 1/2\}}.$$

Here, λ_n, $n = 1 \dots \infty$ are the eigenvalues yet to be determined. Recall that $\psi_{1,n}(t)$ and $\psi_{2,n}(t)$ satisfy the boundary conditions

$$\psi_{1,n}(0) = \psi_{2,n}(1) = 0, \qquad \psi_{1,n}(1/2) = \psi_{2,n}(1/2), \tag{2.38}$$

such that

$$\lambda_n \psi_n(1/2) = \int_0^1 (1 - |2s-1|)\, [1 + \ln(1 - |2s-1|)]\, \psi_n(s)\, ds. \tag{2.39}$$

Taking the derivative of equation (2.37) with respect to t, we obtain

$$\lambda_n \psi_n'(t) = \int_0^1 \frac{\partial K_f(t,s)}{\partial t}\, \psi_n(s)\, ds \quad \text{for} \quad 0 \le t \le 1.$$

Evaluating the derivative of the last equation we obtain

$$\lambda_n \psi_n''(t) + 4\, \psi_n(t) = \frac{4\, C_n}{1 - |2t-1|} \quad \text{for} \quad 0 \le t \le 1,$$

where

$$C_n = \int_0^1 (1 - |2s-1|)\, \psi_n(s)\, ds. \tag{2.40}$$

Let $\nu_n = 1/\sqrt{\lambda_n}$, then the last equation can be written as follow

$$\begin{aligned} \psi_{1,n}''(t) + 4\,\nu_n^2 \psi_{1,n}(t) &= \tfrac{2\,\nu_n^2 C_n}{t} \quad \text{for} \quad t < 1/2, \\ \psi_{2,n}''(t) + 4\,\nu_n^2 \psi_{2,n}(t) &= \tfrac{2\,\nu_n^2 C_n}{1-t} \quad \text{for} \quad t \ge 1/2. \end{aligned}$$

The solution of the homogeneous equation is given by

$$\psi_n(t) = \begin{cases} \psi_{1,n}(t) = A_{1,n} \sin(2\nu_n t) + B_{1,n} \cos(2\nu_n t) & \text{for} \quad t < 1/2, \\ \psi_{2,n}(t) = A_{2,n} \sin(2\nu_n t) + B_{2,n} \cos(2\nu_n t) & \text{for} \quad t \ge 1/2. \end{cases}$$

The Lagrange method provides the relations

$$\begin{aligned} A_{1,n}'(t) \sin(2\nu_n t) + B_{1,n}'(t) \cos(2\nu_n t) &= 0, \\ A_{1,n}'(t) \cos(2\nu_n t) - B_{1,n}'(t) \sin(2\nu_n t) &= \frac{\nu_n C_n}{t}. \end{aligned}$$

Hence, we obtain

$$A_{1,n}(t) = \nu_n C_n \mathrm{Ci}(2\nu_n t), \qquad B_{1,n}(t) = -\nu_n C_n \mathrm{Si}(2\nu_n t),$$

with

$$\mathrm{Ci}(t) = \gamma + \ln(t) + \int_0^t \frac{\cos(s) - 1}{s}\, ds, \quad \mathrm{Si}(t) = \int_0^t \frac{\sin(s)}{s}\, ds.$$

Then, the first equation has a solution of the following form

$$\psi_{1,n}(t) = A_{1,n} \sin(2\nu_n t) + B_{1,n} \cos(2\nu_n t) + \nu_n C_n\, \alpha(2\nu_n t) \quad \text{for} \quad t < 1/2,$$

where

$$\alpha(t) = \mathrm{Ci}(t) \sin(t) - \mathrm{Si}(t) \cos(t).$$

In the same manner, we have for the second equation

$$\begin{aligned} A'_{2,n}(t) \sin(2\nu_n t) + B'_{2,n}(t) \cos(2\nu_n t) &= 0, \\ A'_{2,n}(t) \cos(2\nu_n t) - B'_{2,n}(t) \sin(2\nu_n t) &= \frac{\nu_n C_n}{1-t}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} A_{2,n}(t) &= -\nu_n C_n \left[\mathrm{Ci}(2\nu_n(1-t)) \cos(2\nu_n) + \mathrm{Si}(2\nu_n(1-t)) \sin(2\nu_n)\right], \\ B_{2,n}(t) &= \nu_n C_n \left[\mathrm{Ci}(2\nu_n(1-t)) \sin(2\nu_n) - \mathrm{Si}(2\nu_n(1-t)) \cos(2\nu_n)\right]. \end{aligned}$$

Then, the second equation has a solution of the following form

$$\psi_{2,n}(t) = A_{2,n} \sin(2\nu_n t) + B_{2,n} \cos(2\nu_n t) + \nu_n C_n\, \alpha(2\nu_n(1-t)), \quad t \geq 1/2.$$

Therefore, using the boundary conditions (2.38), we conclude that the general solution is of the following form

$$\begin{aligned} \psi_n(t) = {} & (A_{1,n} 1_{\{t<1/2\}} - A_{2,n} 1_{\{t \geq 1/2\}}) \sin(\nu_n(1 - |2t-1|)) \\ & + \nu_n C_n\, \alpha(\nu_n(1 - |2t-1|)) \quad \text{for} \quad 0 \leq t \leq 1. \end{aligned} \tag{2.41}$$

Now, returning to the equation (2.39), replacing $\psi_n(s)$ by this general form and making the change of variable $u = \nu_n s$, we obtain

$$\begin{aligned} \psi_n(1/2) &= \frac{\nu_n^2}{2} \int_0^1 s\,(1 + \ln(s))\left[(A_{1,n} - A_{2,n}) \sin(\nu_n s) + 2\, C_n \nu_n \alpha(\nu_n s)\right] ds \\ &= \frac{1}{2}(A_{1,n} - A_{2,n}) \int_0^{\nu_n} u\,[1 - \ln(\nu_n) + \ln(u)] \sin(u)\, du \\ &\quad + C_n \nu_n \int_0^{\nu_n} u\,[1 - \ln(\nu_n) + \ln(u)]\, \alpha(u)\, du. \end{aligned}$$

By a simple integration by parts we have

$$\begin{aligned} \psi_n(1/2) = {} & \frac{1}{2}(A_{1,n} - A_{2,n})\left[2\sin(\nu_n) - \nu_n \cos(\nu_n) - \mathrm{Si}(\nu_n)\right] \\ & + C_n \nu_n \left[2\,\alpha(\nu_n) - \nu_n \dot{\alpha}(\nu_n) - G(\nu_n)\right], \end{aligned}$$

where

$$\dot{\alpha}(t) = \frac{d}{dt}\alpha(t) = \mathrm{Si}(t) \sin(t) + \mathrm{Ci}(t) \cos(t), \qquad G(t) = \int_0^t \frac{\alpha(s)}{s}\, ds.$$

Finally we have

$$\begin{aligned}(A_{1,n}-A_{2,n})\,[\sin(\nu_n)-\nu_n\cos(\nu_n)-\mathrm{Si}(\nu_n)]&\\ +2\,C_n\nu_n\,[\alpha(\nu_n)-\nu_n\dot{\alpha}(\nu_n)-\mathrm{G}(\nu_n)]=0.&\end{aligned} \tag{2.42}$$

In the same manner, we obtain for equality (2.40)

$$\begin{aligned}C_n&=\frac{1}{2\,\nu_n^2}\int_0^{\nu_n}u\,[(A_{1,n}-A_{2,n})\sin(u)+2\,C_n\nu_n\alpha(u)]\,du\\&=\frac{1}{2\,\nu_n^2}(A_{1,n}-A_{2,n})\,[\sin(\nu_n)-\nu_n\cos(\nu_n)]\\&\qquad+\frac{1}{\nu_n^2}\,C_n\nu_n\,[\alpha(\nu_n)-\nu_n\dot{\alpha}(\nu_n)+\nu_n]\,.\end{aligned}$$

Hence, we have

$$(A_{1,n}-A_{2,n})\,(\sin(\nu_n)-\nu_n\cos(\nu_n))+2\,C_n\nu_n\,(\alpha(\nu_n)-\nu_n\dot{\alpha}(\nu_n))=0. \tag{2.43}$$

Using the boundary conditions (2.38) and equalities (2.42), (2.43), we get the linear system

$$\begin{bmatrix}\sin(\nu_n) & \sin(\nu_n) & 0\\ \mathrm{Si}(\nu_n) & -\mathrm{Si}(\nu_n) & 2\,\nu_n\mathrm{G}(\nu_n)\\ \sin(\nu_n)-\nu_n\cos(\nu_n) & \nu_n\cos(\nu_n)-\sin(\nu_n) & 2\,\nu_n\,[\alpha(\nu_n)-\nu_n\dot{\alpha}(\nu_n)]\end{bmatrix}$$

$$\times\begin{bmatrix}A_{1,n}\\ A_{2,n}\\ C_n\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}, \tag{2.44}$$

which has non-zero solutions in $(A_{1,n}, A_{2,n}, C_n)$, if and only if

$$\begin{vmatrix}\sin(\nu_n) & \sin(\nu_n) & 0\\ \mathrm{Si}(\nu_n) & -\mathrm{Si}(\nu_n) & 2\,\nu_n\mathrm{G}(\nu_n)\\ \sin(\nu_n)-\nu_n\cos(\nu_n) & \nu_n\cos(\nu_n)-\sin(\nu_n) & 2\,\nu_n\,[\alpha(\nu_n)-\nu_n\dot{\alpha}(\nu_n)]\end{vmatrix}$$

$$=4\,\nu_n\sin(\nu_n)\,f\,(\nu_n)=0. \tag{2.45}$$

Here

$$f\,(\nu_n)=\mathrm{G}(\nu_n)\,[\sin(\nu_n)-\nu_n\cos(\nu_n)]-\mathrm{Si}(\nu_n)\,[\alpha(\nu_n)-\nu_n\dot{\alpha}(\nu_n)]\,. \tag{2.46}$$

Next we consider the function $\psi_n(t)$ of the form (2.41) and $A_{1,n}, A_{2,n}, C_n$ fulfilling (2.44). Now $\nu_{1,n}, \nu_{2,n}, n=1,2,\ldots$ denote respectively solutions of $\sin(\nu_n)=0$ and $f(\nu_n)=0$. We have the following two cases:

Case 1: Let $\nu_n=\nu_{1,n}=n\,\pi$, for some $n=1,2,\ldots$, then $\sin(\nu_{1,n})=0$ and (2.44) implies that $A_{1,n}=A_{2,n}$ and $C_n=0$. The requirement that $\int_0^1\psi_n^2(t)\,dt=1$ yields,

$$\psi_n(t)=\sqrt{2}\,\mathrm{sgn}(1/2-t)\,\sin\left(n\,\pi(1-|2\,t-1|)\right)\quad\text{for}\quad t\in[0,1].$$

Case 2: Let $\nu_n=\nu_{2,n}$, for some $n=1,2,\ldots$, then $f\,(\nu_n)=0$ and (2.44) implies that $A_{1,n}=-C_n\,[\alpha(\nu_{2,n})-\nu_{2,n}\dot{\alpha}(\nu_{2,n})]\,/\,[\sin(\nu_{2,n})-\nu_{2,n}\cos(\nu_{2,n})]=-A_{2,n}$. The requirement that $\int_0^1\psi_n^2(t)\,dt=1$ yields,

$$\begin{aligned}\psi_n(t)=\frac{\sqrt{2}}{\mathrm{Si}(\nu_{2,n})}\Bigg[&\left(\frac{\alpha(\nu_{2,n})}{\nu_{2,n}}-\dot{\alpha}(\nu_{2,n})\right)\sin\left(\nu_{2,n}(1-|2\,t-1|)\right)\\&-\left(\frac{\sin(\nu_{2,n})}{\nu_{2,n}}-\cos(\nu_{2,n})\right)\alpha\left(\nu_{2,n}(1-|2\,t-1|)\right)\Bigg]\quad\text{for}\quad t\in[0,1].\end{aligned}$$

Parameter estimation

3

Sommaire

In this chapter we consider the problem of parameter estimation by the observation of ergodic diffusion process. We suppose that the unknown parameter is two-dimensional and the trend coefficient of the process is discontinuous "sign-type". We describe the asymptotic properties of the maximum likelihood estimator, Bayesian estimator and the estimator of the method of moments in this case.

3.1 Introduction

We consider the problem of parameter estimation by the observation of diffusion process

$$dX_t = -\vartheta_1 \mathrm{sgn}(X_t - \vartheta_2)dt + dW_t, \quad X_0, \quad 0 \leq t \leq T, \tag{3.1}$$

where the unknown parameter $\vartheta = (\vartheta_1, \vartheta_2) \in \Theta = (\alpha_1, \beta_1) \times (\alpha_2, \beta_2)$ with $\alpha_1 > 0$. The problems of parameter estimation for ergodic diffusion processes are well studied. It is shown that under regularity conditions the classical estimators (maximum likelihood, Bayesian and method of moments) are consistent, asymptotically normal (with the rate $\sqrt{T}$). Moreover, the maximum likelihood and Bayesian estimators are efficient (see Kutoyants 2004).

The case of singular estimation, when the Fisher information is infinite is less studied. We can mention here the works by Küchler and Kutoyants (2000), Dachian and Kutoyants (2003), where such problems of parameter estimation were considered in the situations of discontinuous and cusp-type singular us usual, the rate of convergence of the same estimators (in singular case) is better than in regular case.

Statistical estimation for the switching diffusion process first was studied in Kutoyants (2000), where the problem of estimations of the one-dimensional parameter $\vartheta = \vartheta_2$ by observation (3.1) was treated.

In the present work we consider the same model of observation, but we suppose that the both parameters ϑ_1 and ϑ_2 are unknown. We show that the maximum likelihood estimator and Bayesian estimator have different limit distributions with the normalizing rates $\sqrt{T}$ for ϑ_1 and T for ϑ_2. We follow the same method as used in Kutoyants (2004).

The proof is based on the general theorems by Ibragimov and Khasminskii (1981), which allow to describe the asymptotic properties of these estimators through the properties of the properly normalized likelihood ratio process.

3.2 Maximum likelihood and Bayesian estimators

We observe a trajectory $X^T = \{X_t, 0 \leq t \leq T\}$ of the diffusion process given by the stochastic differential equation (3.1). It is easy to see that the conditions of the existence of the solution and ergodicity (see Theorem 1.3 and Theorem 1.7), are fulfilled and (3.1) is an ergodic diffusion process with the stationary density $f(\vartheta, x) = \vartheta_1 \, e^{-2\,\vartheta_1 |x-\vartheta_2|}$, for $x \in \mathbb{R}$.

To construct the maximum likelihood and the Bayesian estimators we introduce the likelihood ratio function

$$L\left(\vartheta, X^T\right) = \frac{d\mathbf{P}_\vartheta^{(T)}}{d\mathbf{P}^{(T)}}\left(X^T\right), \quad \vartheta \in \Theta,$$

(here $\mathbf{P}^{(T)} = \mathbf{P}_{0,0}^{(T)}$) by the formula

$$L\left(\vartheta, X^T\right) = \exp\left\{-\vartheta_1 \int_0^T \operatorname{sgn}(X_t - \vartheta_2)\, dX_t - \frac{\vartheta_1^2}{2}\, T\right\}.$$

The maximum likelihood estimator (MLE) $\hat{\vartheta}_T$ is defined as solution of the equation

$$L\left(\hat{\vartheta}_T, X^T\right) = \sup_{\vartheta\in\Theta} L\left(\vartheta, X^T\right).$$

If this equation has more than one solution, then we can take anyone as MLE.

To introduce the Bayesian estimator (BE) we suppose that the unknown parameter ϑ is a random vector with a known prior density $\{p(\theta), \theta \in \Theta\}$, which is continuous and positive. Using the quadratic loss function, the BE $\tilde{\vartheta}_T$ (which minimizes the mean square error) is the conditional mathematical expectation

$$\tilde{\vartheta}_T = \int_\Theta \theta\, p\left(\theta | X^T\right)\, d\theta = \frac{\int_\Theta \theta\, p\,(\theta)\; L\left(\theta, X^T\right)\, d\theta}{\int_\Theta p\,(\theta)\; L\left(\theta, X^T\right)\, d\theta}.$$

Indeed, even if θ is not random we can use this formula (with some function $p\,(\cdot)$) for construction of an estimator and Theorem 3.2 below is valid for this estimator too.

To describe the limit behavior of the MLE and BE we need two random vectors $\hat{w} = (\hat{v}, \hat{u})$ and $\tilde{w} = (\tilde{v}, \tilde{u})$ defined with the help of the following stochastic process:

$$Z_\vartheta(w) = Z_{\vartheta_1}(v)\, Z_{\vartheta_2}(u), \quad w = (v, u) \in \mathbb{R}^2,$$

where

$$Z_{\vartheta_1}(v) = \exp\left\{v\,\zeta - \frac{v^2}{2}\right\}, \qquad Z_{\vartheta_2}(u) = \exp\left\{2\,\vartheta_1^{3/2}\,W(u) - 2\,\vartheta_1^3\,|u|\right\},$$

as follows:

$$Z_\vartheta(\hat{w}) = \sup_{w\in\mathbb{R}^2} Z_\vartheta(w), \qquad \tilde{w} = \frac{\int_{\mathbb{R}^2} w\,Z_\vartheta\,(w)\;dw}{\int_{\mathbb{R}^2} Z_\vartheta\,(w)\;dw}.$$

Here ζ and $W(\cdot)$ are independent, where ζ denote $\mathcal{N}(0,1)$ random variable. $W(\cdot)$ is a two-sided Wiener process, i.e., $W(u) = W_+(u)$ for $u \geq 0$ and $W(u) = W_-(-u)$ for $u < 0$, where $W_+(u)$, $W_-(u)$, $u \geq 0$ are two independent standard Wiener processes.

Introduce the normalizing matrix

$$\varphi_T = \begin{pmatrix} T^{-1/2} & 0 \\ 0 & T^{-1} \end{pmatrix}.$$

The asymptotic properties of the MLE $\hat{\vartheta}_T = (\hat{\vartheta}_T^{(1)}, \hat{\vartheta}_T^{(2)})$ and the BE $\tilde{\vartheta}_T = (\tilde{\vartheta}_T^{(1)}, \tilde{\vartheta}_T^{(2)})$ are described in the following theorems.

Theorem 3.1 *The MLE $\hat{\vartheta}_T$ constructed by the observations X^T of the diffusion process is consistent, i.e., for any $\nu > 0$*

$$\lim_{T\to\infty} \mathbf{P}_\vartheta^{(T)}\left\{|\hat{\vartheta}_T - \vartheta| > \nu\right\} = 0,$$

the distribution of the random vector $\varphi_T^{-1}(\hat{\vartheta}_T - \vartheta)$ converge to the distribution of the random vector $\hat{w}$ and for any $p > 0$

$$\lim_{T\to\infty} \mathbf{E}_\vartheta \left|\varphi_T^{-1}(\hat{\vartheta}_T - \vartheta)\right|^p = \mathbf{E}_\vartheta\,|\hat{w}|^p\,.$$

Theorem 3.2 *The BE $\tilde{\vartheta}_T$ constructed by the observations X^T of the diffusion process is consistent, the normalized difference $\varphi_T^{-1}(\tilde{\vartheta}_T - \vartheta)$ converges in distribution:*

$$\varphi_T^{-1}(\tilde{\vartheta}_T - \vartheta) \Longrightarrow \tilde{w},$$

and for any $p > 0$

$$\lim_{T\to\infty} \mathbf{E}_\vartheta \left|\varphi_T^{-1}(\tilde{\vartheta}_T - \vartheta)\right|^p = \mathbf{E}_\vartheta\,|\tilde{w}|^p\,.$$

Proof. The proof is based on the general result by Ibragimov and Khasminskii (1981), Theorems 1.10.1 and 1.10.2, so we check the conditions of these theorems with the help of the three lemmas presented below. That is, we show the convergence of marginal distributions of the normalized likelihood ratio $Z_T(\cdot)$ and establish two estimates on the increments $Z_T(\cdot)$ and on its decrease.

