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Tests d'hypothèses asymptotiquement optimaux pour les processus de Poisson non homogènes

Présentée par

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Contents

Introduction

Un processus de Poisson est un modèle mathématique pour un phénomène physique caractérisé par des "événements" aléatoires très localisés de manière que chaque événement peut être considéré comme un point. Le processus de Poisson, grâce à sa simplicité mathématique, peut modeliser un grand nombre des phénomènes dans de différents domaines comme l'astronomie, biologie, fiabilité, géologie, médcine, physique, traitement d'image, etc. Pour la définition du processus de Poisson et ses propriétés on pourra se référer à [15] et [24] et pour une liste riche des applications à [42].

Afin de préciser le sujet de cette thèse nous tenons à rappeler qu'un processus de Poisson $X^{(n)}$ sur un ensemble $\mathbb{A}_n \subseteq \mathbb{R}^d$, où $\mathbb{R}^d$ est l'espace Euclidien d dimensionnel, est une mesure aléatoire à valeurs entières qui est caractérisée par les deux propriétés suivantes:

- Pour chaque famille des sous ensembles Boréliens disjoints $\mathbb{B}_1, \ldots, \mathbb{B}_m$ de $\mathbb{A}_n$, les variables aléatoires $X^{(n)}(\mathbb{B}_1), \ldots, X^{(n)}(\mathbb{B}_m)$ (les nombres des événements) sont indépendantes,

- la variable aléatoire $X^{(n)}(\mathbb{B})$, où $\mathbb{B}$ est un ensemble borélien, suit la loi de Poisson de paramètre $\Lambda_n(\mathbb{B})$, où Λ_n est une mesure $\sigma-$finie (sur $\mathbb{A}_n$).

La mesure Λ_n, qui contrôle la moyenne d'appariton des événements (points aléatoires), est dite *la mesure d'intensité*. Si Λ_n est absolument continue par rapport à la mesure de Lebesgue, la densité $S(x) = \frac{\Lambda_n(\mathrm{d}x)}{\mathrm{d}x}$, $x \in \mathbb{A}_n$, est appelée *la fonction d'intensité*. L'existence d'un processus de Poisson avec une fonction d'intensité $S(\cdot)$ donnée (ou avec une mesure d'intensité Λ_n) sur un espace métrique $\mathbb{A}_n$ est démontrée dans [31].

Dans la pratique, nous observons une réalisation d'un phénomène aléatoire. Supposons que cette observation nous conduit à modéliser le problème en utilisant un processus de Poisson. Alors l'étape suivante consiste à trouver la fonction d'intensité $S(\cdot)$. Si cette fonction peut être paramétrée à l'aide d'un paramètre fini dimensionnel ϑ, nous avons un problème statistique paramétrique. Cela signifie que $S(\cdot) \in \{S(\vartheta,\cdot),\ \vartheta \in \Theta\}$ où $S(\vartheta,x),\ x \in \mathbb{A}_n$ est une fonction connue dépendante d'un paramètre ϑ inconnu dont les valeurs sont dans un ensemble $\Theta \subseteq \mathbb{R}^k$, $k \geq 1$. Dans le cas contraire nous avons un problème statistique non paramétrique. La théorie de l'estimation paramétrique et non paramétrique pour les processus de Poisson non homogènes, ainsi que les propriétés asymptotiques des estimateurs sont bien développées dans [27]. On pourra aussi se référer à [7], [14] et [15].

Soit $X^{(n)}$ un processus de Poisson non homogène observé sur un ensemble $\mathbb{A}_n$ avec la fonction d'intensité $S(\vartheta,x),\ x \in \mathbb{A}_n$ dépendante d'un paramètre $\vartheta \in \Theta$ inconnu. Dans cette thèse, nous traitons certains problèmes de test d'hypothèses sur ϑ au vu de la trajectoire $X^{(n)}$. Nous considérons une hypothèse nulle simple $\mathcal{H}_0 : \vartheta = \vartheta_0$ pour ϑ_0 donné, contre différents types d'alternatives. Dans le Chapitre 2, l'alternative est également simple et dans le Chapitre 3, nous avons une alternative unilatérale composée $\mathcal{H}_1 : \vartheta > \vartheta_0$, lorsque ϑ est unidimensionnel. Le cas où ϑ est un vecteur de paramètres inconnus et $\mathcal{H}_1 : \vartheta \neq \vartheta_0$ sera examinée au Chapitre 4.

Les *statistiques de test* utilisées dans ce travail sont basées sur *l'integrale stohastique* par rapport au processus de Poisson

$$I(f_n) = \int_{\mathbb{A}_n} f_n(x)\ X^{(n)}(\mathrm{d}x) = \sum_{x_i 2\, \mathbb{A}_n} f_n(x_i),$$

qui est définie pour une grande classe de fonctions $f_n(\cdot)$. Ici $\{x_i\}$ sont les événtements (les points aléatoires) du processus de Poisson. Les propriétés de base des intégrales stochastiques, y compris les moments et la fonction de caractéristique sont présentés dans le Chapitre 1. Il est bien connu que dans un problème de test d'hypothèses composées, pour n fixé, à l'exception de cas particuliers, il n'y a pas de test *uniformément le plus puissant* dans la classe des tests avec un seuil de signfication $\alpha \in (0,1)$. À cause de cette limitation de la théorie des échantillons de petites tailles, nous nous tournons vers l'approche asymptotique. Un théorème centrale limite pour une suite $I(f_n)$, $n = 1, 2, \ldots$ d' intégrales stochastiques nous amène au cadre d'approche asymptotique de première ordre. Ces résultats sont tirés du livre de Kutoyants, [27]. Afin d'étudier les propriétés de deuxième ordre, nous avons précisé le théorème centrale limite en utilisant le développement du type d'Edgeworth pour la fonction de répartition $F_n(\cdot)$ de $I(f_n)$ (centrée reduite). Le développement à un terme après le terme gaussien a la forme suivante :

$$F_n(y) = \mathcal{N}(y) + q(y)\, n(y)\, \varepsilon_n + O(\varepsilon_n^2),$$

uniformément par rapport à y, où $\mathcal{N}(\cdot)$ et $n(\cdot)$ désignent les fonctions de répartition et la densité de la loi normale standard et $q(\cdot)$ est un polynôme. Le petit paramètre $\varepsilon_n \to 0$ lorsque $n \to \infty$. Nous rappelons que d'après le théorème central limite, $F_n(y) = \mathcal{N}(y) + o(1)$. Dans la section 1.5, nous avons présenté le développement d'Edgeworth (le cas où $I(f_n)$ est unidimensionnel) pour un et deux termes après le terme gaussien. Le cas général (plusieurs termes) est traité par Kutoyants [27]. Notons que pour calculer "*l'asymptotic deficiency*" d'un test par rapport à un autre, nous avons besoin de développer les fonctions de répartition des statistiques de test à deux termes après le terme gaussien (voir Remarque 9, Chapitre 3). Le développement d'Edgeworth pour un vecteur d'intégrales stochastiques est traité dans le Chapitre 4. L'importance du développement asymptotique des statistiques et de leurs fonctions de répartition dans la théorie des tests d'hypothèses et de l'estimation est bien connue. On pourra se référer à [10], [34], [35], [36] et leurs références. Dans le Chapite 1, section 1.5, comme cas particulier nous avons utilisé ces résultats dans le modèle

$$S(\vartheta, x) = S(\vartheta\, x) + \lambda_0, \quad x \in [0, n], \qquad n = 1, 2 \ldots,$$

où $S(\cdot)$ est une fonction connue, périodique de période $\tau > 0$, non constante et dérivable avec la dérivée $\dot{S}(\cdot)$. Le paramètre $\lambda_0 > 0$ est connu, mais la fréquence $\vartheta > 0$ est inconnue. Pour tester $\mathcal{H}_0 : \vartheta = \vartheta_0$ contre $\mathcal{H}_1 : \vartheta > \vartheta_0$, où ϑ_0 est une valeur connue, on utilise le test basé sur l'intégrale stochastique (fortement non homogène)

$$\Delta_n(\vartheta_0) = \varphi_n \int_0^n x\, \frac{\dot{S}(\vartheta_0\, x)}{S(\vartheta_0, x)}\, \pi(\mathrm{d}x),$$

où $\pi(\mathrm{d}x) = X^{(n)}(\mathrm{d}x) - S(x, \vartheta_0)\,\mathrm{d}x$ est le processus de Poisson centré et

$$\varphi_n^{-2} = \int_0^n x^2 \,\frac{\dot S(\vartheta_0\, x)^2}{S(\vartheta_0, x)}\,\mathrm{d}x \sim d\, n^{3/2}$$

pour une constante $d > 0$. En utilisant le développement d'Edgeworth sous l'hypothèse nulle $\mathcal{H}_0 : \vartheta = \vartheta_0$, on a montré que le risque de première espèce du test

$$\phi_n^{*}\left(X^{(n)}\right) = 1 \quad \text{si} \quad \Delta_n(\vartheta_0) > c_n$$

avec

$$c_n = z_\alpha - \frac{\gamma_{3,n}}{6}(1 - z_\alpha^2) - \frac{\gamma_{3,n}^2}{72}\left(-2\, z_\alpha^5 + 8\, z_\alpha^3 - 12\, z_\alpha\right) - \frac{\gamma_{4,n}}{4!}(3 z_\alpha - z_\alpha^3),$$

est égal à

$$\mathbf{P}_{\vartheta_0}^{(n)}\left\{\Delta_n(\vartheta_0) > c_n\right\} = \alpha + O(n^{-3/2})$$

où $\alpha \in (0, 1)$ est le seuil de signification; z_α désigne $1 - \alpha$ quantile de la loi gaussienne standard; $\mathbf{P}_{\vartheta_0}^{(n)}$ est la distribution du processus sous $\mathcal{H}_0$ et

$$\gamma_{3,n} = \sqrt{\frac{27}{16\, n\, \tau}}\,\frac{\int_0^\tau \frac{\dot S(\vartheta_0\, x)^3}{S(\vartheta_0, x)^2}\,\mathrm{d}x}{\left(\int_0^\tau \frac{\dot S(\vartheta_0\, x)^2}{S(\vartheta_0, x)}\,\mathrm{d}x\right)^{3/2}}\,(1 + o(1))$$

$$\gamma_{4,n} = \frac{9\, \tau}{5\, n}\,\frac{\int_0^\tau \frac{\dot S(\vartheta_0\, x)^4}{S(\vartheta_0, x)^3}\,\mathrm{d}x}{\left(\int_0^\tau \frac{\dot S(\vartheta_0\, x)^2}{S(\vartheta_0, x)}\,\mathrm{d}x\right)^{2}}\,(1 + o(1)).$$

On voit que $\gamma_{3,n} = O(n^{-1/2})$ et $\gamma_{4,n} = O(n^{-1})$. Notons que le théorème centrale limite, propose le test

$$\overline{\phi}_n\left(X^{(n)}\right) = 1 \quad \text{si} \quad \Delta_n(\vartheta_0) > z_\alpha,$$

dont le risque de première espèce est égal à

$$\mathbf{P}_{\vartheta_0}^{(n)}\left\{\Delta_n(\vartheta_0) > z_\alpha\right\} = \alpha + o(1).$$

Le développement sous *l'alternative locale* $\vartheta_u = \vartheta_0 + \varphi_n\, u$, $u > 0$, nous permet de calculer la puissance du test pour les alternatives proches de ϑ_0 qui convergent vers ϑ_0 avec la vitesse de φ_n.

Le risque de deuxième espèce d'un test raisonnable pour une valeur donnée ϑ (fixée) d'alternative , converge vers 0 lorsque $n \to \infty$. Afin de déterminer la vitesse de convergence, nous avons étudié *les grandes déviations* pour une suite d'intégrales stochastiques. Les conditions sont précisées dans le Chapitre 1. La démonstration est essentiellement basée sur le théorème de Gärtner-Ellis ([16], Section 2.3) et un théorème de Cramér ([16], Section 2.2). Ces résultats sont utilisés dans le Chapitre 2. La dernière section du Chapitre 1, a pour but d'introduire les notions de base de tests d'hypothèses.

Dans le Chapitre 2, nous présentons le test le plus puissant (test de Neyman-Pearson) basé sur le rapport de vraisemblance pour deux hypothèses simples $\mathcal{H}_0$: $\vartheta = \vartheta_0$ et $\mathcal{H}_1 : \vartheta = \vartheta_1$. Nous considérons le cas général, i.e., les distributions $\mathbf{P}_{\vartheta_0}^{(n)}$ et $\mathbf{P}_{\vartheta_1}^{(n)}$ du processus sous $\mathcal{H}_0$ et $\mathcal{H}_1$, ne sont pas nécessairement équivalantes et nous tenons également compte des contributions des parties singulières. Puisque le rapport de vraisemblance dépend d'une intégrale stochastique, le calcul des paramètres du test, y compris le seuil (the threshold), la partie randomisée et la puissance, ne sont pas faciles. L'approche traditionnelle asymptotique, en utilisant un théorème centrale limite pour une suite des intégrales stochastiques, nous permet de construire un test (proche du test de Neyman-Pearson) dont le risque de première espèce est égale à $\alpha + o(1)$ lorsque $n \to \infty$. Le risque de deuxième espèce de ce test tend vers zéro. Nous avons modifié le test à l'aide du développement d'Edgeworth en précisant le seuil (the threshold). Le risque de première espèce du test modifié est égal à $\alpha + O(\varepsilon_n^2)$, où $\varepsilon_n \to 0$. Comme ces tests sont *consistants*, i.e., leurs puissances convergent vers 1, alors les vitesses de convergence sont décrites en utilisant des *principes de grandes déviations* (PGD) pour des intégrales stochastiques. Nous avons montré que la puissance tend exponentiellement vers 1 avec une certaine vitesse.

Le Chapitre 3 a pour but de construire un test *efficace de deuxième ordre* pour $\mathcal{H}_0 : \vartheta = \vartheta_0$ contre $\mathcal{H}_1 : \vartheta > \vartheta_0$. Nous rappelons que si la famille des distributions $\left\{\mathbf{P}_{\vartheta}^{(n)}, \vartheta \in \Theta\right\}$ du processus $X^{(n)}$ est *localement asymptotiquement normale* (LAN) au point de ϑ_0, alors le test

$$\bar{\phi}_n\left(X^{(n)}\right) = 1 \quad \text{si} \quad \Delta_n(\vartheta_0) > z_\alpha$$

basé sur la statistique

$$\Delta_n\left(\vartheta_0\right) = I_n(\vartheta_0)^{-1/2} \int_{\mathbb{A}_n} \frac{\dot{S}\left(\vartheta_0, x\right)}{S\left(\vartheta_0, x\right)} \left(X^{(n)}\left(\mathrm{d}x\right) - S\left(\vartheta_0, x\right)\,\mathrm{d}x\right),$$

où $\dot{S}\left(\vartheta, x\right)$ est la dérivée de $S\left(\vartheta, x\right)$ par rapport à ϑ et $I_n(\vartheta_0)$ est l'information de Fisher au point ϑ_0, est *asymptotiquement localement uniformément le plus puissant.* Cela signifie que

$$\mathbf{E}_{\vartheta_u}\bar{\phi}_n\left(X^{(n)}\right) - \mathbf{E}_{\vartheta_u}\tilde{\phi}_n\left(X^{(n)}\right) = o(1),$$

uniformément par rapport à $u \in [0,\, K]$ pour tout $K > 0$ (voir [41]). Ici $\tilde{\phi}_n$ est le test le plus puissant au seuil $\alpha + o(1)$ pour tester $\mathcal{H}_0 : \vartheta = \vartheta_0$ contre l'alternative locale $\mathcal{H}_u$: $\vartheta = \vartheta_0 + I_n(\vartheta_0)^{-1/2}\,u$ (avec $u > 0$) qui tend vers ϑ_0 avec la vitesse de $\varphi_n = I_n(\vartheta_0)^{-1/2}$. L'espérance mathématique par rapport à l'alternative locale $\vartheta_u = \vartheta_0 + I_n(\vartheta_0)^{-1/2}\,u$ est notée $\mathbf{E}_{\vartheta_u}$. On voit que la probabilité d'erreur de première espèce de $\bar{\phi}_n$ est égale à $\alpha + o(1)$. Nous avons introduit le test modifié suivant

$$\phi_n^{\star}\left(X^{(n)}\right) = 1 \quad \text{si} \quad \Delta_n(\vartheta_0) > c_n$$

avec

$$c_n = z_\alpha - \frac{\gamma_{3,n}}{6}\left(1 - z_\alpha^2\right), \qquad \gamma_{3,n} = \varphi_n^3 \int_{\mathbb{A}_n} \frac{\dot{S}\left(\vartheta_0, x\right)^3}{S\left(\vartheta_0, x\right)^2}\,\mathrm{d}x.$$

La probabilité d'erreur de première espèce de $\phi_n^{\star}$ est égale à $\alpha + O(\varepsilon_n^2)$. Ce test est efficace de deuxième ordre, i.e., uniformément par rapport à $u \in [0,\, K]$ pour tout $K > 0$ nous avons

$$\mathbf{E}_{\vartheta_u}\phi_n^{\star}\left(X^{(n)}\right) - \mathbf{E}_{\vartheta_u}\tilde{\phi}_n\left(X^{(n)}\right) = O(\varepsilon_n^2). \tag{1}$$

Le petit paramètre $\varepsilon_n \to 0$ et $\gamma_{3,n} = O(\varepsilon_n)$. La puissance de $\phi_n^{\star}$ approxime la *fonction de puissance enveloppe* (the envelope power function) jusqu'à l'ordre $O(\varepsilon_n^2)$. Cette fonction est définie par $\mathbf{E}_{\vartheta_u}\tilde{\phi}_n\left(X^{(n)}\right)$, comme une fonction de u. Elle donne le maximum accessible de puissance pour tester $\mathcal{H}_0$ contre $\mathcal{H}_1$. Notons que $\tilde{\phi}_n$, le test le plus puissant pour $\mathcal{H}_0$ contre $\mathcal{H}_u$, dépend du paramètre u et qu'il n'est pas un test pour les hypothèses principales. La démonstration de (1) est basée sur les développements (sous l'alternative locale ϑ_u) des fonctions de répartition de $\Delta_n(\vartheta_0)$ et du logarithme du rapport de vraisemblance et de la comparaison des termes correspondants. Nous avons appliqué ces résultats à deux modèles : Modulation de phase et Modulation de fréquence.

Dans le Chapitre 4, nous supposons que $\vartheta \in \mathbb{R}^k, k \geq 2$ est un vecteur de-paramètres inconnus et nous testons $\mathcal{H}_0 : \vartheta = \vartheta_0$ contre l'alternative multilatérale $\mathcal{H}_1 : \vartheta \neq \vartheta_0$. Les résultats sont également valables pour $k = 1$. Pour une telle alternative, il est courant d'utiliser l'une des procédures optimales suivantes : le test de Wald, le test du rapport des vraisemblances et le Rao score test. Puisque ces tests sont asymptotiquement équivalents (voir [22], [28]) et les deux premiers exigent le calcul d'un estimateur efficace (e.g., l'estimateur par le maximum de vraisemblance), nous utilisons le Rao score test. Ce test est basé sur le vecteur des dérivées

$$\Delta_n = \left.\frac{\partial L(\vartheta, X^{(n)})}{\partial \vartheta}\right|_{\vartheta=\vartheta_0},$$

au point ϑ_0, du logarithme du rapport de vraisemblance

$$\begin{aligned} L(\vartheta,\, X^{(n)}) &= \ln \frac{\mathrm{d}\mathbf{P}_{\vartheta}^{(n)}}{\mathrm{d}\mathbf{P}_{\vartheta_0}^{(n)}}(X^n) = \\ &= \int_{\mathbb{A}_n} \ln \frac{S(\vartheta, x)}{S(\vartheta_0, x)} X^{(n)}(\,\mathrm{d}x) - \int_{\mathbb{A}_n} \left[S(\vartheta, x) - S(\vartheta_0, x)\right]\,\mathrm{d}x. \end{aligned}$$

Le Rao score test rejette l'hypothèse nulle, si $\Delta_n'\, I_n^{-1}(\vartheta_0)\Delta_n > c_{n,\alpha}$. La matrice $I_n(\vartheta_0) = \mathbf{E}_{\vartheta_0}\left(\Delta_n\, \Delta_n'\right)$ est l'information de Fisher au point ϑ_0, où Δ_n' désigne la transposition de Δ_n. La normalité asymptotique du vecteur Δ_n sous l'hypothèse nulle (e.g., voir [27]) nous donne l'approximation de premier ordre $c_{n,\alpha} = \chi_{k,\alpha} + o(1)$, où $\chi_{k,\alpha}$ est le $1-\alpha$ quantile de la loi Khi-deux avec k degrés de liberté, *i.e.*, $\mathbf{P}\left\{\xi < \chi_{k,\alpha}\right\} = 1 - \alpha$ et $\xi \sim \chi_k^2$. La puissance du Rao score test sous l'alternative locale $\vartheta_u = \vartheta_0 + I_n(\vartheta_0)^{-1/2}\, u$, $u \neq 0$ admet la représentation suivante

$$\mathbf{P}_{\vartheta_u}^{(n)}\left\{\Delta_n'\, I_n^{-1}(\vartheta_0)\,\Delta_n > c_{n,\alpha}\right\} = \mathbf{P}\left\{\|\zeta + u\|^2 > \chi_{k,\alpha}^2\right\} + o(1),$$

où ζ est un vecteur k dimensionnel de variables aléatoires independentes qui suivent la loi normale standard et $\|\cdot\|$ est la norme euclidienne. Afin d'améliorer le calcul du risque de première espèce, nous avons développé la distribution du vecteur Δ_n d'intégrales stochastiques sous $\mathcal{H}_0$ et nous avons obtenu

$$c_{n,\alpha} = \chi_{k,\alpha} - 2^{\frac{k}{2}}\,\Gamma\left(k/2\right)\left[a_n\,\chi^2_{k,\alpha} + b_n\,\chi^4_{k,\alpha} + c_n\,\chi^6_{k,\alpha}\right] + O(\varepsilon_n^4).$$

Les coefficients a_n, b_n, c_n sont de l'ordre de $O(\varepsilon_n^2)$ et peuvent être calculés explicitement (voir (4.7) pour $k = 2$). Avec cette constante, le risque de première espèce du Rao score test est donné par

$$\mathbf{P}^{(n)}_{\vartheta_0}\left\{\Delta_n' \, I_n^{-1}(\vartheta_0)\Delta_n > c_{n,\alpha}\right\} = \alpha + O(\varepsilon_n^3).$$

En utilisant le développement d'Edgeworth la puissance du test sous l'alternative locale ϑ_u admet la réprésentation suivante :

$$\mathbf{P}^{(n)}_{\vartheta_u}\left\{\Delta_n' \, I_n^{-1}(\vartheta_0)\Delta_n > c_{n,\alpha}\right\} = 1 - \int_{\mathbb{C}} h_n(y)\,\mathrm{d}y + O(\varepsilon_n^3),$$

où $\mathbb{C}$ est un sous ensemble convexe (dépendant de n, α et u) de $\mathbb{R}^k$ délimité par une ellipse et la densité

$$h_n(y) = (2\,\pi)^{-k/2}\exp\left\{-\|y\|^2/2\right\}\left[1 + p_1(y)\,\varepsilon_n + p_2(y)\,\varepsilon_n^2\right].$$

Ici $p_1(\cdot)$ et $p_2(\cdot)$ sont des polynômes. La technique utilisée pour le développement d'Edgeworth dans le cas vectoriel est différente de celle du cas unidimensionnel. La démonstration est une modification de [27], page 135. Nous avons présenté deux formulations différentes. Pour la simplicité d'exposition nous considérons deux termes après le terme gaussien. La première formulation qui est utile pour notre problème de test d'hypothèses, a la forme suivante

$$\mathbf{P}^{(n)}_{\vartheta}\left\{\Delta_n \in \mathbb{C}\right\} = \int_{\mathbb{C}} h_n(y)\,\mathrm{d}y + O(\varepsilon_n^3),$$

uniformément par rapport à tous les ensembles $\mathbb{C}$ convexes dont les mesures de Lebesgue sont inférieures ou égales à $\eta > 0$ donné. Dans la deuxième formulation, pour $2 < r < 3$ arbitraire mais fixé, la distribution de Δ_n admet la représentation

$$\mathbf{P}^{(n)}_{\vartheta}\left\{\Delta_n \in \mathbb{C}\right\} = \int_{\mathbb{C}} h_n(y)\,\mathrm{d}y + O(\varepsilon_n^r),$$

uniformément par rapport à tous les ensembles $\mathbb{C} \subseteq \mathbb{R}^k$ convexes. Nous avons appliqué ces résultats pour un processus de Poisson avec la fonction d'intensité

$$S(\vartheta, x) = \vartheta_1 \, \sin(\vartheta_2 \, x) + \lambda_0, \; x \in [\,0, n\,] \,, \qquad \vartheta = (\vartheta_1, \vartheta_2)$$

où les paramètres ϑ_1 (phase) et ϑ_2 (fréquence) sont inconnus ($\lambda_0 > 0$ est connu). La statistique des dérivées Δ_n est

$$\Delta_n \equiv (\Delta_1, \Delta_2)^{\mathsf{0}} = \left(\frac{\partial L(\vartheta, X^{(n)})}{\partial \vartheta_1}, \, \frac{\partial L(\vartheta, X^{(n)})}{\partial \vartheta_2} \right)^{\mathsf{0}} \Bigg|_{\vartheta = \vartheta_0} =$$
$$= \left(\int_0^n \frac{\sin(\vartheta_{02} \, x)}{S(\vartheta_0, x)} \, \pi(\mathrm{d}x), \, \int_0^n \frac{\vartheta_{01} \, x \, \cos(\vartheta_{02} \, x)}{S(\vartheta_0, x)} \, \pi(\mathrm{d}x) \right)^{\mathsf{0}},$$

où $\pi(\mathrm{d}x) = X^{(n)}(\mathrm{d}x) - S(\vartheta_0, x) \, \mathrm{d}x$ est le processus centré. Nous avons montré que Δ_n remplit les conditions de développement (sous $\mathcal{H}_0$ et $\mathcal{H}_u$). Notons que ce vecteur a deux vitesses différentes pour ces composants : une vitesse classique $\sqrt{n}$ pour le premier composant et une vitesse non classique $n^{3/2}$ pour le deuxième. Les détails se trouvent dans la section 4.2.

Publications

1. Fazli Kh., Kutoyants Yu. A. *Two simple hypotheses testing for Poisson process.* Far East J. of Theoretical Statistics, 2005, 15, 2, 251-290

2. Fazli Kh. *Second order efficient test for inhomogeneous Poisson Processes*, à paraître dans : Statistical Inference for Stochastic Processes.

3. Fazli Kh. *Hypotheses Testing for a Multidimensional Parameter of Inhomogeneous Poisson Processes*, Université du Maine, prépublication 06-1, Janvier 2006, soumis.

Notations

$\chi_{\mathbb{A}}(\cdot)$	The indicator (or characterization) function of the set $\mathbb{A}$, *i.e.*, $\chi_{\mathbb{A}}(x) = 1$ if $x \in \mathbb{A}$, otherwise $\chi_{\mathbb{A}}(x) = 0$.
$\mathbf{P}_{\vartheta}^{(n)}$	The distribution of the Poisson process on the set $\mathbb{A}_n$, when ϑ is the true parameter.
$\mathbf{E}_{\vartheta}$	Mathematical expectation w.r.t. $\mathbf{P}_{\vartheta}^{(n)}$.
$\phi_n(\cdot)$	The test function (decision rule).
$\mathcal{N}(\cdot)$, $n(\cdot)$	The distribution and density functions of the standard Gaussian law, respectively.
α	The level of significance, where $\alpha \in (0,1)$.
z_α	The $1-\alpha$ quantile of the standard gaussian law, *i.e*, $\mathcal{N}(z_\alpha) = 1 - \alpha$.
$\xi \sim \chi_k^2$	The random variable ξ follows the Chi-squared law with k degrees of freedom.
$\chi_{k,\alpha}$	The $1-\alpha$ quantile of χ_k^2, *i.e*, $\mathbf{P}\left\{\xi > \chi_{k,\alpha}\right\} = \alpha$.
$\mathbb{R}$	The set of real numbers.
$\mathcal{K}_\alpha$	The class of tests of level $1-\alpha$ (size α).
$\mathcal{K}_\alpha^0$	The class of tests of asymptotic level $1-\alpha$ (size α).
$a_n = o(b_n)$	The sequence $a_n/b_n \to 0$ as $n \to \infty$.
$a_n = O(b_n)$	The sequence a_n/b_n is bounded.
$a_n \sim b_n$	The sequences $a_n/b_n \to 1$ as $n \to \infty$.

Chapter 1

Preliminaries

1.1 Poisson processes

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the basic probability space with the σ-algebra $\mathcal{F}$ completed by the sets of probability zero and a complete separable metric space $(\mathbb{A}, \varrho)$ (ϱ is a metric on $\mathbb{A}$) with the σ-algebra $\mathfrak{B} = \mathfrak{B}(\mathbb{A})$ of Borel subsets of $\mathbb{A}$. By $\mathcal{M}$ we mean the space of σ-finite measures on $(\mathbb{A}, \mathfrak{B})$ and $\mathcal{M}_0$ denotes the subspace of all integer-valued point measures (trajectories)

$$X = \sum_i \varepsilon_{x_i},$$

where $x_i \in \mathbb{A}$ and ε_x is the unit mass located in x. Let $\mathfrak{B}(\rangle \Rightarrow$ $\langle$ $\sigma-$field with respect to which the mappings:

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- for every $\mathbb{B} \in \mathfrak{B}$ with $\Lambda(\mathbb{B}) < \infty$, the random variable $X(\mathbb{B})$ has a Poisson distribution with parameter $\Lambda(\mathbb{B})$.

The existence of a Poisson process with arbitrary state space and $\sigma-$finite intensity measure was shown in [31].

1.2 Stochastic integrals

Let X be a Poisson process on the set $\mathbb{A}$ with intensity measure Λ and a real measurable function $f : \mathbb{A} \longrightarrow \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers). The test statistics considered in this work are based on the stochastic integral with respect to the Poisson process

$$I(f) \equiv \int_{\mathbb{A}} f(x)\, X(\mathrm{d}x) = \sum_{x_i 2\, \mathbb{A}} f(x_i),$$

where $\{x_i\}$ are the *events* (random points) of the Poisson process. For a simple function f,

$$f(x) = \sum_{j=1}^{m} \alpha_j\, \chi_{\mathbb{A}_j}(x)$$

the stochastic integral (on the set $\mathbb{A}$) is defined by the random variable

$$I(f) \equiv \int_{\mathbb{A}} f(x)\, X(\mathrm{d}x) = \sum_{j=1}^{m} \alpha_j\, X(\mathbb{A}_j)$$

where $\alpha_1, \cdots, \alpha_m$ are some constant numbers, the family $\{\mathbb{A}_1, \cdots, \mathbb{A}_m\}$ is a finite (measurable) partition of $\mathbb{A}$ and $\chi_{\mathbb{A}_j}(\cdot)$ denotes the indicator function of the set $\mathbb{A}_j$, *i. e.*, $\chi_{\mathbb{A}_j}(x) = 1$ if $x \in \mathbb{A}_j$ and $\chi_{\mathbb{A}_j}(x) = 0$ otherwise. Since the random variable $X(\mathbb{A}_j)$ has a Poisson distribution with parameter $\Lambda(\mathbb{A}_j)$, the mathematical expectation of $I(f)$ is equal to

$$\begin{aligned}\mathbf{E}I(f) &= \sum_{j=1}^{m} \alpha_j\, \Lambda(\mathbb{A}_j) = \sum_{j=1}^{m} \int_{\mathbb{A}_j} \alpha_j\, \Lambda(\mathrm{d}x) = \\ &= \sum_{j=1}^{m} \int_{\mathbb{A}_j} f(x)\, \Lambda(\mathrm{d}x) = \int_{\mathbb{A}} f(x)\, \Lambda(\mathrm{d}x)\end{aligned}$$

and similarly its variance is given by

$$\mathbf{E}\left(I(f)-\mathbf{E}I(f)\right)^2=\int_{\mathbb{A}} f(x)^2\,\Lambda(\mathrm{d}x).$$

For nonnegative measurable function f there exists a sequence of nonnegative measurable simple functions such that $f_n(x)\uparrow f(x)$ as $n\to\infty$. Hence the sequence $I(f_n)$ is nondecreasing and we define

$$I(f)=\lim_{n\to\infty} I(f_n).$$

For general measurable function f we have $f(x)=f_+(x)-f_-(x)$ where

$$f_+(x)=\max\left\{f(x),0\right\},\quad f_-(x)=\max\left\{-f(x),0\right\},$$

respectively. These functions are nonnegative and the stochastic integral is defined by

$$I(f)=I(f_+)-I(f_-),$$

unless $I(f_+)=I(f_-)=\infty$. We define the stochastic integral with respect to centered Poisson process $\pi(\mathrm{d}x)=X(\mathrm{d}x)-\Lambda(\mathrm{d}x)$ as

$$I_*(f)\equiv\int_{\mathbb{A}} f(x)\,\pi(\mathrm{d}x)=I(f)-\int_{\mathbb{A}} f(x)\,\Lambda(\mathrm{d}x).$$

Let $\mathbf{L}_p(\Lambda)$, $p\geq 1$ denote the class of all real measurable functions $f:\mathbb{A}\to\mathbb{R}$ such that

$$\int_{\mathbb{A}} |f(x)|^p\,\Lambda(\mathrm{d}x)<\infty.$$

Some properties of the stochastic integrals $I(f)$ and $I_*(f)$ are given in the following lemma.

Lemma 1. *Let $f(\cdot)\in\mathbf{L}_1(\Lambda)$. Then the stochastic integrals $I(f)$ and $I_*(f)$ are well defined and*

$$\mathbf{E}I(f)=\int_{\mathbb{A}} f(x)\,\Lambda(\mathrm{d}x),\quad \mathbf{E}I_*(f)=0.$$

The characteristic functions of $I(f)$ and $I_*(f)$ are equal to

$$\phi(\lambda)=\mathbf{E}\exp\left\{i\lambda\, I(f)\right\}=\exp\left\{\int_{\mathbb{A}}\left[\exp\left\{i\lambda\, f(x)\right\}-1\right]\,\Lambda(\mathrm{d}x)\right.$$

If $f(\cdot), g(\cdot) \in \mathbf{L}_1(\Lambda) \bigcap \mathbf{L}_2(\Lambda)$ *then*

$$\mathbf{E} I_{\square}(f)^2 = \int_{\mathbb{A}} f(x)^2\, \Lambda(\mathrm{d}x), \quad \mathbf{E}\left(I_{\square}(f)\, I_{\square}(g)\right) = \int_{\mathbb{A}} f(x)\, g(x)\, \Lambda(\mathrm{d}x).$$

For a function $f(\cdot) \in \mathbf{L}_1(\Lambda)$ *such that*

$$e^{f(\cdot)} - 1 - f(\cdot) \in \mathbf{L}_1(\Lambda)$$

we have

$$\mathbf{E} \exp\left\{ \int_{\mathbb{A}} f(x)\, \pi(\mathrm{d}x) \right\} = \exp\left\{ \int_{\mathbb{A}} \left[e^{f(x)} - 1 - f(x) \right] \Lambda(\mathrm{d}x) \right\}.$$

Proof. For proof see, for example [27], Lemma 1.1. page 18.

1.3 Likelihood ratio

Let X be a Poisson process with the intensity measure Λ on the set $\mathbb{A}$. In fact X is a random element on the measurable space $(\mathcal{M}_0, \mathfrak{B}(\mathcal{M}_0))$ containing all integer valued measures defined on the set $\mathbb{A}$. Let $\mathbf{P}$ denotes the distribution of the random element X. Let Λ_1 and Λ_2 be two finite measures on $(\mathbb{A}, \mathfrak{B})$, that is $\Lambda_i(\mathbb{A}) < \infty, i = 1, 2$ and $\mathbf{P}_1, \mathbf{P}_2$ be the corresponding distributions of X. The singularity, absolute continuity and equivalence of the measures $\mathbf{P}_1, \mathbf{P}_2$ are denoted by $\mathbf{P}_1 \perp \mathbf{P}_2$, $\mathbf{P}_1 \ll \mathbf{P}_2$ and $\mathbf{P}_1 \sim \mathbf{P}_2$, respectively. If $\Lambda_2 \ll \Lambda_1$ then the Radon-Nykodim derivative

$$S(x) = \frac{\mathrm{d}\Lambda_2}{\mathrm{d}\Lambda_1}(x), \qquad x \in \mathbb{A}$$

is called *the intensity function*. The following theorem gives the likelihood ratio.

Theorem 1. *(M. Brown [8]) If* $\Lambda_2 \ll \Lambda_1$ *then* $\mathbf{P}_2 \ll \mathbf{P}_1$ *and the likelihood ratio*

$$\frac{\mathrm{d}\mathbf{P}_2}{\mathrm{d}\mathbf{P}_1}(X) = \exp\left\{ \int_{\mathbb{A}} \ln S(x)\, X(\mathrm{d}x) - \int_{\mathbb{A}} [S(x) - 1]\, \Lambda_1(\mathrm{d}x) \right\}.$$

If $\Lambda_1 \sim \Lambda_2$ *then* $\mathbf{P}_1 \sim \mathbf{P}_2$.

Proof. For proof see, for example [27], Theorem 1.3., page 28.

As a particular case let $\mathbb{A} \subseteq \mathbb{R}^d$, a subset of the d dimensional Euclidian space and the intensity measures (absolutely continuous with respect to the Lebesgue measure)

$$\Lambda_i(\mathbb{E}) = \int_{\mathbb{E}} S_i(x)\,\mathrm{d}x, \qquad i = 1, 2$$

where $S_1(\cdot)$ and $S_2(\cdot)$ are two (almost everywhere) positive measurable functions and $\mathbb{E}$ a Borel measurable subset of $\mathbb{A}$. Therefore $\Lambda_1 \sim \Lambda_2$ and hence the corresponding probability measures $\mathbf{P}_1$ and $\mathbf{P}_2$ are also equivalent with the likelihood ratio

$$\frac{\mathrm{d}\mathbf{P}_2}{\mathrm{d}\mathbf{P}_1}(X) = \exp\left\{\int_{\mathbb{A}} \ln \frac{S_2(x)}{S_1(x)}\, X(\mathrm{d}x) - \int_{\mathbb{A}} \left[S_2(x) - S_1(x)\right]\,\mathrm{d}x\right\}.$$

The following theorem gives sufficient conditions for absolutely continuity and singularity, when Λ_1, Λ_2 are $\sigma-$finite (not necessarily finite) measures.

Theorem 2. *Let Λ_1 and Λ_2 be two $\sigma-$ finite measures and $\Lambda_2 \ll \Lambda_1$. Then*

$$\mathbf{P}_2 \ll \mathbf{P}_1, \qquad \text{if} \qquad \int_{\mathbb{A}} \left(\sqrt{S(x)} - 1\right)^2 \Lambda_1(\mathrm{d}x) < \infty$$

$$\mathbf{P}_2 \perp \mathbf{P}_1, \qquad \text{if} \qquad \int_{\mathbb{A}} \left(\sqrt{S(x)} - 1\right)^2 \Lambda_1(\mathrm{d}x) = \infty.$$

Proof. See [30].