Introduce the normalized likelihood ratio process

$$Z_T(w) = \frac{d\mathbf{P}_{\vartheta+\varphi_T w}^{(T)}}{d\mathbf{P}_\vartheta^{(T)}}(X^T) = \frac{L(\vartheta + \varphi_T\,w, X^T)}{L(\vartheta, X^T)}, \quad w \in \mathbb{W}_T,$$

where ϑ is the true value and the set

$$\mathbb{W}_T = \left(\sqrt{T}\,(\alpha_1 - \vartheta_1)\,, \sqrt{T}\,(\beta_1 - \vartheta_1)\right) \times (T\,(\alpha_2 - \vartheta_2)\,, T\,(\beta_2 - \vartheta_2))\,.$$

Write the BE ($\vartheta_w = \vartheta + \varphi_T\, w$) as

$$\begin{aligned}\tilde{\vartheta}_T &= \frac{\int_\Theta \theta\, p\,(\theta)\; L\left(\theta, X^T\right)\, d\theta}{\int_\Theta p\,(\theta)\; L\,(\theta, X^T)\; d\theta} = \vartheta + \varphi_T\, \frac{\int_{\mathbb{W}_T} w\, p\,(\vartheta_w)\; L\left(\vartheta_w, X^T\right)\, dw}{\int_{\mathbb{W}_T} p\,(\vartheta_w)\; L\,(\vartheta_w, X^T)\; dw} \\ &= \vartheta + \varphi_T\, \frac{\int_{\mathbb{W}_T} w\, p\,(\vartheta_w)\; \frac{L\left(\vartheta_w, X^T\right)}{L(\vartheta, X^T)}\, dw}{\int_{\mathbb{W}_T} p\,(\vartheta_w)\; \frac{L(\vartheta_w, X^T)}{L(\vartheta, X^T)}\, dw} = \vartheta + \varphi_T\, \frac{\int_{\mathbb{W}_T} w\, p(\vartheta_w)\, Z_T(w) dw}{\int_{\mathbb{W}_T} p(\vartheta_w)\, Z_T(w)\, dw}.\end{aligned}$$

We have the following presentations for the MLE and BE:

$$\hat{w}_T = \left(\sqrt{T}(\hat{\vartheta}_T^{(1)} - \vartheta_1),\, T(\hat{\vartheta}_T^{(2)} - \vartheta_2)\right) = \arg\sup_{w\in\mathbb{W}_T} Z_T(w),$$

and

$$\tilde{w}_T = \left(\sqrt{T}(\tilde{\vartheta}_T^{(1)} - \vartheta_1),\, T(\tilde{\vartheta}_T^{(2)} - \vartheta_2)\right) = \frac{\int_{\mathbb{W}_T} w\, p(\vartheta_w)\, Z_T(w) dw}{\int_{\mathbb{W}_T} p(\vartheta_w)\, Z_T(w)\, dw}.$$

Lemma 3.1 *The marginal (finite-dimensional) distributions of the random functions $Z_T(\cdot)$ converge to the marginal distributions of the random functions $Z_\vartheta(\cdot)$.*

Proof. As before, we put $\vartheta_v = \vartheta_1 + \frac{v}{\sqrt{T}}$, $\vartheta_u = \vartheta_2 + \frac{u}{T}$ and denoted

$$\delta(\vartheta_w, \vartheta, x) = -2\,\vartheta_v\, q_\vartheta(u, x) - \frac{v}{\sqrt{T}}\,\mathrm{sgn}(x - \vartheta_2), \quad \vartheta_w = (\vartheta_v, \vartheta_u), \tag{3.2}$$

where

$$q_\vartheta(u, x) = \mathrm{sgn}(u)\, 1_{\{\vartheta_2 \wedge \vartheta_u \le x \le \vartheta_2 \vee \vartheta_u\}}.$$

The normalized likelihood ratio $Z_T(\cdot)$ with $\mathbf{P}_\vartheta^{(T)}$ probability 1 admits the representation

$$\begin{aligned} Z_T(v, u) &= \exp\left\{\int_0^T \delta(\vartheta_w, \vartheta, X_t)\, dW_t - \frac{1}{2}\int_0^T \delta(\vartheta_w, \vartheta, X_t)^2 dt\right\} \\ &= \exp\left\{2\,\vartheta_v\, I_T(u, \vartheta) - 2\,\vartheta_1 \vartheta_v \int_0^T q_\vartheta(u, X_t)^2 dt + v\,\zeta_T - \frac{v^2}{2}\right\}, \end{aligned} \tag{3.3}$$

where the stochastic integrals:

$$I_T(u, \vartheta) = -\int_0^T q_\vartheta(u, X_t)\, dW_t, \qquad \zeta_T = -\frac{1}{\sqrt{T}}\int_0^T \mathrm{sgn}\,(X_t - \vartheta_2)\, dW_t.$$

We show that for a fixed u

$$\begin{aligned} \mathbf{P}_\vartheta - \lim_{T\to\infty} \int_0^T q_\vartheta(u, X_t)^2 dt &= \mathbf{P}_\vartheta - \lim_{T\to\infty} \int_0^T 1_{\{\vartheta_2\wedge\vartheta_u \le X_t \le \vartheta_2 \vee \vartheta_u\}}\, dt \\ &= \vartheta_1 |u|. \end{aligned} \tag{3.4}$$

Note that the local time $\Lambda_T(\vartheta, x)$ of the diffusion process (3.1) is defined as

$$\Lambda_T(\vartheta, x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^T 1_{\{|X_t - x| < \varepsilon\}}\, dt; \quad x \in \mathbb{R},$$

and admits the Tanaka-Meyer representation

$$\Lambda_T(\vartheta, x) = |X_T - x| - |X_0 - x| - \int_0^T \operatorname{sgn}(X_t - x)\, dX_t.$$

Let us denote $f_T^\circ(\vartheta, x) = \Lambda_T(\vartheta, x)/T$, the local time estimator of the invariant density $f(\vartheta, x)$. Then we can write the equality: for any integrable function $h(\vartheta, \cdot)$ (see Revuz and Yor 1991)

$$\frac{1}{T} \int_0^T h(\vartheta, X_t)\, dt = \int_{\mathbb{R}} h(\vartheta, x)\, f_T^\circ(\vartheta, x)\, dx.$$

Hence we obtain an estimate for the ordinary integral

$$\begin{aligned}
&\mathbf{E}_\vartheta \left(\int_0^T 1_{\{\vartheta_2 \wedge \vartheta_u \le X_t \le \vartheta_2 \vee \vartheta_u\}}\, dt - \vartheta_1 |u| \right)^2 \\
&\quad = \mathbf{E}_\vartheta \left(\int_{\vartheta_2 \wedge \vartheta_u}^{\vartheta_2 \vee \vartheta_u} f_T^\circ(\vartheta, x)\, dx - |u|\, f(\vartheta, \vartheta_2) \right)^2 \\
&\quad \le 2\, T^2 \mathbf{E}_\vartheta \left(\int_{\vartheta_2 \wedge \vartheta_u}^{\vartheta_2 \vee \vartheta_u} \left[f_T^\circ(\vartheta, x) - f(\vartheta, x) \right] dx \right)^2 \\
&\qquad\quad + 2\, T^2 \left(\int_{\vartheta_2 \wedge \vartheta_u}^{\vartheta_2 \vee \vartheta_u} \left[f(\vartheta, x) - f(\vartheta, \vartheta_2) \right] dx \right)^2 \\
&\quad \le 2\, \frac{|u|^2}{T} \sup_{x \in (\alpha_2, \beta_2)} \mathbf{E}_\vartheta \left(\sqrt{T} \left[f_T^\circ(\vartheta, x) - f(\vartheta, x) \right] \right)^2 + 2\, \vartheta_1^2 \frac{|u|^4}{T^2} \\
&\quad \le 2 \left(\frac{16}{T} + 4 \right) \frac{|u|^2}{T} + 2\, \beta_1^2 \frac{|u|^4}{T^2},
\end{aligned} \tag{3.5}$$

where we used the estimates

$$|f(\vartheta, x) - f(\vartheta, \vartheta_2)| \le 2\, \vartheta_1 |x - \vartheta_2|,$$

and

$$\sup_{x \in (\alpha_2, \beta_2)} \mathbf{E}_\vartheta \left(\sqrt{T} \left[f_T^\circ(\vartheta, x) - f(\vartheta, x) \right] \right)^2 \le \frac{16}{T} + 4.$$

The first one is trivial and the second follows from the representation (1.14)

$$\begin{aligned}
\sqrt{T} \left[f_T^\circ(\vartheta, x) - f(\vartheta, x) \right] &= \frac{2\, f(\vartheta, x)}{\sqrt{T}} \int_{X_0}^{X_T} \frac{1_{\{v > x\}} - F(\vartheta, v)}{f(\vartheta, v)}\, dv \\
&\quad - \frac{2\, f(\vartheta, x)}{\sqrt{T}} \int_0^T \frac{1_{\{X_t > x\}} - F(\vartheta, X_t)}{f(\vartheta, X_t)}\, dW_t.
\end{aligned}$$

To simplify the exposition we suppose that the process X^T is stationary. So the direct calculation gives us the estimate, for any $p \geq 1$

$$
\begin{aligned}
\mathbf{E}_\vartheta & \left|\sqrt{T}\left[f_T^\circ(\vartheta,x) - f(\vartheta,x)\right]\right|^{2p} \\
&\leq 2^{4p-1}\, T^{-p}\, f(\vartheta,x)^{2p}\, \mathbf{E}_\vartheta \left| \int_{X_0}^{X_T} \frac{1_{\{v>x\}} - F(\vartheta,v)}{f(\vartheta,v)}\, dv \right|^{2p} \\
&\quad + 2^{4p-1} f(\vartheta,x)^{2p}\, \mathbf{E}_\vartheta \left| \frac{1_{\{\xi>x\}} - F(\vartheta,\xi)}{f(\vartheta,\xi)} \right|^{2p} \\
&\leq \frac{2^{6p-3}\,\Gamma(2p+1)}{(2p-1)^{2p+1}\,\vartheta_1}\, T^{-p}\, f(\vartheta,x) + \frac{2^{4p-2}}{(2p-1)\,\vartheta_1}\, f(\vartheta,x).
\end{aligned}
$$

Therefore, for any $\delta > 0$

$$
\begin{aligned}
\lim_{T\to\infty} \mathbf{P}_\vartheta & \left\{ \left| \int_0^T 1_{\{\vartheta_2\wedge\vartheta_u \leq X_t \leq \vartheta_2\vee\vartheta_u\}}\, dt - \vartheta_1|u| \right| > \delta \right\} \\
&\leq \lim_{T\to\infty} \frac{1}{\delta^2} \mathbf{E}_\vartheta \left(\int_0^T 1_{\{\vartheta_2\wedge\vartheta_u \leq X_t \leq \vartheta_2\vee\vartheta_u\}}\, dt - \vartheta_1|u| \right)^2 = 0.
\end{aligned}
$$

The central limit theorem for stochastic integrals (see Kutoyants, 2004, Theorem 1.19) and convergence (3.3) provide the asymptotic normality

$$
\mathcal{L}_\vartheta\left\{I_T(u,\vartheta)\right\} \Longrightarrow \mathcal{N}\left(0, \vartheta_1|u|\right), \qquad \mathcal{L}_\vartheta\left\{\zeta_T\right\} \Longrightarrow \mathcal{L}_\vartheta\left\{\zeta\right\} = \mathcal{N}(0,1).
$$

Note that the integrals I_T and ζ_T are asymptotically independent because

$$
\lim_{T\to\infty} \mathbf{E}_\vartheta\left(I_T\zeta_T\right) = \lim_{T\to\infty} \sqrt{T} \int_{\vartheta_2\wedge\vartheta_u}^{\vartheta_2\vee\vartheta_u} f(\vartheta,x)\, dx \leq \lim_{T\to\infty} \beta_1 \frac{|u|}{\sqrt{T}} = 0.
$$

These provide the convergence of one-dimensional distributions of $Z_T(v,u)$ to those of $Z_\vartheta(v,u)$ only.

It can be shown by a similar argument that for any $u_l, u_m \in \mathbb{R}$

$$
\mathbf{P}_\vartheta - \lim_{T\to\infty} \int_0^T q_\vartheta(u_l, X_t)\, q_\vartheta(u_m, X_t)\, dt = \begin{cases} 0 & \text{if} \quad u_l\, u_m \leq 0, \\ |u_l| \wedge |u_m| & \text{if} \quad u_l\, u_m > 0. \end{cases} \tag{3.6}
$$

Further, the same central limit theorem and the convergence (3.6) provides the joint asymptotic normality

$$
\mathcal{L}_\vartheta\left\{I_T(u_1,\vartheta),\ldots,I_T(u_k,,\vartheta)\right\} \Longrightarrow \mathcal{L}_\vartheta\left\{\vartheta_1^{1/2}W(u_1),\ldots,\vartheta_1^{1/2}W(u_k)\right\},
$$

for any $k = 1,2,\ldots$. Finally, we obtain the convergence of the finite-dimensional distributions

$$
\mathcal{L}_\vartheta\left\{Z_T(v_1,u_1),\ldots,Z_T(v_k,u_k)\right\} \Longrightarrow \mathcal{L}_\vartheta\left\{Z_\vartheta(v_1,u_1),\ldots,Z_\vartheta(v_k,u_k)\right\}.
$$

□

Lemma 3.2 *There exist constants C_1 and C_2 such that*

$$
\mathbf{E}_\vartheta\left|Z_T^{1/8}(v_1,u_1) - Z_T^{1/8}(v_2,u_2)\right|^4 \leq C_1\left(|v_1-v_2|^4 + |u_1-u_2|^2\right), \tag{3.7}
$$

$$
\mathbf{E}_\vartheta\left|Z_T^{1/2}(v_1,u_1) - Z_T^{1/2}(v_2,u_2)\right|^2 \leq C_2\left(|v_1-v_2|^2 + |u_1-u_2|\right). \tag{3.8}
$$

Proof. Let us denote

$$V_T = \left(\frac{d\mathbf{P}^{(T)}_{\vartheta_{w_1}}}{d\mathbf{P}^{(T)}_{\vartheta_{w_2}}}(X^T) \right)^{1/8} = \left(\frac{Z_T(v_1,u_1)}{Z_T(v_2,u_2)} \right)^{1/8}$$

(here $\vartheta_{w_i} = (\vartheta_{v_i}, \vartheta_{u_i}) = \left(\vartheta_1 + v_i/\sqrt{T}, \vartheta_2 + u_i/T\right), i = 1,2$).

The stochastic process V^T=$\{V_t, 0 \le t \le T\}$ by the Itô formula admits (with $\mathbf{P}^{(T)}_{\vartheta_{w_1}}$ probability 1) the following representation (see Kutoyants, 2004, Lemma 1.13)

$$V_T = 1 - \frac{7}{128}\int_0^T V_t\, \delta(\vartheta_{w_1}, \vartheta_{w_2}, X_t)^2\, dt + \frac{1}{8}\int_0^T V_t\, \delta(\vartheta_{w_1}, \vartheta_{w_2}, X_t)\, dW_t,$$

where

$$\begin{aligned}\delta(\vartheta_{w_1}, \vartheta_{w_2}, X_t) = &-2\,\mathrm{sgn}(u_2-u_1)\,\vartheta_{v_1} 1_{\left\{\vartheta_{u_1}\wedge\vartheta_{u_2} \le X_t \le \vartheta_{u_1}\vee\vartheta_{u_2}\right\}} \\ &- \frac{v_1-v_2}{\sqrt{T}}\,\mathrm{sgn}\left(X_t - \vartheta_{u_2}\right).\end{aligned}$$

To prove the inequality (3.7), we can write

$$\begin{aligned}\mathbf{E}_\vartheta &\left| Z_T^{1/8}(v_1,u_1) - Z_T^{1/8}(v_2,u_2) \right|^4 = \mathbf{E}_\vartheta Z_T^{1/2}(v_2,u_2)\, |V_T - 1|^4 \\ &\le \left(\frac{1}{8}\right)^4 \mathbf{E}_\vartheta Z_T^{1/2}(v_2,u_2) \left(\int_0^T V_t\, \delta(\vartheta_{w_1}, \vartheta_{w_2}, X_t)\, dW_t \right)^4 \\ &\le \left(\frac{1}{8}\right)^4 \left(\mathbf{E}_{\vartheta_{w_2}} \left(\sup_{0\le t\le T} V_t^8 \right) \right)^{1/2} \left(\mathbf{E}_\vartheta \left(\int_0^T \delta(\vartheta_{w_1}, \vartheta_{w_2}, X_t)^2\, dt \right)^4 \right)^{1/2}.\end{aligned}$$

The first expectation is bounded

$$\mathbf{E}_{\omega_2} \left(\sup_{0\le t\le T} V_t^8 \right) \le 1.$$

Indeed,

$$0 < V_t^8 \le 1 + \int_0^t V_s^8\, \delta(\vartheta_{w_1}, \vartheta_{w_2}, X_s)\, dW_s.$$

For the second expectation we have

$$\begin{aligned}\mathbf{E}_\vartheta &\left(\int_0^T \delta(\vartheta_{w_1}, \vartheta_{w_2}, X_t)^2\, dt \right)^4 \\ &\le 8\,(4\,\vartheta_{v_1}\vartheta_{v_2})^2\, \mathbf{E}_\vartheta \left(\int_0^T 1_{\left\{\vartheta_{u_1}\wedge\vartheta_{u_2} \le X_t \le \vartheta_{u_1}\vee\vartheta_{u_2}\right\}}\, dt \right)^4 + 8\,|v_1 - v_2|^8,\end{aligned}$$

and we have the same estimate as above, This gives us the following estimates

$$
\begin{aligned}
&\mathbf{E}_\vartheta \left(\int_0^T 1_{\{\vartheta_{u_1}\wedge\vartheta_{u_2}\le X_t\le \vartheta_{u_1}\vee\vartheta_{u_2}\}}\, dt\right)^4 \\
&\qquad \le 8\,T^2\,\mathbf{E}_\vartheta \left(\int_{\vartheta_{u_1}\wedge\vartheta_{u_2}}^{\vartheta_{u_1}\vee\vartheta_{u_2}} \sqrt{T}\,[f_T^\circ(\vartheta,x)-f(\vartheta,x)]\;dx\right)^4 \\
&\qquad\quad + 8\,T^4\,\mathbf{E}_\vartheta \left(\int_{\vartheta_{u_1}\wedge\vartheta_{u_2}}^{\vartheta_{u_1}\vee\vartheta_{u_2}} f(\vartheta,x)\,dx\right)^4 \\
&\qquad \le \left(\frac{8^4\,\Gamma(5)}{3^5\,T^4}+\frac{8^3}{3\,T^2}\right)|u_1-u_2|^4 + 8\,\beta_1^4\,|u_1-u_2|^4 .
\end{aligned}
$$

Hence choosing $C_2 = \max\left(4\,\beta_1^4, \sqrt{2}/4\right)/8^3$ we obtain the estimate (3.7). For the inequality (3.8) we have

$$
\begin{aligned}
&\mathbf{E}_\vartheta \left| Z_T^{1/2}(v_1,u_1) - Z_T^{1/2}(v_2,u_2)\right|^2 \\
&\quad = \mathbf{E}_\vartheta Z_T(v_2,u_2)\left|V_T^4-1\right|^2 = \mathbf{E}_{\vartheta_{w_2}}\left|V_T^4-1\right|^2 \\
&\quad \le \frac{1}{4}\,\mathbf{E}_{\vartheta_{w_2}}\left(\int_0^T V_t^4\,\delta(\vartheta_{w_1},\vartheta_{w_2},X_t)\,dW_t\right)^2 = \frac{1}{4}\,T\,\mathbf{E}_{\vartheta_{w_1}}\delta(\vartheta_{w_1},\vartheta_{w_2},\xi)^2 \\
&\quad = \vartheta_{v_1}\vartheta_{v_2} T\int_{\vartheta_{u_1}\wedge\vartheta_{u_2}}^{\vartheta_{u_1}\vee\vartheta_{u_2}} f(\vartheta,x)\,dx + \frac{1}{4}\;|v_1-v_2|^2 \\
&\quad \le \beta_1^3\;|u_1-u_2| + \frac{1}{4}\;|v_1-v_2|^2 ,
\end{aligned}
$$

where we used the representation

$$
V_T^4 = 1 - \frac{1}{8}\int_0^T V_t^4\,\delta(\vartheta_{w_1},\vartheta_{w_2},X_t)^2\,dt + \frac{1}{2}\int_0^T V_t^4\,\delta(\vartheta_{w_1},\vartheta_{w_2},X_t)\,dW_t,
$$

Therefore, for $C_1 = \max\left(\beta_1^3, 1/4\right)$ the inequality (3.8) is proved. □

The last estimate is given in the following lemma.

Lemma 3.3 *For any $N>0$ there exist constants $C = C\,(N) > 0$ and $k>0$ such that*

$$
\mathbf{P}_\vartheta^{(T)}\left\{ Z_T(v,u) > e^{-k(v^2+|u|)}\right\} \le \frac{C}{(v^2+|u|)^N}. \tag{3.9}
$$

Proof. In this proof we use the same arguments as in the proof of Theorem 3.17 and 3.18 in Kutoyants (2004). Let us introduce the set

$$
\mathbb{A} = \left\{\omega : \int_0^T \delta(\vartheta_w,\vartheta,X_t)^2\,dt \ge 8\,k\,(v^2+|u|)\right\},
$$

where $\delta(\vartheta_w,\vartheta,X_t)$ is defined above in (3.2) and the number $k>0$ will be chosen later. Then we can write

$$\begin{aligned}
&\mathbf{P}_\vartheta^{(T)}\left\{Z_T(v,u) > e^{-k(v^2+|u|)}\right\}\\
&\quad= \mathbf{P}_\vartheta^{(T)}\left\{\int_0^T \delta(\vartheta_w,\vartheta,X_t)\,dW_t - \frac{1}{2}\int_0^T \delta(\vartheta_w,\vartheta,X_t)^2\,dt \geq -k(v^2+|u|)\right\}\\
&\quad\leq \mathbf{P}_\vartheta^{(T)}\left\{\int_0^T \frac{\delta(\vartheta_w,\vartheta,X_t)}{2}\,dW_t - \int_0^T \frac{\delta(\vartheta_w,\vartheta,X_t)^2}{8}\,dt \geq \frac{k}{2}(v^2+|u|), \mathbb{A}\right\}\\
&\qquad + \mathbf{P}_\vartheta^{(T)}\left\{\mathbb{A}^c\right\}\\
&\quad\leq e^{-\frac{k}{2}(v^2+|u|)} + \mathbf{P}_\vartheta^{(T)}\left\{\mathbb{A}^c\right\},
\end{aligned}$$

where we used the Markov inequality and the equality

$$\mathbf{E}_\vartheta \exp\left\{\int_0^T \frac{\delta(\vartheta_w,\vartheta,X_t)}{2}\,dW_t - \int_0^T \frac{\delta(\vartheta_w,\vartheta,X_t)^2}{8}\,dt\right\} = 1.$$

Further, put $Y_t = \mathbf{E}_\vartheta \delta(\vartheta_w,\vartheta,X_t)^2 - \delta(\vartheta_w,\vartheta,X_t)^2$. Then we have

$$\begin{aligned}
\mathbf{P}_\vartheta^{(T)}\left\{\mathbb{A}^c\right\} &= \mathbf{P}_\vartheta^{(T)}\left\{\int_0^T \delta(\vartheta_w,\vartheta,X_t)^2\,dt < 8\,k\,(v^2+|u|)\right\}\\
&= \mathbf{P}_\vartheta^{(T)}\left\{\int_0^T Y_t\,dt > \int_0^T \mathbf{E}_\vartheta\delta(\vartheta_w,\vartheta,X_t)^2\,dt - 8\,k\,(v^2+|u|)\right\}.
\end{aligned}$$

For the last mathematical expectation we can write

$$\mathbf{E}_\vartheta \int_0^T \delta(\vartheta_w,\vartheta,X_t)^2\,dt = 4\,\vartheta_1\,\vartheta_v\,T\int_{\vartheta_2\wedge\vartheta_u}^{\vartheta_2\vee\vartheta_u} f(\vartheta,x)\,dx + v^2 \geq k_*\,(v^2+|u|),$$

where $k_* = min\left(4\,\alpha_1^3 e^{-2\,\beta_1|\beta_2-\alpha_2|}, 1\right)$. Now we choose $k = k_*/16$ and (3.10) becomes

$$\mathbf{P}_\vartheta^{(T)}\left\{\mathbb{A}^c\right\} \leq \mathbf{P}_\vartheta^{(T)}\left\{\int_0^T Y_t\,dt > \frac{k_*}{2}\,(v^2+|u|)\right\}.$$

For the last expression we consider separately two sets. The first one is

$$\left\{v,u\,:\,v^2+|u| \leq T^{3/4}\right\},$$

and we at first note that

$$\int_{\vartheta_2\wedge\vartheta_u}^{\vartheta_2\vee\vartheta_u} f(\vartheta,x)\,dx = \vartheta_1\,\frac{|u|}{T} + o\left(T^{-1/4}\right).$$

Therefore we consider the main term $\frac{\vartheta_1|u|}{T}$ only. We can write

$$\begin{aligned}
&\mathbf{P}_\vartheta^{(T)}\left\{\int_0^T Y_t\,dt > \frac{k_*}{2}\,(v^2+|u|)\right\}\\
&\quad\leq \mathbf{P}_\vartheta^{(T)}\left\{4\,\vartheta_1\vartheta_v\left|\int_0^T 1_{\{\vartheta_2\wedge\vartheta_u\leq X_t\leq\vartheta_2\vee\vartheta_u\}}\,dt - \vartheta_1\,|u|\right| > \frac{k_*}{2}\,(v^2+|u|)\right\}\\
&\quad\leq \frac{C}{(v^2+|u|)^{2k}}\,\mathbf{E}_\vartheta\left|\int_0^T 1_{\{\vartheta_2\wedge\vartheta_u\leq X_t\leq\vartheta_2\vee\vartheta_u\}}\,dt - \vartheta_1\,|u|\right|^{2k}\\
&\quad\leq \frac{C}{(v^2+|u|)^{2k}}\left(\frac{u^{2k}}{T^k} + \frac{u^{4k}}{T^{2k}}\right) \leq \frac{C}{(v^2+|u|)^k} + \frac{C\,u^{2k}}{(v^2+|u|)^{2k}}\,T^{-k/2}\\
&\quad\leq \frac{C}{(v^2+|u|)^{k/2}}.
\end{aligned}$$

The last estimates were obtained using the same arguments as that for obtaining (3.5) and we used the inequality $(v^2+|u|)^k \leq T^{3k/4}$. On the set

$$\left\{v, u\,:\, T^{3/4} \leq v^2+|u| < \left[(\beta_1-\alpha_1)^2+\beta_2-\alpha_2\right] T\right\},$$

we have the estimate

$$\begin{aligned}
&\mathbf{P}_\vartheta^{(T)}\left\{\int_0^T Y_t\,dt > \frac{k_*}{2}\,(v^2+|u|)\right\}\\
&\quad\leq \mathbf{P}_\vartheta^{(T)}\left\{4\,\vartheta_1\vartheta_v T\left|\int_{\vartheta_2\wedge\vartheta_u}^{\vartheta_2\vee\vartheta_u}\left[f_T^\circ(\vartheta,x)-f(\vartheta,x)\right]dx\right| > \frac{k_*}{2}\,(v^2+|u|)\right\}\\
&\quad\leq \frac{C\,T^k}{(v^2+|u|)^{2k}}\,\mathbf{E}_\vartheta\left|\int_{\vartheta_2\wedge\vartheta_u}^{\vartheta_2\vee\vartheta_u}\sqrt{T}\left[f_T^\circ(\vartheta,x)-f(\vartheta,x)\right]dx\right|^{2k}\\
&\quad\leq \frac{C}{(v^2+|u|)^{2k}}\frac{u^{2k}}{T^k} \leq \frac{C}{T^k} \leq \frac{C}{(v^2+|u|)^k}.
\end{aligned}$$

Therefore we obtain (3.9) with $N = k/2$. □

The properties of the likelihood ratio $Z_T(\cdot)$ established in Lemmas 3.1-3.3 allow us to apply Theorems 3.1 and 3.2 and so to obtain the desired properties of the MLE and BE. □

3.3 Estimator of the method of moments

The traditional definition of the (generalized) method of moments in classical statistics (statistics of i.i.d. observation) is the following. Let $\{X_1,\ldots,X_n\}$ be i.i.d., with a distribution function $F(\vartheta,\cdot)$ of one observation X_j depending on $\vartheta \in \Theta \subset \mathbb{R}^d$. Introduce a vector function $\mathbf{q}(x) = (q_1(x),\ldots,q_d(x))$ and denote by $m(\vartheta) = \mathbf{E}_\vartheta \mathbf{q}(X_1)$ its mathematical expectation. Define the set of values of $m(\vartheta)$ as $\mathbb{M} = \{m : m(\vartheta), \vartheta \in \Theta\}$ and suppose that the function $\mathbf{q}(\cdot)$ is such that the system of equations

$$m(\vartheta) = m, \quad \vartheta \in \Theta$$

has a unique solution of any $m \in \mathbb{M}$. Then the estimator of the method of moments $\bar\vartheta_n$ is defined as the solution of the following system of equations:

$$m\left(\bar\vartheta_n\right) = \frac{1}{n}\sum_{j=1}^n \mathbf{q}(X_j)$$

Of course, if the value of the right hand side is out of the set $\mathbb{M}$ then $\bar\vartheta_n$ is defined by the value of $m\,(\vartheta)$ which is closest one the this sum. This estimator is consistent and asymptotically normal (see, for example, Borovkov 1998). We describe below the behavior of the continuous-time analog of this estimator. The observed process is ergodic diffusion

$$dX_t = S(\vartheta, X_t)\,dt + dW_t, \quad X_0, \quad 0 \leq t \leq T,$$

where $\vartheta \in \Theta \subset \mathbb{R}^d$, Θ is an open bounded set. The function $S(\cdot,\cdot)$ is such that this equation has a unique weak solution and this solution has ergodic properties with the invariant distribution function $F(\vartheta,\cdot)$ for all $\vartheta\in\Theta$.

Let $\mathbf{q}(x)=(q_1(x),\ldots,q_d(x))$ be a vector function and $m(\vartheta)=\mathbf{E}_\vartheta\mathbf{q}(\xi)$ be its mathematical expectation. Here ξ is a "stationary random variable". The set $\mathbb{M}$ contains all values of $m(\vartheta)$ and the function $\mathbf{q}(\cdot)$ is chosen in such a way that the equation

$$m(\vartheta)=m,\quad \vartheta\in\Theta$$

has a unique solution for any $m\in\mathbb{M}$. The estimator of the method of moments (EMM) $\bar\vartheta_T$ is defined by the equality

$$\left|m(\bar\vartheta_T)-\frac{1}{T}\int_0^T\mathbf{q}(X_t)\,dt\right|=\inf_{\vartheta\in\Theta}\left|m(\vartheta)-\frac{1}{T}\int_0^T\mathbf{q}(X_t)\,dt\right|$$

If this equation has more than one solution then any of them can be taken as the EMM $\bar\vartheta_T$. Note that this estimator admits the representation

$$\bar\vartheta_T=\vartheta_T\,1_{\{\hat m_T\in\mathbb{M}\}}+\vartheta_T^\circ\,1_{\{\hat m_T\in\mathbb{M}^c\}}$$

where

$$\hat m_T=\frac{1}{T}\int_0^T\mathbf{q}(X_t)\,dt$$

and ϑ_T is the solution of the system of equations $m(\vartheta_T)=\hat m_T$ when $\hat m_T\in\mathbb{M}$. The value of ϑ_T° corresponds to $m\in\mathbb{M}$ closest to $\hat m_T$, when $\hat m_T\in\mathbb{M}^c$.

Say, in a one-dimensional case with $\Theta\in(\alpha,\beta)$ and monotonically increasing function $m(\cdot)$ we have $\mathbb{M}=(m(\alpha),m(\beta))$ and

$$\bar\vartheta_T=\alpha\,1_{\{\hat m_T\le m(\alpha)\}}+m^{-1}(\hat m_T)1_{\alpha<\{\hat m_T<\beta\}}+\beta\,1_{\{\hat m_T\ge m(\beta)\}}$$

where $m^{-1}(\cdot)$ is inverse to $m(\cdot)$ function.

Below we show for the switching diffusion process model (3.1), *i.e.*, when $S(\vartheta,x)=-\vartheta_1\mathrm{sgn}(x-\vartheta_2)$ that as in classical statistics this EMM is consistent and asymptotically normal. The asymptotic properties of EMM $\bar\vartheta_T=(\bar\vartheta_T^{(1)},\bar\vartheta_T^{(2)})$ are described in the following theorem.

Theorem 3.3 *The EMM $\bar\vartheta_T$ is consistent: for any $\nu>0$*

$$\lim_{T\to\infty}\mathbf{P}_\vartheta^{(T)}\left\{\left|\bar\vartheta_T-\vartheta\right|>\nu\right\}=0,$$

is asymptotically normal

$$\mathcal{L}_\vartheta\left\{\sqrt{T}\left(\bar\vartheta_T-\vartheta\right)\right\}\Longrightarrow\mathcal{N}\left(0,d(\vartheta)^2\right),\quad d(\vartheta)^2=\begin{pmatrix}\frac{7}{2\,\vartheta_1^2} & 0\\ 0 & \frac{5}{4\,\vartheta_1^4}\end{pmatrix},$$

and for any $p>0$ the moments converge

$$\lim_{T\to\infty}\mathbf{E}_\vartheta\left|\sqrt{T}\left(\bar\vartheta_T-\vartheta\right)\right|^p=\mathbf{E}_\vartheta\left|\zeta\right|^p,\quad \mathcal{L}(\zeta)=\mathcal{N}\left(0,d(\vartheta)^2\right).$$

Proof. Let $\mathbf{q}(x) = (x, x^2)$. We have $\mathbf{E}_\vartheta \xi = \vartheta_2$ and $\mathbf{E}_\vartheta \xi^2 = \vartheta_2^2 + 1/2\vartheta_1^2$. Hence the EMM $\bar\vartheta_T = (\bar\vartheta_T^{(1)}, \bar\vartheta_T^{(2)})$ is

$$\bar\vartheta_T^{(1)} = \frac{1}{\sqrt{2|Y_2 - Y_1^2|}}, \qquad \bar\vartheta_T^{(2)} = Y_1,$$

where

$$Y_1 = \frac{1}{T}\int_0^T X_t\, dt \to \vartheta_2, \qquad Y_2 = \frac{1}{T}\int_0^T X_t^2\, dt \to \vartheta_2^2 + \frac{1}{2\vartheta_1^2}.$$

Hence this estimator is consistent. Let us put $\delta_T = T^{-1/2}$ and

$$\zeta_T = \frac{1}{\sqrt{T}}\int_0^T [X_t - \vartheta_2]\, dt, \qquad \eta_T = \frac{1}{\sqrt{T}}\int_0^T \left[X_t^2 - \vartheta_2^2 - \frac{1}{2\vartheta_1^2}\right] dt.$$

Then we can write

$$\begin{aligned}\bar\vartheta_T^{(1)} &= \frac{1}{\sqrt{2\left|\frac{1}{2\vartheta_1^2} + \delta_T\,\eta_T - 2\,\vartheta_2\,\delta_T\,\zeta_T\right|}}\,(1 + o(1)) \\ &= \vartheta_1\left[1 - \delta_T\vartheta_1^2\left(\eta_T - 2\,\vartheta_2\zeta_T\right)\right](1 + o(1)).\end{aligned}$$

Therefore

$$\sqrt{T}\left(\bar\vartheta_T^{(1)} - \vartheta_1\right) = -\vartheta_1^2\frac{1}{\sqrt{T}}\int_0^T \left[X_t^2 - 2\,\vartheta_2 X_t + \vartheta_2^2 - \frac{1}{2\vartheta_1^2}\right] dt\,(1 + o(1)).$$

Note that for $h(\vartheta, x) = x^2 - 2\,\vartheta_2 x + \vartheta_2^2 - 1/2\vartheta_1^2$ we have $\mathbf{E}_\vartheta h(\vartheta, \xi) = 0$. Hence by the Itô formula we have

$$\begin{aligned}\sqrt{T}\left(\bar\vartheta_T^{(1)} - \vartheta_1\right) &= -\vartheta_1^2\frac{1}{\sqrt{T}}\int_0^T h(\vartheta, X_t)dt\,(1 + o(1)) \\ &= \vartheta_1^2\frac{2}{\sqrt{T}}\int_0^T \frac{1}{f(\vartheta, X_t)}\int_{-\infty}^{X_t} h(\vartheta, u)f(\vartheta, u)\,du\,dW_t\,(1 + o(1)).\end{aligned}$$

Finally

$$\mathcal{L}_\vartheta\left\{\sqrt{T}\left(\bar\vartheta_T^{(1)} - \vartheta_1\right)\right\} \Longrightarrow \mathcal{N}\left(0, d_{\vartheta_1}(\vartheta)^2\right),$$

with

$$d_{\vartheta_1}(\vartheta)^2 = 4\,\vartheta_1^4\mathbf{E}_\vartheta\left(\int_{-\infty}^{\xi}\frac{h(\vartheta, u)f(\vartheta, u)}{f(\vartheta, \xi)}\,du\right)^2 = \frac{7}{2\,\vartheta_1^2}.$$