1.4 Central limit theorem

In the framework of the asymptotic approach let $X^{(n)}, n = 1, 2, \ldots$ be a sequence of Poisson processes with intensity measures $\Lambda^{(n)}(\cdot),\ n = 1, 2, \ldots$ observed over the sets $\mathbb{A}_n$ and a sequence $f_n(\cdot), n = 1, 2, \ldots$ of real measurable functions defined on $\mathbb{A}_n$. In this section we consider the asymptotic distribution of stochastic integrals

$$Y_n = \int_{\mathbb{A}_n} f_n(x)\; X^{(n)}(\mathrm{d}x),$$

by proving a *central limit theorem*. Let us denote

$$\mu_n = \mathbf{E}(Y_n) = \int_{\mathbb{A}_n} f_n(x)\ \Lambda^{(n)}(\mathrm{d}x),$$
$$\sigma_n^2 = \mathbf{E}(Y_n - \mu_n)^2 = \int_{\mathbb{A}_n} f_n^2(x)\ \Lambda^{(n)}(\mathrm{d}x),$$
$$\mathbb{A}_n(\delta) = \{x \in \mathbb{A}_n :\ |f_n(x)| > \delta\sigma_n\}, \quad \delta > 0.$$

Theorem 3. *Let*

$$f_n(\cdot) \in \mathbf{L}_1\left(\Lambda^{(n)}\right) \cap \mathbf{L}_2\left(\Lambda^{(n)}\right),$$

$0 < \sigma_n \to \infty$ *and for any* $\delta > 0$ *(Lindeberg condition)*

$$\lim_{n\to\infty} \sigma_n^{-2} \int_{\mathbb{A}_n(\delta)} f_n(x)^2\ \Lambda^{(n)}(\mathrm{d}x) = 0.$$

Then the stochastic integral Y_n *is asymptotically normal*

$$\mathcal{L}\left\{\frac{Y_n - \mu_n}{\sigma_n}\right\} \Rightarrow \mathcal{N}(0,1).$$

Proof. Following [27], page 24, we set

$$\zeta_n \equiv \frac{Y_n - \mu_n}{\sigma_n} = \sigma_n^{-1} \int_{\mathbb{A}_n} f_n(x)\ \pi^{(n)}(\mathrm{d}x),$$

where $\pi^{(n)}(\mathrm{d}x) = X^{(n)}(\mathrm{d}x) - \Lambda^{(n)}(\mathrm{d}x)$ is the centered Poisson process. By the Lemma 1 the characteristic function of the stochastic integral ζ_n is given by

$$\phi_n(\lambda) = \mathbf{E}e^{i\lambda\,\zeta_n} = \exp\left\{\int_{\mathbb{A}_n} \left(e^{i\lambda\,\sigma_n^{-1} f_n(x)} - 1 - i\lambda\,\sigma_n^{-1} f_n(x)\right) \Lambda^{(n)}(\mathrm{d}x)\right\} =$$
$$= e^{-\lambda^2/2} \exp\left\{\int_{\mathbb{A}_n} \left(e^{i\lambda\sigma_n^{-1} f_n(x)} - 1 - i\lambda\,\sigma_n^{-1} f_n(x) + \frac{\lambda^2 \sigma_n^{-2}}{2} f_n^2(x)\right) \Lambda^{(n)}(\mathrm{d}x)\right\}$$

where we used the equality

$$\sigma_n^2 = \int_{\mathbb{A}_n} f_n^2(x)\ \Lambda^{(n)}(\mathrm{d}x).$$

Hence for $\delta > 0$ arbitrary we can write

$$\begin{aligned}
&\ln\phi_n(\lambda) = \\
&= \frac{-\lambda^2}{2} + \int_{\mathbb{A}_n} \left(e^{i\lambda\sigma_n^{-1} f_n(x)} - 1 - i\lambda\,\sigma_n^{-1} f_n(x) + \frac{\lambda^2}{2}\sigma_n^{-2} f_n^2(x)\right) \Lambda^{(n)}(\mathrm{d}x) = \\
&= \frac{-\lambda^2}{2} + \int_{\mathbb{A}_n(\delta)} + \int_{\mathbb{A}_n(\delta)^c} \equiv \frac{-\lambda^2}{2} + I_n(\delta) + J_n(\delta),
\end{aligned}$$

with the obvious notations. We have

$$\left|e^{i\lambda}-1-i\,\lambda\right|\leq\frac{\lambda^2}{2}$$
$$\left|e^{i\lambda}-1-i\,\lambda+\frac{\lambda^2}{2}\right|\leq\frac{\left|\lambda\right|^3}{6},$$

The first formula allows us to write

$$|I_n(\delta)|\leq\lambda^2\sigma_n^{-2}\int_{\mathbb{A}_n(\delta)}f_n(x)^2\Lambda^{(n)}(\mathrm{d}x)$$

and by the second one

$$\begin{aligned}|J_n(\delta)|&=\left|\int_{\mathbb{A}_n(\delta)^c}\frac{\gamma_2\lambda^3f_n^3(x)\sigma_n^{-3}}{3!}\Lambda^{(n)}(\mathrm{d}x)\right|\leq\\&\leq\frac{\left|\lambda\right|^3\sigma_n^{-3}}{3!}\int_{\mathbb{A}_n(\delta)^c}\left|f_n^3(x)\right|\Lambda^{(n)}(\mathrm{d}x)=\frac{\left|\lambda\right|^3\sigma_n^{-3}}{3!}\int_{\mathbb{A}_n(\delta)^c}\left|f_n(x)\right|f_n^2(x)\Lambda^{(n)}(\mathrm{d}x)\\&\leq\frac{\left|\lambda\right|^3\sigma_n^{-2}\delta}{3!}\int_{\mathbb{A}_n}f_n^2(x)\Lambda^{(n)}(\mathrm{d}x)=\frac{\left|\lambda\right|^3\delta}{3!}.\end{aligned}$$

Hence for any $\delta>0$

$$|I_n(\delta)+J_n(\delta)|\leq\lambda^2\sigma_n^{-2}\int_{\mathbb{A}_n(\delta)}f_n^2(x)\Lambda^{(n)}(\mathrm{d}x)+\frac{\left|\lambda\right|^3\delta}{3!}.$$

By tending $n\to\infty$ and using Lindeberg condition and then tending $\delta\to 0$ we obtain

$$\lim_{n\to\infty}\int_{\mathbb{A}_n}\left(e^{i\lambda\sigma_n^{-1}f_n(x)}-1-i\lambda\,\sigma_n^{-1}f_n(x)+\frac{\lambda^2}{2}\sigma_n^{-2}f_n^2(x)\right)\Lambda^{(n)}(\mathrm{d}x)=0.$$

Hence for any λ

$$\lim_{n\to\infty}\ln\phi_n(\lambda)=\frac{-\lambda^2}{2},$$

which completes the proof of the Theorem.

1.5 Edgeworth type expansion for stochastic integrals

Let us consider a sequence of Poisson processes $X^{(n)},n=1,2,\ldots$ with intensity measures $\Lambda^{(n)}$ observed on the sets $\mathbb{A}_n$. We are interested in the Edgeworth expansion of the distribution function of the stochastic integral

$$F_n\,(y) = \mathbf{P}^{(n)} \left\{ \int_{\mathbb{A}_n} f_n(x)\;\pi^{(n)}(\mathrm{d}x) \le y \right\}, \tag{1.1}$$

where $\pi^{(n)}(\mathrm{d}x) = X^{(n)}(\mathrm{d}x) - \Lambda^{(n)}(\mathrm{d}x)$ is the centered Poisson process. As we saw in the preceding section under the Lindeberg condition

$$F_n(y) = \mathcal{N}(y) + o(1),$$

uniformly in $y \in \mathbb{R}$, as $n \to \infty$, where $\mathcal{N}(\cdot)$ denotes the distribution function of the standard Gaussian law. The Edgeworth type expansion refines the central limit theorem by giving the next terms after the Gaussian term which permits us in turn to modify the threshold of the consistent test obtained by the central limit theorem (see Chapter 2). By the help of the expansion we prove also that the second order efficiency of the modified test (see Chapter 3).

Without loss of generality we suppose that

$$\int_{\mathbb{A}_n} f_n^2(x)\;\Lambda^{(n)}(\mathrm{d}x) = 1. \tag{1.2}$$

Let us introduce the following conditions for one term after the Gaussian term :

$\mathcal{A}_1$. *There exists a sequence of real numbers* $\varepsilon_n \to 0$, *as* $n \to \infty$ *and constants* $C_r > 0, r = 3, 4$, *such that*

$$\int_{\mathbb{A}_n} |f_n(x)|^r\;\;\Lambda^{(n)}(\mathrm{d}x) \le C_r\;\varepsilon_n^{r-2}.$$

$\mathcal{A}_2$. *There exist constants* $\gamma \ge 3/2$ *and* $c_0 > 0$ *satisfying the inequality* $\frac{C_3}{3!}\,c_0 + \frac{C_4}{4!}\,c_0^2 - \frac{1}{2} < 0$ *such that*

$$\inf_{c_0\,\varepsilon_n^{-1} \le\, 2|\lambda| \le\, \varepsilon_n^{-2}} \int_{\mathbb{A}_n} \sin^2\left(\lambda f_n(x)\right)\;\Lambda^{(n)}(\mathrm{d}x) \ge \gamma \ln \varepsilon_n^{-1}$$

for all large n.

The first condition (Lindeberg type) imposes some restrictions on the third and fourth cumulants of the stochastic integrals. The second condition implies that the absolute value of the characteristic function of the stochastic integral $\int_{\mathbb{A}_n} f_n(x)\;\pi^{(n)}(\mathrm{d}x)$

is enough small for the values enough far from zero. This is not true for the lattice distributions where the characteristic function is periodic. Observe that the Lindeberg condition of the Theorem 3 follows from $\mathcal{A}_1$ with $r = 3$. Indeed for given $\delta > 0$

$$\int_{\{x \in \mathbb{A}_n :\, |f_n(x)| > \delta\}} f_n^2(x)\ \Lambda^{(n)}(\mathrm{d}x) \le \frac{1}{\delta} \int_{\mathbb{A}_n} |f_n(x)|^3\ \Lambda^{(n)}(\mathrm{d}x) \le \frac{C_3}{\delta}\ \varepsilon_n \to 0.$$

Remind that by (1.2) we have $\sigma_n^2 = 1$. Let us define the normed cumulant

$$\rho_{3,n} = \varepsilon_n^{-1} \int_{\mathbb{A}_n} f_n(x)^3\ \Lambda^{(n)}(\mathrm{d}x)$$

and the function

$$Q_{1,n}(y) = \frac{\rho_{3,n}}{3!\sqrt{2\pi}} \int_{-\infty}^{y} H_3(t)\, e^{-t^2/2}\ \mathrm{d}t = \frac{\rho_{3,n}}{3!\sqrt{2\pi}}\,(1 - y^2)\, e^{-y^2/2},$$

where $H_3(t) = t^3 - 3\,t$ is a Hermite polynomial. Observe that, $|\rho_{3,n}$

Proof. The proof can be found in [21], Chapter XVII.

The functions $F_n(y)$ and
$$G_n(y) = \mathcal{N}(y) + Q_{1,n}(y)\ \varepsilon_n$$
satisfy the conditions of the lemma. Hence for $N = \varepsilon_n^{-2}$ and an arbitrary constant $m > 1$ there exists $C(m)$ such that
$$|F_n(y) - G_n(y)| \leq \frac{m}{2\pi} \int_{-N}^{N} \left| \frac{\Phi_n(\lambda) - \Psi_n(\lambda)}{\lambda} \right| \mathrm{d}\lambda + C(m)\frac{M}{N},$$
where $\Phi_n(\lambda)$ and $\Psi_n(\lambda)$ are Fourier transforms of the functions $F_n(y)$ and $G_n(y)$ respectively and M is an upper bound of
$$G_n'(y) = \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2} + \frac{\rho_{3,n}}{3!\sqrt{2\pi}}\ \varepsilon_n\ (y^3 - 3y)\ e^{-y^2/2}.$$
One can choose M independent from n, because $\rho_{3,n}$ and ε_n are bounded. Hence for the second term in the right hand side of the last inequality we have the estimate ,
$$C(m)\frac{M}{N} \leq C\ \varepsilon_n^2$$
for some positive constant C. Now we estimate the first term. By the Taylor formula and the assumption (1.2), the characteristic function of $F_n(y)$ can be written as
$$\begin{aligned}
\Phi_n(\lambda) &= \int_{-\infty}^{+\infty} e^{\mathrm{i}\lambda y}\ F_n(\mathrm{d}y) = \exp\left\{ \int_{\mathbb{A}_n} \left(e^{\mathrm{i}\lambda f_n(x)} - 1 - \mathrm{i}\lambda f_n(x)\right) \Lambda^{(n)}(\mathrm{d}x) \right\} = \\
&= e^{-\lambda^2/2} \exp\left\{ \int_{\mathbb{A}_n} \left(e^{\mathrm{i}\lambda f_n(x)} - 1 - \mathrm{i}\lambda f_n(x) - \frac{(\mathrm{i}\lambda)^2}{2} f_n(x)^2\right) \Lambda^{(n)}(\mathrm{d}x) \right\} = \\
&= e^{-\lambda^2/2} \exp\left\{ \frac{(\mathrm{i}\lambda)^3}{3!} \int_{\mathbb{A}_n} f_n(x)^3 \Lambda^{(n)}(\mathrm{d}x) + \frac{(\mathrm{i}\lambda)^4}{4!} \int_{\mathbb{A}_n} e^{\mathrm{i}\tilde{\lambda} f_n(x)} f_n(x)^4 \Lambda^{(n)}(\mathrm{d}x) \right\} = \\
&= e^{-\lambda^2/2} \exp\left\{ \frac{(\mathrm{i}\lambda)^3}{3!}\ \rho_{3,n}\ \varepsilon_n + (\mathrm{i}\lambda)^4\ r_n\ \varepsilon_n^2 \right\},
\end{aligned}$$
where
$$r_n = (4!)^{-1}\ \varepsilon_n^{-2} \int_{\mathbb{A}_n} e^{\mathrm{i}\tilde{\lambda} f_n(x)} f_n(x)^4 \Lambda^{(n)}(\mathrm{d}x),$$
and $\tilde{\lambda} \in (0, \lambda)$. Note that $|r_n| \leq C_4/4!$ by $\mathcal{A}_1$.

We define the function
$$y(z) = \frac{(\mathrm{i}\lambda)^3}{3!}\ \rho_{3,n}\ z + (\mathrm{i}\lambda)^4\ r_n\ z^2$$

and obtain the Taylor expansion

$$e^{y(z)} = 1 + \frac{(\mathrm{i}\lambda)^3}{3!}\ \rho_{3,n}\ z + R_n\ z^2.$$

Letting $z = \varepsilon_n$, it follows that

$$R_n = \frac{1}{2!}\ e^{y(\tilde{\varepsilon}_n)} \left[2\ (\mathrm{i}\lambda)^4\ r_n + \left(\frac{(\mathrm{i}\lambda)^3}{3!}\ \rho_{3,n} + 2(\mathrm{i}\lambda)^4\ r_n\ \tilde{\varepsilon}_n \right)^2 \right],$$

where $\tilde{\varepsilon}_n \in (0, \varepsilon_n)$. By considering two cases $|\lambda| \le 1$ and $|\lambda| > 1$, the inequality

$$|R_n| \le C\ e^{y(\tilde{\varepsilon}_n)} \left\{ |\lambda|^4 + |\lambda|^8 \right\},$$

holds with some positive constant C for all λ. By direct calculation the Fourier transform of $G_n(y)$ is equal to

$$\begin{aligned}
\Psi_n(\lambda) &= \int_{\text{¡}1}^{+1} e^{\mathrm{i}\lambda y} \mathrm{d}G_n(y) = \int_{\text{¡}1}^{+1} e^{\mathrm{i}\lambda y} \frac{1}{\sqrt{2\pi}}\ e^{\text{¡}\ y^2/2} \mathrm{d}y + \\
&+ \frac{\rho_{3,n}\ \varepsilon_n}{3!\sqrt{2\pi}} \int_{\text{¡}1}^{+1} e^{\mathrm{i}\lambda y}\ (y^3 - 3y)\ e^{\text{¡}\ y^2/2}\ \mathrm{d}y = \\
&= e^{\text{¡}\ \lambda^2/2} + \frac{\rho_{3,n}\ \varepsilon_n}{3!}\ e^{\text{¡}\ \lambda^2/2} (\mathrm{i}\lambda)^3 = \\
&= e^{\text{¡}\ \lambda^2/2} \left\{ 1 + \frac{\rho_{3,n}}{3!} (\mathrm{i}\lambda)^3\ \varepsilon_n \right\}.
\end{aligned}$$

Therefore for all λ

$$|\Phi_n(\lambda) - \Psi_n(\lambda)| = e^{\text{¡}\ \lambda^2/2}\ |R_n|\ \varepsilon_n^2 \le C e^{\text{¡}\ \lambda^2/2 + y(\tilde{\varepsilon}_n)} \left\{ |\lambda|^4 + |\lambda|^8 \right\}\ \varepsilon_n^2.$$

For n large the inequalities $1 < N_0 \equiv c_0\ \varepsilon_n^{\text{¡}\ 1} < N = \varepsilon_n^{\text{¡}\ 2}$ hold and we can write

$$\int_{\text{¡}\ N}^{N} \left| \frac{\Phi_n(\lambda) - \Psi_n(\lambda)}{\lambda} \right| \mathrm{d}\lambda = \int_{\text{¡}\ N}^{\text{¡}\ N_0} + \int_{\text{¡}\ N_0}^{N_0} + \int_{N_0}^{N},$$

with obvious notations. Since in the second integral $|\lambda\ \tilde{\varepsilon}_n| \le |\lambda\ \varepsilon_n| \le c_0$, then

$$\begin{aligned}
|y(\tilde{\varepsilon}_n)| &= \left| \frac{(\mathrm{i}\lambda)^3}{3!}\ \rho_{3,n}\ \tilde{\varepsilon}_n + (\mathrm{i}\lambda)^4\ r_n\ \tilde{\varepsilon}_n^2 \right| = \lambda^2 \left| \frac{\rho_{3,n}}{3!}\ (\mathrm{i}\lambda\ \tilde{\varepsilon}_n) + r_n\ (\mathrm{i}\lambda\ \tilde{\varepsilon}_n)^2 \right| \le \\
&\le \lambda^2 \left\{ \frac{C_3}{3!}\ c_0 + \frac{C_4}{4!}\ c_0^2 \right\}.
\end{aligned}$$

Therefore

$$\int_{-N_0}^{N_0} \left| \frac{\Phi_n(\lambda) - \Psi_n(\lambda)}{\lambda} \right| \mathrm{d}\lambda \leq$$
$$\leq C\varepsilon_n^2 \int_{-\infty}^{\infty} \exp\left\{ \lambda^2 \left(\frac{C_3}{3!}\, c_0 + \frac{C_4}{4!}\, c_0^2 - \frac{1}{2} \right) \right\} \left(|\lambda|^3 + |\lambda|^7 \right) \mathrm{d}\lambda \leq C_1\, \varepsilon_n^2,$$

for some positive constant C_1. Observe that the exponent of the integral is negative by $\mathcal{A}_2$. In the last integral

$$\int_{N_0}^{N} \left| \frac{\Phi_n(\lambda) - \Psi_n(\lambda)}{\lambda} \right| \mathrm{d}\lambda \leq \int_{N_0}^{N} |\Phi_n(\lambda)|\, |\lambda|^{-1}\, \mathrm{d}\lambda + \int_{N_0}^{N} |\Psi_n(\lambda)|\, |\lambda|^{-1}\, \mathrm{d}\lambda,$$

since $|\lambda| \geq N_0 > 1$, for suitable positive constant C

$$1 + \frac{|\rho_{3,n}|}{3!}\, |\lambda|^3\, \varepsilon_n \leq C\, |\lambda|^3,$$

and consequently

$$\int_{N_0}^{N} |\Psi_n(\lambda)|\, |\lambda|^{-1}\, \mathrm{d}\lambda \leq C \int_{N_0}^{\infty} e^{-\lambda^2/2}\, \lambda^2\, \mathrm{d}\lambda \leq$$
$$\leq C_2\, N_0\, e^{-N_0^2/2} = C_2 c_0\, \varepsilon_n^{-1}\, e^{-c_0^2\, \varepsilon_n^{-2}/2} \leq C\, \varepsilon_n^2.$$

Now consider the integral

$$\int_{N_0}^{N} |\Phi_n(\lambda)|\, |\lambda|^{-1}\, \mathrm{d}\lambda = \int_{N_0}^{N} \exp\left\{ -2 \int_{\mathbb{A}_n} \sin^2\left(\frac{\lambda}{2} f_n(x) \right) \Lambda^{(n)}(\mathrm{d}x) \right\} \lambda^{-1}\, \mathrm{d}\lambda,$$

and note that by $\mathcal{A}_2$

$$\int_{\mathbb{A}_n} \sin^2\left(\frac{\lambda}{2} f_n(x) \right) \Lambda^{(n)}(\mathrm{d}x) \geq \gamma \ln \varepsilon_n^{-1}.$$

Since $\gamma \geq 3/2$, we have

$$\int_{N_0}^{N} |\Phi_n(\lambda)|\, |\lambda|^{-1}\, \mathrm{d}\lambda \leq \int_{N_0}^{N} \exp\left(-2\gamma \ln \varepsilon_n^{-1} \right) \lambda^{-1}\, \mathrm{d}\lambda \leq N_0^{-1}\, \varepsilon_n^{2\gamma}\, (N - N_0) \leq$$
$$\leq c_0^{-1}\, \varepsilon_n\, \varepsilon_n^{2\gamma}\, \varepsilon_n^{-2} = c_0^{-1}\, \varepsilon_n^{2\gamma-1} \leq c_0^{-1}\, \varepsilon_n^2.$$

Hence for the last integral we have

$$\int_{N_0}^{N} \left| \frac{\Phi_n(\lambda) - \Psi_n(\lambda)}{\lambda} \right| \mathrm{d}\lambda \leq C\, \varepsilon_n^2$$

with some positive C. By the same way we obtain a similar estimate for the first integral, which completes the proof of the theorem.

Corollary 5. *Let $0 < \alpha < 1$ be given and suppose that the conditions $\mathcal{A}_1 - \mathcal{A}_2$ are satisfied. Then the equation*

$$F_n(c_n) = 1 - \alpha + O(\varepsilon_n^2),$$

has a solution

$$c_n = z_\alpha - \frac{\gamma_{3,n}}{6}(1 - z_\alpha^2),$$

where z_α the $1-\alpha$ quantile of the standard normal law, i.e., $\mathcal{N}(z_\alpha) = 1 - \alpha$ and

$$\gamma_{3,n} = \int_{\mathbb{A}_n} f_n(x)^3 \, \Lambda^{(n)}(\mathrm{d}x),$$

which is of order ε_n (by $\mathcal{A}_1$ with $r = 3$).

Proof. The distribution function $F_n(y)$ admits the representation

$$F_n(y) - g(y) - 1 + \alpha = O(\varepsilon_n^2),$$

uniformly in y with

$$g(y) = \mathcal{N}(y) + \frac{\gamma_{3,n}}{6}(1 - y^2)\, n(y) - 1 + \alpha$$

where $n(y)$ denotes the standard normal density function. Hence it is sufficient to find c_n such that $g(c_n) = O(\varepsilon_n^2)$. Let

$$h(y) = \mathcal{N}(y) - 1 + \alpha$$

and observe that $h(y) - g(y) = O(\varepsilon_n)$ uniformly in y. Now since $h(z_\alpha) = 0$, then

$$h(c_n) - h(z_\alpha) = h(c_n) - g(c_n) + g(c_n) = O(\varepsilon_n).$$

This implies that $c_n = z_\alpha + O(\varepsilon_n)$. Hence by the Taylor formula we can write

$$g(c_n) = g(z_\alpha) + (c_n - z_\alpha)\, g'(z_\alpha) + O(\varepsilon_n^2).$$

Since $g(c_n) = O(\varepsilon_n^2)$, then we get

$$\begin{aligned} c_n &= z_\alpha - \frac{g(z_\alpha)}{g'(z_\alpha)} + O(\varepsilon_n^2) = \\ &= z_\alpha - \frac{\mathcal{N}(z_\alpha) + \frac{\gamma_{3,n}}{6}(1 - z_\alpha^2)\, n(z_\alpha) - 1 + \alpha}{n(z_\alpha) + O(\varepsilon_n)} + O(\varepsilon_n^2) = \\ &= z_\alpha - \frac{\frac{\gamma_{3,n}}{6}(1 - z_\alpha^2)}{1 + O(\varepsilon_n)} + O(\varepsilon_n^2) = z_\alpha - \frac{\gamma_{3,n}}{6}(1 - z_\alpha^2) + O(\varepsilon_n^2). \end{aligned}$$

Here we used the fact that the parameter $\gamma_{3,n} \to 0$ with the rate ε_n by $\mathcal{A}_1$ with $r = 3$. This implies in turn that

$$\frac{\frac{\gamma_{3,n}}{6}(1 - z_\alpha^2)}{1 + O(\varepsilon_n)} = \frac{\gamma_{3,n}}{6}(1 - z_\alpha^2) + O(\varepsilon_n^2).$$

Corollary 6. *Let* $Y_n = \int_{\mathbb{A}_n} f_n(x)\, \pi^{(n)}(\mathrm{d}x)$ *and the conditions* $\mathcal{A}_1 - \mathcal{A}_2$ *be fulfilled. Then we have the representation*

$$\mathbf{P}^{(n)} \left\{ Y_n^2 \le x \right\} = 1 - 2\mathcal{N}(-\sqrt{x}) + O(\varepsilon_n^2),$$

uniformly in $x \ge 0$. *This follows easily from the theorem 4 and the equality*

$$\mathbf{P}^{(n)} \left\{ Y_n^2 \le x \right\} = \mathbf{P}^{(n)} \left\{ Y_n \le \sqrt{x} \right\} - \mathbf{P}^{(n)} \left\{ Y_n < -\sqrt{x} \right\}.$$

Example 1.1 (Frequency parameter). In this example we consider a strongly inhomogeneous case with a nonclassical rate $n^{\text{¡}\,3/2}$ instead of $n^{\text{¡}\,1/2}$ as in the i.i.d. case. Let $X^{(n)}$ be realization of a nonhomogeneous Poisson process on the real line with positive intensity function

$$S(\vartheta, x) = S(\vartheta\, x) + \beta, \quad x \in [0, n], \qquad n = 1, 2 \ldots,$$

where $S(\cdot)$ is a nonconstant periodic function with period τ and two times differentiable. The "dark current" parameter β is known and positive. The frequency parameter ϑ is unknown. Let ϑ_0 be a known value for ϑ (under a null hypothesis, say) and consider the stochastic integral (normalized score test statistic)

$$\Delta_n(\vartheta_0) = \varphi_n \left. \frac{\partial L(\vartheta, \vartheta_0, X^{(n)})}{\partial \vartheta} \right|_{\vartheta = \vartheta_0} = \varphi_n \int_0^n \frac{x\, \dot{S}(\vartheta_0\, x)}{S(\vartheta_0, x)}\, \pi(\mathrm{d}x),$$

where $\pi(\mathrm{d}x) = X^{(n)}(\mathrm{d}x) - S(x, \vartheta_0)\, \mathrm{d}x$ is the centered Poisson process, $\dot{S}(y)$ is the derivative of $S(\cdot)$ at y, the loglikelihood ratio

$$L(\vartheta, \vartheta_0, X^{(n)}) = \ln \frac{\mathrm{d}\mathbf{P}_\vartheta^{(n)}}{\mathrm{d}\mathbf{P}_{\vartheta_0}^{(n)}}(X^n)$$

and the normalizing factor

$$\varphi_n^{\text{¡}\,2} = \int_0^n \frac{x^2\, \dot{S}(\vartheta_0\, x)^2}{S(\vartheta_0, x)}\, \mathrm{d}x.$$

for some $M > 0$. Therefore we can write

$$I_n(\lambda) \equiv \int_0^n \sin^2\left(\lambda\,\varphi_n \frac{x\,\dot S(\vartheta_0\,x)}{S(\vartheta_0,x)}\right)\, S(\vartheta_0,x)\,\mathrm{d}x \geq$$

$$\geq \sum_{k=\left[\frac{n}{2\tau}\right]}^{\left[\frac{n}{\tau}\right]-1} \int_{b_1+k\tau}^{b_2+k\tau} \sin^2\left(\lambda\,\varphi_n \frac{x\,\dot S(\vartheta_0\,x)}{S(\vartheta_0,x)}\right)\, S(\vartheta_0,x)\,\mathrm{d}x,$$

where $[\,\cdot\,]$ denotes the integer part. By the change of variable $x \to u = \lambda\,\varphi_n\,x\,g(x)$ we get

$$I_n(\lambda) \geq \frac{l}{\lambda\,\varphi_n} \sum_{k=\left[\frac{n}{2\tau}\right]}^{\left[\frac{n}{\tau}\right]-1} \frac{1}{(b_2+k\,\tau)M + g(b_2)} \int_{u_1}^{u_2} \sin^2(u)\;\mathrm{d}u,$$

where

$$u_1 = \lambda\,\varphi_n\,(b_1 + k\,\tau)\,g(b_1+k\,\tau) = \lambda\,\varphi_n\,(b_1+k\,\tau)\,g(b_1) = 0$$
$$u_2 = \lambda\,\varphi_n\,(b_2 + k\,\tau)\,g(b_2+k\,\tau) = \lambda\,\varphi_n\,(b_2+k\,\tau)\,g(b_2).$$

Hence the integral

$$\int_{u_1}^{u_2} \sin^2(u)\;\mathrm{d}u = \frac{1}{2}\left(u_2 - \frac{\sin(2\,u_2)}{2}\right) = \frac{u_2}{2}\left(1 - \frac{\sin(2\,u_2)}{2\,u_2}\right).$$

There exists a constant $D > 0$ such that uniformly for all $\lambda \geq c_0\sqrt{n}/2$ we have $u_2 \geq D$. Therefore the inequality

$$\frac{\sin(2\,u_2)}{2\,u_2} \leq \eta$$

is valid for some constant $0 < \eta < 1$ and we get

$$\inf_{c_0\sqrt{n}<2\,\lambda<n} I_n(\lambda) \geq \frac{l\,(1-\eta)}{\lambda\,\varphi_n} \sum_{k=\left[\frac{n}{2\tau}\right]}^{\left[\frac{n}{\tau}\right]-1} \frac{\lambda\,\varphi_n\,(b_2+k\,\tau)\,g(b_2)}{(b_2+k\,\tau)M + g(b_2)} \geq$$
$$\geq C\,n \geq \gamma \ln\sqrt{n},$$

for some $C > 0$ and all n enough large. Hence by the last Corollary for given $0 < \alpha < 1$, we obtain

$$\mathbf{P}_{\vartheta_0}^{(n)}\left\{\Delta_n(\vartheta_0) < c_n\right\} = 1 - \alpha + O(n^{-1})$$

where

$$c_n = z_\alpha - \frac{\gamma_{3,n}}{6}(1 - z_\alpha^2)$$

with

$$\gamma_{3,n} = \varphi_n^3 \int_0^n \frac{x^3\, \dot S(\vartheta_0\, x)^3}{S(\vartheta_0, x)^2}\, \mathrm{d}x = \sqrt{\frac{27}{16\, n\, \tau}}\, \frac{\int_0^\tau \frac{\dot S(\vartheta_0\, x)^3}{S(\vartheta_0, x)^2}\, \mathrm{d}x}{\left(\int_0^\tau \frac{\dot S(\vartheta_0\, x)^2}{S(\vartheta_0, x)}\, \mathrm{d}x\right)^{3/2}}\, (1 + o(1)).$$

As it can be seen, using the Edgeworth expansion we obtain the constant c_n which refines the classical choice $c_n = z_\alpha$ (up to order $O(n^{-1})$) obtained from the central limit theorem. Remind that by the central limit theorem

$$\mathbf{P}^{(n)}\left\{\Delta_n(\vartheta_0) < z_\alpha\right\} = 1 - \alpha + o(1).$$

Now we consider the Edgeworth expansion for two terms after the Gaussian term. It helps us to describe the large deviations of a vector of stochastic integrals (see corollary 10) from which we use to obtain an Edgeworth type expansion for a vector of stochastic integrals (see remark 12, Chapter 4). It is also useful for calculating the *asymptotic deficiency* in the hypothesis testing problem (see remark 9, Chapter 3). Introduce the conditions:

$\mathcal{B}_1$. *There exists a sequence of real numbers* $\varepsilon_n \to 0$, *as* $n \to \infty$ *and constants* $C_r > 0, r = 3, 4, 5$, *such that*

$$\int_{\mathbb{A}_n} |f_n(x)|^r\ \Lambda^{(n)}(\mathrm{d}x) \leq C_r\, \varepsilon_n^{r-2}.$$

$\mathcal{B}_2$. *There exist constants* $\gamma \geq \frac{5}{2}$ *and* $c_0 > 0$ *satisfying the inequality* $\frac{C_3}{3!}c_0 + \frac{C_4}{4!}c_0^2 + \frac{C_5}{5!}c_0^3 - \frac{1}{2} < 0$ *such that*

$$\inf_{c_0\, \varepsilon_n^{-1} \leq 2|\lambda| \leq \varepsilon_n^{-3}} \int_{\mathbb{A}_n} \sin^2(\lambda f_n(x))\ \Lambda^{(n)}(\mathrm{d}x) \geq \gamma \ln \varepsilon_n^{-1}$$

for all large n.

Below $n(y)$ denotes the density function of the standard Gaussian normal, i.e., $n(y) = (2\,\pi)^{-1/2}\, e^{-y^2/2}$. Let us define the normed cumulants

$$\rho_{r,n} = \varepsilon_n^{-r+2} \int_{\mathbb{A}_n} f_n(x)^r \,\Lambda^{(n)}(\mathrm{d}x) \qquad r = 3, 4$$

and the functions

$$\begin{aligned}
Q_{1,n}(y) =& \frac{\rho_{3,n}}{3!} \int_{-\infty}^{y} H_3(t)\, n(t)\, \mathrm{d}t = \frac{\rho_{3,n}}{3!}(1-y^2)\, n(y) = -\frac{\rho_{3,n}}{3!}\, H_2(y)\, n(y) \\
Q_{2,n}(y) =& \frac{\rho_{4,n}}{4!} \int_{-\infty}^{y} H_4(t)\, n(t)\, \mathrm{d}t + \frac{\rho_{3,n}^2}{72} \int_{-\infty}^{y} H_6(t)\, n(t)\, \mathrm{d}t = \\
=& \frac{\rho_{4,n}}{4!}(3y - y^3)\, n(y) + \frac{\rho_{3,n}^2}{72}(-y^5 + 10\, y^3 - 15\, y)\, n(y) = \\
=& -\frac{\rho_{4,n}}{4!} H_3(y)\, n(y) - \frac{\rho_{3,n}^2}{72} H_5(y)\, n(y),
\end{aligned}$$

where the Hermit polynomials

$$\begin{aligned}
&H_1(t) = t,\ H_2(t) = t^2 - 1,\ H_3(t) = t^3 - 3t,\ H_4(t) = t^4 - 6t^2 + 3, \\
&H_5(t) = t^5 - 10t^3 + 15t,\ H_6(t) = t^6 - 15t^4 + 45t^2 - 15.
\end{aligned}$$

Remind that for $r = 1, 2, \dots,$ the Hermit polynomial $H_r(t)$ is defined by

$$H_r(t) = (-1)^r \left[\frac{d^r}{d\,t^r} \exp\left\{-t^2/2\right\}\right] \exp\left\{t^2/2\right\},$$

and we have

$$\int_{-\infty}^{y} H_r(t)\, e^{-t^2/2}\, \mathrm{d}t = -H_{r-1}(y)\, e^{-y^2/2}.$$

Note that $|\rho_{r,n}| \le C_r$ by $\mathcal{B}_1$. We have the following expansion:

Theorem 7. *Let conditions $\mathcal{B}_1$, $\mathcal{B}_2$ be fulfilled, then*

$$\sup_y \left|F_n(y) - \mathcal{N}(y) - Q_{1,n}(y)\, \varepsilon_n - Q_{2,n}(y)\, \varepsilon_n^2\right| \le C\, \varepsilon_n^3,$$

for some constant $C > 0$ and all n large.

Proof. The functions $F_n(y)$ and

$$G_n(y) = \mathcal{N}(y) + Q_{1,n}(y)\, \varepsilon_n + Q_{2,n}(y)\, \varepsilon_n^2$$

satisfy the conditions of the lemma of Essen. Hence for $N = \varepsilon_n^{-3}$ and an arbitrary constant $m > 1$ there exists $C(m)$ such that

$$|F_n(y) - G_n(y)| \le \frac{m}{2\pi} \int_{-N}^{N} \left| \frac{\Phi_n(\lambda) - \Psi_n(\lambda)}{\lambda} \right| \mathrm{d}\lambda + C(m)\frac{M}{N},$$

where $\Phi_n(\lambda)$ and $\Psi_n(\lambda)$ are Fourier transforms of the functions $F_n(y)$ and $G_n(y)$ respectively and M is an upper bound of $G_n'(y)$. One can choose M independent from n, because $\rho_{r,n}$ and ε_n are bounded. Hence for the second term in the right hand side of the last inequality we have the estimate ,

$$C(m)\frac{M}{N} \le C\,\varepsilon_n^3$$

for some positive constant C. Now we estimate the first term. By Taylor formula and the assumption (1.2), the characteristic function of $F_n(y)$ can be written as

$$\begin{aligned}
\Phi_n(\lambda) &= \int_{-\infty}^{+\infty} e^{\mathrm{i}\lambda y}\, F_n(\mathrm{d}y) = \exp\left\{ \int_{\mathbb{A}_n} \left(e^{\mathrm{i}\lambda f_n(x)} - 1 - \mathrm{i}\lambda f_n(x)\right) \Lambda^{(n)}(\mathrm{d}x) \right\} = \\
&= e^{-\lambda^2/2} \exp\left\{ \int_{\mathbb{A}_n} \left(e^{\mathrm{i}\lambda f_n(x)} - 1 - \mathrm{i}\lambda f_n(x) - \frac{(\mathrm{i}\lambda)^2}{2} f_n(x)^2 \right) \Lambda^{(n)}(\mathrm{d}x) \right\} = \\
&= e^{-\lambda^2/2} \exp\left\{ \frac{(\mathrm{i}\lambda)^3}{3!} \int_{\mathbb{A}_n} f_n(x)^3 \Lambda^{(n)}(\mathrm{d}x) + \frac{(\mathrm{i}\lambda)^4}{4!} \int_{\mathbb{A}_n} f_n(x)^4 \Lambda^{(n)}(\mathrm{d}x) + \right. \\
&\left. + \frac{(\mathrm{i}\lambda)^5}{5!} \int_{\mathbb{A}_n} e^{\mathrm{i}\tilde{\lambda} f_n(x)} f_n(x)^5 \Lambda^{(n)}(\mathrm{d}x) \right\} = \\
&= e^{-\lambda^2/2} \exp\left\{ \frac{(\mathrm{i}\lambda)^3}{3!}\, \rho_{3,n}\, \varepsilon_n + \frac{(\mathrm{i}\lambda)^4}{4!}\, \rho_{4,n}\, \varepsilon_n^2 + (\mathrm{i}\lambda)^5\, r_n\, \varepsilon_n^3 \right\},
\end{aligned}$$

where

$$r_n = (5!)^{-1}\, \varepsilon_n^{-3} \int_{\mathbb{A}_n} e^{\mathrm{i}\tilde{\lambda} f_n(x)} f_n(x)^5 \Lambda^{(n)}(\mathrm{d}x),$$

and $\tilde{\lambda} \in (0, \lambda)$. Note that $|r_n| \le C_5/5!$, by $\mathcal{B}_1$. We define the function

$$y(z) = \frac{\rho_{3,n}(\mathrm{i}\lambda)^3}{3!}\, z + \frac{\rho_{4,n}(\mathrm{i}\lambda)^4}{4!}\, z^2 + (\mathrm{i}\lambda)^5\, r_n\, z^3$$

and obtain the Taylor expansion

$$e^{y(z)} = 1 + \frac{\rho_{3,n}(\mathrm{i}\lambda)^3}{3!}\, z + \left\{ \frac{\rho_{4,n}(\mathrm{i}\lambda)^4}{4!} + \frac{\rho_{3,n}^2(\mathrm{i}\lambda)^6}{72} \right\} z^2 + R_n\, z^3.$$

Observe that if we let $z = \varepsilon_n$, then

$$\Phi_n(\lambda) = e^{\mathsf{i}\ \lambda^2/2 + y(\varepsilon_n)} =$$
$$= e^{\mathsf{i}\ \lambda^2} \left\{ 1 + \frac{\rho_{3,n}(\mathrm{i}\lambda)^3}{3!}\ \varepsilon_n + \left[\frac{\rho_{4,n}(\mathrm{i}\lambda)^4}{4!} + \frac{\rho_{3,n}^2(\mathrm{i}\lambda)^6}{72} \right] \varepsilon_n^2 + R_n\ \varepsilon_n^3 \right\},$$

and

$$R_n = \frac{1}{3!}\ e^{y(\tilde{\varepsilon}_n)} \left\{ \left[\frac{(\mathrm{i}\lambda)^3 \rho_{3,n}}{3!} + \frac{2(\mathrm{i}\lambda)^4 \rho_{4,n}}{4!}\ \tilde{\varepsilon}_n + 3(\mathrm{i}\lambda)^5 r_n \tilde{\varepsilon}_n^2 \right]^3 + \right.$$
$$+ 3 \left[\frac{(\mathrm{i}\lambda)^3 \rho_{3,n}}{3!} + \frac{2(\mathrm{i}\lambda)^4 \rho_{4,n}}{4!}\ \tilde{\varepsilon}_n + 3(\mathrm{i}\lambda)^5 r_n \tilde{\varepsilon}_n^2 \right] \left[\frac{(\mathrm{i}\lambda)^4 \rho_{4,n}}{12} + 6(\mathrm{i}\lambda)^5 r_n \tilde{\varepsilon}_n \right] +$$
$$\left. + \frac{(\mathrm{i}\lambda)^4 \rho_{4,n}}{12} + 6(\mathrm{i}\lambda)^5 r_n \tilde{\varepsilon}_n \right\},$$

where $\tilde{\varepsilon}_n \in (0, \varepsilon_n)$. By considering two cases $|\lambda| \leq 1$ and $|\lambda| > 1$, the inequality

$$|R_n| \leq C\ e^{y(\tilde{\varepsilon}_n)} \left\{ |\lambda|^4 + |\lambda|^{15} \right\},$$

holds with some positive constant C for all λ.

Now we obtain the Fourier transform of $G_n(y)$. Since the Fourier transform of $H_r(y)\, e^{\mathsf{i}\ y^2/2}$ is equal to $\sqrt{2\pi}(\mathrm{i}\lambda)^r \exp\left(-\lambda^2/2\right)$, then

$$\Psi_n(\lambda) = \int_{\mathsf{i}\mathsf{1}}^{+\mathsf{1}} e^{\mathrm{i}\lambda y} \mathrm{d}\, G_n(y) = \int_{\mathsf{i}\mathsf{1}}^{+\mathsf{1}} e^{\mathrm{i}\lambda y}\ G_n^{\mathsf{0}}(y)\ \mathrm{d}y =$$
$$= e^{\mathsf{i}\ \lambda^2/2} \left\{ 1 + \frac{\rho_{3,n}(\mathrm{i}\lambda)^3}{3!}\ \varepsilon_n + \left[\frac{\rho_{4,n}(\mathrm{i}\lambda)^4}{4!} + \frac{\rho_{3,n}^2(\mathrm{i}\lambda)^6}{72} \right] \varepsilon_n^2 \right\}.$$

Therefore for all λ

$$|\Phi_n(\lambda) - \Psi_n(\lambda)| = e^{\mathsf{i}\ \lambda^2/2}\ \ |R_n|\ \ \varepsilon_n^3 \leq C e^{\mathsf{i}\ \lambda^2/2 + y(\tilde{\varepsilon}_n)} \left\{ |\lambda|^4 + |\lambda|^{15} \right\}\ \ \varepsilon_n^3.$$

For n large the inequalities $1 < N_0 \equiv c_0\ \varepsilon_n^{\mathsf{i}\ 1} < N = \varepsilon_n^{\mathsf{i}\ 3}$ hold and we can write

$$\int_{\mathsf{i}\ N}^{N} \left| \frac{\Phi_n(\lambda) - \Psi_n(\lambda)}{\lambda} \right| \mathrm{d}\lambda = \int_{\mathsf{i}\ N}^{\mathsf{i}\ N_0} + \int_{\mathsf{i}\ N_0}^{N_0} + \int_{N_0}^{N},$$

with obvious notations. Since in the second integral $|\lambda\ \tilde{\varepsilon}_n| \leq |\lambda\ \varepsilon_n| \leq c_0$, then

$$|y(\tilde{\varepsilon}_n)| = \left| \frac{\rho_{3,n}(\mathrm{i}\lambda)^3}{3!}\ \tilde{\varepsilon}_n + \frac{\rho_{4,n}(\mathrm{i}\lambda)^4}{4!}\ \tilde{\varepsilon}_n^2 + (\mathrm{i}\ \lambda)^5\ r_n\ \tilde{\varepsilon}_n^3 \right| =$$
$$= \lambda^2 \left| \left\{ \frac{\rho_{3,n}}{3!} (\mathrm{i}\ \lambda\ \tilde{\varepsilon}_n) + \frac{\rho_{4,n}}{4!}\ (\mathrm{i}\ \lambda\ \tilde{\varepsilon}_n)^2 + r_n\ (\mathrm{i}\ \lambda\ \tilde{\varepsilon}_n)^3 \right\} \right| \leq$$
$$\leq \lambda^2 \left\{ \frac{C_3}{3!}\ c_0 + \frac{C_4}{4!}\ c_0^2 + \frac{C_5}{5!}\ c_0^3 \right\}.$$

Therefore

$$
\begin{aligned}
&\int_{-N_0}^{N_0} \left|\frac{\Phi_n(\lambda)-\Psi_n(\lambda)}{\lambda}\right| \mathrm{d}\lambda \le \\
&\le C\varepsilon_n^3 \int_{-\infty}^{\infty} \exp\left\{\lambda^2\left(\frac{C_3}{3!}\,c_0+\frac{C_4}{4!}\,c_0^2+\frac{C_5}{5!}\,c_0^3-\frac{1}{2}\right)\right\}\,\left(|\lambda|^3+|\lambda|^{14}\right)\,\mathrm{d}\lambda \le \\
&\le C_1\,\varepsilon_n^3,
\end{aligned}
$$

for some positive constant C_1. Observe that the exponent in the integral is negative by $\mathcal{B}_2$. In the last integral

$$
\int_{N_0}^{N} \left|\frac{\Phi_n(\lambda)-\Psi_n(\lambda)}{\lambda}\right| \mathrm{d}\lambda \le \int_{N_0}^{N} |\Phi_n(\lambda)|\,|\lambda|^{-1}\,\mathrm{d}\lambda + \int_{N_0}^{N} |\Psi_n(\lambda)|\,|\lambda|^{-1}\,\mathrm{d}\lambda,
$$

since $|\lambda| \ge N_0 > 1$, for suitable positive constant C

$$
1+\frac{|\rho_{3,n}|}{3!}\,|\lambda|^3\;\varepsilon_n+\left(\frac{|\rho_{4,n}|}{4!}\,|\lambda|^4+\frac{\rho_{3,n}^2}{72!}\,|\lambda|^6\right)\;\varepsilon_n^2 \le C\,|\lambda|^6\,,
$$

and consequently

$$
\begin{aligned}
\int_{N_0}^{N} |\Psi_n(\lambda)|\,|\lambda|^{-1}\,\mathrm{d}\lambda &\le C\int_{N_0}^{\infty} e^{-\lambda^2/2}\,\lambda^5\,\mathrm{d}\lambda \le \\
&\le C_2\,N_0^4\,e^{-N_0^2/2} = C_2\,c_0\,\varepsilon_n^{-4}\,e^{-c_0^2\,\varepsilon_n^{-2}/2} \le C\,\varepsilon_n^3.
\end{aligned}
$$

Now consider the integral

$$
\int_{N_0}^{N} |\Phi_n(\lambda)|\,|\lambda|^{-1}\,\mathrm{d}\lambda = \int_{N_0}^{N} \exp\left\{-2\int_{\mathbb{A}_n} \sin^2\left(\frac{\lambda}{2}f_n(x)\right)\Lambda^{(n)}(\mathrm{d}x)\right\}\lambda^{-1}\,\mathrm{d}\lambda,
$$

and note that by $\mathcal{B}_2$

$$
\int_{\mathbb{A}_n} \sin^2\left(\frac{\lambda}{2}f_n(x)\right)\Lambda^{(n)}(\mathrm{d}x) \ge \gamma\ln\varepsilon_n^{-1}.
$$

Since $\gamma \ge 5/2$, we have

$$
\begin{aligned}
\int_{N_0}^{N} |\Phi_n(\lambda)|\,|\lambda|^{-1}\,\mathrm{d}\lambda &\le \int_{N_0}^{N} \exp\left(-2\gamma\ln\varepsilon_n^{-1}\right)\lambda^{-1}\,\mathrm{d}\lambda \le N_0^{-1}\,\varepsilon_n^{2\gamma}\,(N-N_0) \le \\
&\le c_0^{-1}\,\varepsilon_n\,\varepsilon_n^{2\gamma}\,\varepsilon_n^{-3} = c_0^{-1}\,\varepsilon_n^{2\gamma-2} \le c_0^{-1}\,\varepsilon_n^3. \qquad (1.3)
\end{aligned}
$$

Hence for the last integral we have

$$\int_{N_0}^{N} \left| \frac{\Phi_n(\lambda) - \Psi_n(\lambda)}{\lambda} \right| \mathrm{d}\lambda \leq C\, \varepsilon_n^3$$

with some positive C. By the same way we obtain a similar estimate for the first integral, which completes the proof of the theorem.

Due to the Theorem 7, we can refine the corollary 5 as following.

Corollary 8. *Let $0 < \alpha < 1$ be given and the conditions $\mathcal{B}_1 - \mathcal{B}_2$ be fulfilled. Then the equation $F_n(y) = 1 - \alpha + O(\varepsilon_n^3)$ has a solution $y = c_{n,\alpha}$,*

$$c_{n,\alpha} = z_\alpha - \frac{\gamma_{3,n}}{6}(1 - z_\alpha^2) - \frac{\gamma_{3,n}^2}{72}\left(-2\, z_\alpha^5 + 8\, z_\alpha^3 - 12\, z_\alpha\right) - \frac{\gamma_{4,n}}{4!}(3z_\alpha - z_\alpha^3),$$

where the parameters

$$\gamma_{r,n} = \int_{\mathbb{A}_n} f_n(x)^r\ \Lambda^{(n)}(\mathrm{d}x), \qquad r = 3, 4$$

are dominated by ε_n and ε_n^2, respectively by $\mathcal{A}_1$ with $r = 3, 4$.

Proof. *Under the conditions $\mathcal{B}_1 - \mathcal{B}_2$*

$$F_n(y) - g(y) - 1 + \alpha = O(\varepsilon_n^3),$$

uniformly in y, where

$$g(y) = \mathcal{N}(y) + Q_{1,n}(y)\ \varepsilon_n + Q_{2,n}(y)\ \varepsilon_n^2 - 1 + \alpha.$$

Therefore we obtain $c_{n,\alpha}$ such that $g(c_{n,\alpha}) = O(\varepsilon_n^3)$. Let us introduce the function

$$h(y) = \mathcal{N}(y) + Q_{1,n}(y)\ \varepsilon_n - 1 + \alpha$$

for which we have

$$g(y) - h(y) = O(\varepsilon_n^2)$$

uniformly in y. As we saw in the corollary 5 for the solution

$$c_n = z_\alpha - \frac{\gamma_{3,n}}{6}(1 - z_\alpha^2)$$

we have $h(c_n) = O(\varepsilon_n^2)$. *Therefore*

$$h(c_{n,\alpha}) - h(c_n) = h(c_{n,\alpha}) - g(c_{n,\alpha}) + g(c_{n,\alpha}) - h(c_n) = O(\varepsilon_n^2)$$

which implies in turn that

$$c_{n,\alpha} = c_n + O(\varepsilon_n^2).$$

Expanding $g(c_{n,\alpha})$ *about* c_n *gives*

$$g(c_{n,\alpha}) = g(c_n) + (c_{n,\alpha} - c_n)\, g'(c_n) + \frac{(c_{n,\alpha} - c_n)^2}{2}\, g''(\xi_n)$$

where the intermediate point $\xi_n \to z_\alpha$ *as* $n \to \infty$. *Since* $g(c_{n,\alpha}) = O(\varepsilon_n^3)$ *and the last term of the expansion is of order* $O(\varepsilon_n^4)$ *we obtain*

$$c_{n,\alpha} = c_n - \frac{g(c_n)}{g'(c_n)}.$$

By the Taylor formula

$$\mathcal{N}(c_n) = \mathcal{N}(z_\alpha) + (c_n - z_\alpha)\, n(c_n) - \frac{(c_n - z_\alpha)^2}{2}\, c_n\, n(c_n) + O(\varepsilon_n^3).$$

Hence we can write

$$\begin{aligned} g(c_n) = &\left(c_n - z_\alpha + \frac{\gamma_{3,n}}{6}(1 - c_n^2)\right) n(c_n) - \\ &- \frac{(c_n - z_\alpha)^2}{2}\, c_n\, n(c_n) + \frac{\gamma_{4,n}}{4!}(3\, c_n - c_n^3)\, n(c_n) + \\ &+ \frac{\gamma_{3,n}^2}{72}(-c_n^5 + 10\, c_n^3 - 15\, c_n)\, n(c_n) + O(\varepsilon_n^3). \end{aligned}$$

Observe that the term $c_n - z_\alpha + \frac{\gamma_{3,n}}{6}(1 - c_n^2)$ *as well as the three other terms in the right and side are of order* $O(\varepsilon_n^2)$. *On the other hand we have*

$$g'(c_n) = n(c_n) + O(\varepsilon_n)$$

which allows us to write

$$\begin{aligned} \frac{g(c_n)}{g'(c_n)} = &\; c_n - z_\alpha + \frac{\gamma_{3,n}}{6}(1 - c_n^2) - \frac{(c_n - z_\alpha)^2}{2}\, c_n + \frac{\gamma_{4,n}}{4!}(3\, c_n - c_n^3) + \\ &+ \frac{\gamma_{3,n}^2}{72}(-c_n^5 + 10\, c_n^3 - 15\, c_n) + O(\varepsilon_n^3). \end{aligned}$$

By considering the representations

$$
\begin{aligned}
\frac{\gamma_{3,n}}{6}\left(1-c_n^2\right) &= \frac{\gamma_{3,n}}{6}\left(1-z_\alpha^2\right)+\frac{\gamma_{3,n}^2}{18}z_\alpha\left(1-z_\alpha 2\right)+O(\varepsilon_n^3)\\
\left(c_n-z_\alpha\right)^2 c_n &= \frac{\gamma_{3,n}^2}{36}\,z_\alpha\left(1-z_\alpha^2\right)^2+O\left(\varepsilon_n^3\right)\\
\gamma_{4,n}\left(3\,c_n-c_n^3\right) &= \gamma_{4,n}\left(3\,z_\alpha-z_\alpha^3\right)+O(\varepsilon_n^3)\\
\gamma_{3,n}^2\left(-c_n^5+10\,c_n^3-15\,c_n\right) &= \gamma_{3,n}^2\left(-z_\alpha^5+10\,z_\alpha^3-15\,z_\alpha\right)+O(\varepsilon_n^3)
\end{aligned}
$$

we obtain the solution

$$
c_{n,\alpha}=z_\alpha-\frac{\gamma_{3,n}}{6}(1-z_\alpha^2)-\frac{\gamma_{3,n}^2}{72}\left(-2\,z_\alpha^5+8\,z_\alpha^3-12\,z_\alpha\right)-\frac{\gamma_{4,n}}{4!}(3z_\alpha-z_\alpha^3).
$$

Remind that $\gamma_{3,n}$ *is of order* $O(\varepsilon_n)$ *and* $\gamma_{4,n}$ *and* $\gamma_{3,n}^2$ *are of order* $O(\varepsilon_n^2)$.

Example 1.2 (Frequency parameter). The Poisson process with the intensity function given in the preceding example (frequency parameter) satisfies the conditions $\mathcal{B}_1-\mathcal{B}_2$ with $\varepsilon_n=n^{¡\ 1/2}$. Hence

$$
\mathbf{P}_{\vartheta_0}^{(n)}\left\{\Delta_n(\vartheta_0)<c_{n,\alpha}\right\}=1-\alpha+O(n^{¡\ 3/2})
$$

where

$$
c_{n,\alpha}=z_\alpha-\frac{\gamma_{3,n}}{6}(1-z_\alpha^2)-\frac{\gamma_{3,n}^2}{72}\left(-2\,z_\alpha^5+8\,z_\alpha^3-12\,z_\alpha\right)-\frac{\gamma_{4,n}}{4!}(3z_\alpha-z_\alpha^3),
$$

with

$$
\begin{aligned}
\gamma_{3,n} &= \varphi_n^3\int_0^n\frac{x^3\,\dot S(\vartheta_0\,x)^3}{S(\vartheta_0,x)^2}\,\mathrm{d}x=\sqrt{\frac{27}{16\,n\,\tau}}\,\frac{\int_0^\tau\frac{\dot S(\vartheta_0\,x)^3}{S(\vartheta_0,x)^2}\,\mathrm{d}x}{\left(\int_0^\tau\frac{\dot S(\vartheta_0\,x)^2}{S(\vartheta_0,x)}\,\mathrm{d}x\right)^{3/2}}\,(1+o(1))\\
\gamma_{4,n} &= \varphi_n^4\int_0^n\frac{x^4\,\dot S(\vartheta_0\,x)^4}{S(\vartheta_0,x)^3}\,\mathrm{d}x=\frac{9\,\tau}{5\,n}\,\frac{\int_0^\tau\frac{\dot S(\vartheta_0\,x)^4}{S(\vartheta_0,x)^3}\,\mathrm{d}x}{\left(\int_0^\tau\frac{\dot S(\vartheta_0\,x)^2}{S(\vartheta_0,x)}\,\mathrm{d}x\right)^{2}}\,(1+o(1)).
\end{aligned}
$$

The following two corollaries are needed in Chapter 4. Let $2<r<3$ be an arbitrary given number and $\mathbf{f}_n(x)=(f_1(x),\ldots,f_k(x))$ where $f_\nu(x),\nu=1,2,\ldots,k$ are some real measurable functions which satisfy the following more weak version of the condition $\mathcal{B}_2$.

$\mathcal{B}_2^0$. *There exists constants $\gamma \geq 1 + \frac{r}{2}$ such that for all n large*

$$\inf_{c_0\, \varepsilon_n^{-1} \leq\, 2|\lambda| \leq\, \varepsilon_n^{-3}} \int_{\mathbb{A}_n} \sin^2\left(\lambda f_\nu(x)\right)\; \Lambda^{(n)}(\mathrm{d}x) \geq \gamma \ln \varepsilon_n^{-1},$$

where $c_0 > 0$ satisfies the inequality $\frac{C_3}{3!} c_0 + \frac{C_4}{4!} c_0^2 + \frac{C_5}{5!} c_0^3 - \frac{1}{2} < 0.$

Corollary 9. *Under the conditions $\mathcal{B}_1$ and $\mathcal{B}_2^0$, the distribution function of Y_ν^2 of the stochastic integral*

$$Y_\nu = \int_{\mathbb{A}_n} f_\nu(x)\; \pi^{(n)}(\mathrm{d}x).$$

admits the representation

$$\begin{aligned}\mathbf{P}^{(n)}\left\{Y_\nu^2 > x\right\} = 2\,\mathcal{N}(-\sqrt{x}) - \frac{2\,\gamma_{4,n}}{4!}\, H_3(\sqrt{x})\, n(\sqrt{x}) - \\ - \frac{2\,\gamma_{3,n}^2}{72}\, H_5(\sqrt{x})\, n(\sqrt{x}) + O_r(\varepsilon_n^r),\end{aligned}$$

uniformly in $x \geq 0$, where $\gamma_{3,n} = \varepsilon_n\, \rho_{3,n}$ and $\gamma_{4,n} = \varepsilon_n^2\, \rho_{4,n}$.

Proof. Applying the condition $\mathcal{B}_2^0$ instead of $\mathcal{B}_2$ in (1.3) provides the following expansion of the distribution function $F_\nu(\cdot) = F_\nu^{(n)}(\cdot)$ of Y_ν

$$\sup_y \left| F_\nu(y) - \mathcal{N}(y) - Q_{1,n}(y)\, \varepsilon_n - Q_{2,n}(y)\, \varepsilon_n^2 \right| \leq C_r\, \varepsilon_n^r,$$

for some constant $C_r > 0$ and all n large. Hence

$$\mathbf{P}^{(n)}\left\{Y_\nu \leq y\right\} = F_\nu(y) = \mathcal{N}(y) - \frac{\gamma_{3,n}}{3!} H_2(y)\, n(y) - \frac{\gamma_{4,n}}{4!} H_3(y)\, n(y) -$$

$$- \gamma_{3,n}^2$$

Corollary 10. *Let the conditions $\mathcal{B}_1$ and $\mathcal{B}_2^0$ be fulfilled for the components of the random vector $\mathbf{Y}_n = (Y_1, \ldots, Y_k)$ where the stochastic integrals*

$$Y_\nu = \int_{\mathbb{A}_n} f_\nu(x)\, \pi^{(n)}(\mathrm{d}x), \quad \nu = 1, \ldots, k.$$

Then for any $\eta > 0$

$$\mathbf{P}^{(n)} \left\{ \|\mathbf{Y}_n\| > \varepsilon_n^{-\eta} \right\} = O(\varepsilon_n^r),$$

where $\|\mathbf{Y}_n\|^2 = \sum_{\nu=1}^k Y_\nu^2$.

Proof. The proof follows from inequality

$$\mathbf{P}^{(n)} \left\{ \|\mathbf{Y}_n\| > \varepsilon_n^{-\eta} \right\} \leq \sum_{\nu=1}^k \mathbf{P}^{(n)} \left\{ Y_\nu^2 > \frac{\varepsilon_n^{-2\eta}}{k} \right\}$$

and the last corollary for the components of $\mathbf{Y}_n$ with $x = \frac{\varepsilon_n^{-2\eta}}{k}$. Note that the first three term in the right hand side of the expansion of $\mathbf{P}^{(n)} \left\{ Y_\nu^2 > \frac{\varepsilon_n^{-2\eta}}{k} \right\}$ converge exponentially to zero. We remark that the assertion of the corollary holds true under $\mathcal{B}_1 - \mathcal{B}_2$ and with $r = 3$.

1.6 Large deviations principle

Let $X^{(n)}$ be a Poisson process with intensity function $S_n(x)$, $x \in \mathbb{A}_n$ with respect to Lebesgue measure. In this section we consider the *large deviations principle* for the stochastic integral

$$Y_n = \int_{\mathbb{A}_n} f(x)\, X^{(n)}\,(\mathrm{d}x)\,,$$

where $f(\cdot) = f_n(\cdot)$ is a real measurable function. This helps us to determine the rate of convergence of the power of a consistent test which is based on a stochastic integral. The results of this section are needed in the next chapter. Let

$$\mu_n \equiv \mathbf{E}\,(Y_n) = \int_{\mathbb{A}_n} f(x)\, S_n(x)\, \mathrm{d}x,$$

$$\sigma_n^2 \equiv \mathbf{E}\,(Y_n - \mu_n)^2 = \int_{\mathbb{A}_n} f(x)^2\, S_n(x)\, \mathrm{d}x,$$

and suppose for the moment that

$$\frac{\mu_n}{\sigma_n} \to \infty.$$

Let also $\{\nu_n\}$ be a sequence of real numbers such that

$$\lim_{n\to\infty} \frac{\nu_n}{\mu_n} = \gamma < 1.$$

Then by the Chebychev's inequality

$$\mathbf{P}^{(n)}\left\{Y_n < \nu_n\right\} = \mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n} - 1 < \frac{\nu_n}{\mu_n} - 1\right\} \le$$

$$\le \mathbf{P}^{(n)}\left\{\left|\frac{Y_n}{\mu_n} - 1\right| > \left|\frac{\nu_n}{\mu_n} - 1\right|\right\} \le \frac{\mathbf{E}\left(\frac{Y_n}{\mu_n} - 1\right)^2}{\left(\frac{\nu_n}{\mu_n} - 1\right)^2} = \frac{\sigma_n^2}{\mu_n^2\left(\frac{\nu_n}{\mu_n} - 1\right)^2} \to 0.$$

Now one may ask about the rate of this convergence. It can be shown that under certain conditions this probability converges exponentially to zero with a certain rate. Let us introduce the following notations

- logarithmic moment generating function of Y_n :

$$\Lambda_n(\lambda) = \ln \mathbf{E} \exp(\lambda Y_n) = \int_{\mathbb{A}_n} \left[e^{\lambda f(x)} - 1\right] S_n(x)\, \mathrm{d}x,$$

- the limit

$$\Lambda(\lambda) = \lim_{n\to\infty} \frac{\Lambda_n(\lambda)}{\mu_n}, \tag{1.4}$$

- the Fenchel-Legendre transform of $\Lambda(\cdot)$:

$$\Lambda^*(\nu) = \sup_{\lambda\in\mathbb{R}}\{\lambda\nu - \Lambda(\lambda)\},$$

- $D^\circ_{\Lambda^*}$ and D°_{Λ} are the interiors of the sets

$$D_{\Lambda^*} = \{\nu : \quad \Lambda^*(\nu) < \infty\}$$
$$D_{\Lambda} = \{\lambda : \quad \Lambda(\lambda) < \infty\},$$

respectively.

The behavior of the sequence $\mathbf{P}^{(n)}\{Y_n < \nu_n\}$ is described below in the asymptotic of

$$\mu_n \to \infty.$$

The parameter μ_n plays the role of *natural parameter* like n in the i.i.d. case $X_1, \dots, X_n$ and the function $\Lambda(\lambda)$ corresponds to the logarithmic moment generating function of X_1.

Theorem 11. *Let $\mu_n \to \infty$ and the following conditions be fulfilled:*

- *for all λ the limit $\Lambda(\lambda)$ exists, finite or infinite,*
- *for given sequence $\{\nu_n\}$ the limit*

$$\lim_{n\to\infty} \frac{\nu_n}{\mu_n} = \gamma < 1$$

  exists and $\gamma \in D^{\circ}_{\Lambda^*}$,

- *the function $\Lambda(\cdot)$ is differentiable in D°_{Λ} and for any $\nu < 1$ in $D^{\circ}_{\Lambda^*}$ there exists $\eta \in D^{\circ}_{\Lambda}$ such that $\Lambda'(\eta) = \nu$.*

Then we have the representation

$$\mathbf{P}^{(n)}\{Y_n < \nu_n\} = \exp\{-\mu_n \Lambda^*(\gamma)(1 + o(1))\}. \tag{1.5}$$

Proof. We follow below the proof of Gärtner-Ellis theorem (see [16], Section 2.3). First we remind several properties of the functions $\Lambda(\cdot)$ and $\Lambda^*(\cdot)$. As it follows from their definitions these functions are convex (see, e.g., [16], p. 28) and the function $\Lambda^*(\nu) \geq 0$ for all ν because $\Lambda(0) = 0$. By assumptions of Theorem $D^{\circ}_{\Lambda^*}$ is nonempty, because $\gamma \in D^{\circ}_{\Lambda^*}$. This implies implicitly that $\Lambda(\lambda) > -\infty$ for all $\lambda \in R$. Indeed if $\Lambda(\lambda) = -\infty$ for some λ, then $\Lambda^*(\nu) = \infty$ for all $\nu \in R$. If $\nu \in D^{\circ}_{\Lambda^*}$, then it follows from convexity that $\Lambda^*(\cdot)$ is continuous at ν. Now fix $\nu < 1$ and take $\lambda \leq 0$. By Chebychev's inequality :

$$\mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n} \leq \nu\right\} = \mathbf{P}^{(n)}\left\{\exp\left(\lambda\mu_n\left(\frac{Y_n}{\mu_n} - \nu\right)\right) \geq 1\right\} \leq$$
$$\leq \exp(-\lambda\mu_n\nu)\,\mathbf{E}\exp(\lambda Y_n) = \exp\{-\lambda\mu_n\nu + \Lambda_n(\lambda)\}.$$

By taking first $\limsup_{n\to\infty}$ and then $\inf_{\lambda\le 0}$ we obtain

$$\limsup_{n\to\infty} \frac{\ln \mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n} \le \nu\right\}}{\mu_n} \le -\sup_{\lambda\le 0}\{\lambda\nu - \Lambda(\lambda)\}. \tag{1.6}$$

Now we show that for $\nu < 1$,

$$\sup_{\lambda\le 0}\{\lambda\nu - \Lambda(\lambda)\} = \Lambda^*(\nu). \tag{1.7}$$

Let $\lambda \in R$ be arbitrary. By Jensen's inequality

$$\mathbf{E}\exp\{\lambda Y_n\} \ge \exp\{\lambda \mathbf{E} Y_n\} = \exp\{\lambda\,\mu_n\}.$$

Therefore for all $\lambda \in \mathbb{R}$

$$\frac{\Lambda_n(\lambda)}{\mu_n} = \frac{\ln \mathbf{E}\exp\{\lambda Y_n\}}{\mu_n} \ge \lambda.$$

By taking limit as $n \to \infty$, we obtain

$$\Lambda(\lambda) \ge \lambda\ .$$

Here we used the assumption of existence of the limit $\Lambda(\lambda)$ for all $\lambda \in \mathbb{R}$. This inequality implies $\Lambda^*(1) = 0$, because

$$0 \le \Lambda^*(1) = \sup_{\lambda\in R}\{\lambda - \Lambda(\lambda)\} \le 0.$$

Now if $\nu < 1$ and $\lambda > 0$,

$$\lambda\nu - \Lambda(\lambda) \le \lambda - \Lambda(\lambda) \le \Lambda^*(1) = 0.$$

Finally

$$\Lambda^*(\nu) = \sup_{\lambda\in\mathbb{R}}\{\lambda\nu - \Lambda(\lambda)\} = \sup_{\lambda\le 0}\{\lambda\nu - \Lambda(\lambda)\},$$

and from (1.7) we obtain the upper bound:

$$\limsup_{n\to\infty} \frac{\ln \mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n} \le \nu\right\}}{\mu_n} \le -\Lambda^*(\nu). \tag{1.8}$$

Before obtaining the lower bound we show the monotonicity of $\Lambda^*(\nu)$ for $\nu < 1$ and $\nu > 1$. This function is non-increasing for $\nu < 1$. Let $\nu_1 < \nu_2 < 1$ and $\lambda \leq 0$,

$$\lambda\nu_1 - \Lambda(\lambda) \geq \lambda\nu_2 - \Lambda(\lambda).$$

Consequently

$$\sup_{\lambda \leq 0}\{\lambda\nu_1 - \Lambda(\lambda)\} \geq \sup_{\lambda \leq 0}\{\lambda\nu_2 - \Lambda(\lambda)\},$$

and from (1.7)

$$\Lambda^*(\nu_1) \geq \Lambda^*(\nu_2).$$

By a similar argument it can be shown that $\Lambda^*(\nu)$ is nondecreasing for $\nu > 1$. In order to establish the lower bound it suffices to prove that for all $x < \nu$ (in a left neighborhood of ν) for which $x \in D^\circ_{\Lambda^*}$;

$$\liminf_{\delta \to 0} \liminf_{n \to \infty} \frac{\ln \mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n} \in (x-\delta, x+\delta)\right\}}{\mu_n} \geq -\Lambda^*(x), \tag{1.9}$$

because for each $x < \nu$ there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq (-\infty, \nu)$. Hence

$$\liminf_{n \to \infty} \frac{\ln \mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n} \leq \nu\right\}}{\mu_n} \geq \liminf_{n \to \infty} \frac{\ln \mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n} \in (x-\delta, x+\delta)\right\}}{\mu_n}.$$

Since the left hand side does not depend on δ, we can take $\delta \to 0$ and from (1.9), for $x < \nu$ we obtain

$$\liminf_{n \to \infty} \frac{\ln \mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n} \leq \nu\right\}}{\mu_n} \geq -\Lambda^*(x).$$

Hence

$$\liminf_{n \to \infty} \frac{\ln \mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n} \leq \nu\right\}}{\mu_n} \geq \sup_{x<\nu}\{-\Lambda^*(x)\} = -\inf_{x<\nu} \Lambda^*(x) = -\Lambda^*(\nu).$$

The last equality follows from the fact that $\Lambda^*(\cdot)$ is convex and non-increasing for $x < \nu < 1$. Therefore the lower bound is a consequence of the inequality (1.9). To prove it, let $x < \upsilon < 1$ and $x \in D^\circ_{\Lambda^*}$. By assumptions of Theorem there exists $\eta \in D^\circ_\Lambda$ such that $\Lambda'(\eta) = x$. Let $\mathbf{Q}^{(n)}$ be the distribution of the random variable $\frac{Y_n}{\mu_n}$ and $\tilde{\mathbf{Q}}^{(n)}$ a probability measure defined by ;

$$\frac{\mathrm{d}\tilde{\mathbf{Q}}^{(n)}}{\mathrm{d}\mathbf{Q}^{(n)}}(z) = \exp\left(\mu_n\eta z - \Lambda_n(\eta)\right).$$

(Since $\Lambda(\eta) < \infty$, then for n large $\Lambda_n(\eta) < \infty$ and hence the probability measures $\tilde{\mathbf{Q}}^{(n)}$ are well defined.) If we let

$$\tilde{\Lambda}_n(\lambda) = \ln \int_{\mathbb{R}} \exp(\lambda z)\tilde{\mathbf{Q}}^{(n)}(\mathrm{dz})$$
$$\tilde{\Lambda}(\lambda) = \lim_{n\to\infty} \frac{\tilde{\Lambda}_n(\lambda\mu_n)}{\mu_n}$$
$$\tilde{\Lambda}^*(z) = \sup_{\lambda\in\mathbb{R}}\{\lambda z - \tilde{\Lambda}(\lambda)\}.$$

then;

$$\tilde{\Lambda}_n(\lambda) = \Lambda_n(\eta + \frac{\lambda}{\mu_n}) - \Lambda_n(\eta)$$
$$\tilde{\Lambda}(\lambda) = \Lambda(\eta + \lambda) - \Lambda(\eta)$$
$$\tilde{\Lambda}^*(z) = \Lambda^*(z) - \eta z + \Lambda(\eta).$$

We consider some properties of $\tilde{\Lambda}^*$. First note that

$$\tilde{\Lambda}^*(x) = \sup_{\lambda\in\mathbb{R}}\{\lambda x - \tilde{\Lambda}(\lambda)\} = \sup_{\lambda\in\mathbb{R}}\{\lambda x - \Lambda(\lambda + \eta) + \Lambda(\eta)\}.$$

The function $g(\lambda) = \lambda x - \Lambda(\lambda + \eta) + \Lambda(\eta)$ has a maximum at $\lambda = 0$ because $g'(0) = x - \Lambda'(\eta) = 0$ and $g(\lambda)$ is a concave function. Hence $\tilde{\Lambda}^*(x) = g(0) = 0$. For $z < x$ if $\lambda > 0$,

$$\lambda z - \tilde{\Lambda}(\lambda) < \lambda x - \tilde{\Lambda}(\lambda),$$

therefore

$$\sup_{\lambda>0}\{\lambda z - \tilde{\Lambda}(\lambda)\} \leq \sup_{\lambda>0}\{\lambda x - \tilde{\Lambda}(\lambda)\} \leq \tilde{\Lambda}^*(x) = 0.$$

Hence for $z < x$,

$$\tilde{\Lambda}^*(z) = \sup_{\lambda\leq 0}\{\lambda z - \tilde{\Lambda}(\lambda)\}. \tag{1.10}$$

This implies that $\tilde{\Lambda}^*$ is non-increasing for $z < x$. Similarly it can be shown that for $z > x$,

$$\tilde{\Lambda}^*(z) = \sup_{\lambda\geq 0}\{\lambda z - \tilde{\Lambda}(\lambda)\}.$$

Therefore $\tilde{\Lambda}^*$ is nondecreasing for $z > x$.

Turning to the proof of the inequality (1.9) we consider

$$
\begin{aligned}
&\mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n}\in(x-\delta,x+\delta)\right\}=\int_R 1_{(x-\delta,x+\delta)}\mathbf{Q}^{(n)}(\mathrm{dz})=\\
&=\int_R 1_{(x-\delta,x+\delta)}\exp\left(\Lambda_n(\eta)\right)\exp\left(-\mu_n\eta(z-x+x)\right)\tilde{\mathbf{Q}}^{(n)}(\mathrm{dz})\geq\\
&\geq\exp\left(\Lambda_n(\eta)\right)\exp\left(-\mu_n\eta x\right)\ \exp\left(-\mu_n\left|\eta\right|\delta\right)\tilde{\mathbf{Q}}^{(n)}(x-\delta,x+\delta),
\end{aligned}
$$

hence

$$
\frac{\ln\mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n}\in(x-\delta,x+\delta)\right\}}{\mu_n}\geq\frac{\Lambda_n(\eta)}{\mu_n}-\eta x-\left|\eta\right|\delta+\frac{\ln\tilde{\mathbf{Q}}^{(n)}(x-\delta,x+\delta)}{\mu_n}
$$

therefore

$$
\begin{aligned}
&\liminf_{\delta\to 0}\liminf_{n\to\infty}\frac{\ln\mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n}\in(x-\delta,x+\delta)\right\}}{\mu_n}\geq\\
&\qquad\geq\Lambda(\eta)-\eta x+\liminf_{\delta\to 0}\liminf_{n\to\infty}\frac{\ln\tilde{\mathbf{Q}}^{(n)}(x-\delta,x+\delta)}{\mu_n}.
\end{aligned}
$$

Since $\Lambda^*(x)\geq\eta x-\Lambda(\eta)$;

$$
\begin{aligned}
&\liminf_{\delta\to 0}\liminf_{n\to\infty}\frac{\ln\mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n}\in(x-\delta,x+\delta)\right\}}{\mu_n}\geq-\Lambda^*(x)+\\
&\qquad+\liminf_{\delta\to 0}\liminf_{n\to\infty}\frac{\ln\tilde{\mathbf{Q}}^{(n)}(x-\delta,x+\delta)}{\mu_n}.
\end{aligned}
\tag{1.11}
$$

Now we show that the last term in the right hand side of this inequality is equal to zero. It suffices to show that as $n\to\infty$

$$
\tilde{\mathbf{Q}}^{(n)}(-\infty,x-\delta]+\tilde{\mathbf{Q}}^{(n)}[x+\delta,\infty)\to 0.
$$

Let Z_n be a random variable with distribution $\tilde{\mathbf{Q}}^{(n)}$. For $\lambda \le 0$ by Chebychev's inequality

$$\begin{aligned}\tilde{\mathbf{Q}}^{(n)}(-\infty, x-\delta] = \mathbf{P}\{Z_n \le x-\delta\} = \\ &= \mathbf{P}\left\{\exp\left(\lambda\mu_n Z_n\right) \ge \exp\left(\lambda\mu_n(x-\delta)\right)\right\} \le \\ &\le \exp\left(-\lambda\mu_n(x-\delta)\right)\mathbf{E}\exp\left(\lambda\mu_n Z_n\right).\end{aligned}$$

Hence for $\lambda \le 0$;

$$\frac{\ln \tilde{\mathbf{Q}}^{(n)}(-\infty, x-\delta]}{\mu_n} \le -\lambda(x-\delta) + \frac{\tilde{\Lambda}_n(\lambda)}{\mu_n}$$

By considering first $\limsup_{n\to\infty}$ and then $\inf_{\lambda\le 0}$;

$$\limsup_{n\to\infty} \frac{\ln \tilde{\mathbf{Q}}^{(n)}(-\infty, x-\delta]}{\mu_n} \le -\sup_{\lambda\le 0}\{\lambda(x-\delta) - \tilde{\Lambda}(\lambda)\} = -\tilde{\Lambda}^*(x-\delta)$$

where the last equality follows from (1.10). Now we show that $\tilde{\Lambda}^*(x-\delta)$ is positive. Since for δ small $x-\delta \in D^{\circ}_{\Lambda^*}$, then there exists some $\vartheta \in D^{\circ}_{\Lambda}$ such that $\Lambda'(\vartheta) = x-\delta$ and

$$\tilde{\Lambda}^*(x-\delta) = \sup_{\lambda\le 0}\{\lambda(x-\delta) - \tilde{\Lambda}(\lambda)\} = \sup_{\lambda\le 0}\{\lambda(x-\delta) - \Lambda(\lambda+\eta) + \Lambda(\eta)\}$$

Let $g(\lambda) = \lambda(x-\delta) - \Lambda(\lambda+\eta) + \Lambda(\eta)$. Note that $g(0) = 0$, $g'(\lambda) = x-\delta-\Lambda'(\lambda+\eta)$, $g'(0) = x-\delta-\Lambda'(\eta) = -\delta < 0$, and $g'(\vartheta-\eta) = x-\delta-\Lambda'(\vartheta) = 0$. The concavity of g implies that $\vartheta-\eta < 0$, because $g'(\vartheta-\eta) = 0 > g'(0) = -\delta$. Therefore

$$\tilde{\Lambda}^*(x-\delta) = \sup_{\lambda\le 0} g(\lambda) = g(\vartheta-\eta).$$

Since g is concave, $g(0) = 0$, $g'(0) = -\delta < 0$, and $g'(\vartheta-\eta) = 0$, we conclude that $\tilde{\Lambda}^*(x-\delta) = g(\vartheta-\eta) > 0$. Hence

$$\limsup_{n\to\infty} \frac{\ln \tilde{\mathbf{Q}}^{(n)}(-\infty, x-\delta]}{\mu_n} \le -\tilde{\Lambda}^*(x-\delta) < 0.$$

This shows that the sequence $a_n \equiv \tilde{\mathbf{Q}}^{(n)}(-\infty, x-\delta] \to 0$, because otherwise

$$0 < a = \limsup_{n\to\infty} \tilde{\mathbf{Q}}^{(n)}(-\infty, x-\delta] \le 1.$$

Now consider a subsequence $\{a_{n_k}\}$ which $a_{n_k} \to a$. Since $0 < a \leq 1$ and $\mu_n \to \infty$, then

$$\lim_{k\to\infty} \frac{\ln a_{n_k}}{\mu_{n_k}} = 0.$$

Therefore

$$\limsup_{n\to\infty} \frac{\ln a_n}{\mu_n} = \limsup_{n\to\infty} \frac{\ln \tilde{\mathbf{Q}}^{(n)}(-\infty, x-\delta]}{\mu_n} \geq 0$$

which is a contradiction. Similarly it can be shown that $\tilde{\mathbf{Q}}^{(n)}[x+\delta,\infty) \to 0$. Therefore $\tilde{\mathbf{Q}}^{(n)}(x-\delta, x+\delta) \to 1$ and consequently for each $\delta > 0$

$$\liminf_{n\to\infty} \frac{\ln \tilde{\mathbf{Q}}^{(n)}(x-\delta, x+\delta)}{\mu_n} = 0.$$

Therefore the inequality (1.11) implies the inequality (1.9).