The EMM $\bar\vartheta_T^{(2)}$ admit the representation

$$\begin{aligned}\sqrt{T}\left(\bar\vartheta_T^{(2)} - \vartheta_2\right) &= \frac{1}{\sqrt{T}}\int_0^T [X_t - \vartheta_2]\, dt = \frac{1}{\sqrt{T}}\int_0^T g(\vartheta, X_t)dt \\ &= -\frac{2}{\sqrt{T}}\int_0^T \frac{1}{f(\vartheta, X_t)}\int_{-\infty}^{X_t} g(\vartheta, u)f(\vartheta, u)\,du\,dW_t,\end{aligned}$$

and the limit variance is

$$d_{\vartheta_2}(\vartheta)^2 = 4\,\mathbf{E}_\vartheta\left(\int_{-\infty}^{\xi}\frac{g(\vartheta, u)f(\vartheta, u)}{f(\vartheta, \xi)}\,du\right)^2 = \frac{5}{4\,\vartheta_1^4}.$$

These representations of the estimators give us the joint asymptotic normality

$$\mathcal{L}_\vartheta\left\{\sqrt{T}\left(\bar\vartheta_T-\vartheta\right)\right\}\Longrightarrow\mathcal{N}\left(0,d(\vartheta)^2\right),$$

where the matrix

$$d(\vartheta)^2=\left(\begin{array}{cc} d_{\vartheta_1}(\vartheta)^2 & d_{\vartheta_1\vartheta_2}(\vartheta)\\ d_{\vartheta_1\vartheta_2}(\vartheta) & d_{\vartheta_2}(\vartheta)^2\end{array}\right),$$

and

$$d_{\vartheta_1\vartheta_2}(\vartheta)=4\,\vartheta_1^2\mathbf{E}_\vartheta\left(\int_{-\infty}^{\xi}\frac{h(\vartheta,v)f(\vartheta,v)}{f(\vartheta,\xi)}\,dv\int_{-\infty}^{\xi}\frac{g(\vartheta,u)f(\vartheta,u)}{f(\vartheta,\xi)}\,du\right)=0.$$

□

Goodness-of-fit test with composite parametric hypothesis

4

Sommaire

4.1 Introduction

This chapter is a continuation of the study started in chapter 2, where the following goodness of fit tests were studied. Let $X^T = \{X_t, 0 \leq t \leq T\}$ be an observed trajectory of ergodic diffusion process

$$dX_t = S(X_t)\,dt + dW_t, \quad X_0, \quad 0 \leq t \leq T, \tag{4.1}$$

where X_0 is its initial value, independent of the Wiener process $\{W_t, t \geq 0\}$, The trend coefficient $S(\cdot)$ is supposed to be unknown to the observer and he has to verify if $S(x) = S_*(x)$, where $S_*(x) = -\mathrm{sgn}(x)$. The Cramér-von Mises statistics, based on the observation X^T solution of (4.1) are:

$$V_T^2 = T\int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_*}(x)\right]^2 dx, \quad W_T^2 = T\int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_*}(x)\right]^2 dF_{S_*}(x),$$

where $f_T^\circ(x)$ is the estimate of the invariant density function $f_{S_*}(x)$ (see, Section 1.2). It is shown that these statistics converge as $(T \to \infty)$ to the following functionals

$$V_T^2 \Longrightarrow \int_{\mathbb{R}} \eta_f^2(x)\,dx, \quad W_T^2 \Longrightarrow \int_{\mathbb{R}} \eta_f^2(x)\,dF_{S_*}(x),$$

where $\{\eta_f(x), x \in \mathbb{R}\}$ is a Gaussian process with zero mean and covariance function defined in Lemma A.3. Hence the corresponding tests

$$\varphi_T(X^T) = 1_{\{V_T^2 > d_\alpha\}}, \quad \phi_T(X^T) = 1_{\{W_T^2 > c_\alpha\}},$$

with the constants d_α and c_α defined by the equations

$$\mathbf{P}\left\{\int_{\mathbb{R}} \eta_f^2(x)\,dx > d_\alpha\right\} = \alpha, \quad \mathbf{P}\left\{\int_{\mathbb{R}} \eta_f^2(x)\,dF_{S_*}(x) > c_\alpha\right\} = \alpha, \tag{4.2}$$

are of asymptotic size α.

At the present work, we consider the same problem but with the composite basic hypothesis and we describe the limit distributions of these Cramér-von Mises type statistics.

4.2 Model of observation

Let us introduce the simple switching diffusion process solution of the stochastic differential equation

$$dX_t = -\rho\,\mathrm{sgn}(X_t-\theta)\,dt + dW_t, \quad X_0, \quad 0 \le t \le T, \tag{4.3}$$

where the trend coefficient $S_*(x,\vartheta) = -\rho\,\mathrm{sgn}(x-\theta)$ and the unknown parameter $\vartheta = (\rho,\theta) \in \Theta$ where the set

$$\Theta = (\alpha_1,\beta_1)\times(\alpha_2,\beta_2), \quad \text{with} \quad \alpha_1 > 0.$$

It is easy to see that the conditions of the existence of the solution and ergodicity (see, Theorem 1.3 and Theorem 1.7), are fulfilled and (4.3) is an ergodic diffusion process with the stationary density $f_{S_*}(\vartheta,x) = \rho\,e^{-2\rho|x-\theta|}$, for $x\in\mathbb{R}$.

Let us remind the properties of the maximum likelihood estimator and the Bayesian estimator for this model. To construct these estimators we introduce the likelihood ratio function as:

$$L\left(\vartheta, X^T\right) = \frac{d\mathbf{P}_\vartheta^{(T)}}{d\mathbf{P}^{(T)}}\left(X^T\right), \quad \vartheta\in\Theta,$$

(here $\mathbf{P}^{(T)} = \mathbf{P}_{0,0}^{(T)}$) by the formula

$$L\left(\vartheta, X^T\right) = \exp\left\{-\rho\int_0^T \mathrm{sgn}(X_t-\theta)\,dX_t - \frac{\rho^2}{2}\,T\right\}.$$

The MLE is defined as a solution of the equation

$$L\left(\hat\vartheta_T, X^T\right) = \sup_{\vartheta\in\Theta} L\left(\vartheta, X^T\right),$$

and the BE for the quadratic loss function and a priori density $p(y)$, $y\in\Theta$, is the conditional mathematical expectation

$$\hat\vartheta_T = \int_\Theta y\,p\left(y|X^T\right)\,dy = \frac{\int_\Theta y\,p\,(y)\;L\left(y,X^T\right)\,dy}{\int_\Theta p\,(y)\;L\left(y,X^T\right)\,dy}.$$

Introduce the normalized likelihood ratio process

$$Z_T(w) = \frac{L(\vartheta+\varphi_T\,w, X^T)}{L(\vartheta,X^T)}, \quad \varphi_T = \begin{pmatrix} T^{-1/2} & 0 \\ 0 & T^{-1}\end{pmatrix}, \quad w\in\mathbb{W}_T,$$

where ϑ is the true value and the set

$$\mathbb{W}_T = \left(\sqrt{T}\,(\alpha_1 - \rho)\,, \sqrt{T}\,(\beta_1 - \rho)\right) \times \left(T\,(\alpha_2 - \theta)\,, T\,(\beta_2 - \theta)\right).$$

We have the following presentations for the MLE and BE:

$$\hat{w}_T = \arg\sup_{w \in \mathbb{W}_T} Z_T(w), \quad \hat{w}_T = \frac{\int_{\mathbb{W}_T} w\, p(\vartheta + \varphi_T\, w)\, Z_T(w) dw}{\int_{\mathbb{W}_T} p(\vartheta + \varphi_T\, w)\, Z_T(w)\, dw},$$

with

$$\hat{w}_T = (\hat{v}_T, \hat{u}_T) = \left(\sqrt{T}(\hat{\rho}_T - \rho),\, T(\hat{\theta}_T - \theta)\right)$$

In order to describe the limit behavior of the MLE and BE, first we introduce the random vectors $\hat{w} = (\hat{v}, \hat{u})$ defined with the help of the following stochastic process:

$$Z_\vartheta(w) = Z_\rho(v)\, Z_\theta(u), \quad w = (u, v) \in \mathbb{R}^2,$$

where

$$Z_\rho(v) = \exp\left\{v\,\zeta - \frac{v^2}{2}\right\}, \qquad Z_\theta(u) = \exp\left\{2\,\rho^{3/2}\, W(u) - 2\,\rho^3\,|u|\right\},$$

as follows:

$$Z_\vartheta(\hat{w}) = \sup_{w \in \mathbb{R}^2} Z_\vartheta(w), \qquad \tilde{w} = \frac{\int_{\mathbb{R}^2} w\, Z_\vartheta\,(w)\; dw}{\int_{\mathbb{R}^2} Z_\vartheta\,(w)\; dw},$$

where $W(\cdot)$ is a two-sided Wiener process and ζ is i.i.d. $\mathcal{N}(0,1)$ random variable independent of $W(\cdot)$.

Proposition 4.1 *The MLE and BE $\hat{\vartheta}_T$ constructed by the observations X^T of the diffusion process* (4.3) *are consistent, i.e., for any $\nu > 0$*

$$\lim_{T \to \infty} \mathbf{P}_\vartheta^{(T)} \left\{|\hat{\vartheta}_T - \vartheta| > \nu\right\} = 0,$$

the normalized difference $\varphi_T^{-1}\,(\hat{\vartheta}_T - \vartheta)$ converges in distribution:

$$\mathcal{L}_\vartheta \left\{\varphi_T^{-1}\,(\hat{\vartheta}_T - \vartheta)\right\} \Longrightarrow \mathcal{L}_\vartheta\,\{\hat{w}\}\,,$$

and for any $p > 0$

$$\lim_{T \to \infty} \mathbf{E}_\vartheta \left|\varphi_T^{-1}\,(\hat{\vartheta}_T - \vartheta)\right|^p = \mathbf{E}_\vartheta\,|\hat{w}|^p\,.$$

4.3 Composite hypothesis

The testing problem is introduced as follow: Suppose that we observe the process $\{X_t, 0 \le t \le T\}$, solution of the stochastic differential equation (4.1) and we wish to test the composite hypothesis

$$\mathcal{H}_* : \qquad S(\cdot) \in \mathcal{F}_\Theta \qquad \text{where} \qquad \mathcal{F}_\Theta = \{S_*(\cdot, \vartheta), \vartheta \in \Theta\}\,.$$

To construct the tests we use the following statistics constructed with the help of the MLE or BE,

$$V_T^2 = T\int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_*}(x,\hat\vartheta_T)\right]^2 dx,$$
$$W_T^2 = T\int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_*}(x,\hat\vartheta_T)\right]^2 dF_{S_*}(x,\hat\vartheta_T).$$

Fix a number $\alpha\in(0,1)$ and define the class of all the tests of asymptotic level $1-\alpha$ as follows

$$\mathcal{K}_\alpha = \{\phi_T : \lim_{T\to\infty}\sup_{\vartheta\in\Theta}\mathbf{E}_\vartheta\phi_T(X^T)\le\alpha\}.$$

Let us introduce the following Gaussian process

$$\zeta_f(x,\vartheta) = \eta_f(x,\vartheta) - \xi\,\left(1/\rho - 2\left|x-\theta\right|\right) f_{S_*}(x,\vartheta),\quad \xi\sim\mathcal{N}(0,1),$$

and denote by $d_\alpha(\vartheta)$ and $c_\alpha(\vartheta)$ the critical values defined by the equations

$$\mathbf{P}_\vartheta\left\{\int_{\mathbb{R}}\zeta_f^2(x,\vartheta)\,dx > d_\alpha(\vartheta)\right\} = \alpha,$$
$$\mathbf{P}_\vartheta\left\{\int_{\mathbb{R}}\zeta_f^2(x,\vartheta)\,dF_{S_*}(x,\vartheta) > c_\alpha(\vartheta)\right\} = \alpha.$$

Then we have the following Theorem

Theorem 4.1 *The C-vM type tests*

$$\varphi_T(X^T) = 1_{\{V_T^2>d_\alpha(\hat\vartheta)\}},\quad \phi_T(X^T) = 1_{\{W_T^2>c_\alpha(\hat\vartheta)\}},$$

belong to $\mathcal{K}_\alpha$.

Proof. The expansion of the function $f_{S_*}(x,\hat\vartheta_T)$ gives us the representation

$$f_{S_*}(x,\hat\vartheta_T) = f_{S_*}(x,\vartheta) + (\hat\rho_T-\rho)\left[e^{-2\rho|x-\theta|} - 2\tilde\rho_T\left|x-\theta\right| e^{-2\tilde\rho_T|x-\theta|}\right]$$
$$+\left(|x-\hat\theta_T| - |x-\theta|\right)\hat\rho_T\, e^{-2\hat\rho_T|x-\tilde\theta_T|},$$

where $|\tilde\theta_T-\theta|\le|\hat\theta_T-\theta|$ and $|\tilde\rho_T-\rho|\le|\hat\rho_T-\rho|$. Then we use the inequality

$$\left||x-\hat\theta_T| - |x-\theta|\right| \le |\hat\theta_T-\theta|,$$

and from the consistency of these estimators it follows that

$$\sqrt{T}\left|f_{S_*}(x,\hat\vartheta_T) - f_{S_*}(x,\vartheta)\right| \le \left(\frac{|\hat u_T|}{\sqrt T} + |\hat v_T|\,\left|1/\rho - 2\left|x-\theta\right|\right|\right) f_{S_*}(x,\vartheta).$$

With the help of Proposition 4.1 we obtain the following estimate

$$\begin{aligned}
&T\,\mathbf{E}_\vartheta\left(\int_{\mathbb{R}}\left[f_{S_*}(x,\hat\vartheta_T) - f_{S_*}(x,\vartheta)\right]^2 dx\right)\\
&\le \frac{2}{T}\mathbf{E}_\vartheta\left|\hat u_T\right|^2\int_{\mathbb{R}} f_{S_*}^2(x,\vartheta)\,dx + 2\,\mathbf{E}_\vartheta\left|\hat v_T\right|^2\int_{\mathbb{R}}\left(1/\rho - 2\left|x-\theta\right|\right)^2 f_{S_*}^2(x,\vartheta)\,dx\\
&\le \frac{\rho}{T}\mathbf{E}_\vartheta\left|\hat u_T\right|^2 + \frac{1}{2\,\rho}\,\mathbf{E}_\vartheta\left|\hat v_T\right|^2 \le \frac{\beta_1\,C_1}{T} + \frac{C_2}{2\,\alpha_1} \le C\left(\frac{1}{T}+1\right),
\end{aligned}\tag{4.4}$$

and

$$
\begin{aligned}
&\mathbf{E}_\vartheta \left(\int_{\mathbb{R}} \left[f_{S_*}(x,\hat\vartheta_T) - f_{S_*}(x,\vartheta) \right]^2 dF_{S_*}(x,\vartheta) \right) \\
&\le \frac{2}{T} \mathbf{E}_\vartheta \left| \hat u_T \right|^2 \int_{\mathbb{R}} f^3_{S_*}(x,\vartheta)\, dx + 2\, \mathbf{E}_\vartheta \left| \hat v_T \right|^2 \int_{\mathbb{R}} \left(1/\rho - 2\left|x-\theta\right|\right)^2 f^3_{S_*}(x,\vartheta)\, dx \\
&\le \frac{\rho^2}{3\,T} \mathbf{E}_\vartheta \left| \hat u_T \right|^2 + \frac{2}{5}\, \mathbf{E}_\vartheta \left| \hat v_T \right|^2 \le \frac{\beta_1^2\, C_1}{3\,T} + \frac{2\, C_2}{5} \le C \left(\frac{1}{T} + 1 \right). \qquad (4.5)
\end{aligned}
$$

Now, using the representations

$$
V_T^2 = \int_{\mathbb{R}} \left[\eta_T(x,\vartheta) - \sqrt{T}\left(f_{S_*}(x,\hat\vartheta_T) - f_{S_*}(x,\vartheta)\right) \right]^2 dx, \qquad (4.6)
$$

$$
\begin{aligned}
W_T^2 &= \int_{\mathbb{R}} \left[\eta_T(x,\vartheta) - \sqrt{T}\left(f_{S_*}(x,\hat\vartheta_T) - f_{S_*}(x,\vartheta)\right) \right]^2 dF_{S_*}(x,\vartheta) \qquad (4.7) \\
&+ \int_{\mathbb{R}} \left[\eta_T(x,\vartheta) - \sqrt{T}\left(f_{S_*}(x,\hat\vartheta_T) - f_{S_*}(x,\vartheta)\right) \right]^2 \left(f_{S_*}(x,\hat\vartheta_T) - f_{S_*}(x,\vartheta) \right) dx,
\end{aligned}
$$

where $\eta_T(x,\vartheta) = \sqrt{T}\left(f_T^\circ(x) - f_{S_*}(x,\vartheta)\right)$, we obtain

$$
\begin{aligned}
V_T^2 &= \int_{\mathbb{R}} \left[\eta_T(x,\vartheta) - \hat v_T\ \left(1/\rho - 2\left|x-\theta\right|\right) f_{S_*}(x,\vartheta) \right]^2 dx + \delta_T, \\
W_T^2 &= \int_{\mathbb{R}} \left[\eta_T(x,\vartheta) - \hat v_T\ \left(1/\rho - 2\left|x-\theta\right|\right) f_{S_*}(x,\vartheta) \right]^2 dF_{S_*}(x,\vartheta) + \delta_T,
\end{aligned}
$$

where $\delta_T \to 0$ in probability.

Introduce the process

$$
\zeta_T(x,\vartheta) = \eta_T(x,\vartheta) - \hat v_T\ \left(1/\rho - 2\left|x-\theta\right|\right) f_{S_*}(x,\vartheta).
$$

We need the following lemma □

Lemma 4.1 *Under hypothesis $\mathcal{H}_*$, we have, the convergence in distribution*

$$
\begin{aligned}
T \int_{\mathbb{R}} \zeta_T^2(x,\vartheta)\, dx &\Longrightarrow \int_{\mathbb{R}} \zeta_f^2(x,\vartheta)\, dx, \\
T \int_{\mathbb{R}} \zeta_T^2(x,\vartheta)\, dF_{S_*}(x,\vartheta) &\Longrightarrow \int_{\mathbb{R}} \zeta_f^2(x,\vartheta)\, dF_{S_*}(x,\vartheta).
\end{aligned}
$$

Proof. To prove this lemma, we need to prove the following conditions (see Ibragimov and Khasminskii 1981, Theorem A22, p.380)

$$
\begin{aligned}
&(1.I) \quad \sup_{T,\vartheta} \mathbf{E}_\vartheta \int_{\mathbb{R}} \zeta_T^2(x,\vartheta)\, dx < \infty, \quad \sup_{T,\vartheta} \mathbf{E}_\vartheta \int_{\mathbb{R}} \zeta_T^2(x,\vartheta)\, dF_{S_*}(x,\vartheta) < \infty, \\
&(2.I) \quad \sup_{T,\vartheta} \mathbf{E}_\vartheta \left| \zeta_T^2(x,\vartheta) - \zeta_T^2(y,\vartheta) \right| \le C|x-y|^{1/2}, \\
&(3.I) \quad \left(\zeta_T(x_1,\vartheta), \ldots, \zeta_T(x_n,\vartheta)\right) \longrightarrow \left(\zeta_f(x_1,\vartheta), \ldots, \zeta_f(x_n,\vartheta)\right).
\end{aligned}
$$

According to Theorem 2.2, the following conditions are satisfied:

$$
\sup_{T,\vartheta} \mathbf{E}_\vartheta \int_{\mathbb{R}} \eta_T^2(x,\vartheta)\, dx < \infty, \quad \sup_{T,\vartheta} \mathbf{E}_\vartheta \int_{\mathbb{R}} \eta_T^2(x,\vartheta)\, dF_{S_*}(x,\vartheta) < \infty, \qquad (4.8)
$$

$$
\sup_{T,\vartheta} \mathbf{E}_\vartheta \left| \eta_T^2(x,\vartheta) - \eta_T^2(y,\vartheta) \right| \le 2\sqrt{22}\, \beta_1 |x-y|^{1/2}. \qquad (4.9)
$$

Using the condition (4.8) and the estimates (4.4) and (4.5) we obtain $(1.I)$.