We continue the proof of Theorem. By the assumptions of Theorem for $\delta > 0$ and n large

$$\gamma - \delta < \frac{\nu_n}{\mu_n} < \gamma + \delta.$$

Hence

$$\mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n} \leq \gamma - \delta\right\} \leq \mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n} \leq \frac{\nu_n}{\mu_n}\right\} = \mathbf{P}^{(n)}\left\{Y_n \leq \nu_n\right\} \leq$$
$$\leq \mathbf{P}^{(n)}\left\{\frac{Y_n}{\mu_n} \leq \gamma + \delta\right\}.$$

Since $\gamma < 1$, then for δ small enough $\gamma - \delta < \gamma + \delta < 1$, and $\gamma - \delta,\ \gamma + \delta \in D^{\pm}_{\Lambda^*}$. It follows by (19) that

$$-\Lambda^*(\gamma - \delta) \leq \frac{\ln \mathbf{P}^{(n)}\left\{Y_n \leq \nu\right\}}{\mu_n} \leq -\Lambda^*(\gamma + \delta).$$

Recall that since $\gamma \in D^{\pm}_{\Lambda^*}$ the function Λ^* is continuous at a neighborhood of γ. By tending $\delta \to 0$ we obtain

$$\lim_{n\to\infty} \frac{\ln \mathbf{P}^{(n)}\left\{Y_n \leq \nu_n\right\}}{\mu_n} = -\Lambda^*(\gamma).$$

Remark 1. *If $\mu_n \to -\infty$, we replace μ_n by $-\mu_n$ in the definition of $\Lambda(\cdot)$. Let the following conditions be fulfilled:*

- *the limit $\Lambda(\lambda)$ exists,*

- *the limit* $\gamma > 1$ *exists and* $-\gamma \in D^{\pm}_{\Lambda^*}$,

- *for any* $\nu < -1$ *in* $D^{\pm}_{\Lambda^*}$, *there exists* $\eta \in D^{\pm}_{\Lambda}$ *such that* $\Lambda'(\eta) = \nu$.

Then we have the representation

$$\mathbf{P}^{(n)}\left\{Y_n < \nu_n\right\} = \exp\left\{\mu_n \Lambda^*(-\gamma)(1 + o(1))\right\}. \tag{1.12}$$

Remark 2. *Let* $\mu_n \to \infty$ *and the following conditions be fulfilled:*

- *the limit* $\Lambda(\lambda)$ *exists,*

- *the limit*
$$\lim_{n\to\infty} \frac{\nu_n}{\mu_n} = \gamma > 1$$
and $\gamma \in D^{\pm}_{\Lambda^*}$,

- *for any* $\nu > 1$ *in* $D^{\pm}_{\Lambda^*}$, *there exists* $\eta \in D^{\pm}_{\Lambda}$ *such that* $\Lambda'(\eta) = \nu$.

Then we have the representation

$$\mathbf{P}^{(n)}\left\{Y_n > \nu_n\right\} = \exp\left\{-\mu_n \Lambda^*(\gamma)(1 + o(1))\right\}. \tag{1.13}$$

Remark 3. *If* $\mu_n \to -\infty$, *we replace* μ_n *by* $-\mu_n$ *in the definition of* $\Lambda(\cdot)$. *Let the following conditions be fulfilled:*

- *the limit* $\Lambda(\lambda)$ *exists,*

- *the limit* $\gamma < -1$ *exists and* $-\gamma \in D^{\pm}_{\Lambda^*}$,

- *for any* $\nu > -1$ *in* $D^{\pm}_{\Lambda^*}$, *there exists* $\eta \in D^{\pm}_{\Lambda}$ *such that* $\Lambda'(\eta) = \nu$.

Then we have the representation

$$\mathbf{P}^{(n)}\left\{Y_n < \nu_n\right\} = \exp\left\{\mu_n \Lambda^*(-\gamma)(1 + o(1))\right\}. \tag{1.14}$$

1.7 Hypotheses testing

In this section, which is taken essentially from Ingster and Suslina [23], we remind some basic notions of hypotheses testing problem. Let X be a random observation (random element) of a measurable space $(\mathcal{A}, \mathcal{F})$ with distribution $\mathbf{P}_\theta$, where $\theta \in \Theta$ is unknown. The problem is to test the hypotheses

$$\mathcal{H}_0 : \theta \in \Theta_0$$
$$\mathcal{H}_1 : \theta \in \Theta_1$$

where Θ_0 and Θ_1 are two nonempty disjoint sets and $\Theta = \Theta_0 \cup \Theta_1$. We suppose that (identifiability condition)

$$\mathbf{P}_{\theta_0} \neq \mathbf{P}_{\theta_1} \qquad \forall\, \theta_0 \in \Theta_0,\ \forall\, \theta_1 \in \Theta_1.$$

A hypothesis or alternative like $\theta = \theta_0$ or $\theta = \theta_1$ which specifies the exact distribution of X is *simple*, otherwise it is *composite*, say $\theta < \theta_0$, $\theta > \theta_0$ or $\theta \neq \theta_0$. A *decision rule* or a statistical *test* is a measurable mapping

$$\phi : \mathcal{A} \to [\,0\,, 1]\,,$$

defined by the rule "$\phi(x) =$ Probability to reject $\mathcal{H}_0$ (to accept $\mathcal{H}_1$)". A test ϕ for which $\phi(x) \in \{0, 1\}$ for all x, is called a *non-randomized* test with the *critical region* $\{x \in \mathcal{A} : \phi(x) = 1\}$. A statistical decision making (between $\mathcal{H}_0$ and $\mathcal{H}_1$) is based on the available information X via a decision rule $\phi(X)$. Any decision rule $\phi(X)$ contains two types of errors. The *type I error* is to reject the hypothesis $\mathcal{H}_0$ when it is true and the *type II error* is to accept the hypothesis $\mathcal{H}_0$ when it is false. The quality of a test is measured by the error probabilities of types I and II. The type I error probability of a test ϕ is a function with the domain Θ_0 defined by

$$\alpha\,(\phi, \theta) = \mathbf{E}_\theta\,(\phi\,(X))\,, \quad \theta \in \Theta_0,$$

where $\mathbf{E}_\theta$ denotes the mathematical expectation with respect to the probability measure $\mathbf{P}_\vartheta$. The type II error probability is a function with the domain Θ_1 defined by

$$\pi\,(\phi, \theta) = \mathbf{E}_\theta\,(1 - \phi\,(X))\,, \quad \theta \in \Theta_1.$$

The *power function* of ϕ is the probability of true decision which is defined by

$$1-\pi\left(\phi,\theta\right)=\mathbf{E}_{\theta}\left(\phi\left(X\right)\right),\quad \theta\in\Theta_1.$$

It is clear that we are interested in a test ϕ^{*} which provides uniformly minimum error probabilities, i.e., for any other test ϕ

$$\alpha\left(\phi^{*},\theta\right)\leq\alpha\left(\phi,\theta\right),\qquad \theta\in\Theta_0$$
$$\pi\left(\phi^{*},\theta\right)\leq\pi\left(\phi,\theta\right),\qquad \theta\in\Theta_1.$$

But as we will see later there are many difficulties to finding such tests. First we consider a simple Gaussian model, i.e. we observe a random variable X, where

$$X=\upsilon+\xi.$$

Here $\xi\sim\mathcal{N}\left(0,1\right)$ is the standard Gaussian random variable and υ is an unknown real-valued parameter. Our goal is to test the simple hypothesis $\mathcal{H}_0:\upsilon=0$ against a simple alternative $\mathcal{H}_1:\upsilon=\upsilon_1$ where $\upsilon_1>0$ is known. For simplicity in notations the type I and II errors of probability of a test ϕ we write as

$$\alpha\left(\phi\right)=\alpha\left(\phi,0\right),\qquad \pi\left(\phi\right)=\pi\left(\phi,\upsilon_1\right),$$

respectively. We would like to find a test such that these values to be as possible as small. But they can not be decreased simultaneously. For example the test $\phi\equiv0$ has no type I error , while its type II error probability is equal to 1. Two following ways help us to find an "optimal" test in some sense. In the first one, we consider the sum of two errors $\alpha\left(\phi\right)$ and $\pi\left(\phi\right)$, i.e.,

$$\gamma\left(\phi\right)=\alpha\left(\phi\right)+\pi\left(\phi\right)$$

and find a test ϕ that minimize $\gamma\left(\phi\right)$. This leads to the test

$$\phi^{+}\left(X\right)=\chi_{\{X>\upsilon_1/2\}},$$

where χ_A denotes the indicator function of the event A. For this test we have

$$\alpha\left(\phi^{+}\right)=\pi\left(\phi^{+}\right)=\frac{\gamma\left(\phi^{+}\right)}{2}=\mathcal{N}\left(\frac{-\upsilon_1}{2}\right),$$

where $\mathcal{N}(t)$ stands for the distribution function of the standard Gaussian law. Observe that this test depends on the specific value υ_1 of the alternative. In this method the hypothesis and alternative are consider to be equally-important. This criteria can be generalized by considering another functions of $\alpha(\phi)$ and $\pi(\phi)$, say

$$\gamma_t(\phi) = t\,\alpha(\phi) + \pi(\phi)\,,$$

where $t > 0$ is a given number (see [23]). The optimal test $\phi^{(t)}$ is if the form

$$\phi^{(t)}(X) = \chi_{\left\{X > \frac{2\ln t + \upsilon_1^2}{2\,\upsilon_1}\right\}}.$$

Note that the test ϕ^+ corresponds to $t = 1$. Another way to find an optimal test is the Neyman-Pearson approach. Fix $\alpha \in (0,1)$ and introduce the class $\mathcal{K}_\alpha$ of all tests of level α, i.e.,

$$\mathcal{K}_\alpha = \{\phi \;:\quad \alpha(\phi) \le \alpha\}\,.$$

Our goal is to find a *most powerful* test, i.e. a test in the class $\mathcal{K}_\alpha$ which provides minimum type II error probability (maximum power). It is given by

$$\phi_\alpha^+(X) = \chi_{\{X > z_\alpha\}},$$

where z_α is $(1-\alpha)$-quantile of the the Gaussian law, i.e., $\mathcal{N}(z_\alpha) = 1 - \alpha$. For this test are we have

$$\alpha\left(\phi_\alpha^+\right) = \alpha, \quad \pi\left(\phi_\alpha^+, \upsilon_1\right) = \mathcal{N}(z_\alpha - \upsilon_1)\,.$$

Since the test ϕ_α^+ does not depend on the particular alternative $\upsilon_1 > 0$, hence it is a *uniformly most powerful test* of the level α for the composite one-sided alternative $V_1^+ = \{\upsilon > 0\}$. That is for any other test $\phi \in \mathcal{K}_\alpha$ we have

$$\mathbf{E}_\theta(\phi^+) \ge \mathbf{E}_\theta(\phi), \quad \theta \in \Theta_1.$$

Similarly, uniformly most powerful test for alternative of the form $V_1^- = \{\upsilon < 0\}$ is given by

$$\phi_\alpha^-(X) = \chi_{\{X < -z_\alpha\}},$$

and for $\upsilon_1 < 0$

$$\alpha\left(\phi_\alpha^-\right) = \alpha, \quad \pi\left(\phi_\alpha^-, \upsilon_1\right) = \mathcal{N}\left(z_\alpha + \upsilon_1\right).$$

One can ask for uniformly most powerful test for two sided alternative of the form $V_1 = \{\upsilon \neq 0\}$. But in this case there is no such test, because the maximum power for any alternative $\upsilon > 0$ is obtained by the test with critical region $\{X > z_\alpha\}$ and for any alternative $\upsilon < 0$ is obtained by the test with critical region $\{X < -z_\alpha\}$, which are basically different. As a candida let us try the bilateral test

$$\phi_\alpha^\pm(X) = \chi_{\left\{|X| > z_{\alpha/2}\right\}},$$

which belongs to the class $\mathcal{K}_\alpha$. Simple calculations show that

$$\pi\left(\phi_\alpha^\pm, \upsilon_1\right) > \pi\left(\phi_\alpha^+, \upsilon_1\right), \quad \upsilon_1 > 0,$$
$$\pi\left(\phi_\alpha^\pm, \upsilon_1\right) > \pi\left(\phi_\alpha^-, \upsilon_1\right), \quad \upsilon_1 < 0,$$

which prove that $\phi^\pm$ is not uniformly most powerful. Hence for such a class of alternatives we try to find an optimal test in a subclass of $\mathcal{K}_\alpha$, say the class of *unbiased tests*. A test ϕ is called *unbiased* if

$$\sup_{\theta \in \Theta_0} \alpha(\phi, \theta) \leq \inf_{\theta \in \Theta_1} \left\{1 - \pi(\phi, \theta)\right\}.$$

A non-unbiased test can be discarded as a "non desirable" test. In the Gaussian model for testing the simple hypothesis $\mathcal{H}_0 : \upsilon = 0$ against two-sided alternative $\mathcal{H}_1 : \upsilon \neq 0$, among all unbiased tests in the class $\mathcal{K}_\alpha$, the test $\phi_\alpha^\pm$ given above, is uniformly most powerful unbiased test and

$$\alpha\left(\phi_\alpha^\pm\right) = \alpha, \quad \pi\left(\phi_\alpha^\pm, \theta\right) = \mathcal{N}\left(z_{\alpha/2} - |\upsilon_1|\right) - \mathcal{N}\left(-z_{\alpha/2} - |\upsilon_1|\right),$$

for all $\upsilon_1 \neq 0$. Theses results can be extended to the testing hypotheses about the parameter θ, based on the observations

$$Y_i = \theta + \sigma \xi_i, \quad i = 1, \cdots, N$$

where ξ_i are i.i.d. standard Gaussian random variables and $\sigma > 0$ is known. Indeed one can consider the sufficient statistics $X = N^{-1/2}\sigma^{-1}\sum_{i=1}^N Y_i$ and parameter $\upsilon =$

$N^{1/2}\sigma^{-1}\theta$. Observe that for large quantities of $N^{1/2}\sigma^{-1}\theta$ the error probabilities are small.

Now we consider a composite null hypothesis $\mathcal{H}_0 : \theta \in \Theta_0$. In this case the type I error probability is a function of the parameter θ. The quantity

$$\alpha(\phi) = \alpha(\phi, \Theta_0) = \sup_{\theta \in \Theta_0} \alpha(\phi, \theta)$$

is called the *level* of the test ϕ and the class $\mathcal{K}_\alpha$ consists the tests ϕ such that $\alpha(\phi) \leq \alpha$. A test ϕ is called *similar*, if the type I error probability is a constant function, i.e.,

$$\alpha(\phi, \theta) \equiv c, \quad \theta \in \Theta_0.$$

Now consider following Gaussian model

$$X = \upsilon + \xi,\ \upsilon = (\upsilon_1, \ldots, \upsilon_n) \in \mathbb{R}^n, \quad \xi = (\xi_1, \ldots, \xi_n),\ \xi_i \sim N(0,1) \text{ i.i.d.}$$

Here X is an n-dimensional random vector with unknown mean vector υ and unit covariance matrix I_n. Let $r \in \mathbb{R}^n,\ \|r\| = 1$ be a given unit length vector and consider the simple null hypothesis $\mathcal{H}_0 : \upsilon = 0$ against the different alternatives

$$\mathcal{H}_1 : \langle \upsilon, r \rangle > 0, \quad \mathcal{H}_1 : \langle \upsilon, r \rangle < 0, \quad \mathcal{H}_1 : \langle \upsilon, r \rangle \neq 0,$$

where $\langle \upsilon, r \rangle = \sum_{i=1}^n \upsilon_i r_i$ is the inner product and $\|\upsilon\|^2 = \langle \upsilon, \upsilon \rangle$. The test based on the statistics $\langle X, r \rangle$, which is the projection of the observation X to the direction r, provides uniformly most powerful tests for the first and second alternative and uniformly most powerful unbiased test for the third one. These tests are of the following forms

$$\phi_\alpha^+(X) = \chi_{\{\langle X, r\rangle > z_\alpha\}}, \quad \phi_\alpha^-(X) = \chi_{\{\langle X, r\rangle < -z_\alpha\}}, \quad \phi_\alpha^\pm(X) = \chi_{\{|\langle X, r\rangle| > z_{\alpha/2}\}}$$

with the type II error of probabilities

$$\pi\left(\phi_\alpha^+, \upsilon_1\right) = \mathcal{N}\left(z_\alpha - \langle \upsilon, r \rangle\right), \quad \pi\left(\phi_\alpha^-, \upsilon_1\right) = \mathcal{N}\left(z_\alpha + \langle \upsilon, r \rangle\right),$$
$$\pi\left(\phi_\alpha^\pm, \theta\right) = \mathcal{N}\left(z_{\alpha/2} - |\langle \upsilon, r \rangle|\right)$$

These results can be extended to a composite null-hypothesis $\mathcal{H}_0 : \langle \upsilon, r \rangle = 0$ against to the one or two-sided alternatives $\mathcal{H}_1$, as above. The corresponding tests ϕ^+_α

powerful test. But under the asymptotic approach, presented below, it is possible to find an "optimal" test, in some sense.

In the asymptotic approach, instead of a single observation X we consider a family of observations X_ε defined on some measurable spaces $(\mathcal{A}_\varepsilon, \mathcal{F}_\varepsilon)$ with distributions $\left\{\mathbf{P}_\theta^{(\varepsilon)},\ \theta \in \Theta\right\}$ depending on unknown parameter θ, and the asymptotic parameter $\varepsilon \to \varepsilon_0$. The asymptotic parameter ε, for example in the i.i.d. model with sample size N, can be taken as $\varepsilon = 1/N$ or $\varepsilon = N$. The unknown parameter θ and parameter space Θ don't change with ε and we are given a hypotheses testing problem about θ. Let $\phi_\varepsilon = \phi_\varepsilon(X_\varepsilon)$ be a decision function corresponding to the observation X_ε. Therefore instead of one test we have a family of tests and we are interested in its limit properties, as $\varepsilon \to \varepsilon_0$.

For $0 < \alpha < 1$, let $\mathcal{K}_\alpha^0$ denote the class of tests of asymptotic level α, i.e., the class of all (family of) tests ϕ_ε, such that $\alpha\left(\phi_\varepsilon\right) \leq \alpha + o\,(1)$. Under very general conditions, for testing the simple null-hypothesis $\mathcal{H}_0 : \theta = \theta_0$, where the parameter space is one dimensional (there is no nuisance parameter) against the right-hand sided alternative $\mathcal{H}_1 : \theta \in \Theta_0^+ = \Theta \cap (\theta_0, \infty)$, one can construct *asymptotically uniformly most powerful test of asymptotic level* (AUMP) test $\phi_\varepsilon^* = \phi_\varepsilon^*\left(X_\varepsilon\right)$ in the class $\mathcal{K}_\alpha^0$. This means that for any other test $\phi_\varepsilon \in \mathcal{K}_\alpha^0$ we have

$$\liminf_{\varepsilon \to \varepsilon_0} \inf_{\theta \in \Theta_0^+} \left\{\pi\left(\phi_\varepsilon, \theta\right) - \pi\left(\phi_\varepsilon^*, \theta\right)\right\} \geq 0.$$

A slightly weaker notion is the following: a test $\hat{\phi}_\varepsilon \in \mathcal{K}_\alpha^0$ is *locally asymptotically uniformly most powerful* (LAUMP), if for any other sequence of tests $\phi_\varepsilon \in \mathcal{K}_\alpha^0$ and for any $K > 0$;

$$\liminf_{\varepsilon \to \varepsilon_0} \inf_{\theta_0 < \theta \leq \theta_0 + \varphi_\varepsilon K} \left\{\pi\left(\phi_\varepsilon, \theta\right) - \pi\left(\hat{\phi}_\varepsilon, \theta\right)\right\} \geq 0,$$

where the *normalizing factor* $\varphi_\varepsilon = \varphi_\varepsilon\left(\theta_0\right)$ tends to zero as $\varepsilon \to \varepsilon_0$ with a certain rate. Hence at the neighborhood of θ_0, where it is difficult to distinguish between the null hypothesis and the alternative, the test $\hat{\phi}_\varepsilon$ is uniformly most powerful. Let us consider the i.i.d. sample model $X_1, \ldots, X_N,\ N \to \infty$, with common density function (w.r.t. some σ-finite measure) $f(x, \theta),\ \theta \in \Theta$, where Θ is an open interval of real numbers and the hypothesis $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta > \theta_0$. Under the general regularity conditions the test of the form

$$\phi^+_{N,\alpha}\left(X_1, \ldots, X_N\right) = \chi_{\{\Delta_N > z_\alpha\}}$$

based on the score statistic

$$\Delta_N = \frac{1}{\sqrt{N I(\theta_0)}} \sum_{j=1}^{N} \frac{\dot f(X_j, \theta_0)}{f(X_j, \theta_0)}$$

where $\dot f(x, \theta)$ is the derivative of $f(x, \theta)$ with respect to θ and the Fisher information at the point θ given by

$$I(\theta) = \mathbf{E}_\theta \left(\frac{\dot f(X_1, \theta)}{f(X_1, \theta)}\right)^2$$

is LAUMP. The asymptotic of type I and II error probabilities of these tests are given by

$$\alpha\left(\phi^+_{N,\alpha}\right) = \alpha + o\left(1\right), \ \pi\left(\phi^+_{N,\alpha}, \theta\right) = \mathcal{N}\left(z_\alpha - u_N(\theta)\right) + o(1),$$

where $u_N(\theta) = \sqrt{N I(\theta_0)}(\theta - \theta_0)$. If $u_N(\theta) \to \infty$, the type II error probabilities tend to zero. Note that the power of $\phi^+_{N,\alpha}$ for a sequence of *local alternatives* of the form $\theta = \theta_0 + \varphi_N\, h$, with $\varphi_N = N^{-1/2}$ and $h > 0$ has the following non degenerate limit

$$\lim_{N\to\infty} \left(1 - \pi\left(\phi^+_{N,\alpha}, \theta_0 + \varphi_N\, h\right)\right) = 1 - \mathcal{N}\left(z_\alpha - h\sqrt{I(\theta_0)}\right).$$

If, in addition, the sequence $\Delta_N \to \infty$ in probability under $\mathbf{P}^{(N)}_{\theta_N}$, for any sequence θ_N of alternatives with the property that $N^{1/2}\left(\theta_N - \theta_0\right) \to \infty$, then $\phi^+_{N,\alpha}$ is also AUMP at level α (see [29], page 545). This is the case for the double exponential location family $f(x, \theta) = f(x - \theta)$ with

$$f(x) = \frac{1}{2} \exp\left(-\left|x\right|\right).$$

Let $\theta_0 = 0$. Then the test based on the statistic

$$\Delta_N = N^{-1/2} \sum_{i=1}^{N} \operatorname{sign}(X_i)$$

where $\operatorname{sign}(x) = 1$ if $x \geq 0$ and equal to -1 if $x < 0$, is AUMP (see [29], page 546). The similar assertions hold true for the left-hand sided alternatives. For two-hand sided alternative one can construct asymptotically uniformly most powerful unbiased test. When there are nuisance parameters, similar results hold true. Therefore in one-dimensional parameter, under general conditions, the problem of hypothesis testing, asymptotically is close to the Gaussian model.

Chapter 2

Two Simple Hypotheses

2.1 Introduction

Let $X^{(n)}$ be a realization of a nonhomogeneous Poisson process observed on some set $\mathbb{A}_n \subset \mathbb{R}^d$ with intensity function $S(\cdot) = \{S(u), u \in \mathbb{A}_n\}$ (with respect to Lebesgue measure). This intensity function can take one of the following two (known) values $S_1(\cdot)$ or $S_2(\cdot)$ and we have to test the corresponding two simple hypotheses $\mathcal{H}_1$ and $\mathcal{H}_2$. This work is devoted to this problem. If we fix the probability of the error α (to accept $\mathcal{H}_2$ when $\mathcal{H}_1$ is true) then the well-known Neyman-Pearson lemma gives us immeditely the most powerful test in this problem: we have to compare the likelihood ratio with the constant c_α providing the error α (see, e.g. Borovkov [7], [8]). Note that the calculation of the parameters of this test (exact constant c_α and the power) is in general difficult to do. Therefore we consider in our work the traditional asymptotic approach which essentially simplifies these calculus. As usual in such problems, the central limit theorem gives us the asymptotic value of the level c_α and the large deviation principle allows us describe the exponential asymptotics of the power. We do not suppose that the measures are equivalent and study the contribution of the singular parts too. The spatial Poisson processes are widely used in many fields. In particular this is a good mathematical model for the problems of image restoration when the optical signal is weak and statistics of photons is well described by an inhomogeneous Poisson process

[42].

The main contribution of this work is the description of the asymptotic behavior of the Neyman-Pearson test with the help of the asymptotic expansion, which allow us to make precise constant c_α and the large deviations principle (Gärtner-Ellis method [16]) to describe the asymptotics of the power.

2.2 Most powerful test

Suppose that we observe a sample $X^{(n)}$ of a nonhomogeneous Poisson process on the set $\mathbb{A}_n$, a subset of d dimensional Euclidian space $\mathbb{R}^d$. The intensity function $S(\cdot)$ of this process can take one of the following two values: $S_1(\cdot)$ or $S_2(\cdot)$, where $S_i(\cdot), i = 1, 2$ are two known nonnegative functions. Our goal is to test these two simple hypotheses

$$\begin{aligned}
\mathcal{H}_1 : \qquad & S(u) = S_1(u), \qquad u \in \mathbb{A}_n, \\
\mathcal{H}_2 : \qquad & S(u) = S_2(u), \qquad u \in \mathbb{A}_n.
\end{aligned}$$

Let us denote $\mathbb{B}_n$ and $\mathbb{C}_n$ the subsets (possibly empty) of $\mathbb{A}_n$ such that :

$$\begin{aligned}
S_1(u) = 0 \qquad & u \in \mathbb{B}_n, \\
S_2(u) = 0 \qquad & u \in \mathbb{C}_n. \qquad (2.1)
\end{aligned}$$

Without loss of generality we can suppose that $\mathbb{B}_n \cap \mathbb{C}_n = \emptyset$, because we have no observations on this set under both hypotheses.

Let us denote as $\mathbf{P}_1^{(n)}$ and $\mathbf{P}_2^{(n)}$ the probability measures induced on $\left(\mathcal{M}_0^{(n)}, \mathfrak{B}(\mathcal{M}_0^{(n)})\right)$ by the Poisson processes of intensity functions $S_1(\cdot)$ and $S_2(\cdot)$ respectively. The corresponding mathematical expectations are denoted as $\mathbf{E}_1$ and $\mathbf{E}_2$. Note that if $\mathbb{B}_n \neq \emptyset$, then the probability measure $\mathbf{P}_2^{(n)}$ is not absolutely continuous with respect to $\mathbf{P}_1^{(n)}$ on the measurable space $\left(\mathcal{M}_0^{(n)}, \mathfrak{B}(\mathcal{M}_0^{(n)})\right)$, because for the set

$$\mathcal{E} = \left\{ x^{(n)} \in \mathcal{M}_0^{(n)} : \quad x^{(n)}(\mathbb{B}_n) > 0 \right\}$$

we have $\mathbf{P}_1^{(n)}(\mathcal{E}) = 0$, but

$$\mathbf{P}_2^{(n)}(\mathcal{E}) = 1 - \exp\left\{ - \int_{\mathbb{B}_n} S_2(u)\, \mathrm{d}u \right\} > 0.$$

Note that the quantity

$$L_n\left(X^{(n)}\right)=\int_{\mathbb{T}_n}\ln\frac{S_2(u)}{S_1(u)}\,X^{(n)}(\mathrm{d}u)-\int_{\mathbb{T}_n}\left[S_2(u)-S_1(u)\right]\,\mathrm{d}u$$

where the set

$$\mathbb{T}_n=\mathbb{A}_n\cap\mathbb{B}_n^c\cap\mathbb{C}_n^c \tag{2.2}$$

is well defined and we can write for any set $\mathcal{D}\in\mathfrak{B}(\mathcal{M}_0^{(n)})$ the decomposition of Lebesgue

$$\mathbf{P}_2^{(n)}(\mathcal{D})=\int_{\mathsf{D}\backslash\mathsf{F}}e^{L_n\left(x^{(n)}\right)}\,\mathrm{d}\mathbf{P}_1^{(n)}+\mathbf{P}_2^{(n)}\left(\mathcal{D}\cap\mathcal{F}^c\right)$$

where $\mathcal{F}=\left\{x^{(n)}:\quad x^{(n)}\left(\mathbb{B}_n\right)\,+\,x^{(n)}\left(\mathbb{C}_n\right)=0\right\}$. In fact the quantity e^{L_n} is the "likelihood ratio" or more precisely the density of absolutely continuous part of $\mathbf{P}_2^{(n)}$ with respect to $\mathbf{P}_1^{(n)}$.

We suppose in all problems below that

$$\int_{\mathbb{A}_n}\left(\sqrt{S_1\left(u\right)}-\sqrt{S_2\left(u\right)}\right)^2\,\mathrm{d}u<\infty.$$

Remind that if

$$\int_{\mathbb{A}_n}\left(\sqrt{S_1\left(u\right)}-\sqrt{S_2\left(u\right)}\right)^2\,\mathrm{d}u=\infty,$$

then the measures $\mathbf{P}_1^{(n)}$ (under hypothesis) and $\mathbf{P}_2^{(n)}$ (under alternative) are singular (See [30]). Let us fix some $\alpha\in(0,1)$ and denote by $\mathcal{K}_\alpha$ the class of tests of the level α, i.e.

$$\mathcal{K}_\alpha=\left\{\phi:\quad\mathbf{E}_1\phi\left(X^{(n)}\right)=\alpha\right\}.$$

Here and in the sequel $\phi=\phi\left(X^{(n)}\right),\phi_n,\hat{\phi}_n\ldots$ are decision functions (probability to accept the hypothesis $\mathcal{H}_2$).

We suppose that

$$\alpha<\exp\left\{-\int_{\mathbb{C}_n}S_1\left(u\right)\,\mathrm{d}u\right\}\equiv p_n, \tag{2.3}$$

because if $\alpha\geq p_n$, then it is sufficient to put $\phi\left(X^{(n)}\right)=\chi_{\left\{X^{(n)}(\mathbb{C}_n)=0\right\}}$ and this test will be the most powerful in the class $\mathcal{K}_\alpha$ with the power $\beta(\phi)=\mathbf{E}_2\phi(X^{(n)})=1$. Here the indicator function $\chi_{\left\{X^{(n)}(\mathbb{C}_n)=0\right\}} = 1$ if $X^{(n)}\left(\mathbb{C}_n\right) = 0$ and $\chi_{\left\{X^{(n)}(\mathbb{C}_n)=0\right\}} = 0$ if $X^{(n)}\left(\mathbb{C}_n\right) > 0$. Note that since $\mathbf{E}_1\phi(X^{(n)}) = p_n \leq \alpha$, then $\phi \notin \mathcal{K}_\alpha$. But this test is the most "reasonable" one. We can also construct a most powerful test in the class $\mathcal{K}_\alpha$ as follows. Let k denote the smallest nonnegative integer such that

$$\alpha \leq \sum_{j=0}^{k} \mathbf{P}_1^{(n)} \left\{X^{(n)}\left(\mathbb{C}_n\right) = j\right\}$$

and introduce the test

$$\phi^0(X^{(n)}) = \chi_{\left\{X^{(n)}(\mathbb{C}_n)=0\right\}} + q\,\chi_{\left\{X^{(n)}(\mathbb{C}_n)\in\{1,2,\ldots,k\}\right\}}$$

where

$$q = \frac{\alpha - p_n}{\sum_{j=1}^{k} \mathbf{P}_1^{(n)} \left\{X^{(n)}\left(\mathbb{C}_n\right) = j\right\}}.$$

Note that $p_n = \mathbf{P}_1^{(n)} \left\{X^{(n)}\left(\mathbb{C}_n\right) = 0\right\}$. Then we have $q \in [0, 1]$ and $\mathbf{E}_1\phi^0(X^{(n)}) = \alpha$. The power of ϕ^0 is equal to 1. Hence it is the most powerful in $\mathcal{K}_\alpha$.

Let us introduce the test

$$\hat{\phi}(X^{(n)}) = \begin{cases} 1, & \text{if } X^{(n)}(\mathbb{B}_n) > 0 \text{ or } \quad \left(L_n\left(X^{(n)}\right) > c_\alpha \text{ and } X^{(n)}(\mathbb{C}_n) = 0\right), \\ q_\alpha, & \text{if } L_n\left(X^{(n)}\right) = c_\alpha, \quad X^{(n)}(\mathbb{C}_n) = 0, \quad X^{(n)}(\mathbb{B}_n) = 0, \\ 0, & \text{if } X^{(n)}(\mathbb{C}_n) > 0 \text{ or } \quad \left(L_n\left(X^{(n)}\right) < c_\alpha \text{ and } X^{(n)}(\mathbb{B}_n) = 0\right). \end{cases}$$

Here the numbers c_α and q_α satisfy the equation

$$\mathbf{E}_1\hat{\phi}(X^{(n)}) = p_n\,\mathbf{P}_1^{(n)}(L_n\left(X^{(n)}\right) > c_\alpha) + p_n\,q_\alpha\,\mathbf{P}_1^{(n)}(L_n\left(X^{(n)}\right) = c_\alpha) = \alpha. \tag{2.4}$$

According to Neyman-Pearson lemma we have the following proposition.

Theorem 12. *Suppose that the condition* (2.3) *is fulfilled, then the test* $\hat{\phi}$ *is the most powerful in the class* $\mathcal{K}_\alpha$.

Proof. The proof repeats the well known one (see, e.g., [7]), where we just take into account the singular parts. Let us introduce the function

$$\begin{aligned}\varphi(x) = \mathbf{P}_1^{(n)} \left\{L_n(X^{(n)}) > x,\, X^{(n)}(\mathbb{C}_n) = 0\right\}) = \\ = p_n\,\mathbf{P}_1^{(n)} \left\{L_n(X^{(n)}) > x\right\},\end{aligned}$$

and let $c_\alpha = \inf\{x : \varphi(x) \le \alpha\}$. The function $\varphi(\cdot)$ is non-increasing, right-continuous and

$$\lim_{x\to-\infty} \varphi(x) = p_n, \qquad \lim_{x\to\infty} \varphi(x) = 0.$$

Hence c_α is finite, and furthermore

$$\varphi(c_\alpha) \le \alpha \le \varphi(c_\alpha -) = \varphi(c_\alpha) + p_n\, \mathbf{P}_1^{(n)}\left\{L_n(X^{(n)}) = c_\alpha\right\}.$$

If $\varphi(x)$ is continuous at c_α, then $\mathbf{P}_1^{(n)}\left\{L_n(X^{(n)}) = c_\alpha\right\} = 0$, and the equation (2.4) holds with any $q_\alpha \in [0,1]$. But if $\varphi(x)$ has a jump at the point c_α, then to have equality (2.4) we take

$$q_\alpha = \frac{\alpha - \varphi(c_\alpha)}{p_n\, \mathbf{P}_1^{(n)}\{L_n(X^{(n)}) = c_\alpha\}}.$$

Hence the test $\hat\phi \in \mathcal{K}_\alpha$.

Now let ϕ be any other test in the class $\mathcal{K}_\alpha$. We show that the difference

$$\mathbf{E}_2\hat\phi(X^{(n)}) - \mathbf{E}_2\phi(X^{(n)}) = \int \left\{\hat\phi(x^{(n)}) - \phi(x^{(n)})\right\} \mathrm{d}\mathbf{P}_2^{(n)}$$

(in obvious notations) is non negative. Let us introduce the sets

$$G_+ = \left\{x^{(n)} : \quad x^{(n)}(\mathbb{B}_n) > 0\right\}, \qquad G_0 = \left\{x^{(n)} : \quad x^{(n)}(\mathbb{B}_n \cup \mathbb{C}_n) = 0\right\},$$

and note that under hypothesis $\mathcal{H}_2$:

$$\begin{aligned}
\mathbf{E}_2\hat\phi(X^{(n)}) - \mathbf{E}_2\phi(X^{(n)}) &= \int_{G_+\cup\, G_0} \left\{\hat\phi(x^{(n)}) - \phi(x^{(n)})\right\} \mathrm{d}\mathbf{P}_2^{(n)} = \\
&= \int_{G_0} \left\{\hat\phi(x^{(n)}) - \phi(x^{(n)})\right\} e^{L_n(x^{(n)})} \mathrm{d}\mathbf{P}_1^{(n)} + \\
&+ \int_{G_+} \left\{1 - \phi(x^{(n)})\right\} \mathrm{d}\mathbf{P}_2^{(n)} \ge \\
&\ge \int_{G_0} \left\{\hat\phi(x^{(n)}) - \phi(x^{(n)})\right\} e^{L_n(x^{(n)})}\, \mathrm{d}\mathbf{P}_1^{(n)}.
\end{aligned}$$

The set G_0 can be partitioned to the sets ;

$$A_+ = \left\{x^{(n)} : \quad L_n(x^{(n)}) > c_\alpha\right\}, \quad A_- = \left\{x^{(n)} : \quad L_n(x^{(n)}) < c_\alpha\right\},$$
$$A_0 = \left\{x^{(n)} : \quad L_n(x^{(n)}) = c_\alpha\right\}.$$

Therefore

$$
\begin{aligned}
\int_{G_0} &\left\{\hat{\phi}(x^{(n)}) - \phi(x^{(n)})\right\} e^{L_n(x^{(n)})}\, \mathrm{d}\mathbf{P}_1^{(n)} \geq e^{c_\alpha} \int_{A_+} \left\{1 - \phi(x^{(n)})\right\}\, \mathrm{d}\mathbf{P}_1^{(n)} - \\
&- e^{c_\alpha} \int_{A_-} \phi(x^{(n)})\, \mathrm{d}\mathbf{P}_1^{(n)} + e^{c_\alpha} \int_{A_0} \left\{q_\alpha - \phi(x^{(n)})\right\}\, \mathrm{d}\mathbf{P}_1^{(n)} = \\
&= e^{c_\alpha} \left\{ \int_{A_+} \mathrm{d}\mathbf{P}_1^{(n)} + q_\alpha \int_{A_0} \mathrm{d}\mathbf{P}_1^{(n)} - \int_{G_0} \phi(x^{(n)})\, \mathrm{d}\mathbf{P}_1^{(n)} \right\} \geq \\
&\geq e^{c_\alpha} \left(\mathbf{E}_1 \hat{\phi}(X^{(n)}) - \mathbf{E}_1 \phi(X^{(n)}) \right) \geq e^{c_\alpha}(\alpha - \alpha) = 0.
\end{aligned}
$$

Hence the test $\hat{\phi}$ is the most powerful in the class $\mathcal{K}_\alpha$. From the last line of the above proof it can be seen that $\hat{\phi}$ is most powerful in the class of all tests $\phi(\cdot)$ with

$$
\mathbf{E}_1(\phi(X^{(n)})) \leq \alpha.
$$

Example 2.1. Assume that

$$
S_2(u) = c\, S_1(u) > 0, \qquad u \in \mathbb{A}_n,
$$

for some constant $c > 1$. These assumptions impliy that $\mathbb{A}_n = \mathbb{T}_n$, $\mathbb{B}_n = \mathbb{C}_n = \emptyset$ and the inequality $L_n\left(X^{(n)}\right) > c_\alpha$ is equivalent to $X^{(n)}\left(\mathbb{A}_n\right) > c_\alpha^{¤}$ for some suitable number $c_\alpha^{¤}$. Therefore the most powerful test has the following form

$$
\hat{\phi}(X^{(n)}) = \begin{cases} 1, & \text{if } X^{(n)}\left(\mathbb{A}_n\right) > c_\alpha^{¤}, \\ q_\alpha, & \text{if } X^{(n)}\left(\mathbb{A}_n\right) = c_\alpha^{¤}, \\ 0, & \text{if } X^{(n)}\left(\mathbb{A}_n\right) < c_\alpha^{¤}. \end{cases}
$$

Remind that $X^{(n)}\left(\mathbb{A}_n\right)$ is Poisson random variable with the parameter $\int_{\mathbb{A}_n} S_i(u)\, \mathrm{d}u$ under $\mathcal{H}_i$, $i = 1, 2$. Since $p_n = 1$, therefore for any $0 < \alpha < 1$ there exists a most powerful test in the class $\mathcal{K}_\alpha$. It can be easily seen that for

$$
\alpha = 1 - \exp\left\{ - \int_{\mathbb{A}_n} S_1(u)\, \mathrm{d}u \right\}
$$

we have $c_\alpha^{¤} = q_\alpha = 0$. Therefore the most powerful test has the form

$$
\hat{\phi}(X^{(n)}) = \begin{cases} 1, & \text{if } X^{(n)}\left(\mathbb{A}_n\right) > 0, \\ 0, & \text{if } X^{(n)}\left(\mathbb{A}_n\right) = 0. \end{cases}
$$

Example 2.2. We observe a realization $X^{(n)}$ of a Poisson process with periodic intensity function $S(\vartheta, x) = e^{\vartheta \sin x}$ $x \in [0, n]$. The amplitude parameter ϑ is unknown and we want to test the simple hypotheses $\mathcal{H}_1 : \vartheta = \vartheta_1$ against $\mathcal{H}_2 : \vartheta = \vartheta_2$, for two given values $\vartheta_1 < \vartheta_2$. Since the intensity function is positive, then $\mathbb{A}_n = \mathbb{T}_n$, $\mathbb{B}_n = \mathbb{C}_n = \emptyset$ and hence the most powerful test at level α is of the form

$$\hat{\phi}(X^{(n)}) = \begin{cases} 1, & \text{if} \quad Y_n > c_n, \\ q_n, & \text{if} \quad Y_n = c_n, \\ 0, & \text{if} \quad Y_n < c_n, \end{cases}$$

where the random variable

$$Y_n = \int_0^n \ln \frac{S(\vartheta_2, x)}{S(\vartheta_1, x)} \, \mathrm{d}X^{(n)}(x) = (\vartheta_2 - \vartheta_1) \int_0^n \sin x \, \mathrm{d}X^{(n)}(x)$$

and the constants $c_n = c_n(\vartheta_1, \vartheta_2, \alpha)$, $q_n = q_n(\vartheta_1, \vartheta_2, \alpha)$ satisfy the equation $\alpha = \mathbf{E}_{\vartheta_0} \hat{\phi}(X^{(n)})$. Because of the stochastic integral Y_n it is not easy to give the explicit forms of c_n and q_n. But asymptotic normality of Y_n under $\mathcal{H}_1$ provides the following first order approximation $\tilde{c}_n$ of c_n :

$$\tilde{c}_n = (\vartheta_2 - \vartheta_1) \int_0^n \sin x \, e^{\vartheta_1 \sin x} \, \mathrm{d}x + z_\alpha \, (\vartheta_2 - \vartheta_1) \left(\int_0^n \sin^2 x \, e^{\vartheta_1 \sin x} \, \mathrm{d}x \right)^{1/2},$$

in the sense that for the non randomized test $\tilde{\phi}_n(X^{(n)}) = \chi_{\{Y_n > \tilde{c}_n\}}$ we have

$$\mathbf{E}_{\vartheta_0} \tilde{\phi}_n(X^{(n)}) = \alpha + o(1),$$

as $n \to \infty$. In the next section we consider the asymptotic approach and by correcting the threshold $\tilde{c}_n$ with the help of Edgeworth expansion we improve the order of approximation.

Theorem 13. *Let $\alpha \leq \hat{p}_1$, $\sigma_{1,n} \to \infty$ and for any $\delta > 0$ the condition*

$$\lim_{n\to\infty} \sigma_{1,n}^{-2} \int_{\mathbb{T}_n(\delta)} l\,(u)^2\ S_1(u)\,\mathrm{d}u = 0 \tag{2.5}$$

is fulfilled. Then the test

$$\hat{\phi}_n\left(X^{(n)}\right) = \max\left(\chi_{\left\{Y_n,\ c_{n,\alpha}, X^{(n)}(\mathbb{C}_n)=0\right\}}, \chi_{\left\{X^{(n)}(\mathbb{B}_n)>0\right\}}\right) \tag{2.6}$$

with

$$c_{n,\alpha} = \sigma_{1,n}\ z_{\alpha/\hat{p}_1} + \mu_{1,n}, \tag{2.7}$$

belongs to the class $\mathcal{K}_\alpha^0$. Moreover, if $\hat{p}_2 = 0$ or

$$\eta_n = \frac{\mu_{2,n} - \mu_{1,n}}{\sigma_{2,n}} - \frac{\sigma_{1,n}}{\sigma_{2,n}}\ z_{\alpha/\hat{p}_1} \longrightarrow \infty, \tag{2.8}$$

then the test $\hat{\phi}_n$ is consistent.

Proof. We have

$$\begin{aligned}\mathbf{E}_1\hat{\phi}_n\left(X^n\right) &= \mathbf{P}_1^{(n)}\left\{Y_n > c_{n,\alpha}, X^{(n)}\left(\mathbb{C}_n\right) = 0\right\} = \\ &= p_n\ \mathbf{P}_1^{(n)}\left\{\frac{Y_n - \mu_{1,n}}{\sigma_{1,n}} > z_{\alpha/\hat{p}_1}\right\}.\end{aligned}$$

The statistic Y_n is by the central limit theorem for stochastic integrals (see, e.g., [27], Theorem 1.1) asymptotically normal:

$$\mathcal{L}_1\left\{\frac{Y_n - \mu_{1,n}}{\sigma_{1,n}}\right\} \Longrightarrow \mathcal{N}(0,1).$$

Remind that (2.5) is Lindeberg condition (under $\mathcal{H}_1$). Hence

$$p_n\ \mathbf{P}_1^{(n)}\left\{\frac{Y_n - \mu_{1,n}}{\sigma_{1,n}} > z_{\alpha/\hat{p}_1}\right\} \longrightarrow \hat{p}_1\ \mathbf{P}\left\{\zeta > z_{\alpha/\hat{p}_1}\right\} = \alpha$$

and the test $\hat{\phi}_n$ belongs to $\mathcal{K}_\alpha^0$.

Let us denote

$$q_n = \mathbf{P}_2^{(n)}\left\{X^{(n)}\left(\mathbb{B}_n\right) > 0\right\} = 1 - \exp\left\{-\int_{\mathbb{B}_n} S_2\left(u\right)\ \mathrm{d}u\right\}.$$

Then we can write

$$
\begin{aligned}
\beta(\hat{\phi}_n) &= \mathbf{E}_2 \hat{\phi}_n \left(X^{(n)}\right) = q_n + \\
&\quad + \mathbf{P}_2^{(n)} \left\{ \frac{Y_n - \mu_{2,n}}{\sigma_{2,n}} \geq \frac{\mu_{1,n} - \mu_{2,n}}{\sigma_{2,n}} + \frac{\sigma_{1,n}}{\sigma_{2,n}}\, z_{\alpha/\hat{p}_1} \right\} (1 - q_n) \geq \\
&\geq q_n + (1 - q_n) \left(1 - \mathbf{P}_2^{(n)} \left\{ \left| \frac{Y_n - \mu_{2,n}}{\sigma_{2,n}} \right| \geq \eta_n \right\} \right) \geq \\
&\geq q_n + (1 - q_n) \left(1 - \eta_n^{-2} \right) \longrightarrow 1,
\end{aligned}
$$

because either $q_n \to 1 - \hat{p}_2 = 1$ or $\eta_n \to \infty$. Therefore the test is consistent.

As we have seen the level $c_{n,\alpha}$ is obtained from the asymptotical normality of Y_n (under $\mathcal{H}_1$). In fact $c_{n,\alpha}$ is a solution of the equation

$$
\mathbf{P}_1^{(n)} \left\{ Y_n \leq c_{n,\alpha} \right\} = \mathbf{P}_1^{(n)} \left\{ \frac{Y_n - \mu_{1,n}}{\sigma_{1,n}} \leq \frac{c_{n,\alpha} - \mu_{1,n}}{\sigma_{1,n}} \right\} \simeq \mathcal{N}\left(z_{\alpha/\hat{p}_1} \right) = 1 - \frac{\alpha}{\hat{p}_1},
$$

where $\mathcal{N}(\cdot)$ is the distribution function of the standard normal law. For the finite values of n we have the error $\mathbf{E}_1 \hat{\phi}_n \left(X^{(n)}\right) = \alpha + o(1)$. The rate of approximation $o(1)$ can be improved using the Edgeworth type expansion of the distribution function of the statistic $\sigma_{1,n}^{-1} (Y_n - \mu_{1,n})$ under hypothesis $\mathcal{H}_1$:

$$
F_n(y) = \mathbf{P}_1^{(n)} \left\{ \sigma_{1,n}^{-1} \int_{\mathbb{T}_n} l(u)\ \pi(\mathrm{d}u) < y \right\}. \tag{2.9}
$$

Here $\pi(\mathrm{d}u) = X^{(n)}(\mathrm{d}u) - S_1(u)\,\mathrm{d}u$, is the centered Poisson process.

For simplicity of exposition we take just one after Gaussian term of this expansion and suppose that $p_n = \hat{p}_1$. Note that having asymptotic expansion of the distribution function by the powers of small parameter, say, α_n this error can be reduced up to the order $\varepsilon_n^k, k = 1, 2, \ldots$ (see [5], Chapter 4 for error approximation and [27], Theorem 3.4 for the Edgeworth expansion of $F_n(y)$).

The expansion is obtained under the following two conditions:

$\mathcal{E}_1$. *There exists a sequence of real numbers $\varepsilon_n \to 0$, as $n \to \infty$ and constants $C_r > 0, r = 3, 4$, such that*

$$
\int_{\mathbb{T}_n} |l(u)|^r\ S_1(u)\,\mathrm{d}u \leq C_r\, \sigma_{1,n}^r\, \varepsilon_n^{r-2},
$$

for $r = 3, 4$.

$\mathcal{E}_2$. *The following inequality*

$$\inf_{c_0\varepsilon_n^{-1}<|\lambda|<\frac{c_0\varepsilon_n^{-2}}{2}} \int_{\mathbb{T}_n} \sin^2\left(\frac{\lambda\, l\,(u)}{\sigma_{1,n}}\right)\ S_1(u)\ \mathrm{d}u \geq \gamma \ln \varepsilon_n^{-1}$$

holds for small values of $c_0 > 0$ *and* $\gamma \geq 3/2$.

Note that in the condition $\mathcal{E}_2$ it is sufficient that the constant c_0 satisfies the inequality $\frac{C_3}{3}c_0 + \frac{C_4}{6}c_0^2 - \frac{1}{2} < 0$. Let us denote

$$\gamma_{3,n} = \int_{\mathbb{T}_n} \frac{l(u)^3}{\sigma_{1,n}^3}\ S_1(u)\ \mathrm{d}u.$$

Theorem 14. *Let* $\alpha \leq \hat{p}_1$, $\sigma_{1,n} \to \infty$ *and the conditions* $\mathcal{E}_1$, $\mathcal{E}_2$ *be fulfilled. Then for the test*

$$\hat{\phi}_n\left(X^{(n)}\right) = \max\left(\chi_{\left\{Y_n,\ b_{n,\alpha}, X^{(n)}(\mathbb{C}_n)=0\right\}}, \chi_{\left\{X^{(n)}(\mathbb{B}_n)>0\right\}}\right) \tag{2.10}$$

with

$$b_{n,\alpha} = \sigma_{1,n}\ \left[z_{\alpha/\hat{p}_1} - \frac{\gamma_{3,n}}{6}\left(1 - z^2_{\alpha/\hat{p}_1}\right)\right] + \mu_{1,n}, \tag{2.11}$$

we have

$$\mathbf{E}_1\hat{\phi}_n\left(X^{(n)}\right) = \alpha + O\left(\varepsilon_n^2\right). \tag{2.12}$$

Proof. To verify (2.12) we apply Theorem 13 to the statistic Y_n, *i.e.*, we put

$$f_n(u) = \sigma_{1,n}^{-1}\ \ln\left(\frac{S_2(u)}{S_1(u)}\right)$$

in (1.1) and consider the distribution function (2.9)

$$F_n(y) = \mathbf{P}_1\left\{\frac{Y_n - \mu_{1,n}}{\sigma_{1,n}} < y\right\}.$$

According to Theorem 4 we have the estimate

$$\sup_y \left|F_n(y) - \mathcal{N}(y) - \frac{\gamma_{3,n}\,(1-y^2)}{6\,\sqrt{2\pi}}\ e^{-y^2/2}\right| \leq C\ \varepsilon_n^2,$$

with some positive constant C. Hence, if we find a constant b_n such that

$$\mathcal{N}(b_n) + \frac{\gamma_{3,n}\ (1-b_n^2)}{6\sqrt{2\pi}}\ e^{\mathrm{i}\ b_n^2/2} = 1 - \frac{\alpha}{\hat{p}_1}, \tag{2.13}$$

then by choosing $b_{n,\alpha} = \sigma_{1,n} b_n + \mu_{1,n}$ we have

$$\mathbf{E}_1 \hat{\phi}_n \left(X^{(n)}\right) = p_n\ \left[1 - F_n(b_n)\right] = \alpha + O\left(\varepsilon_n^2\right).$$

Remind that by condition $\mathcal{E}_1$ we have the convergence

$$\gamma_{3,n} \longrightarrow 0$$

with the rate ε_n. Therefore using Taylor formula we obtain the representation

$$b_n = z_{\alpha/\hat{p}_1} - \frac{\gamma_{3,n}\left(1 - z_{\alpha/\hat{p}_1}^2\right)}{6} + O\left(\varepsilon_n^2\right).$$

Hence we can put (2.11) and this provides us (2.12).

Of course, this test is consistent under the same additional conditions $\hat{p}_2 = 0$ or (2.8). Note that condition $\mathcal{E}_1$ $(r = 3)$ is sufficient for (2.5).

Remark 4. *Let the condition $\mathcal{E}_1$ holds also for $r = 5$ and condition $\mathcal{E}_2$ for some $\gamma \geq 5/2$. Then by Corollary 8 for $b_{n,\alpha} = \sigma_{1,n}\, b_n^0 + \mu_{1,n}$ with*

$$b_n^0 = z_\alpha - \frac{\gamma_{3,n}}{6}(1 - z_\alpha^2) - \frac{\gamma_{3,n}^2}{72}\left(-2\, z_\alpha^5 + 8\, z_\alpha^3 - 12\, z_\alpha\right) - \frac{\gamma_{4,n}}{4!}(3\, z_\alpha - z_\alpha^3)$$

where

$$\gamma_{j,n} = \int_{\mathbb{T}_n} \frac{l(u)^j}{\sigma_{1,n}^j}\ S_1(u)\, \mathrm{d}u, \qquad j = 3, 4,$$

we have

$$\mathbf{E}_1 \hat{\phi}_n \left(X^{(n)}\right) = \alpha + O\left(\varepsilon_n^3\right).$$

Note that the sequences

$$\gamma_{3,n}, \gamma_{4,n} \longrightarrow 0,$$

with the rates ε_n and ε_n^2 respectively.

Remark 5. *For $\alpha > \hat{p}_1$ we put $\hat{\phi}_n\left(X^{(n)}\right) = \chi_{\left\{X^{(n)}(\mathbb{C}_n)=0\right\}}$. Then for n large we have $\mathbf{E}_1\hat{\phi}_n\left(X^{(n)}\right) < \alpha$ with the power $\mathbf{E}_2\hat{\phi}_n\left(X^{(n)}\right) = 1$ for all n.*

Example 2. Suppose that the Poisson processes have periodic structure, *i.e.*, the intensity functions are periodic with the same period and the set $\mathbb{A}_n = \cup_{l=1}^{N}\mathbb{V}_l$, where $\mathbb{V}_l = \mathbb{U}_l \cup \mathbb{W}_l$ corresponds to one period. Here $N = \frac{n}{\tau}$ is the number of periods observed and $\mathbb{W}_l = \{u : S_2(u) = 0, u \in \mathbb{V}_l\}$. First note that if

$$\kappa = \int_{\mathbb{W}_1} S_1(u)\ \mathrm{d}u > 0$$

then

$$\mathbf{E}_1\hat{\phi}_n\left(X^{(n)}\right) \le e^{\frac{-\kappa n}{\tau}} \longrightarrow 0.$$

Therefore, we put $\kappa = 0$ and study the case $\mathbb{V}_l = \mathbb{U}_l$. Hence, we have

$$\sigma_{1,n}^2 = \sum_{l=1}^{N}\int_{\mathbb{U}_l} l(u)^2\ S_1(u)\ \mathrm{d}u = \sigma_1^2\ n$$

with obvious notation. It easy to see that in this case we can take $\varepsilon_n = n^{-1/2}$ and the condition $\mathcal{E}_1$ became

$$\int_{\mathbb{U}_1} |l(u)|^3\ S_1(u)\ \mathrm{d}u \le C_3, \qquad \int_{\mathbb{U}_1} |l(u)|^4\ S_1(u)\ \mathrm{d}u \le C_4.$$

The condition $\mathcal{E}_2$ is equivalent to

$$\inf_{c_0 \le \nu \le \sqrt{\frac{n}{\tau}}} \frac{n}{\tau}\int_{\mathbb{U}_l} \sin^2(\nu l(u))\ S_1(u)\ \mathrm{d}u = \frac{n}{\tau}\int_{\mathbb{U}_l} S_1(u)\ \mathrm{d}u -$$
$$- \sup_{c_0 \le \nu \le \sqrt{\frac{n}{\tau}}} \frac{n}{\tau}\int_{\mathbb{U}_l} \cos(2\nu l(u))\ S_1(u)\ \mathrm{d}u \ge \frac{3\ \ln n}{4}.$$

The last condition can be fulfilled in many examples with nonconstant function $|l(u)|$. Remind that under mild conditions

$$\lim_{\nu\to\infty}\int_{\mathbb{U}_l} \cos(2\nu l(u))\ S_1(u)\ \mathrm{d}u = 0.$$

Therefore if

$$\int_{\mathbb{U}_l} l(u)^2\, S_1(u)\ \mathrm{d}u > 0,$$

we can choose

$$b_{n,\alpha} = z_\alpha - \frac{\int_{\mathbb{U}_l} l\,(u)^3\, S_1\,(u)\ \mathrm{d}u}{\left(\int_{\mathbb{U}_l} l\,(u)^2\, S_1\,(u)\ \mathrm{d}u\right)^{3/2}}\ \frac{1 - z_\alpha^2}{6\sqrt{\frac{n}{\tau}}}$$

and this leads to $\mathbf{E}_1 \hat{\phi}_n \left(X^{(n)}\right) = \alpha + O\left(\frac{1}{n}\right)$.

Note that, if, say, $S_2\,(u) = h\,S_1\,(u)$ for some values $u \in \mathbb{U}_l$ and $S_1\,(u) = h\,S_2\,(u)$ for all other $u \in \mathbb{U}_l$, then obviously

$$\inf_{c_0 \cdot\ \nu \cdot\ \sqrt{n}} \frac{n}{\tau} \int_{\mathbb{U}_l} \sin^2\left(\nu l\,(u)\right)\ S_1\,(u)\ \mathrm{d}u = 0.$$

Example 3. We observe the Poisson process $X^{(n)}$ on the circle

$$\mathbb{A}_n = \{u : \quad |u| \leq n\} \subset \mathbb{R}^2.$$

We have to test the following two hypotheses concerning the intensity function $S\,(\cdot)$ of the Poisson process

$$\begin{aligned}
&\mathcal{H}_1 : \qquad S\,(u) = S_1(u) \equiv e^{a\cos(\omega r)}\ \chi_{\{u \in \mathbb{B}_n^c\}} \qquad && u \in \mathbb{A}_n,\\
&\mathcal{H}_2 : \qquad S\,(u) = S_2(u) \equiv e^{b\cos(\omega r)} && u \in \mathbb{A}_n,
\end{aligned}$$

where $u = (x, y)$, $r = \sqrt{x^2 + y^2}$ the frequency $\omega > 0$ and a, b are two given constants such that $ab > 0$. Under $\mathcal{H}_2$ the intensity function is positive, but under $\mathcal{H}_1$ it is zero on the set

$$\mathbb{B}_\omega > \ \cos(\omega r$$

Recall that for periodic function $\varphi(\cdot)$ with period τ we have

$$\lim_{n\to\infty} \frac{\int_0^n r\varphi(r)\,\mathrm{d}r}{n^2} = \frac{\int_0^\tau \varphi(r)\,\mathrm{d}r}{2\tau}. \tag{2.14}$$

Therefore

$$\lim_{n\to\infty} \frac{\int_0^n r\cos^2(\omega r)\, e^{a\cos(\omega r)}\,\mathrm{d}r}{n^2} = \frac{\omega\int_0^\tau \cos^2(\omega r) e^{a\cos(\omega r)}\,\mathrm{d}r}{4\pi},$$

where $\tau = \frac{2\pi}{\omega}$. Hence

$$\lim_{n\to\infty} \frac{\sigma_{1,n}^2}{n^2} = \frac{3\omega}{8}(b-a)^2 \int_0^\tau \cos^2(\omega r) e^{a\cos(\omega r)}\,\mathrm{d}r \equiv c_*^2.$$

Now we verify the condition $\mathcal{E}_1$.

$$\int_{\mathbb{T}_n} \left|\sigma_{1,n}^{-1} l(u)\right|^j S_1(u)\,\mathrm{d}u \le \int_{\mathbb{A}_n} \left|\sigma_{1,n}^{-1}(b-a)\cos(\omega r)\right|^j e^{a\cos(\omega r)}\,\mathrm{d}u =$$
$$= C\sigma_{1,n}^{-j} \int_0^n r\left|\cos(\omega r)\right|^j e^{a\cos(\omega r)}\,\mathrm{d}r \le C' \frac{n^2}{n^j} \le \frac{C_j}{n^{j-2}},$$

for some positive constants C_3, C_4. Therefore the condition $\mathcal{E}_1$ holds with $\varepsilon_n = n^{-1}$. For condition $\mathcal{E}_2$ we consider the following quantity :

$$\inf_{\lambda > c_0 n} \int_{\mathbb{T}_n} \sin^2\left(\frac{\lambda(b-a)\cos(\omega r)}{\sigma_{1,n}}\right) e^{a\cos(\omega r)}\,\mathrm{d}u.$$

Since the region $\mathbb{D}_n \equiv \{u : \quad 0 \le r \le n,\ \theta_1 \le \theta \le \theta_2\}$ is a subset of $\mathbb{T}_n$ for some constants $\theta_1 < \theta_2$, then

$$\int_{\mathbb{T}_n} \sin^2\left(\frac{\lambda(b-a)\cos(\omega r)}{\sigma_{1,n}}\right) e^{a\cos(\omega r)}\,\mathrm{d}u \ge$$
$$\ge \int_{\mathbb{D}_n} \sin^2\left(\frac{\lambda(b-a)\cos(\omega r)}{\sigma_{1,n}}\right) e^{a\cos(\omega r)}\,\mathrm{d}u =$$
$$= (\theta_2 - \theta_1)\int_0^n r\sin^2\left(\frac{\lambda(b-a)\cos(\omega r)}{\sigma_{1,n}}\right) e^{a\cos(\omega r)}\,\mathrm{d}r \ge$$
$$\ge (\theta_2 - \theta_1) e^{-|a|} \int_0^n r\sin^2\left(\frac{\lambda(b-a)\cos(\omega r)}{\sigma_{1,n}}\right)\mathrm{d}r.$$

Let

$$g_n(\lambda) = \int_0^n r\sin^2\left(\frac{\lambda(b-a)\cos(\omega r)}{\sigma_{1,n}}\right)\mathrm{d}r, \quad k = \left[\frac{\pi}{\omega}\right], \quad m = \left[\frac{n}{k}\right].$$

First suppose that m is even. Since the integrand is nonnegative, we get

$$\begin{aligned}
g_n(\lambda) &\geq \int_{\frac{(m-1)\pi}{\omega}}^{\frac{m\pi}{\omega}} r\sin^2\left(\frac{\lambda(b-a)\cos(\omega r)}{\sigma_{1,n}}\right)\,\mathrm{d}r \geq \\
&\geq \frac{(m-1)\pi}{\omega}\int_{\frac{(m-1)\pi}{\omega}}^{\frac{m\pi}{\omega}} \sin^2\left(\frac{\lambda(b-a)\cos(\omega r)}{\sigma_{1,n}}\right)\,\mathrm{d}r \geq \\
&\geq \frac{(m-1)\pi}{\omega}\int_{\frac{(m-1)\pi}{\omega}}^{\frac{m\pi}{\omega}} \sin^2\left(\frac{\lambda(b-a)\cos(\omega r)}{\sigma_{1,n}}\right)(-\sin(\omega r))\,\mathrm{d}r = \\
&= \frac{(m-1)\pi}{2\omega^2}\left\{1-\frac{\sin\left(\frac{2\lambda(b-a)}{\sigma_{1,n}}\right)}{\frac{2\lambda(b-a)}{\sigma_{1,n}}}\right\} = \frac{(m-1)\pi}{2\omega^2}\left\{1-\frac{\sin\left(\frac{2\lambda|b-a|}{\sigma_{1,n}}\right)}{\frac{2\lambda|b-a|}{\sigma_{1,n}}}\right\}.
\end{aligned}$$

The first equality is obtained by transforming the variable $r \to \frac{\lambda(b-a)\cos(\omega r)}{\sigma_{1,n}}$, and by considering that m is even. Since $\sigma_{1,n} \sim c_* n$, then there exists a constant $d > 0$, such that for all n enough large, $\frac{2c_0|b-a|n}{\sigma_{1,n}} \leq d$. Hence

$$\sup_{\lambda \geq c_0 n} \frac{\sin\left(\frac{2\lambda|b-a|}{\sigma_{1,n}}\right)}{\frac{2\lambda|b-a|}{\sigma_{1,n}}} \geq \sup_{\frac{2\lambda|b-a|}{\sigma_{1,n}} \geq d} \frac{\sin\left(\frac{2\lambda|b-a|}{\sigma_{1,n}}\right)}{\frac{2\lambda|b-a|}{\sigma_{1,n}}} = \sup_{x \geq d}\frac{\sin x}{x} = B < 1,$$

and consequently for n large

$$\inf_{\lambda \geq c_0 n} g_n(\lambda) \geq \frac{(m-1)\pi}{2\omega^2}(1-B) = C\left(\left[\frac{n}{k}\right]-1\right) \geq C' n \geq \gamma\ln n = \gamma\ln\varepsilon_n^{-1}.$$

For m odd we obtain this result by a similar argument. Therefore by Theorem 4 the distribution function $F_n(y)$ of Y_n(under $\mathcal{H}_1$) has the expansion

$$F_n(y) = \mathcal{N}(y) + \frac{\mu_{3,n}(1-y^2)e^{-y^2/2}}{6\sqrt{2\pi}\,\sigma_{1,n}^3} + O\left(\frac{1}{n^2}\right),$$

where

$$\begin{aligned}
\mu_{3,n} &= \int_{\mathbb{T}_n} l(u)^3 S_1(u)\,\mathrm{d}u = (b-a)^3\int_{\mathbb{T}_n}\cos^3(\omega r)e^{a\cos(\omega r)}\,\mathrm{d}u = \\
&= \frac{3\pi(b-a)^3}{2}\int_0^n r\cos^3(\omega r)e^{a\cos(\omega r)}\,\mathrm{d}r \equiv d_* n^2(1+o(1)).
\end{aligned}$$

Similarly

$$\begin{aligned}
\mu_{1,n} &= \int_{\mathbb{T}_n} l(u) S_1(u)\,\mathrm{d}u = \\
&= \frac{3\pi(b-a)}{2}\int_0^n r\cos(\omega r)e^{a\cos(\omega r)}\,\mathrm{d}r \equiv h_* n^2(1+o(1)),
\end{aligned}$$

where the constants d_* and h_*

$$
\begin{aligned}
d_* &= \frac{3\omega\,(b-a)^3}{8}\int_0^\tau \cos^3(\omega r)\, e^{a\cos(\omega r)}\,\mathrm{d}r \\
h_* &= \frac{3\omega\,(b-a)}{8}\int_0^\tau \cos(\omega r)\, e^{a\cos(\omega r)}\,\mathrm{d}r.
\end{aligned}
$$

Therefore the threshold $b_{n,\alpha}$ in Theorem 14 can be taken as

$$
b_{n,\alpha} = c_*\, n\left[z_\alpha - \frac{d_*}{6\, c_*^3\, n}\left(1 - z_\alpha^2\right)\right] + h_* n^2.
$$

2.4 Asymptotic behavior of power

To study the convergence of the power $\beta(\hat\phi_n) \to 1$ we use (as usual in such situations) the *large deviations principle* [16] for stochastic integral Y_n. We need the further notation:

- logarithmic moment generating function of Y_n (under $\mathcal{H}_2$):

$$
\Lambda_n(\lambda) = \ln \mathbf{E}_2 \exp(\lambda Y_n) = \int_{\mathbb{T}_n}\left[\left(\frac{S_2(u)}{S_1(u)}\right)^\lambda - 1\right] S_2(u)\,\mathrm{d}u,
$$

- the limits

$$
\Lambda(\lambda) = \lim_{n\to\infty}\frac{\Lambda_n(\lambda)}{\mu_{2,n}}, \tag{2.15}
$$

$$
\gamma = \lim_{n\to\infty}\frac{\mu_{1,n} + z_{\alpha/\hat p_1}\sigma_{1,n}}{\mu_{2,n}} \tag{2.16}
$$

$$
\psi = \lim_{n\to\infty}\frac{\int_{\mathbb{B}_n} S_2(u)\,\mathrm{d}u}{\mu_{2,n}}, \tag{2.17}
$$

- the Fenchel-Legendre transform of $\Lambda(\cdot)$:

$$
\Lambda^*(\nu) = \sup_{\lambda\in\mathbb{R}}\{\lambda\nu - \Lambda(\lambda)\},
$$

- $D^{\circ}_{\Lambda^*}$ is the interior of the set

$$
D_{\Lambda^*} = \{\nu : \quad \Lambda^*(\nu) < \infty\}.
$$

The behavior of the power is described below in the asymptotic

$$\mu_{2,n} \longrightarrow \infty.$$

Hence, $\mu_{2,n}$ plays the role of *natural parameter* like n in the i.i.d. case. Moreover, we suppose that $\psi < \infty$, because if $\psi = \infty$, then the power function can be written as

$$\beta_n\left(\hat{\phi}_n\right) = 1 - \exp\left\{-\int_{\mathbb{B}_n} S_2(u)\,\mathrm{d}u(1 + o(1))\right\}.$$

This representation easily follows from (2.19), the proof of the theorem below.

Theorem 15. *Let the conditions of Theorem 13 and the following conditions be fulfilled:*

- *for all λ the limit $\Lambda(\lambda)$ exists, finite or infinite,*
- *the limit $\gamma < 1$ exists and $\gamma \in D^{\pm}_{\Lambda^*}$,*
- *the function $\Lambda(\cdot)$ is differentiable in $D^{\pm}_{\Lambda}$ and for any $\nu < 1$ in $D^{\pm}_{\Lambda^*}$, there exists $\eta \in D^{\pm}_{\Lambda}$ such that $\Lambda^{0}(\eta) = \nu$.*

Then the power of the test $\hat{\phi}_n$ in (2.5)-(2.6) *admits the representation*

$$\beta_n\left(\hat{\phi}_n\right) = \mathbf{E}_2\hat{\phi}_n(X^{(n)}) = 1 - \exp\left\{-\mu_{2,n}(\psi + \Lambda^{\square}(\gamma))(1 + o(1))\right\}. \tag{2.18}$$

Proof. Recall that the test $\hat{\phi}_n$ belongs to the class $\mathcal{K}^{0}_{\alpha}$. The probability of error of the second kind of $\hat{\phi}_n$ is given by

$$1 - \beta_n\left(\hat{\phi}_n\right) = \mathbf{E}_2(1 - \hat{\phi}_n(X^{(n)})) = \mathbf{P}^{(n)}_2\left\{Y_n \leq c_{n,\alpha}\,, X^{(n)}(\mathbb{B}_n) = 0\right\} =$$
$$= \mathbf{P}^{(n)}_2\left\{Y_n \leq \mu_{1,n} + z_{\alpha/\hat{p}_1}\sigma_{1,n}\right\}\ \exp\left\{-\int_{\mathbb{B}_n} S_2(u)\,\mathrm{d}u\right\}.$$

Note that the random variables Y_n and $X^{(n)}(\mathbb{B}_n)$ are independent, because $X^{(n)}$ is a Poisson process and the sets $\mathbb{T}_n$ and $\mathbb{B}_n$ are disjoint. By taking logarithm and dividing two sides of this equation by $\mu_{2,n}$, we obtain

$$\frac{\ln\left(1 - \beta_n\left(\hat{\phi}_n\right)\right)}{\mu_{2,n}} = \frac{\ln \mathbf{P}^{(n)}_2\left\{Y_n \leq \nu_n\right\}}{\mu_{2,n}} - \frac{\int_{\mathbb{B}_n} S_2(u)\,\mathrm{d}u}{\mu_{2,n}}, \tag{2.19}$$

where $\nu_n = \mu_{1,n} + z_{\alpha/\hat{p}_1}\sigma_{1,n}$. Note that the last term converges to $\psi < \infty$ and since $\frac{\nu_n}{\mu_{2,n}} \to \gamma < 1$, $\gamma \in D^{\pm}_{\Lambda^*}$ we can apply Theorem 11, chapter 1 and obtain

$$\lim_{n\to\infty} \frac{\ln\left(1 - \beta_n\left(\hat{\phi}_n\right)\right)}{\mu_{2,n}} = -\Lambda^*(\gamma) - \psi.$$

Therefore we have the desired representation (2.18).

Remark 6. *Let the conditions of Theorem 15 and $\mathcal{E}_1 - \mathcal{E}_2$ be fulfilled. Also suppose that there exists $C > 0$ such that for all n*

$$\left|\frac{\sigma_{1,n}}{\mu_{2,n}}\right| \leq C.$$

Then by a similar argument as for Theorem 15, we obtain the same representation for the power of the test $\hat{\phi}_n$ in (10)-(11), because by considering the condition $\mathcal{E}_1$ for $r = 3$,

$$\lim_{n\to\infty} \frac{b_{n,\alpha}}{\mu_{2,n}} = \lim_{n\to\infty} \frac{c_{n,\alpha}}{\mu_{2,n}} = \gamma.$$

Remark 7. *If $\mu_{2,n} \to -\infty$, we replace $\mu_{2,n}$ by $-\mu_{2,n}$ in the definition of $\Lambda(\cdot)$. Let the conditions of Theorem 13 and the following conditions be fulfilled:*

- *for all λ the limit $\Lambda(\lambda)$*

where the intensities $S_1(\cdot) \geq 0$, $S_2(\cdot) > 0$ and they are periodic with the same period $\tau > 0$. As we will see $\mu_{i,n} \sim n$, $\sigma_{i,n}^2 \sim n$ for $i = 1, 2$. Hence

$$\eta_n = \frac{\mu_{2,n} - \mu_{1,n}}{\sigma_{2,n}} - \frac{\sigma_{1,n}}{\sigma_{2,n}} z_\alpha \to \infty,$$

and consequently the tests (2.6)- (2.10) are consistent. The sets

$$\begin{aligned} \mathbb{T}_n &= \{0 \leq u \leq n : \quad S_1(u) > 0\} \\ \mathbb{B}_n &= \{0 \leq u \leq n : \quad S_1(u) = 0\}. \end{aligned}$$

In what follows we suppose that

$$I \equiv \int_{\mathbb{T}_1} l\,(u)\, S_2(u)\, \mathrm{d}u \neq 0,$$

where $l\,(u) = \ln\left(\frac{S_2(u)}{S_1(u)}\right)$. Let $\varphi(.)$ be a periodic function with period τ. In what follows we use frequently the formula

$$\lim_{n\to\infty} \frac{\int_{\mathbb{T}_n} \varphi(u)\, \mathrm{d}u}{n} = \frac{\int_{\mathbb{T}_1} \varphi(u)\, \mathrm{d}u}{\tau}. \tag{2.20}$$

This formula implies that

$$\lim_{n\to\infty} \frac{\mu_{2,n}}{n} = \lim_{n\to\infty} \frac{\int_{\mathbb{T}_n} l\,(u)\, S_2(u)\, \mathrm{d}u}{n} = \frac{\int_{\mathbb{T}_1} l\,(u)\, S_2(u)\, \mathrm{d}u}{\tau} = \frac{I}{\tau},$$

and also

$$\lim_{n\to\infty} \frac{\Lambda_n(\lambda)}{n} = \lim_{n\to\infty} \frac{\int_{\mathbb{T}_n} \left[\left(\frac{S_2(u)}{S_1(u)}\right)^\lambda - 1\right] S_2(u)\, \mathrm{d}u}{n} =$$

$$= \frac{\int_{\mathbb{T}_1} \left[\left(\frac{S_2(u)}{S_1(u)}\right)^\lambda - 1\right] S_2(u)\, \mathrm{d}u}{\tau}.$$

The precedent equality shows that $\mu_{2,n} \to \infty$, if $I > 0$ and $\mu_{2,n} \to -\infty$, if $I < 0$. Hence for all real numbers λ we have

$$\Lambda(\lambda) = \lim_{n\to\infty} \frac{\Lambda_n(\lambda)}{|\mu_{2,n}|} = \frac{\int_{\mathbb{T}_1} \left[\left(\frac{S_2(u)}{S_1(u)}\right)^\lambda - 1\right] S_2(u)\, \mathrm{d}u}{|I|}.$$

Note that I does not depend on λ and the function $\Lambda(\lambda)$ has derivative

$$\Lambda'(\lambda) = \frac{\int_{\mathbb{T}_1} l\,(u) \left(\frac{S_2(u)}{S_1(u)}\right)^{\lambda} S_2(u)\,\mathrm{d}u}{|I|},$$

for all real numbers λ. Observe that Λ' is increasing, $\Lambda'(0) = 1$ if $I > 0$ and $\Lambda'(0) = -1$ if $I < 0$. We need the asymptotic behavior of $\Lambda'(\lambda)$ when $\lambda \to -\infty$. For this let

$$\mathbb{T}_{1+} \equiv \{u \in \mathbb{T}_1 : \quad S_1(u) \le S_2(u)\}\,,\ \mathbb{T}_{1-} \equiv \{u \in \mathbb{T}_1 : \quad S_2(u) < S_1(u)\}$$

$$H\,(\lambda) \equiv \int_{\mathbb{T}_1} l\,(u) \left(\frac{S_2(u)}{S_1(u)}\right)^{\lambda} S_2(u)\,\mathrm{d}u = \int_{\mathbb{T}_{1+}} + \int_{\mathbb{T}_{1-}} \equiv H_+\,(\lambda) + H_-\,(\lambda)\,,$$

with obvious notations. If the Lebesgue measures of the sets $\mathbb{T}_{1+}$ and $\mathbb{T}_{1-}$ are positive then by Monotone Convergence Theorem :

$$\lim_{\lambda\to-\infty} H_+\,(\lambda) = 0, \qquad \lim_{\lambda\to-\infty} H_-\,(\lambda) = -\infty.$$

In terms of the Lebesgue measures of these sets there are three different cases(by $\ell(\mathbb{A})$ we mean the the Lebesgue measure of the measurable set $\mathbb{A}$).

a) $\ell(\mathbb{T}_{1-}) = 0$.