By a direct calculation and using the condition (4.9) we obtain the following estimate

$$\sup_{T,\vartheta} \mathbf{E}_\vartheta \left|\zeta_T^2(x,\vartheta) - \zeta_T^2(y,\vartheta)\right| \leq 2\sqrt{22}\,\beta_1 |x-y|^{1/2} + 4\,\beta_1\,\mathbf{E}_\vartheta\left|\hat{\vartheta}_T\right|^2 |x-y|$$
$$+ 4\,\beta_1 \left(\mathbf{E}_\vartheta \left|\hat{\vartheta}_T\right|^2\right)^{1/2} |x-y| \leq C\,|x-y|^{1/2}.$$

The convergence of finite-dimensional distributions follows directly from the representation (1.14)

$$\zeta_T(x,\vartheta) = -\frac{2\,f_{S_*}(x,\vartheta)}{\sqrt{T}} \int_0^T \left\{ \frac{1_{\{X_t>x\}} - F_{S_*}(X_t,\vartheta)}{f_{S_*}(X_t,\vartheta)} \right.$$
$$\left. - \left(\frac{1}{2\rho} - |x-\theta|\right) \operatorname{sgn}\left(X_t-\theta\right) \right\} dW_t + o(1),$$
$$= -\frac{2\,f_{S_*}(x,\vartheta)}{\sqrt{T}} \int_0^T h_{S_*}(x,X_t,\vartheta)\,dW_t + o(1),$$

where

$$\hat{\vartheta}_T = -\frac{1}{\sqrt{T}} \int_0^T \operatorname{sgn}\left(X_t-\theta\right)\,dW_t,$$

and from the central limit theorem for stochastic integrals (see Kutoyants, 2004, Theorem 1.19) because for any $\mathbf{x} = \{x_1,\ldots,x_n\}$ and $\boldsymbol{\lambda} = \{\lambda_1,\ldots,\lambda_n\}$ we have

$$\sum_{i=1}^n \lambda_i\,\zeta_T(x_i,\vartheta) = \sum_{i=1}^n \lambda_i\,\frac{2\,f_{S_*}(x_i,\vartheta)}{\sqrt{T}} \int_0^T h_{S_*}(x_i,X_t,\vartheta)\,dW_t$$
$$= \frac{1}{\sqrt{T}} \int_0^T \sum_{i=1}^n \lambda_i\,2\,f_{S_*}(x_i,\vartheta)\,h_{S_*}(x_i,X_t,\vartheta)\,dW_t$$
$$= \frac{1}{\sqrt{T}} \int_0^T q_{S_*}(\boldsymbol{\lambda},\mathbf{x},X_t,\vartheta)\,dW_t,$$

where $q_{S_*}(\boldsymbol{\lambda},\mathbf{x},X_t,\vartheta)$ is a square-integrable function. Therefore

$$\mathcal{L}_\vartheta\left\{\sum_{i=1}^n \lambda_i\,\zeta_T(x_i,\vartheta)\right\} \Longrightarrow \mathcal{N}\left(0,R_{S_*}\right) = \mathcal{L}_\vartheta\left\{\sum_{i=1}^n \lambda_i\,\zeta_f(x_i,\vartheta)\right\},$$

where

$$R_{S_*} = \mathbf{E}_\vartheta q_{S_*}(\boldsymbol{\lambda},\mathbf{x},\xi,\vartheta)^2 = \sum_{i,j} \lambda_i\lambda_j R_{S_*}(x_i,x_j)$$

with

$$R_{S_*}(x_i,x_j) = \mathbf{E}_\vartheta\left[\zeta_f(x_i,\vartheta)\,\zeta_f(x_j,\vartheta)\right]$$
$$= 4\,f_{S_*}(x_i,\vartheta) f_{S_*}(x_j,\vartheta)\mathbf{E}_\vartheta\left[h_{S_*}(x_i,\xi,\vartheta)\,h_{S_*}(x_j,\xi,\vartheta)\right].$$

Further, by a direct calculation, we have

$$\mathbf{E}_\vartheta \left|\zeta_T(x,\vartheta)\right|^2 \leq 2\,e^{-2\rho\,|x-\theta|}.$$

Therefore, by this inequality, for any constant $L \geq \beta_2$, the integral $\int_{|x|>L} \zeta_T^2(x,\vartheta)\,dx$ and $\int_{|x|>L} \zeta_T^2(x,\vartheta)\,dF_{S_*}(x,\vartheta)$ tends to zero at infinity which completes the proof.

□

Now, we consider the case of the one-dimensional parameter $\vartheta = \theta$, i.e., we suppose that the trend coefficient $S_*(x,\vartheta) = -\rho\,\mathrm{sgn}(x-\vartheta)$ where ρ is known and $\vartheta \in \Theta = (\alpha,\beta)$ is an unknown parameter.

Note that in this case, the choice of the thresholds c_α and d_α does not depend on the hypothesis $\mathcal{H}_*$ (asymptotically distribution-free) and these constants are solutions of the equation (4.2).

Theorem 4.2 *The C-vM type tests*

$$\varphi_T(X^T) = 1_{\{V_T^2 > d_\alpha\}}, \quad \phi_T(X^T) = 1_{\{W_T^2 > c_\alpha\}},$$

belong to $\mathcal{K}_\alpha$.

Proof. Here the expansion of $f_{S_*}(x,\hat\vartheta_T)$ gives us the representation

$$f_{S_*}(x,\hat\vartheta_T) = \left(|x-\hat\vartheta_T| - |x-\vartheta|\right) f_{S_*}(x,\tilde\vartheta_T),$$

where $|\tilde\vartheta_T - \vartheta| \leq |\hat\vartheta_T - \vartheta|$ and from the consistency of this estimator it follows that

$$\sqrt{T}\left|f_{S_*}(x,\hat\vartheta_T) - f_{S_*}(x,\vartheta)\right| \leq \frac{|\hat u_T|}{\sqrt{T}} f_{S_*}(x,\vartheta).$$

Hence, using the representations (4.6) and (4.7), we obtain

$$V_T^2 = \int_{\mathbb{R}} \eta_T^2(x,\vartheta)\,dx + \delta_T, \quad W_T^2 = \int_{\mathbb{R}} \eta_T^2(x,\vartheta)\,dF_{S_*}(x,\vartheta) + \delta_T,$$

where $\delta_T \to 0$ in probability and we have the convergence

$$V_T^2 \Longrightarrow \int_{\mathbb{R}} \eta_f^2(x,\vartheta)\,dx, \quad W_T^2 \Longrightarrow \int_{\mathbb{R}} \eta_f^2(x,\vartheta)\,dF_{S_*}(x,\vartheta).$$

Therefore, we obtain the result because ϑ is a translation parameter. $\square$

On limit distributions of some goodness-of-fit tests statistics in general case

5

Sommaire

In this chapter we consider two Cramér-von Mises type tests for an ergodic diffusion process based on the empirical distribution function and local time estimator of invariant density. It is shown that the limiting distributions of proposed tests statistics are defined by integral type functionals of continuous Gaussian processes. For some particular cases, we provide the explicit distributions of limit distribution of these statistics.

5.1 Introduction

We consider the problem of goodness-of-fit tests for ergodic diffusion process, *i.e.*, the observations $\{X_t, 0 \leq t \leq T\}$ are given by the homogeneous stochastic differential equation

$$dX_t = S(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0, \quad 0 \leq t \leq T, \tag{5.1}$$

where the initial random value X_0 is independent of the Wiener process $\{W_t, t \geq 0\}$, the diffusion coefficient $\sigma(\cdot)^2$ is supposed to be known.

The problem consists in testing a simple hypothesis about the trend coefficient $S(\cdot)$. Namely, in our setup the basic hypothesis is: $\mathcal{H}_0 : S(\cdot) = S_0(\cdot)$, where $S_0(\cdot)$ is some known function.

To test $\mathcal{H}_0$ hypothesis we consider two tests statistics of Cramér-von Mises type. The first one is based on the empirical distribution function

$$\hat{F}_T(x) = \frac{1}{T}\int_0^T 1_{\{X_t<x\}}\,dt.$$

The corresponding statistic is:

$$\Delta_T = T \int_{\mathbb{R}} H(x) \left[\hat{F}_T(x) - F_{S_0}(x)\right]^2 dF_{S_0}(x),$$

with

$$H(x) = \frac{\Phi'(x)}{f_{S_0}(x)\left[F_{S_0}(x) - 1\right]^2} \psi(\Phi(x)),$$

where $\psi(\cdot)$ is a continuous positive function and

$$\Phi(x) = \int_{-\infty}^{x} \frac{F_{S_0}(y)^2}{\sigma(y)^2 f_{S_0}(y)} \mathrm{d}y + \left(\frac{F_{S_0}(x)}{F_{S_0}(x) - 1}\right)^2 \int_{x}^{\infty} \frac{\left(F_{S_0}(y) - 1\right)^2}{\sigma(y)^2 f_{S_0}(y)} \mathrm{d}y.$$

The second test is based on local time estimator $f_T^{\circ}(x)$ of invariant density, which can be written as

$$f_T^{\circ}(x) = \frac{\Lambda_T(x)}{T\,\sigma^2(x)} = \frac{|X_T - x| - |X_0 - x|}{T\,\sigma^2(x)} - \frac{1}{T\,\sigma^2(x)} \int_0^T \mathrm{sgn}(X_t - x)\, dX_t.$$

Here $\Lambda_T(x)$ is the local time of diffusion process. The corresponding statistic is:

$$\delta_T = T \int_{\mu}^{\infty} h(x) \left[f_T^{\circ}(x) - f_{S_0}(x)\right]^2 dF_{S_0}(x),$$

where μ is the median of invariant law, that is $F_0(\mu) = 1/2$ and

$$h(x) = \frac{2F_{S_0}(x) - 1}{4\phi(\mu)^2 \sigma(x)^2 f_{S_0}(x)^4}\ \psi\left(\phi(x)/\phi(\mu)\right)\ 1_{\{x \geq \mu\}},$$

with

$$\phi(x) = \int_{-\infty}^{\infty} \frac{\left(1_{\{y > x\}} - F_{S_0}(y)\right)^2}{\sigma(y)^2 f_{S_0}(y)}\, \mathrm{d}y.$$

The goodness of fit tests are $\hat{\Psi}_T = 1_{\{\Delta_T > c_\alpha\}}$ and $\tilde{\Psi}_T = 1_{\{\delta_T > d_\alpha\}}$. These tests with weight functions $H(x) = 1$ and $h(x) = 1$ were proposed by Dachian and Kutoyants (2007), but they are not asymptotically distribution-free.

Obviously, both statistics converge in distribution under the null hypothesis $\mathcal{H}_0$ to quadratic functionals of Gaussian processes (see Negri 1998; Kutoyants 2004). But, due to quite complicate structure of covariance of these processes, the choice of the thresholds d_α, c_α for these tests in the case when $H(x) = 1$ and $h(x) = 1$ are rather complicated. To avoid such difficulty, Kutoyants (2009) has introduced these weight functions to make these tests asymptotically distribution-free. He has shown that under hypothesis $\mathcal{H}_0$, for certain choice of weight functions $H(x)$ and $h(x)$, the limit distributions of proposed statistics are defined by integral type functionals of continuous Gaussian processes

$$\Delta_T \Longrightarrow \int_0^{\infty} \psi(t)\, W_t^2\, dt, \quad \delta_T \Longrightarrow \int_0^{\infty} \psi(t)\, W_{t+1}^2\, dt, \tag{5.2}$$

where $\{W_t, t \geq 0\}$ is a standard Wiener process.

The goodness-of-fit test problem in the classical case of i.i.d. observations has been studied by Anderson and Darling (1952, 1954). In this case they have introduced an analogue of statistics Δ_T and δ_T and they have also studied an asymptotic behavior of these statistics, see also Pettitt (1976), Shorack and Wellner (1986).

This problem received considerable attention over the past years, leading to interesting statistical applications (see, e.g., Nikitin 1995; Rodrigues and Viollaz 1995; Scott 1999 and the references therein).

Recently, some important progress has been made in this framework by Henze and Nikitin (2000), Deheuvels and Martynov (2003). Actually, they have established connections between some Gaussian processes and special functions, via K-L expansions.

In the present work, for some particular cases, we provide explicit distributions of random variables in (5.2) via direct calculation of Laplace transforms.

5.2 Main results

Let $\{W_t, t \geq 0\}$ be a standard Wiener process. Fix $\beta = \frac{1}{2\nu} + 1 > 1$ and denote by $\{z_{\nu,n}, n \geq 1\}$ the sequence of positive zeros of the Bessel function $J_\nu(\cdot)$ of first order and index ν.

Theorem 5.1 *The following equalities hold:*

$$\mathbf{E}\exp\left(\frac{u}{2}\int_0^\infty (t+1)^{-2\beta} W_t^2 dt\right) = \frac{(\nu\sqrt{u})^{\nu/2}}{\sqrt{\Gamma(\nu+1) J_\nu(2\nu\sqrt{u})}}, \quad u < \frac{z_{\nu,1}^2}{4\nu^2}, \tag{5.3}$$

$$D(z) = \prod_{n=1}^\infty \left(1 - \frac{4\nu^2 z}{z_{\nu,n}^2}\right) = \Gamma(\nu+1)\,\frac{J_\nu(2\nu\sqrt{z})}{(\nu\sqrt{z})^\nu}, \quad z \in \mathbb{C}. \tag{5.4}$$

A direct consequence of Theorem (5.1) is the following Corollary:

Corollary 5.1 *The following identity holds*

$$\int_0^\infty (t+1)^{-2\beta} W_t^2\, dt \stackrel{law}{=} 4\nu^2 \sum_{n=1}^\infty \frac{\xi_n^2}{z_{\nu,n}^2},$$

where $\{\xi_n, n \geq 1\}$, are i.i.d. $\mathcal{N}(0,1)$ random variables.

Remark 5.1 *In fact, in the paper by Deheuvels and Martynov (2003) it was proved that*

$$\mathbf{E}\exp\left(\frac{u}{2}\int_0^1 t^{2(\beta-2)} B_t^2 dt\right) = \left(\Gamma(\nu+1)\,\frac{J_\nu(2\nu\sqrt{u})}{(\nu\sqrt{u})^\nu}\right)^{-\frac{1}{2}}, \quad u < \frac{z_{\nu,1}^2}{4\nu^2},$$

where $\{B_t, 0 \leq t \leq 1\}$ is a standard Brownian bridge. Thus, the following equality holds:

$$\int_0^\infty (t+1)^{-2\beta} W_t^2\, dt \stackrel{law}{=} \int_0^1 t^{2(\beta-2)} B_t^2\, dt.$$

Let $\{z_{0,n}, n \geq 1\}$ and $\{\delta_n, n \geq 1\}$ be sequences of positive zeros of the Bessel function $J_0(\cdot)$ and solutions of equation

$$J_0(\delta_n) - \delta_n J_1(\delta_n) = 0,$$

respectively.

Theorem 5.2 *The following equalities hold:*

$$\mathbf{E}\exp\left(\frac{u}{2}\int_0^\infty e^{-2t}W_t^2dt\right) = \left(J_0(\sqrt{u})\right)^{-\frac{1}{2}}, \quad u < z_{0,1}^2, \tag{5.5}$$

$$D(z) = \prod_{n=1}^\infty \left(1 - \frac{z}{z_{0,n}^2}\right) = J_0(\sqrt{z}), \quad z \in \mathbb{C}, \tag{5.6}$$

$$\mathbf{E}\exp\left(\frac{u}{2}\int_0^\infty e^{-2t}W_{t+1}^2dt\right) = \left(J_0(\sqrt{u}) - \sqrt{u}\,J_1(\sqrt{u})\right)^{-\frac{1}{2}}, \quad u < \delta_1^2, \tag{5.7}$$

$$D(z) = \prod_{n=1}^\infty \left(1 - \frac{z}{\delta_n^2}\right) = J_0(\sqrt{z}) - \sqrt{z}\,J_1(\sqrt{z}), \quad z \in \mathbb{C}. \tag{5.8}$$

The direct consequence of Theorem (5.2) is the following Corollary:

Corollary 5.2 *The following equalities hold:*

$$\int_0^\infty e^{-2t}W_{t+1}^2\,dt \overset{law}{=} \sum_{n=1}^\infty \frac{\xi_n^2}{\delta_n^2}, \quad \int_0^\infty e^{-2t}W_t^2\,dt \overset{law}{=} \sum_{n=1}^\infty \frac{\xi_n^2}{z_{0,n}^2},$$

where $\{\xi_n, n \geq 1\}$, are i.i.d. $\mathcal{N}(0,1)$ random variables.

To complete the representation of limiting distributions we write also the result of Deheuvels and Martynov (2003).

In fact, we propose the other proof for this result. Fix $\beta = \frac{1}{2\nu} + 1 > 1$ and denote by $\{z_{\nu-1,n}, n \geq 1\}$ the sequence of positive zeros of the Bessel function $J_{\nu-1}(\cdot)$ of first order and index $\nu - 1$.

Theorem 5.3 *(Deheuvels and Martynov 2003). The following equalities hold:*

$$\mathbf{E}\exp\left(\frac{u}{2}\int_0^\infty (t+1)^{-2\beta}W_{t+1}^2dt\right) = \frac{(\nu\sqrt{u})^{\frac{\nu}{2}-\frac{1}{2}}}{\sqrt{\Gamma(\nu)J_{\nu-1}(2\nu\sqrt{u})}}, \quad u < \frac{z_{\nu-1,1}^2}{4\nu^2}, \tag{5.9}$$

$$D(z) = \prod_{n=1}^\infty \left(1 - \frac{4\nu^2 z}{z_{\nu-1,n}^2}\right) = \Gamma(\nu)\,\frac{J_{\nu-1}(2\nu\sqrt{z})}{(\nu\sqrt{z})^{\nu-1}}, \quad z \in \mathbb{C}. \tag{5.10}$$

Let us denote by A^2 and B^2 the random variables

$$A^2 = \int_0^\infty e^{-2t}W_t^2\,dt, \quad B^2 = \int_0^\infty e^{-2t}W_{t+1}^2\,dt.$$

The following Tables 5.1 and 5.2 provide some values of quantiles of distributions of A^2, B^2. Calculations were based on the explicit form of the Fredholm determinant and Smirnov formula (1.28)

$$\begin{aligned} D(u) &= \prod_{n=1}^\infty (1 - \lambda_n u) = \prod_{n=1}^\infty \left(1 - \frac{u}{\sigma_n}\right), \quad u \geq 0 \quad \text{with} \quad \lambda_n = \frac{1}{\sigma_n}, \\ F(x) &= 1 + \frac{1}{\pi}\sum_{n=1}^\infty (-1)^n \int_{\sigma_{2n-1}}^{\sigma_{2n}} \frac{e^{-xu/2}}{u\sqrt{|D(u)|}}\,du, \quad x \geq 0, \end{aligned}$$

where $\{\lambda_n, n \geq 1\}$ are the eigenvalues of a covariance operator, associated to Wiener processes $\{W_t, t \geq 0\}$ and $\{W_{t+1}, t \geq 0\}$ in $L^2([0, +\infty), e^{-t}\,dt)$.