In this case $S_1(u) \le S_2(u)$, almost everywhere and hence

$$I > 0,\ H\,(\lambda) = H_+\,(\lambda)\,,\ \Lambda'(0) = 1,\ \lim_{\lambda\to-\infty} \Lambda'(\lambda) = \lim_{\lambda\to-\infty} H_+\,(\lambda) = 0.$$

Therefore the equation $\upsilon - \Lambda'(\lambda) = 0$ has no solution for $\upsilon < 0$. Hence

Λ¤)6(for)]D[()6u6.641TD[( )]TJ sup11.95Tf6.830TD[(.2-96.660TD[(lim)]TJ F *b)* $\ell(\mathbb{T}_{1+}) = 0$.

In this case;

$$I < 0,\ H(\lambda) = H_{-}(\lambda),\ \Lambda'(0) = -1,\ \lim_{\lambda \to -\infty} \Lambda'(\lambda) = \lim_{\lambda \to -\infty} H_{-}(\lambda) = -\infty.$$

By a similar argument it can be shown that $D^{\circ}_{\Lambda^*} = (-\infty, 0)$, $\gamma > 1$ and hence $-\gamma \in D^{\circ}_{\Lambda^*}$.

c) $\ell(\mathbb{T}_{1+}) > 0,\ \ell(\mathbb{T}_{1-}) > 0$.

In this case the sign of the I depends to the values of the functions S_1 and S_2 on $\mathbb{T}_1$. Therefore

$$\Lambda'(\lambda) = \frac{H(\lambda)}{|I|}.$$

Since

$$\lim_{\lambda \to -\infty} H(\lambda) = \lim_{\lambda \to -\infty} H_{+}(\lambda) + \lim_{\lambda \to -\infty} H_{-}(\lambda) = -\infty,$$

then

$$\lim_{\lambda \to -\infty} \Lambda'(\lambda) = -\infty.$$

Hence it can be easily seen that in this case $D_{\Lambda^*} = D^{\circ}_{\Lambda^*} = (-\infty, +\infty)$. Therefore $\gamma, -\gamma \in D^{\circ}_{\Lambda^*}$.

Now we compute ψ and $\Lambda^*(\gamma)$. By (2.20);

$$\psi = \lim_{n \to \infty} \frac{\int_{\mathbb{B}_n} S_2(u)\,\mathrm{d}u}{\mu_{2,n}} = \frac{\int_{\mathbb{F}_1} S_2(u)\,\mathrm{d}u}{I},$$

where

$$\mathbb{F}_1 = \{0 \le u \le \tau : \quad S_1(u) = 0\}.$$

Suppose first that $I > 0$. Therefore $\mu_{2,n} \to \infty$. It can be easily seen that in this case

$$\Lambda^*(\gamma) = \sup_{\lambda \in \mathbb{R}} \{\lambda\gamma - \Lambda(\lambda)\} = -\gamma - \Lambda(-1) =$$
$$= \frac{\int_{\mathbb{T}_1} [S_2(u) - S_1(u) - l(u)\, S_1(u)]\,\mathrm{d}u}{\int_{\mathbb{T}_1} l(u)\, S_2(u)\,\mathrm{d}u}.$$

But if $I<0$, then $\mu_{2,n}\to-\infty$ and

$$\begin{aligned}\Lambda^{*}(-\gamma)=\sup_{\lambda\in\mathbb{R}}\{-\lambda\gamma-\Lambda(\lambda)\}=\gamma-\Lambda(-1)=&\\ =\frac{\int_{\mathbb{T}_1}[S_1(u)-S_2(u)-l(u)\,S_1(u)]\,\mathrm{d}u}{\int_{\mathbb{T}_1}l(u)\,S_2(u)\,\mathrm{d}u}&.\end{aligned}$$

Therefore the power of the test $\hat{\phi}_n$

- by Theorem (15) admits the representation

$$\begin{aligned}\beta_n\left(\hat{\phi}_n\right)=&1-\exp\{-\mu_{2,n}(\psi+\Lambda^{*}(\gamma))(1+\circ(1))\}=\\ &=1-\exp\left\{-n\left(I\frac{(\psi+\Lambda^{*}(\gamma))}{\tau}+\circ(n)\right)(1+\circ(1))\right\},\end{aligned}$$

  if $I>0$ and

- by Remark 1 admits the representation

$$\begin{aligned}\beta_n\left(\hat{\phi}_n\right)=&1-\exp\{\mu_{2,n}(-\psi+\Lambda^{*}(-\gamma))(1+\circ(1))\}=\\ &=1-\exp\left\{n\left(I\frac{(-\psi+\Lambda^{*}(-\gamma))}{\tau}+\circ(n)\right)(1+\circ(1))\right\},,\end{aligned}$$

  if $I<0$.

Example 5. In the example (3), since

$$\eta_n=\frac{\mu_{2,n}-\mu_{1,n}}{\sigma_{2,n}}-\frac{\sigma_{1,n}}{\sigma_{2,n}}z_\alpha\sim C\,n\to\infty,$$

for some $C>0$, then the tests (2.6)-(2.10) are consistent (the set $\mathbb{C}_n=\emptyset$, because $S_2(.)>0$). Therefore we obtain the asymptotic behavior of its power function with the help of large deviations principle. For the logarithmic moment generating function of Y_n(under $\mathcal{H}_2$) we have

$$\begin{aligned}&\Lambda_n(\lambda)=\int_{\mathbb{T}_n}\left[\left(\frac{S_2(u)}{S_1(u)}\right)^{\lambda}-1\right]S_2(u)\,\mathrm{d}u=\\ &=\int_{\mathbb{A}_n}\left[e^{\lambda(b-a)\cos(\omega r)}-1\right]e^{b\cos(\omega r)}\,\mathrm{d}u-\int_{\mathbb{B}_n}\left[e^{\lambda(b-a)\cos(\omega r)}-1\right]e^{b\cos(\omega r)}\,\mathrm{d}u=\\ &=\frac{3\pi}{2}\int_0^n r\left[e^{\lambda(b-a)\cos(\omega r)}-1\right]e^{b\cos(\omega r)}\,\mathrm{d}r\end{aligned}$$

and similarly

$$\mu_{2,n} = \int_{\mathbb{T}_n} \ln\left(\frac{S_2(u)}{S_1(u)}\right) S_2(u)\,\mathrm{d}u = \frac{3\pi(b-a)}{2}\int_0^n r\cos(\omega r)e^{b\cos(\omega r)}\,\mathrm{d}r.$$

Therefore by (2.14), for all λ;

$$\Lambda(\lambda) = \lim_{n\to\infty}\frac{\Lambda_n(\lambda)}{|\mu_{2,n}|} = \frac{\int_0^\tau \left[e^{\lambda(b-a)\cos(\omega r)} - 1\right] e^{b\cos(\omega r)}\,\mathrm{d}r}{\left|(b-a)\int_0^\tau \cos(\omega r)\, e^{b\cos(\omega r)}\,\mathrm{d}r\right|},$$

and also

$$\lim_{n\to\infty}\frac{\mu_{2,n}}{n^2} = \frac{3\omega(b-a)}{8}\int_0^\tau \cos(\omega r)\, e^{b\cos(\omega r)}\,\mathrm{d}r.$$

Similarly we get

$$\lim_{n\to\infty}\frac{\mu_{1,n}}{n^2} = \frac{3\omega(b-a)}{8}\int_0^\tau \cos(\omega r)\, e^{a\cos(\omega r)}\,\mathrm{d}r.$$

Note that the function

$$f(x) = \int_0^\tau \cos(\omega r)\, e^{x\cos(\omega r)}\,\mathrm{d}r, \quad -\infty < x < \infty$$

is strictly increasing ($f' > 0$) and $f(0) = 0$. Hence if $ab < 0$, then $\mu_{1,n} \to -\infty$ but $\mu_{2,n} \to \infty$. If $ab > 0$, they have the same limit. For definiteness suppose that $0 < a < b$. Therefore $\mu_{2,n} \to \infty$ and

$$\Lambda'(\lambda) = \frac{\int_0^\tau \cos(\omega r)\, e^{\lambda(b-a)\cos(\omega r)} e^{b\cos(\omega r)}\,\mathrm{d}r}{\int_0^\tau \cos(\omega r)\, e^{b\cos(\omega r)}\,\mathrm{d}r}.$$

This function is strictly increasing, $\Lambda'(0) = 1$ and $\lim_{\lambda\to-\infty} \Lambda'(\lambda) = 0$. Furthermore it can be easily seen that

$$\Lambda^*(\nu) = \sup_{\lambda\in\mathbb{R}}\{\lambda\nu - \Lambda(\lambda)\} = \infty \quad \text{if} \quad \nu < 0,$$

and $\Lambda^*(\nu) < \infty$ for $\nu > 0$. Hence $D^{\circ}_{\Lambda^*} = (0, \infty)$. Furthermore for any $\nu < 1$ for which $\nu \in D^{\circ}_{\Lambda^*}$ (*i.e.*, $0 < \nu < 1$) there exist some η such that $\Lambda'(\eta) = \nu$. Now we compute the parameters $\gamma, \Lambda^*(\gamma)$ and ψ. By (2.14);

$$\gamma = \lim_{n\to\infty}\frac{\mu_{1,n} + z_\alpha\sigma_{1,n}}{\mu_{2,n}} = \frac{\int_0^\tau \cos(\omega r)\, e^{a\cos(\omega r)}\,\mathrm{d}r}{\int_0^\tau \cos(\omega r)\, e^{b\cos(\omega r)}\,\mathrm{d}r}.$$

Therefore $0 < \gamma < 1$ and hence $\gamma \in D^{\circ}_{\Lambda^*}$. The value of $\Lambda^{\star}(\gamma)$ is equal to

$$\begin{aligned}
\Lambda^{\star}(\gamma) &= \sup_{\lambda\in\mathbb{R}} \{\lambda\gamma - \Lambda(\lambda)\} = -\gamma - \Lambda(-1) = \\
&= \frac{\int_0^\tau \left[e^{b\cos(\omega r)} - e^{a\cos(\omega r)} - (b-a)\cos(\omega r)\, e^{a\cos(\omega r)}\right] \mathrm{d}r}{(b-a)\int_0^\tau \cos(\omega r)\, e^{b\cos(\omega r)}\,\mathrm{d}r}.
\end{aligned}$$

Finally for ψ, we have

$$\psi = \lim_{n\to\infty} \frac{\int_{\mathbb{B}_n} S_2(u)\,\mathrm{d}u}{\mu_{2,n}} = \frac{\int_0^\tau e^{b\cos(\omega r)}\,\mathrm{d}r}{3(b-a)\int_0^\tau \cos(\omega r)\, e^{b\cos(\omega r)}\,\mathrm{d}r}.$$

Therefore in the case $0 < a < b$, the power of the test $\hat{\phi}_n$, admits the representation

$$\begin{aligned}
\beta_n\left(\hat{\phi}_n\right) &= 1 - \exp\left\{-\mu_{2,n}\left(\psi + \Lambda^{\star}(\gamma)\right)(1+o(1))\right\} = \\
&= 1 - \exp\left\{-c\,n^2\,\left(\psi + \Lambda^{\star}(\gamma)\right)\,(1+o(1))\right\},
\end{aligned}$$

where

$$c = \frac{3\omega\,(b-a)}{8}\int_0^\tau \cos(\omega r) e^{b\cos(\omega r)}\,\mathrm{d}r.$$

This representation also holds if $b < a < 0$. In the cases $0 < b < a$ and $a < b < 0$, where $\mu_{2,n} \to -\infty$ we have

$$\begin{aligned}
\beta_n\left(\hat{\phi}_n\right) &= 1 - \exp\left\{\mu_{2,n}\Lambda^{\star}(-\gamma)(1+o(1))\right\} = \\
&= 1 - \exp\left\{c\,n^2\,\left(-\psi + \Lambda^{\star}(-\gamma)\right)\,(1+o(1))\right\}.
\end{aligned}$$

Here $\Lambda^{\star}(-\gamma) = \gamma - \Lambda(-1)$.

Chapter 3

Second Order Efficiency

3.1 Introduction

In this chapter we consider a realization $X^{(n)}$ of a Poisson process on the set $\mathbb{A}_n$, a subset of d dimensional Euclidian space $\mathbb{R}^d$ with intensity function $S(\vartheta, x), x \in \mathbb{A}_n$ depending on the unknown real parameter $\vartheta \in \Theta \subseteq \mathbb{R}$. Let $\mathbf{P}_\vartheta^{(n)}$ denotes the distribution of the random element $X^{(n)}$. The problem is to test the hypotheses

$$\mathcal{H}_0 : \ \vartheta = \vartheta_0$$
$$\mathcal{H}_1 : \ \vartheta > \vartheta_0$$

where $\vartheta_0 \in \Theta$ is a given value. Here we have a simple null hypothesis against an one sided composite alternative. Let us fix an $\alpha \in (0,1)$ and define the class $\mathcal{K}_\alpha^{(n)}$ of tests at level $1-\alpha$ (size α), i.e.,

$$\mathcal{K}_\alpha^{(n)} = \left\{ \phi_n : \quad \mathbf{E}_{\vartheta_0} \phi_n \left(X^{(n)} \right) = \alpha \right\},$$

where $\mathbf{E}_\vartheta$ denotes the mathematical expectation with respect to the probability measure $\mathbf{P}_\vartheta^{(n)}$. With fixed n, generally speaking, there is no uniformly most powerful test in $\mathcal{K}_\alpha^{(n)}$ (see [35]). Therefore we turn to the asymptotic approach and introduce the class $\mathcal{K}_\alpha^{\prime}$ of sequence of tests of asymptotic level $1-\alpha$, i.e.,

$$\mathcal{K}_\alpha^{\prime} = \left\{ \phi_n : \quad \lim_{n \to \infty} \mathbf{E}_{\vartheta_0} \phi_n \left(X^{(n)} \right) = \alpha \right\}.$$

It is well known that if $n \to \infty$ for any given value ϑ of the alternative the power of any reasonable (consistent) test tends to 1 (see [35], [5]). Therefore to compare the different tests, we need to know the rates of convergence which is a complicated problem related to the large deviations principle (see [1], [2], [9]). In this work we use the Pitman's approach (see [37], [33]). In this approach, instead of a fixed alternative ϑ we consider a sequence of so-called *local alternatives* (*closed* or *contiguous alternatives*) which converges to ϑ_0 with a certain rate and hence it is difficult to distinguish between the null hypothesis and alternative. More precisely let $\{\varphi_n\}$ be a sequence of nonnegative numbers which converges to zero with such a rate that the likelihood ratio

$$Z_n(u) = \frac{\mathrm{d}\mathbf{P}^{(n)}_{\vartheta_0+\varphi_n u}}{\mathrm{d}\mathbf{P}^{(n)}_{\vartheta_0}}\left(X^{(n)}\right)$$

has a nondegenerate limit ($\neq 0, \infty$) for any u with $\vartheta_0 + \varphi_n u \in \Theta$. By the Neyman-Pearson Lemma the most powerful test for $\mathcal{H}_0 : \ \vartheta = \vartheta_0$ against the local alternative $\mathcal{H}_u : \ \vartheta = \vartheta_0 + \varphi_n u$ with $u > 0$, is given by

$$\tilde{\phi}_n\left(X^{(n)}\right) = \begin{cases} 1, & \text{if } \Lambda_n(u) > b_n(u) \\ 0, & \text{if } \Lambda_n(u) < b_n(u) \end{cases}$$

where $\Lambda_n(u) = \ln Z_n(u)$ and the constant $b_n(u)$ together with the contribution of the randomized part provide the size $\mathbf{E}_{\vartheta_0}\tilde{\phi}_n\left(X^{(n)}\right) = \alpha$. The power of $\tilde{\phi}_n$ as a function of u is called the *envelope power function* . For any fixed n it is the supremum of the power at the local alternative $\vartheta_u = \vartheta_0 + \varphi_n u$ over all the tests at level $1-\alpha$, i.e.

$$\mathbf{E}_{\vartheta_u}\tilde{\phi}_n\left(X^{(n)}\right) = \sup_{\phi_n \in \mathcal{K}^{(n)}_\alpha} \mathbf{E}_{\vartheta_u}\phi_n\left(X^{(n)}\right).$$

Note that $\tilde{\phi}_n$ is not a test for the main hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ because it depends on the parameter u. A sequence of tests $\bar{\phi}_n \in \mathcal{K}^0_\alpha$ is called *first order efficient* if for any $u > 0$

$$\mathbf{E}_{\vartheta_u}\tilde{\phi}_n\left(X^{(n)}\right) - \mathbf{E}_{\vartheta_u}\bar{\phi}_n\left(X^{(n)}\right) = o(1) \tag{3.1}$$

as $n \to \infty$. For n large, hence the power of $\bar{\phi}_n$ approximates the envelope power function up to order $o(1)$. In order to obtain a first order efficient test we suppose that the family of distributions $\left\{\mathbf{P}_\vartheta^{(n)},\, \vartheta\in\Theta\right\}$ is *locally asymptotically normal* (LAN) at the point ϑ_0, i.e., the likelihood ratio admits the representation

$$Z_n(u)=\exp\left\{u\,\Delta_n\left(\vartheta_0\right)-\frac{1}{2}u^2+r_n\left(\vartheta_0,u,X^{(n)}\right)\right\}$$

where the random variable $\Delta_n\left(\vartheta_0\right)=\Delta_n\left(\vartheta_0,X^{(n)}\right)$ under $\mathbf{P}_{\vartheta_0}^{(n)}$ is asymptotically normal

$$\mathcal{L}_{\vartheta_0}\left\{\Delta_n\left(\vartheta_0\right)\right\}\Rightarrow\mathcal{N}\left(0,1\right)$$

and the reminder term r_n converges to zero in $\mathbf{P}_{\vartheta_0}^{(n)}$-probability.For a family $\left\{\mathbf{P}_\vartheta^{(n)},\, \vartheta\in\Theta\right\}$ of distributions related to a Poisson process with intensity functions $\left\{S\left(\vartheta,\cdot\right),\,\vartheta\in\Theta\right\}$ the conditions of LAN are obtained by Yu. A. Kutoyants, [25]. Under certain regularity conditions (see section 3) the family is LAN with

$$\Delta_n\left(\vartheta_0\right)=\varphi_n\int_{\mathbb{A}_n}\frac{\dot S\left(\vartheta_0,x\right)}{S\left(\vartheta_0,x\right)}\,\pi^{(n)}\left(\mathrm{d}x\right),\qquad \varphi_n^{-2}=\int_{\mathbb{A}_n}\frac{\dot S\left(\vartheta_0,x\right)^2}{S\left(\vartheta_0,x\right)}\,\mathrm{d}x$$

where $\pi^{(n)}\left(\mathrm{d}x\right)=X^{(n)}\left(\mathrm{d}x\right)-S\left(\vartheta_0,x\right)\mathrm{d}x$ is the centered Poisson process and $\dot S\left(\vartheta,x\right)$ is the derivative of $S\left(\vartheta,x\right)$ with respect to ϑ. Let us consider nonrandomized test

$$\bar\phi_n\left(X^{(n)}\right)=\begin{cases}1, & \text{if } \Delta_n\left(\vartheta_0\right)>z_\alpha\\ 0, & \text{if } \Delta_n\left(\vartheta_0\right)\le z_\alpha\end{cases}$$

based on the statistics $\Delta_n\left(\vartheta_0\right)$, where z_α is $1-\alpha$ quantile of standard Gaussian law, i.e. $\mathbf{P}\left\{\zeta>z_\alpha\right\}=1-\alpha$ and $\zeta\sim\mathcal{N}(0,1)$. It is well known that the test $\bar\phi_n\in\mathcal{K}_\alpha^{0}$ is *locally asymptotically uniformly most powerful* (LAUMP) test and (3.1) is satisfied uniformly in $u\in[0,K]$ for any $K>0$ (see [41]). The first order efficiency is related just with the asymptotic normality of $\Lambda_n(u)$ and $\Delta_n(\vartheta_0)$ which follow from the LAN representation. Therefore the refinement of the central limit theorem, by taking into account another terms after the Gaussian term, improves the situation. This can be done by the Edgeworth type expansion of the distribution function of the stochastic integral $\Delta_n(\vartheta_0)$ under $\mathcal{H}_0$ which allows us to correct the threshold z_α by a constant c_n (see (3.5)) such that $\mathbf{E}_{\vartheta_0}\phi_n^{\star}\left(X^{(n)}\right)=\alpha+O(\varepsilon_n^2)$ where the sequence $\varepsilon_n\to0$, and construct a *second order efficient* test, that is a test $\phi_n^{\star}$ with

$$\mathbf{E}_{\vartheta_u}\tilde\phi_n\left(X^{(n)}\right)-\mathbf{E}_{\vartheta_u}\phi_n^{\star}\left(X^{(n)}\right)=o(\varepsilon_n).\tag{3.2}$$

The power function of a second order efficient test approximates the envelope power function up to order $o(\varepsilon_n)$ (or typically $O(\varepsilon_n^2)$). More details about the higher order properties and asymptotic expansions can be found in [5], [6], [10], [36], and [13].

The expansion of the distribution function of a stochastic integral just for one term after the Gaussian term is obtained under certain regularity conditions (see Theorem 19). The general case is due to Kutoyants [27].

In this chapter our aim is to construct a second order efficient test (Theorem 26) for the hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ such that (3.2) is satisfied uniformly in $u \in [0, K]$ for any $K > 0$.

Let us remind that if for two parameter values ϑ_1, ϑ_2 the intensity measures

$$\Lambda_{\vartheta_i}^{(n)}(\mathbb{E}) = \int_{\mathbb{E}} S(\vartheta_i, x)\,(\mathrm{d}x), \qquad \mathbb{E} \subseteq \mathbb{A}_n,\ i = 1, 2$$

are equivalent then the corresponding probability measures $\mathbf{P}_{\vartheta_1}^{(n)}$ and $\mathbf{P}_{\vartheta_2}^{(n)}$ are equivalent and the likelihood ratio is given by

$$\begin{aligned} &\frac{\mathrm{d}\mathbf{P}_{\vartheta_2}^{(n)}}{\mathrm{d}\mathbf{P}_{\vartheta_1}^{(n)}}\left(X^{(n)}\right) = \\ &= \exp\left\{\int_{\mathbb{A}_n} \ln\frac{S(\vartheta_2, x)}{S(\vartheta_1, x)}\, X^{(n)}(\mathrm{d}x) - \int_{\mathbb{A}_n} \left[S(\vartheta_2, x) - S(\vartheta_1, x)\right]\,\mathrm{d}x\right\}. \end{aligned}$$

We suppose that the family $\left\{\mathbf{P}_{\vartheta}^{(n)},\, \vartheta \in \Theta\right\}$ of distributions of the random element $X^{(n)}$ is *locally asymptotically normal* at the point ϑ_0. This is introduced in the following section.

3.2 Local asymptotic normality

The parametric family of probability measures $\left\{\mathbf{P}_{\vartheta}^{(n)},\, \vartheta \in \Theta\right\}$ is called *locally asymptotically normal* (LAN) at the point ϑ_0 if there exists a sequence $\varphi_n = \varphi_n(\vartheta_0)$ such that for any $u \in U_n = \{u :\ \vartheta_0 + \varphi_n u \in \Theta\}$ the likelihood ratio admits the representation

$$\frac{\mathrm{d}\mathbf{P}_{\vartheta_0+\varphi_n u}^{(n)}}{\mathrm{d}\mathbf{P}_{\vartheta_0}^{(n)}}\left(X^{(n)}\right) = \exp\left\{u\,\Delta_n(\vartheta_0) - \frac{1}{2}u^2 + r_n\left(\vartheta_0, u, X^{(n)}\right)\right\},$$

where the distribution of the random variable $\Delta_n(\vartheta_0) = \Delta_n\left(\vartheta_0, X^{(n)}\right)$ under $\mathbf{P}_{\vartheta_0}^{(n)}$ is asymptotically normal

$$\mathcal{L}_{\vartheta_0}\left\{\Delta_n(\vartheta_0)\right\} \Rightarrow \mathcal{N}(0,1)$$

and the reminder term

$$\mathbf{P}_{\vartheta_0}^{(n)} - \lim_{n\to\infty} r_n\left(\vartheta_0, u, X^{(n)}\right) = 0,$$

that is the sequence $r_n\left(\vartheta_0, u, X^{(n)}\right)$ converges to zero in $\mathbf{P}_{\vartheta_0}^{(n)}$-probability. Below we present the conditions of LAN at ϑ_0 in terms of intensity function which are obtained by Yu. A. Kutoyants (see [27], p. 46).

Let us denote the intensity measure by $\Lambda_\vartheta^{(n)}(\mathrm{d}x) = S(\vartheta, x)\,\mathrm{d}x$ and the derivative of $S(\vartheta, x)$ with respect to ϑ by $\dot{S}(\vartheta, x)$. We introduce the following regularity conditions:

$\mathcal{A}_1$. The function $S(\vartheta, x)$ is differentiable at the point $\vartheta = \vartheta_0$. The Fisher information at ϑ_0

$$I_n(\vartheta_0) = \int_{\mathbb{A}_n} \frac{\dot{S}(\vartheta_0, x)^2}{S(\vartheta_0, x)}\,\mathrm{d}x > 0$$

and the normalizing factor $\varphi_n = I_n(\vartheta_0)^{-1/2} \to 0$ as $n \to \infty$.

$\mathcal{A}_2$. For any $\delta > 0$ the Lindeberg condition holds

$$\lim_{n\to\infty} \varphi_n \int_{\mathbb{A}_n(\delta)} \frac{\dot{S}(\vartheta_0, x)^2}{S(\vartheta_0, x)}\,\mathrm{d}x = 0$$

where the set $\mathbb{A}_n(\delta) = \left\{x \in \mathbb{A}_n : \left|\varphi_n \frac{\dot{S}(\vartheta_0,x)}{S(\vartheta_0,x)}\right| > \delta\right\}$.

Let

$$\vartheta_u = \vartheta_0 + \varphi_n u, \quad u \in U_n = \left\{u :\ \vartheta_0 + \varphi_n u \in \Theta\right\}.$$

$\mathcal{A}_3$. The intensity measures $\left\{\Lambda_{\vartheta_u}^{(n)},\ u \in U_n\right\}, n = 1, 2, \cdots$ are equivalent and $\Lambda_{\vartheta_u}^{(n)}(\mathbb{A}_n) < \infty$.

$\mathcal{A}_4$. For any real bounded sequence $\{u_n\}$ with $u_n \in U_n$,

$$\lim_{n\to\infty} \int_{\mathbb{A}_n} \left[\ln \frac{S(\vartheta_{u_n}, x)}{S(\vartheta_0, x)} - \frac{u_n \varphi_n \dot{S}(\vartheta_0, x)}{S(\vartheta_0, x)}\right]^2 S(\vartheta_0, x)\,\mathrm{d}x = 0 \tag{3.3}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{A}_n}\left[\frac{S(\vartheta_{u_n},x)}{S(\vartheta_0,x)}-1-\ln\frac{S(\vartheta_{u_n},x)}{S(\vartheta_0,x)}-\right.$$
$$\left.-\frac{1}{2}\left(\frac{u_n\varphi_n\dot S(\vartheta_0,x)}{S(\vartheta_0,x)}\right)^2\right]S(\vartheta_0,x)\,\mathrm{d}x=0. \tag{3.4}$$

We give some sufficient conditions under which $\mathcal{A}_2$ and $\mathcal{A}_4$ are satisfied (see condition $\mathcal{C}_3$ and Corollary 4). Introduce the statistic

$$\Delta_n(\vartheta_0)=\varphi_n\int_{\mathbb{A}_n}\frac{\dot S(\vartheta_0,x)}{S(\vartheta_0,x)}\,\pi^{(n)}(\mathrm{d}x),$$

where $\pi^{(n)}(\mathrm{d}x)=X^{(n)}(\mathrm{d}x)-S(\vartheta_0,x)\,\mathrm{d}x$ is the centered Poisson process.

Theorem 16. *Let the conditions $\mathcal{A}_1-\mathcal{A}_4$ be fulfilled; then the family of probability measures $\left\{\mathbf{P}_\vartheta^{(n)},\ \vartheta\in\Theta\right\}$ is LAN at ϑ_0, i.e., the likelihood ratio admits the representation*

$$\frac{\mathrm{d}\mathbf{P}^{(n)}_{\vartheta_0+\varphi_n u}}{\mathrm{d}\mathbf{P}^{(n)}_{\vartheta_0}}\left(X^{(n)}\right)=\exp\left\{u\,\Delta_n(\vartheta_0)-\frac{1}{2}u^2+r_n\left(\vartheta_0,u,X^{(n)}\right)\right\}$$

where the random variable $\Delta_n(\vartheta_0)$ under $\mathbf{P}^{(n)}_{\vartheta_0}$, is asymptotically normal

$$\mathcal{L}_{\vartheta_0}\left\{\Delta_n(\vartheta_0)\right\}\Rightarrow\mathcal{N}(0,1)$$

and for any bounded sequence $\{u_n\}$ with $u_n\in U_n$

$$\mathbf{P}^{(n)}_{\vartheta_0}-\lim_{n\to\infty}r_n\left(\vartheta_0,u_n,X^{(n)}\right)=0.$$

Proof. See [27], Theorem 2.1, page 47.

3.3 Locally asymptotically uniformly most powerful test

Let $\beta_n(u,\phi_n)$ denote the power of the test ϕ_n at the local alternative $\vartheta_u=\vartheta_0+\varphi_n u$, that is

$$\beta_n(u,\phi_n)=\mathbf{E}_{\vartheta_u}\phi_n(X^{(n)}).$$

To compare the different tests we use the following well known definition:

Definition 17. *We call a test $\phi_n^{*} \in \mathcal{K}_\alpha^{0}$ locally asymptotically uniformly most powerful (LAUMP) in the class $\mathcal{K}_\alpha^{0}$ if for any other test $\phi_n \in \mathcal{K}_\alpha^{0}$ we have :*

$$\lim_{n\to\infty} \inf_{0\leq u\leq K} \left[\beta_n\left(u,\phi_n^{*}\right) - \beta_n\left(u,\phi_n\right)\right] \geq 0,$$

for any $K>0$.

Introduce the test

$$\bar{\phi}_n\left(X^{(n)}\right) = \chi_{\{\Delta_n(\vartheta_0)>z_\alpha\}}$$

where z_α is $1-\alpha$ quantile of standard Gaussian law, i.e., $\mathbf{P}\left\{\zeta > z_\alpha\right\} = 1-\alpha$ and $\zeta \sim \mathcal{N}\left(0,1\right)$.

Theorem 18. *Let the conditions $\mathcal{A}_1 - \mathcal{A}_4$ be satisfied. Then the test $\bar{\phi}_n \in \mathcal{K}_\alpha^{0}$ is locally asymptotically uniformly most powerful in the class $\mathcal{K}_\alpha^{0}$ and its power function*

$$\beta_n\left(u,\bar{\phi}_n\right) = \mathbf{P}\left\{\zeta > z_\alpha - u\right\} + o\left(1\right)$$

for any $u>0$.

Proof. See [41]. Indeed it can be shown that for any $K>0$,

$$\sup_{0\leq u\leq K}\left|\beta_n\left(u,\bar{\phi}_n\right) - \beta_n\left(u,\tilde{\phi}_n\right)\right| = o(1)$$

as $n\to\infty$, where $\tilde{\phi}_n$ is the most powerful test of size α for testing the simple hypotheses $\mathcal{H}_0$ and $\mathcal{H}_u$.

The asymptotic normality of $\Delta_n\left(\vartheta_0\right)$ under null hypothesis, which follows from the LAN representation, implies that

$$\mathbf{E}_{\vartheta_0}\bar{\phi}_n\left(X^{(n)}\right) = \mathbf{P}_{\vartheta_0}^{(n)}\left\{\Delta_n(\vartheta_0) > z_\alpha\right\} = \alpha + o(1).$$

The rate of approximation can be improved by using the Edgeworth type expansion of the distribution function of stochastic integral $\Delta_n\left(\vartheta_0\right)$ under $\mathcal{H}_0$. Below we give the conditions of the expansion for just one term after the Gaussian term for distribution function of stochastic integral

$$F_n(y) = \mathbf{P}_{\vartheta}^{(n)}\left\{\int_{\mathbb{A}_n} f_n\left(x\right)\ \pi^{(n)}(\mathrm{d}x) < y\right\},$$

where $\pi^{(n)}(\mathrm{d}x) = X^{(n)}(\mathrm{d}x) - S(\vartheta, x)\,\mathrm{d}x$ is the centered Poisson process. We suppose that

$$\int_{\mathbb{A}_n} f_n(x)^2\; S(\vartheta, x)\,\mathrm{d}x = 1.$$

The expansion is obtained under the following two conditions:

$\mathcal{B}_1$. *There exists a sequence of real numbers* $\varepsilon_n \to 0$, *as* $n \to \infty$ *and constants* $C_r > 0, r = 3, 4$, *such that*

$$\int_{\mathbb{A}_n} |f_n(x)|^r\; S(\vartheta, x)\,\mathrm{d}x \le C_r\, \varepsilon_n^{r-2}.$$

$\mathcal{B}_2$. *There exist constants* $\gamma \ge 3/2$ *and* $c_0 > 0$ *satisfying the inequality* $\frac{C_3}{3!}c_0 + \frac{C_4}{4!}c_0^2 - \frac{1}{2} < 0$ *such that*

$$\inf_{\frac{c_0\varepsilon_n^{-1}}{2} < t < \frac{\varepsilon_n^{-2}}{2}} \int_{\mathbb{A}_n} \sin^2\left(t f_n(x)\right)\; S(\vartheta, x)\,\mathrm{d}x \ge \gamma \ln \varepsilon_n^{-1}$$

for all large n.

Theorem 19. *Let the conditions* $\mathcal{B}_1$, $\mathcal{B}_2$ *be fulfilled, then*

$$\sup_y \left| F_n(y) - \mathcal{N}(y) - \frac{\int_{\mathbb{A}_n} f_n(x)^3\; S(\vartheta, x)\,\mathrm{d}x}{6\sqrt{2\pi}} (1 - y^2)\, e^{-y^2/2} \right| \le C\, \varepsilon_n^2,$$

for some constant C *and all* n *large. Here* $\mathcal{N}(y)$ *is the distribution function of standard Gaussian law.*

Proof. See [19] . The proof is a slight modification of a general theorem given by Kutoyants, where the expansion is obtained by the powers of ε_n up to order ε_n^k, $k = 1, 2, ...$ (see ([27]), page 131).

Let us suppose that the conditions $\mathcal{B}_1$ and $\mathcal{B}_2$ are satisfied under the null hypothesis $\vartheta = \vartheta_0$ for the stochastic integral $\Delta_n(\vartheta_0)$ with a sequence $\varepsilon_n \to 0$ and

$$f_n(x) = \varphi_n \frac{\dot{S}(\vartheta_0, x)}{S(\vartheta_0, x)}, \quad x \in \mathbb{A}_n.$$

The problem is to find a constant c_n such that

$$\mathbf{P}_{\vartheta_0}^{(n)}\left\{\Delta_n(\vartheta_0) > c_n\right\} = \alpha + O(\varepsilon_n^2).$$

By Theorem 19 we can write

$$1-\alpha = \mathcal{N}(c_n) + \frac{\gamma_{3,n}}{6\sqrt{2\pi}}(1-c_n^2)\, e^{-c_n^2/2} + O(\varepsilon_n^2),$$

where the parameter

$$\gamma_{3,n} = \varphi_n^3 \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)^3}{S\left(\vartheta_0, x\right)^2}\, \mathrm{d}x.$$

Observe that $\gamma_{3,n}$ converges to zero with the rate ε_n by $\mathcal{B}_1$. Substituting $1-\alpha$ for $\mathcal{N}(z_\alpha)$ and using Taylor formula yields

$$c_n = z_\alpha - \frac{\gamma_{3,n}}{6}\left(1 - z_\alpha^2\right).$$

Hence for the test

$$\phi_n^{\star}\left(X^{(n)}\right) = \chi_{\left\{\Delta_n(\vartheta_0) > c_n\right\}}$$

we have

$$\mathbf{E}_{\vartheta_0}\phi_n^{\star}\left(X^{(n)}\right) = \alpha + O(\varepsilon_n^2).$$

Theorem 20. *Let in addition of $\mathcal{A}_1 - \mathcal{A}_4$, the conditions $\mathcal{B}_1 - \mathcal{B}_2$ for the stochastic integral $\Delta_n(\vartheta_0)$ under $\mathcal{H}_0 : \vartheta = \vartheta_0$ be fulfilled. Then the test $\phi_n^{\star}$ with $\mathbf{E}_{\vartheta_0}\phi_n^{\star}\left(X^{(n)}\right) = \alpha + O(\varepsilon_n^2)$ is locally asymptotically uniformly most powerful in the class $\mathcal{K}_\alpha^{0}$ and its power function*

$$\beta_n\left(u, \phi_n^{\star}\right) = \mathbf{P}\left\{\zeta > z_\alpha - u\right\} + o\left(1\right)$$

for any $u > 0$, where $\zeta \sim \mathcal{N}\left(0, 1\right)$.

3.4 Second order efficient test

Let us consider the simple null hypothesis $\mathcal{H}_0 : \vartheta = \vartheta_0$ against the local alternative $\mathcal{H}_u : \vartheta = \vartheta_0 + \varphi_n u$, for $u > 0$. For each n, by the Neyman-Pearson Lemma, the most powerful test in the class $\mathcal{K}_\alpha^{(n)}$ is given by

$$\tilde\phi_n\left(u\right) = \chi_{\left\{\Lambda_n(u) > b_n(u)\right\}} + q_n(u)\ \chi_{\left\{\Lambda_n(u) = b_n(u)\right\}},$$

where $\Lambda_n(u)$ is the logarithm of the likelihood ratio and the numbers $b_n(u)$ and $q_n(u)$ satisfy the equation $\mathbf{E}_{\vartheta_0}\tilde{\phi}_n\left(X^{(n)}\right) = \alpha$. The asymptotic normality of $\Lambda_n(u)$ under $\mathcal{H}_0$, which follows from the LAN property, implies that $b_n(u) \to u\, z_\alpha - \frac{u^2}{2}$. On the other hand by the Le Cam's Third Lemma $\Lambda_n(u)$ is asymptotically normal under the local alternative $\vartheta_u = \vartheta_0 + \varphi_n u$

$$\mathcal{L}_{\vartheta_u}\left\{\Lambda_n(u)\right\} \Rightarrow \mathcal{N}\left(\frac{u^2}{2}, u^2\right).$$

Hence for any $u > 0$

$$\beta_n\left(u, \tilde{\phi}_n\right) = \mathbf{P}\left\{\zeta > z_\alpha - u\right\} + o\left(1\right)$$

where $\zeta \sim \mathcal{N}\left(0, 1\right)$. Therefore the test $\phi_n^{\star}$ is *first order asymptotically efficient*, i.e.,

$$\beta_n\left(u, \phi_n^{\star}\right) - \beta_n(u, \tilde{\phi}_n) = o(1).$$

Indeed the convergence is uniform in $u \in [0, K]$ for any $K > 0$. Under certain regularity conditions $\phi_n^{\star}$ is also *second order efficient*, i.e.

$$\beta_n\left(u, \phi_n^{\star}\right) - \beta_n(u, \tilde{\phi}_n) = o(\varepsilon_n)$$

(or typically $O(\varepsilon_n^2)$) where the sequence $\varepsilon_n \to 0$. Note that we can not apply the most powerful test $\tilde{\phi}_n$ for the main problem $\mathcal{H}_0$ against $\mathcal{H}_1$, because $\Lambda_n(u)$ depends on the parameter u. The power function $\beta_n\left(u, \tilde{\phi}_n\right)$, as a function of u, is called *envelope power function*. It is the maxiuum 0TD[[(n)]TJ F311.95T3(˜)79.4

$\mathcal{C}_1$. The conditions $\mathcal{A}_1 - \mathcal{A}_4$ of LAN at ϑ_0 are satisfied and $S(\vartheta, x)$ is two times differentiable with respect to ϑ in a right neighborhood of ϑ_0.

$\mathcal{C}_2$. The conditions $\mathcal{B}_1$ and $\mathcal{B}_2$ of the expansions of $\Delta_n(\vartheta_0)$ and $\Lambda_n(u)$ under $\mathcal{H}_0$ and $\mathcal{H}_u$ are satisfied with a sequence $\varepsilon_n \to 0$,

$\mathcal{C}_3$. There exists some functions $f(x), g(x), h(x),\ x \in \mathbb{A}_n$ not depending on ϑ such that $S(\vartheta, x) \geq f(x)$, $\left|\dot{S}(\vartheta, x)\right| \leq g(x)$, and $\left|\ddot{S}(\vartheta, x)\right| \leq h(x)$ for all $x \in \mathbb{A}_n$ and all ϑ in a right neighborhood of ϑ_0. We suppose also that

$$\varphi_n^k \int_{\mathbb{A}_n} \frac{|g(x)|^k}{f(x)^{k-1}}\, \mathrm{d}x = O(\varepsilon_n^{k-2}), \quad k = 2, 3, 4$$
$$\varphi_n^4 \int_{\mathbb{A}_n} \frac{h(x)^2}{f(x)}\, \mathrm{d}x = O(\varepsilon_n^2).$$

Example 1. Let $X^{(n)}$ be a realization of a Poisson process on the set $\mathbb{A}_n = [0, n]$ with positive intensity function $S(\vartheta, x) = \vartheta S(x) + \lambda$ (amplitude parameter) or $S(\vartheta, x) = S(\vartheta + x) + \lambda$ (phase parameter), where $S(\cdot)$ is a two times differentiable periodic function and $\lambda > 0$ (dark current) is a known constant. In both cases the condition $\mathcal{C}_3$ is satisfied with $\varphi_n = \varepsilon_n = n^{-1/2}$. For the frequency modulation model $S(\vartheta, x) = S(\vartheta\, x) + \lambda$ we have $\varphi_n = n^{-3/2}$ and $\varepsilon_n = n^{-1/2}$.

The condition $\mathcal{C}_3$ yields the following results:

Corollary 21. *$k = 3$ in $\mathcal{C}_3$ follows from $k = 2, 4$. Because by the Cauchy-Schwartz inequality*

$$\varphi_n^3 \int_{\mathbb{A}_n} \frac{|g(x)|^3}{f(x)^2}\, \mathrm{d}x \leq \varphi_n^3 \sqrt{\int_{\mathbb{A}_n} \frac{|g(x)|^2}{f(x)}\, \mathrm{d}x \int_{\mathbb{A}_n} \frac{|g(x)|^4}{f(x)^3}\, \mathrm{d}x} = O(\varepsilon_n).$$

Corollary 22.

$$\varphi_n^3 \int_{\mathbb{A}_n} \frac{|g(x)h(x)|}{f(x)}\, \mathrm{d}x \leq \varphi_n^3 \sqrt{\int_{\mathbb{A}_n} \frac{|g(x)|^2}{f(x)}\, \mathrm{d}x \int_{\mathbb{A}_n} \frac{|h(x)|^2}{f(x)}\, \mathrm{d}x} = O(\varepsilon_n).$$

Corollary 23.

$$\varphi_n^4 \int_{\mathbb{A}_n} \frac{|g(x)^2 h(x)|}{f(x)^2}\,\mathrm{d}x \le \varphi_n^4 \sqrt{\int_{\mathbb{A}_n} \frac{|g(x)|^4}{f(x)^3}\,\mathrm{d}x \int_{\mathbb{A}_n} \frac{|h(x)|^2}{f(x)}\,\mathrm{d}x} = O(\varepsilon_n^2).$$

Corollary 24. *The conditions $\mathcal{A}_2$ and $\mathcal{A}_4$ of LAN follow from $\mathcal{C}_3$.*

Proof. For any $\delta > 0$

$$\varphi_n^2 \int_{\mathbb{A}_n(\delta)} \frac{\dot S(\vartheta_0,x)^2}{S(\vartheta_0,x)}\,\mathrm{d}x = \varphi_n^3 \int_{\mathbb{A}_n(\delta)} \frac{\dot S(\vartheta_0,x)^3}{S(\vartheta_0,x)^2} \frac{S(\vartheta_0,x)}{\varphi_n \dot S(\vartheta_0,x)}\,\mathrm{d}x \le$$
$$\le \delta^{-1} \varphi_n^3 \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_0,x)^3}{S(\vartheta_0,x)^2}\,\mathrm{d}x = O(\varepsilon_n).$$

This proves the Lindeberge condition $\mathcal{A}_2$. For $\mathcal{A}_4$ by the Taylor formula we write

$$S(\vartheta_{u_n},x) = S(\vartheta_0,x) + \varphi_n u_n \dot S(\vartheta_n,x),$$

where $\vartheta_n = \vartheta_n(u_n,x)$ and $\vartheta_0 < \vartheta_n < \vartheta_{u_n}$. Also we have

$$\ln(1+y) = y - \frac{y^2}{2(1+\xi)^2}, \quad y = \frac{\varphi_n u_n \dot S(\vartheta_n,x)}{S(\vartheta_0,x)}$$

where $\xi = \xi(y)$ is some point between 0 and y. If $y \ge 0$ then $1+\xi \ge 1$ and if $-1 < y < 0$ then $1+\xi \ge 1+y \ge 0$. Hence for any bounded sequence $\{u_n\}$

$$\int_{\mathbb{A}_n} \left[\ln \frac{S(\vartheta_{u_n},x)}{S(\vartheta_0,x)} - \frac{u_n\varphi_n \dot S(\vartheta_0,x)}{S(\vartheta_0,x)}\right]^2 S(\vartheta_0,x)\,\mathrm{d}x =$$
$$= \int_{\mathbb{A}_n} \left[\frac{u_n\varphi_n\left(\dot S(\vartheta_n,x) - \dot S(\vartheta_0,x)\right)}{S(\vartheta_0,x)} - \frac{u_n^2\varphi_n^2 \dot S(\vartheta_n,x)^2}{S(\vartheta_0,x)^2(1+\xi)^2}\right]^2 S(\vartheta_0,x)\,\mathrm{d}x =$$
$$= O(\varepsilon_n^2)$$

by $\mathcal{C}_3$ and the elementary inequality $(a+b)^2 \le 2a^2 + 2b^2$ which proves (3.3). Similarly we can show that (3.4) holds.

Corollary 25. *The condition $\mathcal{B}_1$ of expansion for $\Delta_n(\vartheta_0)$ and $\Lambda_n(u)$ under $\mathcal{H}_0$ and $\mathcal{H}_u$ follows from $\mathcal{C}_3$.*

Proof. For $\Delta_n(\vartheta_0)$ under $\mathcal{H}_0$ and $\mathcal{H}_u$, the inequalities

$$\varphi_n^r \int_{\mathbb{A}_n} \frac{\left|\dot S\left(\vartheta_0, x\right)\right|^r}{S\left(\vartheta_0, x\right)^{r-1}}\, \mathrm{d}x \leq C_r \varepsilon_n^{r-2}, \quad \varphi_n^r \int_{\mathbb{A}_n} \frac{\left|\dot S\left(\vartheta_0, x\right)\right|^r}{S\left(\vartheta_0, x\right)^r} S\left(\vartheta_u, x\right)\, \mathrm{d}x \leq C'_r \varepsilon_n^{r-2}$$

for $r = 3, 4$ follow easily from $\mathcal{C}_3$. For $\Lambda_n(u)$ using Taylor expansion

$$\ln(1+y) = \frac{y}{1+\xi}, \quad y = \frac{\varphi_n u \dot S(\vartheta_n, x)}{S(\vartheta_0, x)}$$

and $\mathcal{C}_3$ give us

$$\int_{\mathbb{A}_n} \left| \ln \frac{S\left(\vartheta_u, x\right)}{S\left(\vartheta_0, x\right)}\right|^r S\left(\vartheta_0, x\right) \leq D_r \varepsilon_n^{r-2}, \quad \int_{\mathbb{A}_n} \left| \ln \frac{S\left(\vartheta_u, x\right)}{S\left(\vartheta_0, x\right)}\right|^r S\left(\vartheta_u, x\right) \leq D'_r \varepsilon_n^{r-2}$$

for $r = 3, 4$.

Let us write again the test $\phi_n^*\left(X^{(n)}\right) = \chi_{\{\Delta_n(\vartheta_0) > c_n\}}$ based on the score statistic

$$\Delta_n\left(\vartheta_0\right) = \varphi_n \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)}{S\left(\vartheta_0, x\right)}\, \pi^{(n)}\left(\mathrm{d}x\right)$$

where

$$c_n = z_\alpha - \frac{\gamma_{3,n}}{6}\left(1 - z_\alpha^2\right), \quad \gamma_{3,n} = \varphi_n^3 \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)^3}{S\left(\vartheta_0, x\right)^2}\, \mathrm{d}x \tag{3.5}$$

and the most powerful test at level $1-\alpha$ for $\mathcal{H}_0$ against $\mathcal{H}_u$

$$\tilde\phi_n\left(u\right) = \chi_{\{\Lambda_n(u) > b_n(u)\}} + q_n(u)\, \chi_{\{\Lambda_n(u) = b_n(u)\}},$$

where the numbers $b_n(u)$ and $q_n(u)$ provide the given size $\mathbf{E}_{\vartheta_0} \tilde\phi_n\left(X^{(n)}\right) = \alpha$. Remind that $\mathbf{E}_{\vartheta_0}\phi_n^*\left(X^{(n)}\right) = \alpha + O(\varepsilon_n^2)$. We have the following theorem:

Theorem 26. *Let the conditions $\mathcal{C}_1 - \mathcal{C}_3$ be satisfied. Then for the hypothesis $\mathcal{H}_0 : \vartheta = \vartheta_0$ against $\mathcal{H}_1 : \vartheta > \vartheta_0$ the test $\phi_n^* \in \mathcal{K}'_\alpha$ is second order efficient, i.e., it satisfies*

$$\sup_{0 \leq u \leq K} \left|\beta_n\left(u, \phi_n^*\right) - \beta_n(u, \tilde\phi_n)\right| = O(\varepsilon_n^2),$$

for any $K > 0$.

Proof. First note that for $u = 0$

$$\beta_n\left(0, \phi_n^*\right) - \beta_n(0, \tilde\phi_n) = \mathbf{E}_{\vartheta_0}\phi_n^*(X^{(n)}) - \mathbf{E}_{\vartheta_0}\tilde\phi_n(X^{(n)}) = O(\varepsilon_n^2).$$

Therefore we exclude $u = 0$ and show that only

$$\sup_{0<u\leq K} \left|\beta_n\left(u, \phi_n^\star\right) - \beta_n(u, \tilde{\phi}_n)\right| = O(\varepsilon_n^2).$$

The expansion of the distribution function of $\Lambda_n(u)$ under $\mathcal{H}_0$ gives

$$\mathbf{P}_{\vartheta_0}^{(n)}\left\{\Lambda_n(u) \leq b_n(u)\right\} = \mathcal{N}\left(\frac{b_n(u) - \mu_n}{\sigma_n}\right) + \frac{\gamma_{3,n}^{0}}{6} h(z_\alpha) + O(\varepsilon_n^2),$$

where z_α is the $1-\alpha$ quantile of the Gaussian law and

$$\begin{aligned}
h(y) &= \frac{(1-y^2)}{\sqrt{2\,\pi}}\, e^{-y^2/2},\\
\mu_n &= \mathbf{E}_{\vartheta_0}\Lambda_n(u) = \int_{\mathbb{A}_n}\left[\ln\frac{S\left(\vartheta_u, x\right)}{S\left(\vartheta_0, x\right)} - \frac{S\left(\vartheta_u, x\right)}{S\left(\vartheta_0, x\right)} + 1\right] S\left(\vartheta_0, x\right) \mathrm{d}x,\\
\sigma_n^2 &= \mathbf{E}_{\vartheta_0}\left(\Lambda_n(u) - \mu_n\right)^2 = \int_{\mathbb{A}_n}\left(\ln\frac{S\left(\vartheta_u, x\right)}{S\left(\vartheta_0, x\right)}\right)^2 S\left(\vartheta_0, x\right) \mathrm{d}x,\\
\gamma_{3,n}^{0} &= \frac{1}{\sigma_n^3}\int_{\mathbb{A}_n}\left(\ln\frac{S\left(\vartheta_u, x\right)}{S\left(\vartheta_0, x\right)}\right)^3 S\left(\vartheta_0, x\right) \mathrm{d}x.
\end{aligned}$$

Note that by the LAN property we have $\mu_n \to -\frac{u^2}{2}$ and $\sigma_n^2 \to u^2$. The expansion of $\Lambda_n(u)$ under $\mathcal{H}_0$ allows us to write (as we saw for c_n) the explicit form of

$$b_n(u) = \mu_n + \sigma_n\left[z_\alpha - \frac{\gamma_{3,n}^{0}}{6}(1 - z_\alpha^2)\right]$$

up to order $O(\varepsilon_n^2)$. The powers of $\phi_n^\star$ and $\tilde{\phi}_n$ (at ϑ_u) are given by

$$\begin{aligned}
\beta_n\left(u, \phi_n^\star\right) &= \mathbf{P}_{\vartheta_u}^{(n)}\left\{\Delta_n(\vartheta_0) > c_n\right\}\\
\beta_n\left(u, \tilde{\phi}_n\right) &= \mathbf{P}_{\vartheta_u}^{(n)}\left\{\Lambda_n(u) > b_n(u)\right\} + q_n(u)\mathbf{P}_{\vartheta_u}^{(n)}\left\{\Lambda_n(u) = b_n(u)\right\}.
\end{aligned}$$

In order to treat the difference of the powers we introduce the following notations

$$
\begin{aligned}
m_n(u) &= \mathbf{E}_{\vartheta_u}\Delta_n\left(\vartheta_0\right) = \varphi_n \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0,x\right)}{S\left(\vartheta_0,x\right)}\left[S\left(\vartheta_u,x\right)-S\left(\vartheta_0,x\right)\right]\,\mathrm{d}x,\\
\eta_n^2 &= \mathbf{E}_{\vartheta_u}\left(\Delta_n(\vartheta_0)-m_n(u)\right)^2 = \varphi_n^2 \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0,x\right)^2}{S\left(\vartheta_0,x\right)^2} S\left(\vartheta_u,x\right)\,\mathrm{d}x,\\
\mu_n(u) &= \mathbf{E}_{\vartheta_u}\Lambda_n(u) = \int_{\mathbb{A}_n}\left[\ln\frac{S\left(\vartheta_u,x\right)}{S\left(\vartheta_0,x\right)}S\left(\vartheta_u,x\right)-S\left(\vartheta_u,x\right)+S\left(\vartheta_0,x\right)\right]\mathrm{d}x,\\
\sigma_n^2(u) &= \mathbf{E}_{\vartheta_u}\left(\Lambda_n(u)-\mu_n(u)\right)^2 = \int_{\mathbb{A}_n}\left(\ln\frac{S\left(\vartheta_u,x\right)}{S\left(\vartheta_0,x\right)}\right)^2 S\left(\vartheta_u,x\right)\mathrm{d}x,\\
\gamma_{3,n}(u) &= \frac{\varphi_n^3}{\eta_n^3}\int_{\mathbb{A}_n}\frac{\dot S\left(\vartheta_0,x\right)^3}{S\left(\vartheta_0,x\right)^3}S\left(\vartheta_u,x\right)\,\mathrm{d}x,\\
\gamma^{0}_{3,n}(u) &= \frac{1}{\sigma_n(u)^3}\int_{\mathbb{A}_n}\left(\ln\frac{S\left(\vartheta_u,x\right)}{S\left(\vartheta_0,x\right)}\right)^3 S\left(\vartheta_u,x\right)\mathrm{d}x.
\end{aligned}
$$

The asymptotic normality of $\Lambda_n(u)$ and $\Delta_n\left(\vartheta_0\right)$ under $\mathcal{H}_u$ (which follow from the Le Cam's Third Lemma) imply that $m_n(u)\to u, \eta_n^2\to 1, \mu_n(u)\to \frac{u^2}{2}$ and $\sigma_n^2(u)\to u^2$.

Let us first consider the contribution of randomized part of $\tilde\phi_n$. The expansion of the distribution function of $\Lambda_n(u)$ under $\mathcal{H}_u$ gives

$$
\begin{aligned}
&q_n(u)\,\mathbf{P}^{(n)}_{\vartheta_u}\left\{\Lambda_n(u)=b_n(u)\right\}\leq\\
&\leq \mathbf{P}^{(n)}_{\vartheta_u}\left\{\Lambda_n(u)<b_n(u)+\varepsilon_n^2\,\sigma_n(u)\right\}-\mathbf{P}^{(n)}_{\vartheta_u}\left\{\Lambda_n(u)<b_n(u)-\varepsilon_n^2\,\sigma_n(u)\right\}=\\
&=\mathcal{N}\left(y_2\right)-\mathcal{N}\left(y_1\right)+\frac{\gamma^{0}_{3,n}(u)}{6}\left[h\left(y_2\right)-h\left(y_1\right)\right]+O(\varepsilon_n^2)
\end{aligned}
$$

where

$$
y_1=\frac{b_n(u)-\mu_n(u)-\varepsilon_n^2\,\sigma_n(u)}{\sigma_n(u)},\quad y_2=\frac{b_n(u)-\mu_n(u)+\varepsilon_n^2\,\sigma_n(u)}{\sigma_n(u)}.
$$

For $\gamma^{0}_{3,n}(u)$ if we use the Taylor expansion

$$
\ln(1+y)=\frac{y}{1+\xi},\quad y=\frac{\varphi_n u\dot S(\vartheta_n,x)}{S(\vartheta_0,x)}
$$

we obtain

$$
\gamma^{0}_{3,n}(u)=\frac{\varphi_n^3u^3}{\sigma_n(u)^3}\int_{\mathbb{A}_n}\frac{\dot S(\vartheta_n,x)^3S(\vartheta_u,x)}{S(\vartheta_0,x)^3(1+\xi)^3}\,\mathrm{d}x.
$$

As we will show later (see (3.15))

$$
\sup_{0<u\cdot\ K}\left(\frac{\sigma_n(u)}{u}-1\right)=O(\varepsilon_n).
$$

Therefore from $\mathcal{C}_3$

$$\sup_{0<u\le K}\left|\mathring{\gamma}_{3,n}(u)\right| \le \varphi_n^3 \int_{\mathbb{A}_n} \frac{\left|g(x)\right|^3}{f(x)^2}\,\mathrm{d}x = O(\varepsilon_n).$$

This relation together with the fact that $y_2 - y_1 = 2\,\varepsilon_n^2$ imply that

$$\sup_{0<u\le K} q_n(u)\,\mathbf{P}_{\vartheta_u}^{(n)}\left\{\Lambda_n(u) = b_n(u)\right\} = O(\varepsilon_n^2).$$

Therefore we exclude the randomized term. Now the Edgeworth expansions of the distribution functions of $\Lambda_n(u)$ and $\Delta_n(\vartheta_0)$ under $\mathcal{H}_u$ can be written as

$$\beta_n\left(u,\phi_n^{*}\right) = \mathcal{N}\left(\frac{m_n(u)-c_n}{\eta_n}\right) - \frac{\gamma_{3,n}(u)}{6} h\left(\frac{m_n(u)-c_n}{\eta_n}\right) + O(\varepsilon_n^2)$$

$$\beta_n\left(u,\tilde{\phi}_n\right) = \mathcal{N}\left(\frac{\mu_n(u)-b_n(u)}{\sigma_n(u)}\right) - \frac{\mathring{\gamma}_{3,n}(u)}{6} h\left(\frac{\mu_n(u)-b_n(u)}{\sigma_n(u)}\right) + O(\varepsilon_n^2),$$

where the terms of order $O(\varepsilon_n^2)$ don't depend on u (see Theorem 19). As we show later

$$\frac{m_n(u)-c_n}{\eta_n} = u - z_\alpha + O(\varepsilon_n),$$

$$\frac{\mu_n(u)-b_n(u)}{\sigma_n(u)} = u - z_\alpha + O(\varepsilon_n).$$

Hence the power functions admit the representations

$$\beta_n\left(u,\phi_n^{*}\right) = \mathcal{N}\left(\frac{m_n(u)-c_n}{\eta_n}\right) - \frac{\gamma_{3,n}(u)}{6} h(u-z_\alpha) + O(\varepsilon_n^2) \tag{3.6}$$

$$\beta_n\left(u,\tilde{\phi}_n\right) = \mathcal{N}\left(\frac{\mu_n(u)-b_n(u)}{\sigma_n(u)}\right) - \frac{\mathring{\gamma}_{3,n}(u)}{6} h(u-z_\alpha) + O(\varepsilon_n^2),$$

Therefore in order to prove the theorem it is sufficient to show that

$$\sup_{0<u\le K}\left|\gamma_{3,n}(u) - \mathring{\gamma}_{3,n}(u)\right| = O(\varepsilon_n^2),$$

$$\sup_{0<u\le K}\left|\frac{m_n(u)-c_n}{\eta_n} - \frac{\mu_n(u)-b_n(u)}{\sigma_n(u)}\right| = O(\varepsilon_n^2).$$

For the first difference we have

$$\begin{aligned}
&\gamma_{3,n}(u) - \mathring{\gamma}_{3,n}(u) = \qquad\qquad (3.7)\\
&= \frac{1}{\eta_n^3\,u^3}\int_{\mathbb{A}_n}\left[\left(\frac{\varphi_n\,u\,\dot{S}\left(\vartheta_0,x\right)}{S\left(\vartheta_0,x\right)}\right)^3 - \left(\ln\frac{S\left(\vartheta_u,x\right)}{S\left(\vartheta_0,x\right)}\right)^3\right] S\left(\vartheta_u,x\right)\,\mathrm{d}x +\\
&+ \frac{\sigma_n^3(u) - \eta_n^3\,u^3}{\eta_n^3\,u^3\,\sigma_n^3(u)}\int_{\mathbb{A}_n}\left(\ln\frac{S\left(\vartheta_u,x\right)}{S\left(\vartheta_0,x\right)}\right)^3 S\left(\vartheta_u,x\right)\mathrm{d}x.
\end{aligned}$$

then the condition $\mathcal{C}_3$ implies that $|B_n| = O(\varepsilon_n^2)$ uniformly in u. Therefore the first integral (and hence the first term) in the right hand side of (3.7) is $O(\varepsilon_n^2)$ uniformly in u. By (3.8) the second integral can be written as

$$\int_{\mathbb{A}_n} \left(\ln \frac{S(\vartheta_u, x)}{S(\vartheta_0, x)}\right)^3 S(\vartheta_u, x)\, \mathrm{d}x =$$
$$= \varphi_n^3 u^3 \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n, x)^3 S(\vartheta_u, x)}{S(\vartheta_0, x)^3}\, \mathrm{d}x + B_n = O(\varepsilon_n)$$

by $\mathcal{C}_3$. As we will show later (see (3.15)) uniformly in u, $\sigma_n(u) - u = O(\varepsilon_n)$ and $\eta_n - 1 = O(\varepsilon_n)$. Hence $\sigma_n^3(u) - \eta_n^3 u^3 = O(\varepsilon_n)$, uniformly in u. Therefore the second term is also $O(\varepsilon_n^2)$ and consequently

$$\sup_{0<u\leq K} \left|\gamma_{3,n}(u) - \gamma_{3,n}^0(u)\right| = O(\varepsilon_n^2).$$

Now we consider the difference

$$\frac{m_n(u) - c_n}{\eta_n} - \frac{\mu_n(u) - b_n(u)}{\sigma_n(u)} =$$
$$= \frac{\sigma_n(u)\,(m_n(u) - c_n) - \eta_n\,(\mu_n(u) - b_n(u))}{u} \frac{u}{\eta_n \sigma_n(u)}.$$

The relations (3.15) yield

$$\sup_{0<u\leq K} \left|\frac{u}{\eta_n \sigma_n(u)} - 1\right| = O(\varepsilon_n^2).$$

Hence it remains to show that

$$\sup_{0<u\leq K} \frac{\sigma_n(u)\,(m_n(u) - c_n) - \eta_n\,(\mu_n(u) - b_n(u))}{u} = O(\varepsilon_n^2).$$

We obtain the expansions for $\sigma_n(u), \sigma_n$, η_n up to order $O(\varepsilon_n^2)$. The Taylor formula

$$\ln(1 + y) = y - \frac{y^2}{2(1+\xi)^2} \tag{3.9}$$

and the condition $\mathcal{C}_3$ give

$$\sigma_n^2(u) = \varphi_n^2 u^2 \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n, x)^2 S(\vartheta_u, x)}{S(\vartheta_0, x)^2}\, \mathrm{d}x - \varphi_n^3 u^3 \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n, x)^3 S(\vartheta_u, x)}{S(\vartheta_0, x)^3 (1+\xi)^2}\, \mathrm{d}x +$$
$$+ \frac{\varphi_n^4 u^4}{4} \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n, x)^4 S(\vartheta_u, x)}{S(\vartheta_0, x)^4 (1+\xi)^4}\, \mathrm{d}x \equiv I_n - J_n + K_n$$

with obvious notations. The term I_n can be written as

$$\begin{aligned}I_n &= \varphi_n^2\, u^2 \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n,x)^2}{S\left(\vartheta_0,x\right)}\,\mathrm{d}x + \varphi_n^3\, u^3 \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n,x)^3}{S\left(\vartheta_0,x\right)^2}\,\mathrm{d}x = \\ &= u^2 + \varphi_n^2\, u^2 \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n,x)^2 - \dot S(\vartheta_0,x)^2}{S\left(\vartheta_0,x\right)}\,\mathrm{d}x + \varphi_n^3\, u^3 \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n,x)^3}{S\left(\vartheta_0,x\right)^2}\,\mathrm{d}x,\end{aligned}$$

where we used the equality

$$\varphi_n^{-2} = \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0,x\right)^2}{S\left(\vartheta_0,x\right)}\,\mathrm{d}x.$$

For J_n we have

$$\begin{aligned}J_n &= \varphi_n^3\, u^3 \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n,x)^3 S(\vartheta_u,x)}{S\left(\vartheta_0,x\right)^3}\,\mathrm{d}x + \\ &+ \varphi_n^3\, u^3 \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n,x)^3 S(\vartheta_u,x)}{S\left(\vartheta_0,x\right)^3}\left[\frac{1}{(1+\xi)^2} - 1\right]\,\mathrm{d}x = \\ &= \varphi_n^3\, u^3 \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n,x)^3}{S\left(\vartheta_0,x\right)^2}\,\mathrm{d}x + O(\varepsilon_n^2)\end{aligned}$$

uniformly in u, where we used $\mathcal{C}_3$. Similarly it can be shown that $K_n = O(\varepsilon_n^2)$ uniformly in u. Therefore

$$\sigma_n^2(u) = u^2 + \varphi_n^2\, u^2 \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_n,x\right)^2 - \dot S\left(\vartheta_0,x\right)^2}{S\left(\vartheta_0,x\right)}\,\mathrm{d}x + O(\varepsilon_n^2)$$

and by a similar way we obtain

$$\begin{aligned}\sigma_n^2 &= u^2 + \varphi_n^2\, u^2 \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_n,x\right)^2 - \dot S\left(\vartheta_0,x\right)^2}{S\left(\vartheta_0,x\right)}\,\mathrm{d}x - \\ &- \varphi_n^3\, u^3 \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0,x\right)^3}{S\left(\vartheta_0,x\right)^2}\,\mathrm{d}x + O(\varepsilon_n^2)\end{aligned}$$

uniformly in u. Also note that in the expansions of $\sigma_n^2(u)$ and σ_n^2 given above, the terms of order $O(\varepsilon_n^2)$ satisfy

$$\sup_{0<u\leq K} \frac{O(\varepsilon_n^2)}{u} = O(\varepsilon_n^2). \tag{3.10}$$

Now we consider $\sigma_n(u) = u\sqrt{1+t}$ where

$$t = \varphi_n^2 \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_n,x\right)^2 - \dot S\left(\vartheta_0,x\right)^2}{S\left(\vartheta_0,x\right)}\,\mathrm{d}x + O(\varepsilon_n^2).$$

By the Taylor formula

$$\sqrt{1+t} = 1 + \frac{t}{2} - \frac{t^2}{8(1+\psi)^{3/2}} \tag{3.11}$$

where $\psi = \psi(t)$ is some point between 0 and t. If $t > 0$ then $1 + \psi \geq 1$ and if $-1 < t < 0$ then $1 + \psi \geq 1 + t \geq 0$. Since $t = O(\varepsilon_n)$, by Corollary 2, then there exists some constant $C > 0$ such that $1 + \psi > C$ for all t. Therefore the third term in the right hand side of (3.11) is of order $O(\varepsilon_n^2)$ uniformly in u. Now $\mathcal{C}_3$ yields

$$\sigma_n(u) = u + \frac{\varphi_n^2\, u}{2} \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_n, x\right)^2 - \dot S\left(\vartheta_0, x\right)^2}{S\left(\vartheta_0, x\right)}\ \mathrm{d}x + O(\varepsilon_n^2), \tag{3.12}$$

uniformly in u. Similarly for $\sigma_n = u\sqrt{1+t}$ by (3.11) with

$$t = \varphi_n^2 \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_n, x\right)^2 - \dot S\left(\vartheta_0, x\right)^2}{S\left(\vartheta_0, x\right)}\ \mathrm{d}x + \varphi_n^3\, u \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)^3}{S\left(\vartheta_0, x\right)^2}\ \mathrm{d}x + O(\varepsilon_n^2)$$

we obtain uniformly in u

$$\sigma_n = u + \frac{\varphi_n^2\, u}{2} \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_n, x\right)^2 - \dot S\left(\vartheta_0, x\right)^2}{S\left(\vartheta_0, x\right)}\ \mathrm{d}x - \frac{\varphi_n^3\, u^2}{2} \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)^3}{S\left(\vartheta_0, x\right)^2}\ \mathrm{d}x + O(\varepsilon_n^2). \tag{3.13}$$

For the parameter η_n we have

$$\begin{aligned}
\eta_n^2 &= \varphi_n^2 \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)^2}{S\left(\vartheta_0, x\right)^2} S\left(\vartheta_u, x\right)\ \mathrm{d}x = \\
&= 1 + \varphi_n^2 \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)^2}{S\left(\vartheta_0, x\right)^2} \left(S\left(\vartheta_u, x\right) - S\left(\vartheta_0, x\right)\right)\ \mathrm{d}x = \\
&= 1 + \varphi_n^3 u \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)^2 S\left(\vartheta_n, x\right)}{S\left(\vartheta_0, x\right)^2}\ \mathrm{d}x = \\
&= 1 + \varphi_n^3 u \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)^3}{S\left(\vartheta_0, x\right)^2}\ \mathrm{d}x + \\
&+ \varphi_n^3 u \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)^2 \left(\dot S\left(\vartheta_n, x\right) - \dot S\left(\vartheta_0, x\right)\right)}{S\left(\vartheta_0, x\right)^2}\ \mathrm{d}x = \\
&= 1 + \varphi_n^3 u \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)^3}{S\left(\vartheta_0, x\right)^2}\ \mathrm{d}x + O(\varepsilon_n^2),
\end{aligned}$$

uniformly in u by $\mathcal{C}_3$. The term of order $O(\varepsilon_n^2)$ satisfies (3.10). Note that by Corollary

3

$$\varphi_n^3 u \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_0,x)^2 \left|\dot S(\vartheta_n,x)-\dot S(\vartheta_0,x)\right|}{S(\vartheta_0,x)^2}\,\mathrm{d}x \leq$$
$$\leq \varphi_n^4 u^2 \int_{\mathbb{A}_n} \frac{|g(x)^2 h(x)|}{f(x)^2}\,\mathrm{d}x = O(\varepsilon_n^2).$$

Therefore (3.11) implies that

$$\eta_n = 1 + \frac{\varphi_n^3 u}{2}\int_{\mathbb{A}_n} \frac{\dot S(\vartheta_0,x)^3}{S(\vartheta_0,x)^2}\,\mathrm{d}x + O(\varepsilon_n^2). \tag{3.14}$$

Observe that by (3.12)-(3.14) and $\mathcal{C}_3$ we conclude that uniformly in u

$$\frac{\sigma_n(u)}{u} - 1 = O(\varepsilon_n),\ \frac{\sigma_n}{u} - 1 = O(\varepsilon_n),\ \eta_n - 1 = O(\varepsilon_n). \tag{3.15}$$

For the parameter $\gamma_{3,n}^{0}$ from (3.8) we get

$$\gamma_{3,n}^{0} = \frac{1}{\sigma_n^3}\int_{\mathbb{A}_n}\left(\ln\frac{S(\vartheta_u,x)}{S(\vartheta_0,x)}\right)^3 S(\vartheta_0,x)\,\mathrm{d}x = \frac{\varphi_n^3 u^3}{\sigma_n^3}\int_{\mathbb{A}_n}\frac{\dot S(\vartheta_0,x)^3}{S(\vartheta_0,x)^2}\,\mathrm{d}x+$$
$$+\frac{\varphi_n^3 u^3}{\sigma_n^3}\int_{\mathbb{A}_n}\frac{\dot S(\vartheta_n,x)^3-\dot S(\vartheta_0,x)^3}{S(\vartheta_0,x)^2}\,\mathrm{d}x+$$
$$+\frac{\varphi_n^3 u^3}{\sigma_n^3}\int_{\mathbb{A}_n}\frac{\dot S(\vartheta_n,x)^3}{S(\vartheta_0,x)^2}\left[\frac{1}{(1+\xi)^3}-1\right]\mathrm{d}x \equiv C_n + D_n + E_n$$

with obvious notations. By Corollary 3 we have $D_n = O(\varepsilon_n^2)$ and $E_n = O(\varepsilon_n^2)$ by $\mathcal{C}_3$, both uniformly in u. For C_n we write

$$C_n = \varphi_n^3\int_{\mathbb{A}_n}\frac{\dot S(\vartheta_0,x)^3}{S(\vartheta_0,x)^2}\,\mathrm{d}x + \frac{\varphi_n^3(u^3-\sigma_n^3)}{\sigma_n^3}\int_{\mathbb{A}_n}\frac{\dot S(\vartheta_0,x)^3}{S(\vartheta_0,x)^2}\,\mathrm{d}x =$$
$$= \varphi_n^3\int_{\mathbb{A}_n}\frac{\dot S(\vartheta_0,x)^3}{S(\vartheta_0,x)^2}\,\mathrm{d}x + O(\varepsilon_n^2) = \gamma_{3,n} + O(\varepsilon_n^2)$$

uniformly in u, by (3.15) and $\mathcal{C}_3$. Therefore

$$\sup_{0<u\cdot K}\left(\gamma_{3,n}^{0} - \gamma_{3,n}\right) = O(\varepsilon_n^2).$$

Now we consider the difference

$$\mu_n(u) - \mu_n = \int_{\mathbb{A}_n}\ln\frac{S(\vartheta_u,x)}{S(\vartheta_0,x)}\left[S(\vartheta_u,x)-S(\vartheta_0,x)\right]\mathrm{d}x =$$
$$= \varphi_n u\int_{\mathbb{A}_n}\dot S(\vartheta_n,x)\ln\frac{S(\vartheta_u,x)}{S(\vartheta_0,x)}\,\mathrm{d}x$$

which from (3.9) and $\mathcal{C}_3$

$$\mu_n(u) - \mu_n = u^2 + \varphi_n^2 u^2 \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n,x)^2 - \dot S(\vartheta_0,x)^2}{S(\vartheta_0,x)}\,\mathrm{d}x - \\ - \frac{\varphi_n^3 u^3}{2}\int_{\mathbb{A}_n} \frac{\dot S(\vartheta_0,x)^3}{S(\vartheta_0,x)^2}\,\mathrm{d}x + O(\varepsilon_n^2)$$

where $O(\varepsilon_n^2)$ satisfies (3.10). This relation together with $\gamma_{3,n}^0 = \gamma_{3,n} + O(\varepsilon_n^2)$ and (3.13) yield

$$\mu_n(u) - b_n(u) = \mu_n(u) - \mu_n - \sigma_n\left[z_\varepsilon - \frac{\gamma_{3,n}^0}{6}(1 - z_\varepsilon^2)\right] = \\ = u^2 - u\, z_\alpha + \frac{u\gamma_{3,n}}{6}(1 - z_\alpha^2) + \\ + \varphi_n^2\, u^2 \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n,x)^2 - \dot S(\vartheta_0,x)^2}{S(\vartheta_0,x)}\,\mathrm{d}x - \frac{\varphi_n^3\, u^3}{2}\int_{\mathbb{A}_n} \frac{\dot S(\vartheta_0,x)^3}{S(\vartheta_0,x)^2}\,\mathrm{d}x - \\ - \frac{\varphi_n^2\, u\, z_\alpha}{2}\int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n,x)^2 - \dot S(\vartheta_0,x)^2}{S(\vartheta_0,x)}\,\mathrm{d}x + O(\varepsilon_n^2)$$

and the terms of order $O(\varepsilon_n^2)$ satisfy (3.10). It is easy to see that

$$m_n(u) - c_n = u + \varphi_n^2\, u \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_0,x)\left(\dot S(\vartheta_n,x) - \dot S(\vartheta_0,x)\right)}{S(\vartheta_0,x)}\,\mathrm{d}x - c_n = \\ = u - z_\alpha + \frac{\gamma_{3,n}}{6}(1 - z_\alpha^2) + \varphi_n^2\, u \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_0,x)\left(\dot S(\vartheta_n,x) - \dot S(\vartheta_0,x)\right)}{S(\vartheta_0,x)}\,\mathrm{d}x$$

uniformly in u. Finally by (3.12) and (3.14) we get

$$\eta_n\left(\mu_n(u) - b_n(u)\right) - \sigma_n(u)\left(m_n(u) - c_n\right) = \\ = \varphi_n^2\, u^2 \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n,x)^2 - \dot S(\vartheta_0,x)^2}{S(\vartheta_0,x)}\,\mathrm{d}x - \\ - \varphi_n^2\, u^2 \int_{\mathbb{A}_n} \frac{\dot S(\vartheta_0,x)\left(\dot S(\vartheta_n,x) - \dot S(\vartheta_0,x)\right)}{S(\vartheta_0,x)}\,\mathrm{d}x - \\ - \frac{\varphi_n^2\, u^2}{2}\int_{\mathbb{A}_n} \frac{\dot S(\vartheta_n,x)^2 - \dot S(\vartheta_0,x)^2}{S(\vartheta_0,x)}\,\mathrm{d}x + O(\varepsilon_n^2) = \\ = \frac{\varphi_n^2\, u^2}{2}\int_{\mathbb{A}_n} \frac{\left(\dot S(\vartheta_n,x) - \dot S(\vartheta_0,x)\right)^2}{S(\vartheta_0,x)}\,\mathrm{d}x + O(\varepsilon_n^2).$$

The terms of order $O(\varepsilon_n^2)$ satisfy (3.10). Note that the term including integral in the last line is dominated by

$$C\varphi_n^4 u^4 \int_{\mathbb{A}_n} \frac{h(x)^2}{f(x)}\,\mathrm{d}x = O(\varepsilon_n^2).$$

Hence we have

$$\sup_{0<u\leq K} \frac{\eta_n\left(\mu_n(u)-b_n(u)\right)-\sigma_n(u)\left(m_n(u)-c_n\right)}{u} = O(\varepsilon_n^2),$$

which completes the proof of the theorem.

Example 2. (Amplitude Modulation.) Suppose that we observe a realization $X^{(n)}$ of a Poisson process on the set $\mathbb{A}_n=[0,n]$, $n=1,2,\cdots$ with the intensity function

$$S(\vartheta,x)=\vartheta\, S(x)+\lambda, \qquad \vartheta>0$$

where λ is a known positive constant (dark-current) and $S(x)$ is a known, nonconstant, differentiable and periodic function with period $\tau>0$. We have two hypotheses $\mathcal{H}_0:\vartheta=\vartheta_0$ against $\mathcal{H}_1:\vartheta>\vartheta_0$, where $\vartheta_0>0$. The intensity function $S(\vartheta,x)$ is supposed to be positive in a right neighborhood of ϑ_0 and all x. We want to show that for the test

$$\phi_n^{*}\left(X^{(n)}\right)=\chi_{\{\Delta_n(\vartheta_0)>c_n\}}$$

with threshold

$$c_n=z_\alpha-\frac{\int_0^\tau \frac{S(x)^3}{(\vartheta_0 S(x)+\lambda)^2}\,\mathrm{d}x}{6\left(\int_0^\tau \frac{S(x)^2}{\vartheta_0 S(x)+\lambda}\,\mathrm{d}x\right)^{3/2}}\sqrt{\frac{\tau}{n}}(1-z_\alpha^2)$$

we have

$$\sup_{0\leq u\leq K}\left|\beta_n\left(u,\phi_n^{*}\right)-\beta_n(u,\tilde{\phi}_n)\right|=O(n^{-1})$$

as $n\to\infty$ for any $K>0$.