It follows from Theorem 5.2 that the distribution of A^2 is

$$\mathbf{P}\left(\int_0^\infty A^2 > x\right) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\int_{\sigma_{2n-1}}^{\sigma_{2n}} \frac{e^{-xu/2}}{u\sqrt{-J_0(\sqrt{u})}}\,du$$
$$= \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\int_{z_{0,2n-1}}^{z_{0,2n}} \frac{e^{-xu^2/2}}{u\sqrt{-J_0(u)}}\,du, \quad x \geq 0,$$

where $z_{0,n} = \sqrt{\sigma_n}$, for $n \geq 1$ and the distribution of B^2 is

$$\mathbf{P}\left(\int_0^\infty B^2 > x\right) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\int_{\sigma_{2n-1}}^{\sigma_{2n}} \frac{e^{-xu/2}}{u\sqrt{|J_0(\sqrt{u}) - \sqrt{u}J_1(\sqrt{u})|}}\,du$$
$$= \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\int_{\delta_{2n-1}}^{\delta_{2n}} \frac{e^{-xu^2/2}}{u\sqrt{|J_0(u) - u\,J_1(u)|}}\,du, \quad x \geq 0,$$

where $\delta_n = \sqrt{\sigma_n}$, for $n \geq 1$.

$\mathbf{P}\left(A^2 > x\right)$	0.10	0.05	0.01	0.005	0.001
x	0.552	0.747	1.229	1.445	1.954

Table 5.1 – *Values of some quantiles of distribution of* $\int_0^\infty e^{-2t} W_t^2\,dt$.

$\mathbf{P}\left(B^2 > x\right)$	0.10	0.05	0.01	0.005	0.001
x	1.832	2.552	4.323	5.113	6.982

Table 5.2 – *Values of some quantiles of distribution of* $\int_0^\infty e^{-2t} W_{t+1}^2\,dt$.

5.3 Proofs of Theorems 5.1-5.3

In fact, proofs of Theorems 5.1, 5.2 and 5.3 are based on the approach of Kleptsyna and Le Breton (2002).

Suppose that $\{Z_t, t \geq 0\}$ is a centered continuous Gaussian process with covariance function $K(t,s) = \mathbf{E}Z_t Z_s$, for $0 \leq t, s \leq T$. Then we have the following result:

Theorem 5.4 *For any $T \geq 0$, the following equality holds :*

$$\mathbf{E}\exp\left(\frac{u}{2}\int_0^T Z_s^2\,Q(s)\,ds\right) = \exp\left(\frac{u}{2}\int_0^T \gamma(s,s)\,Q(s)\,ds\right), \quad u \leq 0, \quad (5.11)$$

where $Q(s)$ is a non-negative continuous deterministic function: $[0,+\infty) \to [0,+\infty)$ and $\gamma(t,s)$ is a unique solution of the Riccati-Volterra type integral equation

$$\gamma(t,s) = K(t,s) + u\int_0^s \gamma(t,r)\,\gamma(s,r)\,Q(r)\,dr, \quad 0 \leq s \leq t \leq T, \quad (5.12)$$

such that $\gamma(s,s) = \gamma(s)$, for $s \geq 0$.

In the sequel we suppose that in addition

$$\int_0^{\infty} K(s,s)\, Q(s)\, ds < \infty. \tag{5.13}$$

Obviously, in our case, condition (5.13) is satisfied. Note that due to the inequality

$$K^2(t,s) \le K(s,s)K(t,t),$$

the following condition is satisfied

$$\int_0^{\infty} \int_0^{\infty} K^2(t,s)\, Q(t)\, Q(s)\, ds\, dt < \infty. \tag{5.14}$$

Further, the Lebesgue theorem implies that for any $u \le 0$

$$\mathbf{E} \exp\left(\frac{u}{2} \int_0^{\infty} Z_s^2\, Q(s)\, ds\right) = \exp\left(\frac{u}{2} \int_0^{\infty} \gamma(s,s)\, Q(s)\, ds\right). \tag{5.15}$$

The condition (5.14) implies the compactness of covariance operator and due to Hilbert-Schmidt theorem (see, e.g., Kolmogorov and Fomine 1977, p.459), the equality (5.15) is valid for some positive u, namely

$$\mathbf{E} \exp\left(\frac{u}{2} \int_0^{\infty} Z_s^2\, Q(s)\, ds\right) = \exp\left(\frac{u}{2} \int_0^{\infty} \gamma(s,s)\, Q(s)\, ds\right), \quad u < \frac{1}{\lambda_1},$$

where λ_1 is the first eigenvalue of the covariance operator of the process $\{Z_t, t \ge 0\}$ which belongs to $L^2([0,+\infty), \sqrt{Q(t)}\, dt)$.

Note also that the standard arguments imply that the multiplicity m_n of each positive eigenvalue is finite (see, e.g., Kolmogorov and Fomine 1977, p.467).

Condition (5.13), due to the Weierstrass theorem, imply that $\prod_{n=1}^{\infty} (1 - \lambda_n z)^{m_n}$ is an analytic function in $\mathbb{C}$.

Proof of Theorem 5.1

In this case $K(t,s) = s$, for $s \le t$ and the equation (5.12) reduces to following one

$$\gamma(t,s) = s + u \int_0^{s} \gamma(t,r)\, \gamma(s,r)\, (1+r)^{-2\beta}\, dr, \quad 0 \le s \le t \le T.$$

It is clear that $\gamma(t,s)$ does not depend on t, hence $\gamma(t,s) = \gamma(s)$, where $\gamma(s)$ is the solution of the Riccati differential equation

$$\gamma'(s) = 1 + u\, \gamma(s)^2\, (1+s)^{-2\beta}, \quad \gamma(0) = 0.$$

It is well known, that the solution of this equation can be represented in the form $\gamma(s) = \varphi_1^{-1}(s)\varphi_2(s)$, where the pair (φ_1, φ_2) is a solution of the Cauchy problem

$$\begin{cases} \varphi_1'(s) = -u\,(1+s)^{-2\beta}\varphi_2(s), & \varphi_1(0) = 1, \\ \varphi_2'(s) = \varphi_1(s), & \varphi_2(0) = 0. \end{cases} \tag{5.16}$$

We claim that, for any u, the solution of system (5.16), is the following:

$$\begin{aligned}
\varphi_1(s) &= \nu\pi\,\sqrt{u}\,Y_\nu\left(2\nu\sqrt{u}\right)\,(1+s)^{-\frac{1}{2\nu}-\frac{1}{2}}\,J_{\nu+1}\left(2\nu\sqrt{u}\,(1+s)^{-\frac{1}{2\nu}}\right)\\
&\quad -\nu\pi\,\sqrt{u}\,J_\nu\left(2\nu\sqrt{u}\right)\,(1+s)^{-\frac{1}{2\nu}-\frac{1}{2}}\,Y_{\nu+1}\left(2\nu\sqrt{u}\,(1+s)^{-\frac{1}{2\nu}}\right),\\
\varphi_2(s) &= \nu\pi\,Y_\nu\left(2\nu\sqrt{u}\right)\,(1+s)^{\frac{1}{2}}\,J_\nu\left(2\nu\sqrt{u}\,(s+1)^{-\frac{1}{2\nu}}\right)\\
&\quad -\nu\pi\,J_\nu\left(2\nu\sqrt{u}\right)\,(1+s)^{\frac{1}{2}}\,Y_\nu\left(2\nu\sqrt{u}\,(s+1)^{-\frac{1}{2\nu}}\right).
\end{aligned}$$

Indeed, equation (5.16) implies that

$$\varphi_2''(s)+u\,(1+s)^{-2\beta}\varphi_2(s)=0, \tag{5.17}$$

and

$$\varphi_2(0)=0 \quad \text{and} \quad \varphi_2'(0)=1. \tag{5.18}$$

Hence, for some constants A and B, equation (5.17) has a solution given by

$$\begin{aligned}
\varphi_2(s) &= \sqrt{s+1}\left\{A\,J_{\frac{1}{2(\beta-1)}}\left(\frac{\sqrt{u}\,(s+1)^{1-\beta}}{\beta-1}\right)+B\,Y_{\frac{1}{2(\beta-1)}}\left(\frac{\sqrt{u}\,(s+1)^{1-\beta}}{\beta-1}\right)\right\}\\
&= \sqrt{s+1}\left\{A\,J_\nu\left(2\nu\sqrt{u}\,(s+1)^{-\frac{1}{2\nu}}\right)+B\,Y_\nu\left(2\nu\sqrt{u}\,(s+1)^{-\frac{1}{2\nu}}\right)\right\}.
\end{aligned}$$

Now, taking derivative of last equation with respect to s, we obtain

$$\begin{aligned}
\varphi_2'(s) &= \frac{A\sqrt{u}\,(s+1)^{-\frac{1}{2\nu}}}{\sqrt{s+1}}\left\{\frac{J_\nu\left(2\nu\sqrt{u}\,(s+1)^{-\frac{1}{2\nu}}\right)}{2\sqrt{u}\,(s+1)^{-\frac{1}{2\nu}}}-J_\nu'\left(2\nu\sqrt{u}\,(s+1)^{-\frac{1}{2\nu}}\right)\right\}\\
&\quad+\frac{B\sqrt{u}\,(s+1)^{-\frac{1}{2\nu}}}{\sqrt{s+1}}\left\{\frac{Y_\nu\left(2\nu\sqrt{u}\,(s+1)^{-\frac{1}{2\nu}}\right)}{2\sqrt{u}\,(s+1)^{-\frac{1}{2\nu}}}-Y_\nu'\left(2\nu\sqrt{u}\,(s+1)^{-\frac{1}{2\nu}}\right)\right\}.
\end{aligned}$$

By (A.4), for $\nu>0$, we have the recurrence relations:

$$J_\nu'(x)=\frac{\nu\,J_\nu(x)}{x}-J_{\nu+1}(x), \quad Y_\nu'(x)=\frac{\nu\,Y_\nu(x)}{x}-Y_{\nu+1}(x).$$

Therefore, we get

$$\begin{aligned}
\varphi_1(s)=\varphi_2'(s) &= A\,\sqrt{u}\,(1+s)^{-\frac{1}{2\nu}-\frac{1}{2}}\,J_{\nu+1}\left(2\nu\sqrt{u}\,(1+s)^{-\frac{1}{2\nu}}\right)\\
&\quad +B\,\sqrt{u}\,(1+s)^{-\frac{1}{2\nu}-\frac{1}{2}}\,Y_{\nu+1}\left(2\nu\sqrt{u}\,(1+s)^{-\frac{1}{2\nu}}\right).
\end{aligned}$$

Next, using the initial conditions (5.18), we obtain the linear system

$$\begin{cases}
A\,\sqrt{u}\,J_{\nu+1}\left(2\nu\sqrt{u}\right)+B\,\sqrt{u}\,Y_{\nu+1}\left(2\nu\sqrt{u}\right)=1,\\[2ex]
A\,J_\nu\left(2\nu\sqrt{u}\right)+B\,Y_\nu\left(2\nu\sqrt{u}\right)=0.
\end{cases}$$

From Wronski's determinant property (A.11), we have

$$J_\nu\left(2\nu\sqrt{u}\right)Y_{\nu+1}\left(2\nu\sqrt{u}\right)-J_{\nu+1}\left(2\nu\sqrt{u}\right)Y_\nu\left(2\nu\sqrt{u}\right)=-\frac{1}{\nu\pi\sqrt{u}}. \tag{5.19}$$

Finally,

$$A = \nu\pi\, Y_\nu\left(2\nu\sqrt{u}\right) \quad \text{and} \quad B = -\nu\pi\, J_\nu\left(2\nu\sqrt{u}\right).$$

Moreover, for $u < \frac{z_{\nu,1}^2}{4\nu^2}$, we have

$$\mathbf{E}\exp\left(\frac{u}{2}\int_0^T W_s^2\,(s+1)^{-2\beta}\,ds\right) = \left(\varphi_1(T)\right)^{-1/2}. \tag{5.20}$$

Indeed, to see (5.20), it is sufficient to multiply both sides of the first equation of (5.16) by $\varphi_1^{-1}(s)$ and integrate it further with respect to s. Then, we obtain

$$-\ln(\varphi_1(T)) = u\int_0^T \gamma(s)\,(s+1)^{-2\beta}\,ds,$$

due to initial condition $\varphi_1(0) = 1$. Therefore, we get

$$\exp\left(\frac{u}{2}\int_0^\infty \gamma(s)\,(s+1)^{-2\beta}\,ds\right) = \left(\lim_{T\to\infty}\varphi_1(T)\right)^{-1/2}.$$

In fact, the following properties of the Bessel functions in the neighborhood of 0,

$$J_\nu(x) \sim \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1)}, \quad Y_\nu(x) \sim -\frac{\Gamma(\nu)}{\pi}\left(\frac{x}{2}\right)^{-\nu}, \tag{5.21}$$

imply that, as $T \to \infty$,

$$\begin{aligned}\varphi_1(T) \simeq{}& \frac{\nu\pi\,\sqrt{u}\,Y_\nu\left(2\nu\sqrt{u}\right)}{\Gamma(\nu+2)}(1+T)^{-\frac{1}{2\nu}-\frac{1}{2}}\left(\nu\sqrt{u}\,(1+T)^{-\frac{1}{2\nu}}\right)^{\nu+1}\\ &+\nu\,\sqrt{u}\,J_\nu\left(2\nu\sqrt{u}\right)\,\Gamma(\nu+1)\,(1+T)^{-\frac{1}{2\nu}-\frac{1}{2}}\left(\nu\sqrt{u}\,(1+T)^{-\frac{1}{2\nu}}\right)^{-\nu-1}\\ &\longrightarrow \Gamma(\nu+1)J_\nu\left(2\nu\sqrt{u}\right)\left(\nu\sqrt{u}\right)^{-\nu}.\end{aligned}$$

Thus, we obtain

$$\mathbf{E}\exp\left(\frac{u}{2}\int_0^\infty W_s^2\,(s+1)^{-2\beta}\,ds\right) = \left(\Gamma(\nu+1)\frac{J_\nu\left(2\nu\sqrt{u}\right)}{\left(\nu\sqrt{u}\right)^\nu}\right)^{-1/2}, \quad u < \frac{z_{\nu,1}^2}{4\nu^2}.$$

To prove the second statement of Theorem 5.1 it is sufficient to note that for $u < \frac{1}{\lambda_1}$,

$$\prod_{n=1}^\infty (1-\lambda_n\,u) = \left(\mathbf{E}\exp\left(\frac{u}{2}\int_0^\infty W_s^2\,(s+1)^{-2\beta}\,ds\right)\right)^{-2},$$

where $\{\lambda_n, n \geq 1\}$ is a sequence of the eigenvalues, which exists due to the Hilbert-Schmidt theorem. Hence, for $u < \frac{z_{\nu,1}^2}{4\nu^2}$, we have

$$\prod_{n=1}^\infty (1-\lambda_n\,u) = \Gamma(\nu+1)\frac{J_\nu\left(2\nu\sqrt{u}\right)}{\left(\nu\sqrt{u}\right)^\nu}.$$

Now, condition (5.13) and the analytical properties of involved functions imply that, for any $u \in \mathbb{C}$

$$\prod_{n=1}^\infty (1-\lambda_n\,u) = \Gamma(\nu+1)\frac{J_\nu\left(2\nu\sqrt{u}\right)}{\left(\nu\sqrt{u}\right)^\nu},$$

$$\lambda_n = \frac{4\nu^2}{z_{\nu,1}^2},$$

which achieves the proof of Theorem 5.1.

Proof of Theorem 5.2

In a similar way, to prove the equality (5.5), it is sufficient to solve the equation

$$\gamma(t,s) = s + u\int_0^s \gamma(t,r)\,\gamma(s,r)\,e^{-2r}\,dr, \quad 0 \le s \le t \le T,$$

which reduces to the following one

$$\gamma'(s) = 1 + u\,\gamma^2(s)\,e^{-2r}, \quad \gamma(0) = 0.$$

The solution γ of latter can be represented in form $\gamma(s) = \varphi_1^{-1}(s)\varphi_2(s)$, where the pair (φ_1, φ_2) is a solution of the Cauchy problem

$$\begin{cases} \varphi_1'(s) = -u\,e^{-2s}\varphi_2(s), \quad \varphi_1(0) = 1, \\ \varphi_2'(s) = \varphi_1(s), \quad \varphi_2(0) = 0. \end{cases} \tag{5.22}$$

Solving this equation, we obtain

$$\varphi_1(s) = \frac{\pi}{2}\sqrt{u}\,e^{-s}\left\{Y_0\left(\sqrt{u}\right)\,J_1\left(\sqrt{u}\,e^{-s}\right) - J_0\left(\sqrt{u}\right)\,Y_1\left(\sqrt{u}\,e^{-s}\right)\right\},$$
$$\varphi_2(s) = \frac{\pi}{2}\,Y_0\left(\sqrt{u}\right)\,J_0\left(\sqrt{u}\,e^{-s}\right) - \frac{\pi}{2}\,J_0\left(\sqrt{u}\right)\,Y_0\left(\sqrt{u}\,e^{-s}\right),$$

Indeed, equation (5.22) implies that

$$\varphi_2''(s) + u\,e^{-2s}\varphi_2(s) = 0, \quad \varphi_2(0) = 0,\ \varphi_2'(0) = 1.$$

Hence, for some constants A, B,

$$\varphi_2(s) = A\,J_0\left(\sqrt{u}\,e^{-s}\right) + B\,Y_0\left(\sqrt{u}\,e^{-s}\right).$$

By (A.7), $J_1(x) = -J_0'(x)$ and $Y_1(x) = -Y_0'(x)$. Thus, we get

$$\varphi_1(s) = \varphi_2'(s) = \sqrt{u}\,e^{-s}\left\{A\,J_1\left(\sqrt{u}\,e^{-s}\right) + B\,Y_1\left(\sqrt{u}\,e^{-s}\right)\right\}.$$

Using initial conditions, we obtain

$$\begin{cases} A\,J_0\left(\sqrt{u}\right) + B\,Y_0\left(\sqrt{u}\right) = 0 \\ \\ A\,\sqrt{u}\,J_1\left(\sqrt{u}\right) + B\,\sqrt{u}\,Y_1\left(\sqrt{u}\right) = 1. \end{cases}$$

Finally,

$$A = \frac{\pi}{2}\,Y_0\left(\sqrt{u}\right) \quad \text{and} \quad B = -\frac{\pi}{2}\,J_0\left(\sqrt{u}\right),$$

due to Wronski's determinant property:

$$\sqrt{u}\left(J_0\left(\sqrt{u}\right)Y_1\left(\sqrt{u}\right) - J_1\left(\sqrt{u}\right)Y_0\left(\sqrt{u}\right)\right) = -\frac{2}{\pi}.$$

Then, by (A.3), as $x \downarrow 0$, $J_1(x) \sim x/2$ and $Y_1(x) \sim -2/(\pi\,x)$. Thus,

$$\lim_{T\to\infty} \varphi_1(T) = J_0\left(\sqrt{u}\right).$$

Hence, for $u < z_{1,0}^2$ we obtained, the following equality:

$$\mathbf{E}\exp\left(\frac{u}{2}\int_0^\infty W_s^2 e^{-2s}\,ds\right) = \left(J_0(\sqrt{u})\right)^{-1/2}$$

and for any $u \in \mathbb{C}$

$$\prod_{n=1}^{\infty} (1 - \lambda_n u) = J_0\left(\sqrt{u}\right),$$
$$\lambda_n = \frac{1}{z_{n,0}^2}.$$

To prove equality (5.7) it is sufficient to solve the equation

$$\gamma(t,s) = 1 + s + u \int_0^s \gamma(t,r)\,\gamma(s,r)\,e^{-2r}\,dr, \quad 0 \le s \le t \le T,$$

which reduces to the following one

$$\gamma'(s) = 1 + u\,\gamma^2(s)\,e^{-2r}, \quad \gamma(0) = 1.$$

In a similar way, we obtained that for $u < \delta_1^2$,

$$\mathbf{E}\exp\left(\frac{u}{2}\int_0^{\infty} W_{s+1}^2 e^{-2s}\,ds\right) = \left(J_0(\sqrt{u}) - \sqrt{u}J_1(\sqrt{u})\right)^{-1/2},$$

and hence, for any $u \in \mathbb{C}$

$$\prod_{n=1}^{\infty} (1 - \lambda_n u) = J_0(\sqrt{u}) - \sqrt{u}J_1(\sqrt{u}),$$
$$\lambda_n = \frac{1}{\delta_n^2},$$

which achieves the proof of Theorem 5.2.

Proof of Theorem 5.3

We emphasize that equality (5.9) is the result of Deheuvels and Martynov (2003) with simple change variable $s = 1/(1+t)$. We only show the other proof.

Again, in a similar way, to prove the equality (5.9), it is sufficient to solve the equation

$$\gamma(t,s) = s + 1 + u \int_0^s \gamma(t,r)\,\gamma(s,r)\,(1+r)^{-2\beta}\,dr, \quad 0 \le s \le t \le T,$$

which reduces to the following one

$$\gamma'(s) = 1 + u\,\gamma^2(s)\,(1+s)^{-2\beta}, \quad \gamma(0) = 1.$$

The solution γ of latter can be represented in form $\gamma(s) = \varphi_1^{-1}(s)\varphi_2(s)$, where the pair (φ_1, φ_2) is a solution of the Cauchy problem

$$\begin{cases} \varphi_1'(s) = -u\,(1+s)^{-2\beta}\varphi_2(s), & \varphi_1(0) = 1, \\ \varphi_2'(s) = \varphi_1(s), & \varphi_2(0) = 1. \end{cases}$$

This equation implies that

$$\varphi_2''(s) + u\,(1+s)^{-2\beta}\varphi_2(s) = 0, \quad \varphi_2(0) = 1, \quad \varphi_2'(0) = 1.$$

Hence, for some constants A and B,

$$\varphi_1(s) = A\sqrt{u}\,(1+s)^{-\frac{1}{2\nu}-\frac{1}{2}}\,J_{\nu+1}\left(2\nu\sqrt{u}\,(1+s)^{-\frac{1}{2\nu}}\right) + B\sqrt{u}\,(1+s)^{-\frac{1}{2\nu}-\frac{1}{2}}\,Y_{\nu+1}\left(2\nu\sqrt{u}\,(1+s)^{-\frac{1}{2\nu}}\right),$$
$$\varphi_2(s) = \sqrt{s+1}\left\{A\,J_\nu\left(2\nu\sqrt{u}\,(s+1)^{-\frac{1}{2\nu}}\right) + B\,Y_\nu\left(2\nu\sqrt{u}\,(s+1)^{-\frac{1}{2\nu}}\right)\right\}.$$

Properties of the Bessel functions in the neighborhood of 0 (5.21) we gives, as $T \to \infty$,

$$\begin{aligned}\varphi_1(T) \ &\simeq\ \frac{A\sqrt{u}}{\Gamma(\nu+2)}(1+T)^{-\frac{1}{2\nu}-\frac{1}{2}}\left(\nu\sqrt{u}\,(1+T)^{-\frac{1}{2\nu}}\right)^{\nu+1} \\ &\quad -\frac{B\sqrt{u}\,\Gamma(\nu+1)}{\pi}(1+T)^{-\frac{1}{2\nu}-\frac{1}{2}}\left(\nu\sqrt{u}\,(1+T)^{-\frac{1}{2\nu}}\right)^{-\nu-1} \\ &\quad \longrightarrow -\frac{B\Gamma(\nu+1)}{\nu\pi}\left(\nu\sqrt{u}\right)^{-\nu} = -\frac{B\Gamma(\nu)}{\pi}\left(\nu\sqrt{u}\right)^{-\nu}.\end{aligned}$$

Using initial conditions, we get the linear system

$$\begin{cases} A\sqrt{u}\,J_{\nu+1}\left(2\nu\sqrt{u}\right) + B\sqrt{u}\,Y_{\nu+1}\left(2\nu\sqrt{u}\right) = 1 \\ A\,J_\nu\left(2\nu\sqrt{u}\right) + B\,Y_\nu\left(2\nu\sqrt{u}\right) = 1. \end{cases}$$

From Wronski determinant (5.19), we have

$$B = -\nu\pi\left(J_\nu\left(2\nu\sqrt{u}\right) - \sqrt{u}\,J_\nu\left(2\nu\sqrt{u}\right)\right) = -\nu\pi\,\sqrt{u}J_{\nu-1}\left(2\nu\sqrt{u}\right).$$

Hence, for $u < \frac{z^2_{\nu-1,1}}{4\nu^2}$ we obtained, the following equality:

$$\mathbf{E}\exp\left(\frac{u}{2}\int_0^\infty W^2_{s+1}\,(s+1)^{-2\beta}\,ds\right) = \left(\Gamma(\nu)\frac{J_{\nu-1}\left(2\nu\sqrt{u}\right)}{\left(\nu\sqrt{u}\right)^{\nu-1}}\right)^{-1/2},$$

and for any $u \in \mathbb{C}$

$$\prod_{n=1}^{\infty}(1-\lambda_n\,u) = \Gamma(\nu)\frac{J_{\nu-1}\left(2\nu\sqrt{u}\right)}{\left(\nu\sqrt{u}\right)^{\nu-1}},$$
$$\lambda_n = \frac{4\nu^2}{z^2_{\nu-1,1}},$$

which achieves the proof of Theorem 5.3.

Annexes

A

Sommaire

A.1 Some useful properties of Bessel functions

For any real $\nu \in \mathbb{R}$, the Bessel functions are solutions $y(x)$ of Bessel's differential equation

$$x^2\, y''(x) + x\, y'(x) + (x^2 - \nu^2)\, y(x) = 0. \tag{A.1}$$

We recall some important properties of Bessel functions (see, e.g., Korenev 2002 and Watson 1952, for more details). The Bessel function of first kind $J_\nu(\cdot)$ is defined by the series representation

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{x}{2})^{2n}}{\Gamma(\nu+n+1)\,\Gamma(n+1)}, \quad x > 0.$$

Actually, this function can be defined also in $\mathbb{C} \setminus (-\infty, 0)$ by an analytical extension.

For non-integer ν, the functions $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent, and are therefore the two solutions of the differential equation (A.1). On the other hand, for integer order $\nu = n$, the following relationship is valid (note that the Gamma function becomes infinite for negative integer arguments)

$$J_{-n}(x) = (-1)^n\, J_n(x).$$

This means that the two solutions are no longer linearly independent. In this case, the second linearly independent solution is then found to be the Bessel function of the second kind, denoted by $Y_\nu(\cdot)$ are solutions of the differential equation (A.1). For non-integer ν, it is related to $J_\nu(\cdot)$ by

$$Y_\nu(x) = \frac{J_\nu(x)\, \cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}, \quad x > 0. \tag{A.2}$$

The right-hand side goes to the correct limiting value $Y_n(x)$ as ν goes to some integer n, more precisely, the following representation holds:

$$\begin{aligned} Y_\nu(x) = {} & \frac{2}{\pi} J_\nu(x) \left(\gamma + \ln\left(\frac{x}{2}\right)\right) - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(n-k-1)!}{k!} \left(\frac{2}{x}\right)^{n-2k} \\ & - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{k!(n+k)!} \left[\frac{\Gamma'(n+k+1)}{(n+k)!} + \frac{\Gamma'(k+1)}{k!}\right]. \end{aligned}$$

The both Bessel functions look qualitatively like simple power laws, with the asymptotic forms, as $x \downarrow 0$ (see, e.g., Jackson 1962, p.81)

$$J_\nu(x) \sim \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1)} \quad \nu \geq 0, \qquad Y_\nu(x) \sim -\frac{\Gamma(\nu)}{\pi} \left(\frac{x}{2}\right)^{-\nu} \quad \nu > 0. \tag{A.3}$$

In particular

$$Y_0(x) \sim \frac{2}{\pi}\left(\gamma + \ln\left(\frac{x}{2}\right)\right),$$

where $\gamma = 0.577215\ldots$ is the Euler's constant and we have the following properties (see, e.g., Jahnke and Emde 1945; Watson 1952, p.45-66 and p.478-521).

For a real postif $\nu > 0$, we have the recurrence relations

$$J'_\nu(x) = \frac{\nu\, J_\nu(x)}{x} - J_{\nu+1}(x), \quad Y'_\nu(x) = \frac{\nu\, Y_\nu(x)}{x} - Y_{\nu+1}(x) \tag{A.4}$$

$$J'_\nu(x) = J_{\nu-1}(x) - \frac{\nu\, J_\nu(x)}{x}, \quad Y'_\nu(x) = Y_{\nu-1}(x) - \frac{\nu\, Y_\nu(x)}{x} \tag{A.5}$$

$$J_{\nu+1}(x) + J_{\nu-1}(x) = \frac{2\nu}{x} J_\nu(x), \quad Y_{\nu+1}(x) + Y_{\nu-1}(x) = \frac{2\nu}{x} Y_\nu(x) \tag{A.6}$$

In deduced

$$J_1(x) = -J'_0(x), \qquad Y_1(x) = -Y'_0(x) \tag{A.7}$$

$$\frac{d}{dx}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x), \qquad \frac{d}{dx}(x^\nu Y_\nu(x)) = x^\nu Y_{\nu-1}(x) \tag{A.8}$$

The Wronski determinant for the functions $J_\nu(x)$ and $Y_\nu(x)$ is equal to (see, e.g., Korenev 2002)

$$J_\nu(x) Y'_\nu(x) - J'_\nu(x) Y_\nu(x) = \frac{2}{\pi\, x}. \tag{A.9}$$

By combining (A.9) with (A.5) and (A.4), we obtain the formulas

$$J_\nu(x) Y_{\nu-1}(x) - J_{\nu-1}(x) Y_\nu(x) = \frac{2}{\pi\, x}, \tag{A.10}$$

$$J_\nu(x) Y_{\nu+1}(x) - J_{\nu+1}(x) Y_\nu(x) = -\frac{2}{\pi\, x}. \tag{A.11}$$

For $\nu > -1$ and $a, b > 0$ $(a \neq b)$, by a double integration by part, and using the above equalities, we have

$$\begin{aligned} \int_0^1 x\, J_\nu(ax)\, J_\nu(bx)\, dx &= \frac{a\, J_\nu(b)\, J'_\nu(a) - b\, J_\nu(a)\, J'_{\nu+1}(b)}{b^2 - a^2} \\ &= \frac{a\, b}{2\nu} \left\{ \frac{J_{\nu-1}(a)\, J_{\nu+1}(b) - J_{\nu-1}(b)\, J_{\nu+1}(a)}{b^2 - a^2} \right\} \end{aligned} \tag{A.12}$$

$$\begin{aligned} &\int_0^1 x\, Y_\nu(ax)\, Y_\nu(bx)\, dx \\ &\qquad = \frac{a\, b}{2\nu} \left\{ \frac{x^2 \left(Y_{\nu-1}(ax)\, Y_{\nu+1}(bx) - Y_{\nu-1}(bx)\, Y_{\nu+1}(ax) \right)}{b^2 - a^2} \right\} \Bigg|_0^1 \end{aligned} \tag{A.13}$$

and

$$\begin{aligned} \int_0^1 x\, J_\nu^2(ax)\, dx &= \frac{1}{2} \left\{ J_\nu^2(a) - J_{\nu-1}(a)\, J_{\nu+1}(a) \right\} \\ &= \frac{1}{2} \left\{ J_\nu^2(a) + J_{\nu+1}^2(a) - \frac{2\,\nu}{a} J_\nu(a)\, J_{\nu+1}(a) \right\} \\ &= \frac{1}{2} \left\{ J_\nu^2(a) + J_{\nu-1}^2(a) - \frac{2\,\nu}{a} J_\nu(a)\, J_{\nu-1}(a) \right\} \end{aligned} \tag{A.14}$$

$$\int_0^1 x\, Y_\nu^2(ax)\, dx = \frac{1}{2} x^2 \left\{ Y_\nu^2(ax) - Y_{\nu-1}(ax)\, Y_{\nu+1}(ax) \right\} \Big|_0^1. \tag{A.15}$$

For each $\nu > -1$, the positive zeros of $J_\nu(\cdot)$ (solution of $J_\nu(z) = 0$) form an infinity sequence, denoted by $0 < z_{\nu,1} < z_{\nu,2} < \ldots$, and we have the two Rayleigh extensions of Euler's formula (see, e.g., Watson 1952, p.502 or Lehmer 1943-1945, p.405-407)

$$\sum_{n=1}^{\infty} \frac{1}{z_{\nu,n}^2} = \frac{1}{4(\nu+1)}, \quad \sum_{n=1}^{\infty} \frac{1}{z_{\nu,n}^4} = \frac{1}{16(\nu+1)^2(\nu+2)}. \tag{A.16}$$

An alternative definition of the Bessel function $J_\nu(\cdot)$ makes use of Euler's formula (see, e.g., Watson 1952, p.498)

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{z_{\nu,n}^2}\right) \quad \text{for} \quad z > 0. \tag{A.17}$$

A.2 Auxiliary results

Lemma A.1 *Let $\psi_n(\cdot)$, $n \geq 1$ be defined by equation (2.35). Then, for all $n \geq 1$ and $m \geq 1$, we have*

$$\int_{\mathbb{R}} \psi_n(y)\, \psi_m(y)\, dy = \begin{cases} 0 & \text{if} \quad n \neq m, \\ 1 & \text{if} \quad n = m. \end{cases}$$

Proof. Let us carry out the change of variable $x = e^{-y}$, we have

$$\int_{\mathbb{R}} \psi_n(y)\, \psi_m(y)\, dy = 2\left[J_0\left(\nu_{1,n}\right) J_0\left(\nu_{1,m}\right)\right]^{-1} \int_0^1 x\, J_1(\nu_{1,n}\, x)\, J_1(\nu_{1,m}\, x)\, dx.$$

For all $n \geq 1$ and $m \geq 1$, we take $a = \nu_{1,n}$ and $b = \nu_{1,m}$. We know that $J_1(\nu_{1,n}) = J_1(\nu_{1,m}) = 0$, so we use (A.12) and (A.14), we obtain the conclusion. □

Lemma A.2 *Let $\psi_n(\cdot)$, $n \geq 1$ be defined by equation (2.36). Then, for all $n \geq 1$ and $m \geq 1$, we have*

$$\int_{\mathbb{R}} \psi_n(y)\, \psi_m(y)\, dy = \begin{cases} 0 & \text{if} \quad n \neq m, \\ 1 & \text{if} \quad n = m. \end{cases}$$

Proof. Let us carry out the change of variable $x = e^{-y}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \psi_n(y)\, \psi_m(y)\, dy \\
&= 2\left[\left(J_1(\nu_{2,n}) - \frac{J_2\left(\nu_{2,n}\right) Y_1(\nu_{2,n})}{\alpha\left(\nu_{2,n}\right)}\right)\left(J_1(\nu_{2,m}) - \frac{J_2\left(\nu_{2,m}\right) Y_1(\nu_{2,m})}{\alpha\left(\nu_{2,m}\right)}\right)\right]^{-1} \times \\
&\left\{\frac{J_2\left(\nu_{2,n}\right) J_2\left(\nu_{2,m}\right)}{\alpha\left(\nu_{2,n}\right) \alpha\left(\nu_{2,m}\right)} \int_0^1 x \left(Y_1\left(\nu_{2,n}\, x\right) + \frac{2}{\pi\, \nu_{2,n}\, x}\right)\left(Y_1\left(\nu_{2,m}\, x\right) + \frac{2}{\pi\, \nu_{2,m}\, x}\right) dx \right. \\
&\quad - \frac{J_2\left(\nu_{2,n}\right)}{\alpha\left(\nu_{2,n}\right)} \int_0^1 x \left(Y_1\left(\nu_{2,n}\, x\right) + \frac{2}{\pi\, \nu_{2,n}\, x}\right) J_1\left(\nu_{2,m}\, x\right) dx \\
&\quad - \frac{J_2\left(\nu_{2,m}\right)}{\alpha\left(\nu_{2,m}\right)} \int_0^1 x \left(Y_1\left(\nu_{2,m}\, x\right) + \frac{2}{\pi\, \nu_{2,m}\, x}\right) J_1\left(\nu_{2,n}\, x\right) dx \\
&\quad \left. + \int_0^1 x\, J_1\left(\nu_{2,n} x\right) J_1\left(\nu_{2,m} x\right)\, dx \right\}.
\end{aligned}$$

Now, we take $a = \nu_{2,n}$, for $n = m \geq 1$ and we use equalities (A.14), (A.15) we obtain

$$\int_0^1 x\, J_1^2(\nu_{2,n}x)\, dx = \frac{1}{2}\left\{J_1^2(\nu_{2,n}) - J_0(\nu_{2,n})\, J_2(\nu_{2,n})\right\},$$

and

$$\int_0^1 x\left(Y_1\left(\nu_{2,n}\, x\right) + \frac{2}{\pi\, \nu_{2,n}\, x}\right)^2 dx$$
$$= \frac{1}{2}Y_1^2(\nu_{2,n}) - \frac{1}{2}Y_0(\nu_{2,n})\alpha(\nu_{2,n}) - \frac{2}{\pi\, \nu_{2,n}^2}\beta(\nu_{2,n}),$$

and

$$\int_0^1 x\left(Y_1\left(\nu_{2,n}\, x\right) + \frac{2}{\pi\, \nu_{2,n}\, x}\right) J_1\left(\nu_{2,n}\, x\right) dx = \frac{1}{2}J_1(\nu_{2,n})Y_1(\nu_{2,n})$$
$$- \frac{1}{4}J_2(\nu_{2,n})Y_0(\nu_{2,n}) - \frac{1}{4}J_0(\nu_{2,n})\alpha\left(\nu_{2,n}\right) - \frac{1}{\pi\, \nu_{2,n}^2}\left(J_0\left(\nu_{2,n}\right) - 1\right).$$

Remind that $f\left(\nu_{2,n}\right) = 0$, for $n \geq 1$, therefore

$$\int_{\mathbb{R}} \psi_n^2(y)\, dy = \left[J_1(\nu_{2,n}) - \frac{J_2\left(\nu_{2,n}\right) Y_1(\nu_{2,n})}{\alpha\left(\nu_{2,n}\right)}\right]^{-2}$$
$$\times\Bigg\{J_1^2(\nu_{2,n}) - J_0(\nu_{2,n})\, J_2(\nu_{2,n}) - \frac{J_2\left(\nu_{2,n}\right)}{\alpha\left(\nu_{2,n}\right)}\Bigg(2\, J_1(\nu_{2,n})Y_1(\nu_{2,n})$$
$$- J_2(\nu_{2,n})Y_0(\nu_{2,n}) - J_0(\nu_{2,n})\alpha\left(\nu_{2,n}\right) - \frac{4}{\pi\, \nu_{2,n}^2}\left(J_0\left(\nu_{2,n}\right) - 1\right)\Bigg)$$
$$+ \frac{J_2^2\left(\nu_{2,n}\right)}{\alpha^2\left(\nu_{2,n}\right)}\left(Y_1^2(\nu_{2,n}) - Y_0(\nu_{2,n})\alpha\left(\nu_{2,n}\right) - \frac{4}{\pi\, \nu_{2,n}^2}\beta\left(\nu_{2,n}\right)\right)\Bigg\}$$
$$= \left[J_1(\nu_{2,n}) - \frac{J_2\left(\nu_{2,n}\right) Y_1(\nu_{2,n})}{\alpha\left(\nu_{2,n}\right)}\right]^{-2}\Bigg\{J_1^2(\nu_{2,n}) + \frac{J_2^2\left(\nu_{2,n}\right) Y_1^2(\nu_{2,n})}{\alpha^2\left(\nu_{2,n}\right)}$$
$$- 2\,\frac{J_2\left(\nu_{2,n}\right) Y_1(\nu_{2,n})}{\alpha\left(\nu_{2,n}\right)}J_1(\nu_{2,n}) + \frac{4}{\pi\, \nu_{2,n}^2}\frac{J_2\left(\nu_{2,n}\right)}{\alpha^2\left(\nu_{2,n}\right)}f\left(\nu_{2,n}\right)\Bigg\} = 1.$$

Next, we take $a = \nu_{2,n}$, $b = \nu_{2,m}$, for $n \neq m \geq 1$ and we use equalities (A.12), (A.13) we obtain

$$\int_0^1 x\, J_1\left(\nu_{2,n}x\right) J_1\left(\nu_{2,m}x\right)\, dx = \frac{\nu_{2,n}\nu_{2,m}\left[J_0(\nu_{2,n})\, J_2(\nu_{2,m}) - J_0(\nu_{2,m})\, J_2(\nu_{2,n})\right]}{2\left(\nu_{2,m}^2 - \nu_{2,n}^2\right)},$$

and

$$\int_0^1 x\left(Y_1\left(\nu_{2,n}\, x\right) + \frac{2}{\pi\, \nu_{2,n}\, x}\right)\left(Y_1\left(\nu_{2,m}\, x\right) + \frac{2}{\pi\, \nu_{2,m}\, x}\right) dx$$
$$= \frac{\nu_{2,n}\nu_{2,m}}{2\left(\nu_{2,m}^2 - \nu_{2,n}^2\right)}\Bigg\{Y_0(\nu_{2,n})\alpha\left(\nu_{2,m}\right) - Y_0(\nu_{2,m}x)\alpha\left(\nu_{2,n}\right) - \frac{4}{\pi\nu_{2,n}^2}\beta\left(\nu_{2,n}\right)$$
$$+ \frac{4}{\pi\nu_{2,m}^2}\beta\left(\nu_{2,m}\right)\Bigg\},$$

and

$$\int_0^1 x \left(Y_1 \left(\nu_{2,n}\, x \right) + \frac{2}{\pi\, \nu_{2,n}\, x} \right) J_1 \left(\nu_{2,m}\, x \right) dx$$
$$= \frac{\nu_{2,n}\nu_{2,m}}{2\left(\nu_{2,m}^2 - \nu_{2,n}^2\right)} \Bigg\{ Y_0(\nu_{2,n})\, J_2(\nu_{2,m}) - J_0(\nu_{2,m})\alpha\left(\nu_{2,n}\right)$$
$$+ \frac{4}{\pi\, \nu_{2,m}^2} \left(J_0 \left(\nu_{2,m} \right) - 1 \right) \Bigg\},$$

and

$$\int_0^1 x \left(Y_1 \left(\nu_{2,m}\, x \right) + \frac{2}{\pi\, \nu_{2,m}\, x} \right) J_1 \left(\nu_{2,n}\, x \right) dx$$
$$= \frac{\nu_{2,m}\nu_{2,n}}{2\left(\nu_{2,n}^2 - \nu_{2,m}^2\right)} \Bigg\{ Y_0(\nu_{2,m})\, J_2(\nu_{2,n}) - J_0(\nu_{2,n})\alpha\left(\nu_{2,m}\right)$$
$$+ \frac{4}{\pi\, \nu_{2,n}^2} \left(J_0 \left(\nu_{2,n} \right) - 1 \right) \Bigg\}.$$

For $n \geq 1$ and $m \geq 1$, we know that $f\left(\nu_{2,n}\right) = f\left(\nu_{2,m}\right) = 0$, hence

$$\int_{\mathbb{R}} \psi_n(y)\, \psi_m(y)\, dy$$
$$= \left[\left(J_1(\nu_{2,n}) - \frac{J_2\left(\nu_{2,n}\right) Y_1(\nu_{2,n})}{\alpha\left(\nu_{2,n}\right)} \right) \left(J_1(\nu_{2,m}) - \frac{J_2\left(\nu_{2,m}\right) Y_1(\nu_{2,m})}{\alpha\left(\nu_{2,m}\right)} \right) \right]^{-1}$$
$$\times \frac{4\, \nu_{2,n}\nu_{2,m}}{\pi\left(\nu_{2,m}^2 - \nu_{2,n}^2\right)} \left\{ \frac{J_2\left(\nu_{2,n}\right)}{\nu_{2,m}^2 \alpha\left(\nu_{2,n}\right) \alpha\left(\nu_{2,m}\right)} f\left(\nu_{2,m}\right) \right.$$
$$\left. - \frac{J_2\left(\nu_{2,m}\right)}{\nu_{2,n}^2 \alpha\left(\nu_{2,n}\right) \alpha\left(\nu_{2,m}\right)} f\left(\nu_{2,n}\right) \right\} = 0.$$

□

Lemma A.3 *The covariance function of $(\eta_f(x), x \in \mathbb{R})$ (under hypothesis $\mathcal{H}_0$) is given by*

$$R_f(x,y) = 2\left(1_{\{x\vee y<0\}} e^{2(x\wedge y)} + 1_{\{x\wedge y\geq 0\}} e^{-2(x\vee y)}\right)$$
$$- \left(2(|x| + |y|) + \operatorname{sgn}(xy)\right) e^{-2(|x|+|y|)}.$$

Of course,

$$R_f(x,x) = 2\, e^{-2|x|} - (4\,|x| + 1)\, e^{-4|x|}.$$

Proof. In view of Theorem 2.1 the covariance function of $(\eta_f(x), x \in \mathbb{R})$ can be written as follows:

$$R_f(x,y) = 4\, f_{S_0}(x)\, f_{S_0}(y) \left\{ \int_{-\infty}^{x\wedge y} \left(\frac{F_{S_0}(v)}{f_{S_0}(v)} \right)^2 f_{S_0}(v)\, dv \right.$$
$$\left. - \int_{x\wedge y}^{x\vee y} \frac{F_{S_0}(v)\left(1 - F_{S_0}(v)\right)}{f_{S_0}(v)}\, dv + \int_{x\vee y}^{\infty} \left(\frac{1 - F_{S_0}(v)}{f_{S_0}(v)} \right)^2 f_{S_0}(v)\, dv \right\}.$$

We first consider the case where $x \wedge y > 0$, we have

$$R_f(x,y) = 4\, f_{S_0}(x)\, f_{S_0}(y) \left\{ \int_{-\infty}^{0} \left(\frac{F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv \right.$$
$$+ \int_0^{x\wedge y} \left(\frac{F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv - \int_{x\wedge y}^{x\vee y} \frac{F_{S_0}(v)\,(1-F_{S_0}(v))}{f_{S_0}(v)}\, dv$$
$$\left. + \int_{x\vee y}^{\infty} \left(\frac{1-F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv \right\}.$$

Then

$$R_f(x,y) \;=\; 2\, e^{-2(x\vee y)} - 2(x+y)\, e^{-2(x+y)} - e^{-2(x+y)},$$

where

$$\int_{-\infty}^{0} \left(\frac{F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv = \frac{1}{8},$$
$$\int_0^{x\wedge y} \left(\frac{F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv = \frac{e^{2(x\wedge y)}-1}{2} - (x\wedge y) - \frac{e^{-2(x\wedge y)}-1}{8},$$
$$\int_{x\wedge y}^{x\vee y} \frac{F_{S_0}(v)\big(1-F_{S_0}(v)\big)}{f_{S_0}(v)}\, dv = \frac{x\vee y - x\wedge y}{2} + \frac{e^{-2(x\vee y)} - e^{-2(x\wedge y)}}{8},$$
$$\int_{x\vee y}^{\infty} \left(\frac{1-F({}_{S_0}v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv = \frac{1}{8}\, e^{-2(x\vee y)}.$$

In the case $x \vee y < 0$, by symmetry and similarly to the case $x \wedge y > 0$, we obtain

$$R_f(x,y) \;=\; 2e^{2(x\wedge y)} + 2(x+y)e^{2(x+y)} - e^{2(x+y)}.$$

Finally, in the case where $x \vee y > 0$ and $x \wedge y < 0$, we have

$$R_f(x,y) = 4\, f_{S_0}(x)\, f_{S_0}(y) \left\{ \int_{-\infty}^{x\wedge y} \left(\frac{F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv \right.$$
$$- \int_{x\wedge y}^{0} \frac{F_{S_0}(v)\,(1-F_{S_0}(v))}{f_{S_0}(v)}\, dv - \int_0^{x\vee y} \frac{F_{S_0}(v)\,(1-F_{S_0}(v))}{f_{S_0}(v)}\, dv$$
$$\left. + \int_{x\vee y}^{\infty} \left(\frac{1-F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv \right\}.$$

Then

$$R_f(x,y) \;=\; 2(x\wedge y - x\vee y)e^{2(x\wedge y - x\vee y)} + e^{2(x\wedge y - x\vee y)},$$

where

$$\int_{-\infty}^{x\wedge y} \left(\frac{F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv = \frac{1}{8}\, e^{2(x\wedge y)},$$
$$\int_{x\wedge y}^{0} \frac{F_{S_0}(v)\big(1-F_{S_0}(v)\big)}{f_{S_0}(v)}\, dv = -\frac{x\wedge y}{2} + \frac{e^{2(x\wedge y)}-1}{8},$$
$$\int_0^{x\vee y} \frac{F_{S_0}(v)\big(1-F_{S_0}(v)\big)}{f_{S_0}(v)}\, dv = \frac{x\vee y}{2} + \frac{e^{-2(x\vee y)}-1}{8},$$
$$\int_{x\vee y}^{\infty} \left(\frac{1-F_{S_0}(v)}{f_{S_0}(v)}\right)^2 f_{S_0}(v)\, dv = \frac{1}{8}\, e^{-2(x\vee y)}.$$

So we obtain the result for all $x, y \in \mathbb{R}$. $\square$

Lemma A.4 *Under hypothesis $\mathcal{H}_0$ we have the following inequality*

$$\int_{\mathbb{R}} \mathbf{E}_{S_0}\, G(\xi, x)^2\, \nu(dx) < \infty.$$

Proof. We have

$$\begin{aligned}
\mathbf{E}_{S_0}\, G(\xi, x)^2 &= 4\, f_{S_0}(x)^2 \int_{\mathbb{R}} \left(\int_0^y \frac{\mathbb{1}_{\{v>x\}} - F_{S_0}(v)}{f_{S_0}(v)}\, dv \right)^2 f_{S_0}(y)\, dy \\
&= 4\, f_{S_0}(x)^2 \int_{-\infty}^x \left(\int_0^y \frac{F_{S_0}(v)}{f_{S_0}(v)}\, dv \right)^2 f_{S_0}(y)\, dy \\
&\quad + 4\, f_{S_0}(x)^2 \int_x^\infty \left(\int_0^y \frac{1 - F_{S_0}(v)}{f_{S_0}(v)}\, dv \right)^2 f_{S_0}(y)\, dy.
\end{aligned}$$

By a direct calculation, we obtain

$$\mathbf{E}_{S_0}\, G(\xi, x)^2 \le \frac{1}{2} e^{-2|x|} + e^{-4|x|},$$

and

$$\int_{\mathbb{R}} \mathbf{E}_{S_0}\, G(\xi, x)^2\, dx \le \frac{1}{2} < \infty, \quad \int_{\mathbb{R}} \mathbf{E}_{S_0}\, G(\xi, x)^2\, dF_{S_0}(x) \le \frac{7}{24} < \infty.$$

□

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Notations

$a \wedge b$	The minimum of real numbers a and b.
$a \vee b$	The maximum of real numbers a and b.
$b_\varepsilon = o_\varepsilon(a_\varepsilon)$, $\varepsilon \to \varepsilon_0$	The sequence b_ε is asymptotically smaller than a_ε, i.e., $b_\varepsilon / a_\varepsilon \to 0$ as $\varepsilon \to \varepsilon_0$.
$b_\varepsilon = O_\varepsilon(a_\varepsilon)$, $\varepsilon \to \varepsilon_0$	The sequence b_ε is asymptotically bounded by the sequence a_ε, i.e., $\lvert b_\varepsilon\rvert \leq C\lvert a_\varepsilon\rvert$, $\forall \varepsilon$.
$b_\varepsilon \sim a_\varepsilon$	The sequences b_ε and a_ε are asymptotically equivalent, i.e., $b_\varepsilon / a_\varepsilon \to 1$ as $\varepsilon \to \varepsilon_0$.
$L^2[a, b]$	The set of all square integrable functions on $[a, b]$.
$1_A(x)$	The indicator (or characteristic) function of the set $A \subset \mathbb{R}$, i.e., $1_A(x) = 1$ if $x \in A$, otherwise $1_A(x) = 0$.
$\operatorname{sgn}(x)$	The sign function of a real number x, i.e., $\operatorname{sgn}(x) = -1$ if $x < 0$, $\operatorname{sgn}(x) = 0$ if $x = 0$, otherwise $\operatorname{sgn}(x) = 1$.
$\mathbf{P}$	Probability measures.
$\mathbf{E}$	The mathematical expectation w.r.t. $\mathbf{P}$.
$\xi \sim \mathbf{P}$	The random variable ξ follows the law $\mathbf{P}$, i.e., the probability of the event $\xi \in A$ is equal to $\mathbf{P}(A)$.
$\phi_T(\cdot)$	The test function (decision rule).
α	The level of significance, where $\alpha \in (0, 1)$.
$\mathcal{K}_\alpha$	The class of tests of asymptotic level $1 - \alpha$ (size α).

Titre Test d'ajustement d'un processus de diffusion ergodique à changement de régime

Résumé Nous considérons les tests d'ajustement de type Cramér-von Mises pour tester l'hypothèse que le processus de diffusion observé est un "switching diffusion", c'est-à-dire un processus de diffusion à changement de régime dont la dérive est de type signe. Ces tests sont basés sur la fonction de répartition empirique et la densité empirique. Il est montré que les distributions limites des tests statistiques proposés sont définis par des fonctionnelles de type intégrale des processus Gaussiens continus. Nous établissons les développements de Karhunen-Loève des processus limites correspondants. Ces développements nous permettent de simplifier le problème du calcul des seuils. Nous étudions le comportement de ces statistiques sous les alternatives et nous montrons que ces tests sont consistants. Pour traiter les hypothèses de base composite nous avons besoin de connaître le comportement asymptotique des estimateurs statistiques des paramètres inconnus, c'est pourquoi nous considérons le problème de l'estimation des paramètres pour le processus de diffusion à changement de régime. Nous supposons que le paramètre inconnu est à deux dimensions et nous décrivons les propriétés asymptotiques de l'estimateur de maximum de vraisemblance et de l'estimateur bayésien dans ce cas. L'utilisation de ces estimateurs nous ramène à construire les tests de type Cramér-von Mises correspondants et à étudier leurs distributions limites. Enfin, nous considérons deux tests de type Cramér-von Mises de processus de diffusion ergodiques dans le cas général. Il est montré que pour le choix de certaines des fonctions de poids ces tests sont asymptotiquement " distribution-free ". Pour certains cas particuliers, nous établissons les expressions explicites des distributions limites de ces statistiques par le calcul direct de la transformée de Laplace.

Mots-clés Tests d'ajustement, tests de Cramér-von Mises, développement de Karhunen-Loève, transformée de Laplace, processus Gaussiens, fonctions de Bessel, processus de diffusion ergodiques, propriétés asymptotiques, estimateur du maximum de vraisemblance, estimateur Bayésien, estimateur de la méthode des moments.

Title Goodness-of-fit test for switching ergodic diffusion process

Abstract We consider the Cramér-von Mises goodness-of-fit type tests to test the hypothesis that the observed diffusion process is "switching diffusion", *i.e.*, this process has sign-type trend coefficient. These tests are based on empirical distribution and empirical density functions. It is shown that the limit distributions of the proposed test statistics are defined by the integral type functionals of continuous Gaussian processes. We provide the Karhunen-Loéve expansion of the corresponding limiting processes. This expansion allows us to simplify the problem of the calculation of the thresholds. We study the behavior of these statistics under alternatives and we show that these tests are consistent. To treat the composite basic hypotheses we need to know the asymptotic behavior of statistical estimators of the unknown parameters. That is why we consider the problem of parameter estimation for switching diffusion process. We suppose that the unknown parameter is two-dimensional and we describe the asymptotic properties of the maximum likelihood and Bayesian estimators in this case. Using these estimators we construct the corresponding Cramér-von Mises type tests and study their limit distributions. Finally we consider two Cramér-von Mises type tests for ergodic diffusion process in the general case. It is shown that for a certain choice of weight functions these tests are asymptotically distribution-free. For some particular cases, we provide the explicit expressions of the limit distributions of these statistics via direct calculation of Laplace transforms.

Keywords Goodness-of-fit tests, Cramér-von Mises tests, Karhunen-Loève expansion, Laplace transforms, Gaussian processes, Bessel functions, ergodic diffusion process, asymptotic properties, maximum likelihood estimator, Bayesian estimator, estimator of the method of moments.