Since the intensity functions are positive, then the intensity measures are equivalent. The Fisher information:

$$\begin{aligned}
I_n(\vartheta_0) &= \int_{\mathbb{A}_n}\frac{\dot{S}\left(\vartheta_0,x\right)^2}{S\left(\vartheta_0,x\right)}\,\mathrm{d}x=\int_0^n \frac{S(x)^2}{\vartheta_0 S(x)+\lambda}\,\mathrm{d}x=\\
&=\frac{n}{\tau}\int_0^\tau \frac{S(x)^2}{\vartheta_0 S(x)+\lambda}\,\mathrm{d}x(1+O(n^{-1}))\equiv nA^2(1+O(n^{-1}))>0,
\end{aligned}$$

where the constant

$$A^2 = \frac{1}{\tau}\int_0^\tau \frac{S(x)^2}{\vartheta_0 S(x)+\lambda}\,\mathrm{d}x.$$

Therefore $\varphi_n = I_n(\vartheta_0)^{-1/2} = n^{-1/2}\,A^{-1}(1+O(n^{-1})) \to 0$ because $\vartheta_0 > 0$. Note that the Fisher information $I_n(0) = 0$. By considering the periodicity of $S(\vartheta, x)$ and the fact that $\dot S(\vartheta,x) = S(x)$, $\ddot S(\vartheta,x) = 0$ for all x, the condition $\mathcal{C}_3$ is satisfied with $\varepsilon_n = \varphi_n$. Hence by Corollary 4 we have LAN at ϑ_0 and $\mathcal{C}_1$ is satisfied. For $\mathcal{C}_2$ by Corollary 5 it remains only to consider the condition $\mathcal{B}_2$ of the expansions. First we verify the random variable

$$\Delta_n\left(\vartheta_0\right) = \varphi_n \int_0^n \frac{S(x)}{\vartheta_0 S(x)+\lambda}\,\pi^{(n)}\left(\mathrm{d}x\right) \equiv \int_0^n f_n(x)\,\pi^{(n)}\left(\mathrm{d}x\right)$$

with obvious notation. For $\mathcal{B}_2$ under $\mathcal{H}_0$ we show that the inequality

$$\inf_{\frac{c_0\varphi_n^{-1}}{2}<t<\frac{\varphi_n^{-2}}{2}} \int_0^n \sin^2\left(t f_n(x)\right)\, S(\vartheta_0, x)\,\mathrm{d}x \ge \gamma \ln \varphi_n^{-1} \tag{3.16}$$

holds for small values of $c_0 > 0$ and some $\gamma \ge 3/2$. By the periodicity of the integrand it suffices to show that

$$\inf_{\frac{c_0\varphi_n^{-1}}{2}<t<\frac{\varphi_n^{-2}}{2}} n\int_0^\tau \sin^2\left(\frac{t\varphi_n S(x)}{\vartheta_0 S(x)+\lambda}\right)\,(\vartheta_0 S(x)+\lambda)\,\mathrm{d}x \ge \gamma \ln \varphi_n^{-1}.$$

Since $S(x)$ is periodic, nonconstant and differentiable then there exists some constants $0 \le a < b \le \tau$ such that $S(x)$ is strictly increasing on $[a,b]$. Hence for $S'(x)$, the derivative of $S(x)$ there exists a constant $E > 0$ such that $0 < S'(x) \le E$ for $x \in [a,b]$. Therefore we obtain

$$\begin{aligned} I_n(t) \equiv n\int_0^\tau \sin^2\left(\frac{t\,\varphi_n\,S(x)}{\vartheta_0 S(x)+\lambda}\right)\,(\vartheta_0 S(x)+\lambda)\,\mathrm{d}x \ge \\ \ge n\int_a^b \sin^2\left(\frac{t\,\varphi_n\,S(x)}{\vartheta_0 .S(x)+\lambda}\right)(\vartheta_0 S(x)+\lambda)\,\mathrm{d}x. \end{aligned}$$

Now the substitution $x \to y = \frac{t\,\varphi_n\,S(x)}{\vartheta_0 S(x)+\lambda}$ in the last integral gives

$$I_n(t) \ge \frac{n}{t\,\varphi_n}\int_{y_1(n)}^{y_2(n)} \frac{(\vartheta_0 S(x)+\lambda)^3}{S'(x)}\,\sin^2(y)\,\mathrm{d}y \ge \frac{C\,n}{t\,\varphi_n}\int_{y_1(n)}^{y_2(n)} \sin^2(y)\,\mathrm{d}y$$

for some constant $C > 0$ and

$$y_2(n) - y_1(n) = \frac{t\,\varphi_n\,S(b)}{\vartheta_0 S(b) + \lambda} - \frac{t\,\varphi_n\,S(a)}{\vartheta_0 S(a) + \lambda} = \frac{t\,\varphi_n\,\lambda\,(S(b) - S(a))}{(\vartheta_0 S(b) + \lambda)(\vartheta_0 S(a) + \lambda)} \geq$$
$$\geq \frac{c_0\,\lambda\,(S(b) - S(a))}{2(\vartheta_0 S(b) + \lambda)(\vartheta_0 S(a) + \lambda)} = d > 0.$$

Since d doesn't depend on n, then

$$\int_{y_1(n)}^{y_2(n)} \sin^2(y)\,\mathrm{d}y \geq D$$

for some constant $D > 0$. This implies that $I_n(t) \geq \frac{C'\,n}{t\,\varphi_n}$ and hence

$$\inf_{\frac{c_0 \varphi_n^{-1}}{2} < t < \frac{\varphi_n^{-2}}{2}} I_n(t) \geq C''\sqrt{n} \geq \gamma \ln \varphi_n^{-1}.$$

For the condition $\mathcal{B}_2$ under $\mathcal{H}_u$ we replace $S(\vartheta_0, x)$ with $S(\vartheta_u, x)$ and $f_n(x)$ with

$$g_n(x) = \frac{1}{\eta_n} f_n(x) = \frac{\varphi_n S(x)}{\eta_n\,(\vartheta_0 S(x) + \lambda)}.$$

The corresponding inequality can be obtained by a similar way as under $\mathcal{H}_0$. Now we consider the condition $\mathcal{B}_2$ for $\Lambda_n(u)$ under the null hypothesis. We will show that (by periodicity)

$$\inf_{\frac{c_0 \varphi_n^{-1}}{2} < t < \frac{\varphi_n^{-2}}{2}} n \int_0^\tau \sin^2\left(\frac{t}{\sigma_n} \ln \frac{S(\vartheta_u, x)}{S(\vartheta_0, x)}\right) S(\vartheta_0, x)\,\mathrm{d}x \geq \gamma \ln \varphi_n^{-1}.$$

We can write

$$J_n(t) \equiv n \int_0^\tau \sin^2\left(\frac{t}{\sigma_n} \ln \frac{S(\vartheta_u, x)}{S(\vartheta_0, x)}\right) S(\vartheta_0, x)\,\mathrm{d}x \geq$$
$$\geq n \int_a^b \sin^2\left(\frac{t}{\sigma_n} \ln \frac{S(\vartheta_u, x)}{S(\vartheta_0, x)}\right) S(\vartheta_0, x)\,\mathrm{d}x.$$

By the change of variable $x \to y = \frac{t}{\sigma_n} \ln \frac{S(\vartheta_u,x)}{S(\vartheta_0,x)}$ we get

$$J_n(t) \geq \frac{C\,n^{3/2}}{t} \int_{y_1(n)}^{y_2(n)} \sin^2(y)\,\mathrm{d}y \tag{3.17}$$

for some constant $C > 0$ and

$$y_2(n) - y_1(n) = \frac{t}{\sigma_n} \ln \frac{(\vartheta_u S(b) + \lambda)(\vartheta_0 S(a) + \lambda)}{(\vartheta_u S(a) + \lambda)(\vartheta_0 S(b) + \lambda)} =$$
$$= \frac{t}{\sigma_n} \ln \left(1 + \frac{\varphi_n \lambda \, u \, (S(b) - S(a))}{(\vartheta_u S(a) + \lambda)(\vartheta_0 S(b) + \lambda)} \right) \geq \frac{t}{\sigma_n} \ln \left(1 + \varphi_n \, h\right)$$

for some constant $h > 0$. The inequality $\ln(1 + x) \geq x - \frac{x^2}{2}$ for $x \geq 0$ yields

$$y_2(n) - y_1(n) \geq \frac{t}{\sigma_n} (\varphi_n \, h - \frac{\varphi_n^2 \, h^2}{2}) \geq C' t \, \varphi_n \geq \frac{C' c_0}{2} > 0$$

because $t > \frac{c_0 \varphi_n^{-1}}{2}$. From (3.17) we have $J_n(t) \geq \frac{C'' \, n^{3/2}}{t}$ and hence

$$\inf_{\frac{c_0 \varphi_n^{-1}}{2} < t < \frac{\varphi_n^{-2}}{2}} J_n(t) \geq C \sqrt{n} \geq \gamma \ln \varphi_n^{-1},$$

as desired. For the condition $\mathcal{B}_2$ under $\mathcal{H}_u$ by a similar way we can show that

$$\inf_{\frac{c_0 \varphi_n^{-1}}{2} < t < \frac{\varphi_n^{-2}}{2}} n \int_0^\tau \sin^2 \left(\frac{t}{\sigma_n(u)} \ln \frac{S(\vartheta_u, x)}{S(\vartheta_0, x)} \right) S(\vartheta_u, x) \, \mathrm{d}x \geq \gamma \ln \varphi_n^{-1}$$

where $\sigma_n^2(u) \to u^2$ is the variance of $\Lambda_n(u)$ under $\mathcal{H}_u$. All the conditions of Theorem 26 are satisfied. Hence the test $\phi^\star$ has the desired properties. Note that the parameter

$$\gamma_{3,n} = \varphi_n^3 \int_0^n \frac{\dot{S}\left(\vartheta_0, x\right)^3}{S\left(\vartheta_0, x\right)^2} \, \mathrm{d}x = \frac{B}{A^3 \sqrt{n}} (1 + O(n^{-1}))$$

where

$$B = \frac{1}{\tau} \int_0^\tau \frac{S(x)^3}{\left(\vartheta_0 S(x) + \lambda\right)^2} \, \mathrm{d}x.$$

Therefore the threshold

$$c_n = z_\alpha - \frac{\gamma_{3,n}}{6} \left(1 - z_\alpha^2\right) = z_\alpha - \frac{B(1 - z_\alpha^2)}{6 \, A^3 \sqrt{n}}.$$

Example 3. (Frequency modulation.) In this example we consider a strongly nonhomogeneous case with nonclassical rate $n^{-3/2}$ (instead of $n^{-1/2}$ as in the i.i.d. case). Suppose that we observe a realization $X^{(n)}$ of a Poisson process on the set $\mathbb{A}_n = [0, n]$, $n = 1, 2, \cdots$ with periodic intensity function

$$S(\vartheta, x) = e^{\sin(\vartheta \, x)}, \qquad \vartheta > 0.$$

The hypothesis $\mathcal{H}_0 : \vartheta = \vartheta_0$ against $\mathcal{H}_1 : \vartheta > \vartheta_0$ is given and we want to show that the test $\phi_n^\star\left(X^{(n)}\right) = \chi_{\{\Delta_n(\vartheta_0)>c_n\}}$ based on the score statistic

$$\Delta_n\left(\vartheta_0\right) = \varphi_n \int_0^n \frac{\dot{S}\left(\vartheta_0, x\right)}{S\left(\vartheta_0, x\right)}\, \pi^{(n)}\left(\mathrm{d}x\right) = \varphi_n \int_0^n x\, \cos(\vartheta_0\, x)\, \pi^{(n)}\left(\mathrm{d}x\right)$$

is second order efficient. Here $\pi^{(n)}(\mathrm{d}x) = X^{(n)}(\mathrm{d}x) - S(\vartheta_0, x)\,\mathrm{d}x$, the threshold

$$c_n = z_\alpha - \frac{\gamma_{3,n}}{6}\left(1 - z_\alpha^2\right)$$

where

$$\gamma_{3,n} = \varphi_n^3 \int_0^n x^3\, \cos^3(\vartheta_0\, x)\, e($$

where $[\cdot]$ denotes the integer part. Now we us the change of variable $x \to y = t\,\varphi_n\, x\, \cos(\vartheta_0\, x)$. In the interval $[b_1 + k\tau,\, b_2 + k\tau]$ the derivative of y (with respect to x) satisfies

$$0 < y' \leq t\,\varphi_n\, \left(b_2 + k\,\tau + \cos(\vartheta_0 b_2)\right).$$

Therefore we get

$$I_n(t) \geq \frac{C_1}{t\,\varphi_n} \sum_{k=\left[\frac{n}{2\tau}\right]}^{\left[\frac{n}{\tau}\right]-1} \frac{1}{b_2 + k\,\tau + \cos(\vartheta_0\, b_2)} \int_{y_1(n)}^{y_2(n)} \sin^2(y)\ \mathrm{d}y,$$

where the integral bounds

$$y_1(n) = t\,\varphi_n\,(b_1 + k\,\tau)\ \cos(\vartheta_0(b_1 + k\,\tau)) = 0$$
$$y_2(n) = t\,\varphi_n\,(b_2 + k\,\tau)\ \cos(\vartheta_0(b_2 + k\,\tau)) = t\,\varphi_n\,(b_2 + k\,\tau).$$

Hence the integral

$$\int_{y_1(n)}^{y_2(n)} \sin^2(y)\ \mathrm{d}y = \frac{y_2(n)}{2}\left(1 - \frac{\sin(2\,y_2(n))}{2\,y_2(n)}\right).$$

There exists a constant $D > 0$ such that uniformly for all $t \geq c_0\sqrt{n}/2$ we have $y_2(n) \geq D$. Therefore the inequality

$$\frac{\sin(2\,y_2(n))}{2\,y_2(n)} \leq \eta$$

is satisfied for some constant $0 < \eta < 1$. Finally we can write

$$\inf_{t > \frac{c_0\sqrt{n}}{2}} I_n(t) \geq C_2 \sum_{k=\left[\frac{n}{2\tau}\right]}^{\left[\frac{n}{\tau}\right]-1} \frac{b_2 + k\,\tau}{b_2 + k\,\tau + \cos(\vartheta_0\, b_2)} \geq C_3\, n \geq \gamma \ln \sqrt{n},$$

for some constant $C_3 > 0$ and all n enough large.

Now we verify $\mathcal{B}_2$ for (normalized) loglikelihood ratio $\Lambda_n(u)$ under $\mathcal{H}_0$. We show that

$$\inf_{\frac{c_0\,\varepsilon_n^{-1}}{2} < t < \frac{\varepsilon_n^{-2}}{2}} \int_0^n \sin^2\left(\frac{t}{\sigma_n} \ln \frac{S(\vartheta_u, x)}{S(\vartheta_0, x)}\right) S(\vartheta_0, x)\ \mathrm{d}x \geq \gamma \ln \varepsilon_n^{-1}.$$

Using Taylor formula we get

$$
\begin{aligned}
J_n(t) \equiv \int_0^n \sin^2\left(\frac{t}{\sigma_n}\ln\frac{S(\vartheta_u,x)}{S(\vartheta_0,x)}\right) S(\vartheta_0,x)\,\mathrm{d}x = \\
= \int_0^n \sin^2\left(\frac{t}{\sigma_n}(\sin(\vartheta_u\,x)-\sin(\vartheta_0\,x))\right) S(\vartheta_0,x)\,\mathrm{d}x = \\
= \int_0^n \sin^2\left(t\,\varphi_n\,x\,\frac{u}{\sigma_n}\cos(\vartheta_n\,x)\right) S(\vartheta_0,x)\,\mathrm{d}x.
\end{aligned}
$$

for some intermediate point $\vartheta_n \to \vartheta_0$. Remind that also the $\sigma_n \to u$ as $n \to \infty$. As we see for the last integral we have essentially the same situation as in the left hand side of (3.18). Therefore the change of variable $x \to y = \frac{t}{\sigma_n}(\sin(\vartheta_u\,x)-\sin(\vartheta_0\,x))$ and using a similar argument as for $I_n(t)$ leads to the same result. The condition $\mathcal{B}_2$ for $\Delta_n(\vartheta_0)$ and $\Lambda_n(u)$ under the local alternative $\vartheta_u = \vartheta_0 + \varphi_n u$ can be treated similarly. We omit the details. Hence all the conditions of Theorem 26 are satisfied and the test $\phi_n^{\star}$ is second order efficient.

Remark 8. *Let the intensity function be of the form*

$$
S(\vartheta,x) = \sin(\vartheta\,x) + \lambda, \qquad \vartheta > 0
$$

where $\lambda > 1$ is a known positive parameter (dark-current), or more generally

$$
S(\vartheta,x) = S(\vartheta\,x) + \lambda
$$

where $S(\cdot)$ is a periodic function with the same properties as in Example 2. For such a class of intensities by a similar way (with appropriate small changes) as in the last example we can construct a second order efficient test.

Remark 9. *The* asymptotic deficiency *of a test can be calculated by the same method as for proving the second order efficiency. For this purpose we consider two terms after the Gaussian term in the Edgeworth expansions of the distributions functions of the score statistic and the loglikelihood ratio.*

Chapter 4

Hypotheses Testing for a Multidimensional Parameter

4.1 Introduction

Let $X^{(n)}$ be a realization of a nonhomogeneous Poisson process observed on some subset $\mathbb{A}_n$ of the d dimensional Euclidean space $\mathbb{R}^d$ with intensity function $S(\vartheta, x)$, $x \in \mathbb{A}_n$ (with respect to the Lebesgue measure). The intensity function depends on the unknown parameter $\vartheta \in \Theta$ where the parameter space $\Theta \subseteq \mathbb{R}^k,\ k \geq 2$. Based on $X^{(n)}$ we want to test the hypotheses

$$\mathscr{H}_0 : \vartheta = \vartheta_0$$
$$\mathscr{H}_1 : \vartheta \neq \vartheta_0,$$

for given ϑ_0. By a test (decision rule) ϕ_n we mean a measurable function $\phi_n\left(X^{(n)}\right) =$ *probability to reject* $\mathscr{H}_0$, from the space of the observations (trajectories) $X^{(n)}$ of the process to the interval $[0, 1]$. Let $\mathbf{P}_\vartheta^{(n)}$ denotes the distribution of the random element $X^{(n)}$ and $\mathbf{E}_\vartheta$ the mathematical expectation with respect to $\mathbf{P}_\vartheta^{(n)}$. Let us fix $0 < \alpha < 1$. For the class of alternatives $\mathscr{H}_1 : \vartheta \neq \vartheta_0$ doesn't as a rule exist uniformly most powerful test (UMP) neither in the class $\mathcal{K}_\alpha^{(n)}$ of tests of level $1 - \alpha$, i.e.,

$$\mathcal{K}_\alpha^{(n)} = \left\{\phi_n : \quad \mathbf{E}_{\vartheta_0}\phi_n\left(X^{(n)}\right) = \alpha\right\},$$

nor asymptomatically uniformly most powerful test (AUMP) in the class $\mathcal{K}'_\alpha$ of sequence of tests of asymptotic level $1-\alpha$, i.e.,

$$\mathcal{K}'_\alpha = \left\{\phi_n : \quad \lim_{n\to\infty} \mathbf{E}_{\vartheta_0}\phi_n\left(X^{(n)}\right) = \alpha\right\},$$

see [7] or [29]. For testing $\mathscr{H}_0$ against $\mathscr{H}_1$ it is common to use the classical large sample test procedures including *likelihood ratio*, *Wald* and *Rao score* tests. They were introduced by Wilks [45], Wald [44] and Rao [38], respectively. It is well known that under certain conditions these tests are asymptotically equivalent in some sense (see [28]). The likelihood ratio test and the Wald test require calculating an *efficient* estimator $\hat{\vartheta}_n$ of ϑ (say MLE). This is not the case for the Rao score test which is based on the derivative of loglikelihood ratio with respect to ϑ at $\vartheta = \vartheta_0$. For a comparison of the three tests see [32] and [14]. Below we use the Rao score test.

Based on the observation $X^{(n)}$ of the Poisson process, if the parameter ϑ is one dimensional, then for $\mathscr{H}_0 : \vartheta = \vartheta_0$ against the one sided alternative $\mathscr{H}_1 : \vartheta > \vartheta_0$ (or $\mathscr{H}_1 : \vartheta < \vartheta_0$) a *locally asymptotically uniformly most powerful* (LAUMP) test is given in the Chapter 3, which under certain regularity conditions is *second order efficient*. Indeed we use the *Rao score test*

$$\phi_n^\star\left(X^{(n)}\right) = \begin{cases} 1, & \text{if } \Delta_n\left(\vartheta_0\right) > c_n \\ 0, & \text{if } \Delta_n\left(\vartheta_0\right) \le c_n \end{cases}$$

based on the score statistic

$$\Delta_n\left(\vartheta_0\right) = \varphi_n \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)}{S\left(\vartheta_0, x\right)}\, \pi^{(n)}\left(\mathrm{d}x\right), \quad \varphi_n^{-2} = \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)^2}{S\left(\vartheta_0, x\right)}\, \mathrm{d}x$$

where $\pi^{(n)}\left(\mathrm{d}x\right) = X^{(n)}\left(\mathrm{d}x\right) - S\left(\vartheta_0, x\right)\mathrm{d}x$ is the centered Poisson process and $\dot S\left(\vartheta, x\right)$ is the derivative of $S\left(\vartheta, x\right)$ with respect to ϑ. The (refined) threshold

$$c_n = z_\alpha - \frac{\gamma_{3,n}}{6}\left(1 - z_\alpha^2\right), \qquad \gamma_{3,n} = \varphi_n^3 \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)^3}{S\left(\vartheta_0, x\right)^2}\, \mathrm{d}x$$

is obtained by using the Edgeworth type expansion for distribution function of the stochastic integral $\Delta_n\left(\vartheta_0\right)$ under $\mathscr{H}_0$. Here z_α is $1-\alpha$ quantile of standard Gaussian law, i.e. $\mathbf{P}\left\{\zeta > z_\alpha\right\} = 1-\alpha$ and $\zeta \sim \mathcal{N}(0,1)$. The type I probability of error of the test is equal to $\mathbf{E}_{\vartheta_0}\phi_n^\star\left(X^{(n)}\right) = \alpha + O(\varepsilon_n^2)$, where the sequence $\varepsilon_n \to 0$ as $n\to\infty$. This improves the classical approximations obtained by the central limit theorem where the rate of approximation is of order $o(1)$ as $n\to\infty$. Note also that $\gamma_{3,n}$ converges to zero with the rate ε_n. The test $\phi_n^\star$ is second order efficient, i.e., for any $K>0$

$$\sup_{0\leq u\leq K}\left|\mathbf{E}_u\phi_n^\star\left(X^{(n)}\right) - \mathbf{E}_u\tilde{\phi}_n\left(X^{(n)}\right)\right| = O(\varepsilon_n^2),$$

where $\tilde{\phi}_n$ is the most powerful test at level $1-\alpha$ for the simple hypothesis $\mathscr{H}_0 : \vartheta = \vartheta_0$ against the simple local alternative $\mathcal{H}_u : \vartheta = \vartheta_0 + \varphi_n\, u$, for $u>0$ and $\mathbf{E}_u$ denotes the mathematical expectation under the local alternative.

The aim of the present chapter is to consider the Rao score test for testing the null hypothesis $\mathscr{H}_0 : \vartheta = \vartheta_0$ against the multi-sided alternative $\mathscr{H}_1 : \vartheta \neq \vartheta_0$ where $k\geq 2$. We use the Edgeworth type expansion for the distribution function of a vector of stochastic integrals related to a nonhomogeneous Poisson process to study the Roa score test (see section 1.4). The expansion under the null hypothesis allows us to refine the classic threshold obtained by the central limit theorem. The type I probability error of the improved test is nearer to the given level of significance with respect to the classic one. By using the Edgeworth expansion we describe the asymptotic behavior of the power of the Rao score test under the *local alternative*, i.e., a sequence of alternatives which converges to ϑ_0 with a certain rate. The rates of convergence can be different for components of the parameter (see the example frequency and amplitude modulation). The asymptotic expansion of the estimators and their distribution functions (univariate case) for inhomogeneous Poisson processes are obtained by Yu. A. Kutoyants [27] and for diffusion processes with small noise by Yu. A. Kutoyants [26] and N. Yoshida [46]. The importance of such expansions are well known in statistical inference; see [12], [35], [36] and the references therein.

4.2 Rao score test

Let $X^{(n)}$ be (a realization of) a nonhomogeneous Poisson process with intensity function $S(\vartheta, x)$, $x \in \mathbb{A}_n$ which depends on the unknown parameter $\vartheta = (\vartheta_1, \ldots, \vartheta_k) \in \Theta \subseteq \mathbb{R}^k$, $k \geq 2$. We suppose that the intensity function is positive and it is differentiable with respect to ϑ. Our goal is to test the hypotheses

$$\mathscr{H}_0 : \vartheta = \vartheta_0$$
$$\mathscr{H}_1 : \vartheta \neq \vartheta_0,$$

for given $\vartheta_0 = (\vartheta_{01}, \ldots, \vartheta_{0k})$. Let $\mathbf{P}_\vartheta^{(n)}$ denotes the distribution of the random element $X^{(n)}$ and $\mathbf{E}_\vartheta$ the mathematical expectation with respect to $\mathbf{P}_\vartheta^{(n)}$. Since the intensity functions are positive, then the measures $\mathbf{P}_\vartheta^{(n)}$ and $\mathbf{P}_{\vartheta_0}^{(n)}$ are equivalent and the loglikelihood ratio is given by

$$\begin{aligned} L(\vartheta,\, X^{(n)}) &= \ln \frac{\mathrm{d}\mathbf{P}_\vartheta^{(n)}}{\mathrm{d}\mathbf{P}_{\vartheta_0}^{(n)}}(X^n) = \\ &= \int_{\mathbb{A}_n} \ln \frac{S(\vartheta, x)}{S(\vartheta_0, x)} X^{(n)}(\,\mathrm{d}x) - \int_{\mathbb{A}_n} [S(\vartheta, x) - S(\vartheta_0, x)]\,\mathrm{d}x. \end{aligned}$$

For proof see [27]. The Rao *score* test is based on the test statistic vector

$$\begin{aligned} \Delta_n \equiv &(\Delta_1, \ldots, \Delta_k)^{\mathsf{0}} = \left(\frac{\partial L(\vartheta, X^{(n)})}{\partial \vartheta_1}, \ldots, \frac{\partial L(\vartheta, X^{(n)})}{\partial \vartheta_k}\right)^{\mathsf{0}}\Bigg|_{\vartheta=\vartheta_0} = \\ &= \left(\int_{\mathbb{A}_n} \frac{\dot{S}_1(\vartheta_0, x)}{S(\vartheta_0, x)}\,\pi(\mathrm{d}x), \ldots \int_{\mathbb{A}_n} \frac{\dot{S}_k(\vartheta_0, x)}{S(\vartheta_0, x)}\,\pi(\mathrm{d}x)\right)^{\mathsf{0}}, \end{aligned}$$

where $\pi(\mathrm{d}x) = X^{(n)}(\mathrm{d}x) - S(\vartheta_0, x)\,\mathrm{d}x$ is the centered Poisson process and $\dot{S}_\nu(\vartheta_0, x), \nu = 1, \ldots, k$ denotes the partial derivative of the intensity function with respect to ϑ_ν at $\vartheta = \vartheta_0$. The Rao score test rejects $\mathscr{H}_0$ if $\Delta_n^{\mathsf{0}}\, I_n^{\text{¡}1}(\vartheta_0)\Delta_n$ is enough large, i.e., $\Delta_n^{\mathsf{0}}\, I_n^{\text{¡}1}(\vartheta_0)\Delta_n > c$ for some $c = c_{n,\alpha} > 0$ depending on n and the level of significance $\alpha \in (0, 1)$. Here the *Fisher information* matrix $I_n(\vartheta_0) = \mathbf{E}_{\vartheta_0}(\Delta_n\,\Delta_n^{\mathsf{0}})$. A first order approximation of the threshold follows from the asymptotic normality of the vector $I_n^{\text{¡}1/2}(\vartheta_0)\,\Delta_n$ under $\mathscr{H}_0$,

$$\mathcal{L}_{\vartheta_0}\left\{I_n^{\text{¡}1/2}(\vartheta_0)\,\Delta_n\right\} \Rightarrow \mathcal{N}(0, I)\,,$$

where I is the $k \times k$ identity matrix. For the asymptotic normality of a vector of stochastic integrals see, for example, [27] page 24. Hence the threshold $c_{n,\alpha} = \chi_{k,\alpha} + o(1)$ satisfies

$$\mathbf{P}_{\vartheta_0}^{(n)} \left\{ \Delta_n' I_n^{-1}(\vartheta_0)\Delta_n > c_{n,\alpha} \right\} = \alpha + o(1), \tag{4.1}$$

where $\chi_{k,\alpha}$ is $1-\alpha$ quantile of the Chi-squared distribution with k degrees of freedom, i.e., $\mathbf{P}\left\{\xi > \chi_{k,\alpha}\right\} = \alpha$ with $\xi \sim \chi_k^2$. In order to improve the rate of approximation we use the Edgeworth type expansion of distribution of the normalized vector Δ_n (under $\mathscr{H}_0$), *i.e.*,

$$Y_n = I_n^{-1/2}(\vartheta_0)\,\Delta_n \equiv (Y_1, \ldots, Y_k)'$$

where the stochastic integrals

$$Y_\nu = \int_{\mathbb{A}_n} f_\nu(x)\,\pi(\mathrm{d}x), \qquad \nu = 1, \ldots, k.$$

Let $H_m(\cdot)$ denotes the Hermit polynomials defined by

$$H_m(t) = (-1)^m \left[\frac{d^m}{d\,t^m} \exp\left\{-t^2/2\right\}\right] \exp\left\{t^2/2\right\}, \quad m = 1, 2, \ldots$$

(For $m = 1, \ldots, 6$ see section 4.3.) Below we use the following notations

$$\mathbf{f}_n(x) = (f_1(x), \ldots, f_k(x))', \quad \lambda = (\lambda_1, \ldots, \lambda_k)' \in \mathbb{R}^k$$

and the inner product $\langle \lambda, \mathbf{f}_n(x)\rangle = \sum_1^k \lambda_\nu f_\nu(x)$. The expansion is obtained under the following conditions.

$\mathscr{B}_1$. *There exists a sequence of real numbers $\varepsilon_n \to 0$, as $n \to \infty$ and constants $C_p > 0$, $p = 3, 4, 5$ such that*

$$\int_{\mathbb{A}_n} \left(|f_1(x)|^p + \cdots + |f_k(x)|^p\right) S(\vartheta_0, x)\,\mathrm{d}x \leq C_p\,\varepsilon_n^{p-2}.$$

$\mathscr{B}_2$. *There exists $\gamma \geq \frac{3+3k}{2}$ such that the inequality*

$$\inf_{2\|\lambda\| \leq c_0\varepsilon_n^{-1}} \int_{\mathbb{A}_n} \sin^2\left(\langle\lambda, \mathbf{f}_n(x)\rangle\right) S(\vartheta_0, x)\,\mathrm{d}x \geq \gamma \ln \varepsilon_n^{-1}$$

holds for small values of $c_0 > 0$.

We have the following theorem.

Theorem 27. *Let the conditions $\mathscr{B}_1 - \mathscr{B}_2$ be fulfilled. Then the type I probability of error of the Rao score test*

$$\phi_n^{¤}\left(X^{(n)}\right) = \begin{cases} 1, & \text{if } \Delta_n^{0} I_n^{¡\,1}(\vartheta_0 \\ & \end{cases}$$

The coefficients Q_n and R_n are dominated by ε_n and ε_n^2, respectively (see section 4.3) and they are given by

$$\frac{1}{\nu_1!\ldots\nu_k!}\int_{\mathbb{A}_n} f_1^{\nu_1}(x)\ldots f_k^{\nu_k}(x)\,S(\vartheta_0,x)\,\mathrm{d}x = \begin{cases} Q_n(\nu_1,\cdots,\nu_k), & \text{if } \sum\nu_j = 3,\\ R_n(\nu_1,\cdots,\nu_k), & \text{if } \sum\nu_j = 4.\end{cases}$$

Using (4.2) we can write

$$\mathbf{E}_{\vartheta_0}\phi_n^*\left(X^{(n)}\right) = 1 - \int_{\mathbb{C}_\alpha^{(n)}} h_n(y)\,\mathrm{d}y + O(\varepsilon_n^3) = \alpha + O(\varepsilon_n^3).$$

Hence we find $z = \sqrt{c_{n,\alpha}}$ such that

$$\int_{\|y\|\le z} h_n(y)\,\mathrm{d}y = 1-\alpha.$$

This equation by using the fact that for the Hermit polynomials

$$\int_{-z}^{z} H_m(t)\,e^{-t^2/2}\,\mathrm{d}t = \begin{cases} -2\,H_{m-1}(z)\,e^{-z^2/2} & m \text{ even},\\ 0 & m \text{ odd},\end{cases}$$

leads to the equation

$$\kappa_n(z) \equiv \mathbf{P}\left\{\xi > z^2\right\} - \left(a_n\,z^k + b_n\,z^{k+2} + c_n\,z^{k+4}\right)\,e^{-z^2/2} - \alpha = 0$$

where the random variable $\xi \sim \chi_k^2$ and the coefficients a_n, b_n, c_n (depending on k) are of order $O(\varepsilon_n^2)$ and they can be calculated explicitly for each k. For $k = 2$ see the example. Since the coefficients a_n, b_n and c_n are of order $O(\varepsilon_n^2)$ then,

$$z^2 = c_{n,\alpha} = \chi_{k,\alpha} + O(\varepsilon_n^2).$$

Now the Taylor expansion of $\kappa_n\left(\sqrt{c_{n,\alpha}}\right)$ about the point $\sqrt{\chi_{k,\alpha}}$ gives

$$\kappa_n\left(\sqrt{c_{n,\alpha}}\right) = \kappa_n\left(\sqrt{\chi_{k,\alpha}}\right) + \left(\sqrt{c_{n,\alpha}} - \sqrt{\chi_{k,\alpha}}\right)\,\kappa_n'\left(\sqrt{\chi_{k,\alpha}}\right) + O(\varepsilon_n^4).$$

By using the fact that $\kappa_n\left(\sqrt{c_{n,\alpha}}\right) = 0$ we obtain

$$\sqrt{c_{n,\alpha}} = \sqrt{\chi_{k,\alpha}} - \frac{\kappa_n\left(\sqrt{\chi_{k,\alpha}}\right)}{\kappa_n'\left(\sqrt{\chi_{k,\alpha}}\right)} + O(\varepsilon_n^4),$$

which yields

$$c_{n,\alpha} = \chi_{k,\alpha} - 2\,\sqrt{\chi_{k,\alpha}}\,\frac{\kappa_n\left(\sqrt{\chi_{k,\alpha}}\right)}{\kappa'_n\left(\sqrt{\chi_{k,\alpha}}\right)} + O(\varepsilon_n^4).$$

Letting $t = \sqrt{\chi_{k,\alpha}}$ we can write

$$\begin{aligned}\frac{\kappa_n\,(t)}{\kappa'_n\,(t)} &= \frac{\mathbf{P}\left\{\xi > t^2\right\} - \left(a_n\,t^k + b_n\,t^{k+2} + c_n\,t^{k+4}\right)\,e^{-\,t^2/2} - \alpha}{-2^{1-\,k/2}\,\Gamma(k/2)^{-\,1}\,t^{k-\,1}\,e^{-\,t^2/2} + O(\varepsilon_n^2)} = \\ &= 2^{k/2-\,1}\,\Gamma(k/2)\,\left(a_n\,t + b_n\,t^3 + c_n\,t^5\right) + O(\varepsilon_n^4).\end{aligned}$$

Finally we get the solution

$$c_{n,\alpha} = \chi_{k,\alpha} - 2^{k/2}\,\Gamma\,(k/2)\left[a_n\,\chi_{k,\alpha}^2 + b_n\,\chi_{k,\alpha}^4 + c_n\,\chi_{k,\alpha}^6\right] + O(\varepsilon_n^4).$$

Now the probability of error of the first kind of the Rao score test with the above $c_{n,\alpha}$ is equal to

$$\mathbf{E}_{\vartheta_0}\phi_n^*\left(X^{(n)}\right) = \mathbf{P}_{\vartheta_0}^{(n)}\left\{\Delta'_n\,I_n^{-\,1}(\vartheta_0)\Delta_n > c_{n,\alpha}\right\} = \alpha + O(\varepsilon_n^3).$$

This means that the threshold $c_{n,\alpha}$ obtained with the help of Edgeworth expansion refines the classic first order solution $c_{n,\alpha} = \chi_{k,\alpha} + o(1)$ up to order $O(\varepsilon_n^3)$ (compare with (4.1)).

Now we describe the power of the Rao score test at *local alternative*

$$\vartheta_u = \vartheta_0 + I_n(\vartheta_0)^{-\,1/2}\,u$$

where the vector $u = (u_1, \ldots, u_k)' \neq 0$. The power is equal to

$$\beta_n(u) = \mathbf{P}_{\vartheta_u}^{(n)}\left\{\Delta'_n\,I_n^{-\,1}(\vartheta_0)\,\Delta_n > c_{n,\alpha}\right\} = \mathbf{P}_{\vartheta_u}^{(n)}\left\{Y'_nY_n > c_{n,\alpha}\right\}.$$

Let $\mu_n(u)$ and $\mathbf{\Sigma}_n$ denote the mean vector and the covariance matrix of Y_n under the local alternative, *i.e.*,

$$\mu_n(u) = \mathbf{E}_{\vartheta_u}Y_n, \qquad \mathbf{\Sigma}_n = \mathbf{E}_{\vartheta_u}\left(Y_n - \mu_n(u)\right)\left(Y_n - \mu_n(u)\right)'.$$

Let us suppose that the conditions $\mathscr{B}_1 - \mathscr{B}_2$ of the expansion are fulfilled for the functions $g_1(\cdot),\ldots,g_k(\cdot)$ instead of $f_1(\cdot),\ldots,f_k(\cdot)$, where

$$\left(\int_{\mathbb{A}_n} g_1(x)\,\pi_u(\mathrm{d}x),\ldots,\int_{\mathbb{A}_n} g_k(x)\,\pi_u(\mathrm{d}x)\right)' \equiv W_n = \mathbf{\Sigma}_n^{-1/2}\left(Y_n - \mu_n(u)\right).$$

Here $\pi_u(\mathrm{d}x) = X^{(n)}(\mathrm{d}x) - S(\vartheta_u,x)\,\mathrm{d}x$. Then Edgeworth expansion allows us to obtain the following representation

$$\begin{aligned}\beta_n(u) &= 1 - \mathbf{P}_{\vartheta_u}^{(n)}\left\{Y_n \in \mathbb{C}_\alpha^{(n)}\right\} = 1 - \mathbf{P}_{\vartheta_u}^{(n)}\left\{W_n \in \mathbf{\Sigma}_n^{-1/2}\left(\mathbb{C}_\alpha^{(n)} - \mu_n(u)\right)\right\} = \\ &= 1 - \int_{\mathbf{\Sigma}_n^{-1/2}\left(\mathbb{C}_\alpha^{(n)} - \mu_n(u)\right)} h_n^{(u)}(y)\,\mathrm{d}y + O(\varepsilon_n^3),\end{aligned}$$

where the density $h_n^{(u)}(\cdot)$ has the same form as $h_n(\cdot)$ with the corresponding coefficients (depending on u) given by

$$\frac{1}{\nu_1!\ldots\nu_k!}\int_{\mathbb{A}_n} g_1^{\nu_1}(x)\ldots g_k^{\nu_k}(x)\,S(\vartheta_u,x)\,\mathrm{d}x = \begin{cases} Q_n^{(u)}(\nu_1,\ldots\nu_k), & \text{if } \sum\nu_j = 3,\\ R_n^{(u)}(\nu_1,\ldots\nu_k), & \text{if } \sum\nu_j = 4,\end{cases}$$

and the convex set

$$\mathbf{\Sigma}_n^{-1/2}\left(\mathbb{C}_\alpha^{(n)} - \mu_n(u)\right) = \left\{\mathbf{\Sigma}_n^{-1/2}\left(x - \mu_n(u)\right),\ x \in \mathbb{C}_\alpha^{(n)}\right\}$$

is the interior of the ellipse

$$y'\mathbf{\Sigma}_n\, y + 2\,\mu_n'(u)\,\mathbf{\Sigma}_n^{1/2}\, y + \mu_n'(u)\,\mu_n(u) - c_{n,\alpha} = 0.$$

Therefore we have the following theorem.

Theorem 28. *Let the functions $g_1(\cdot),\ldots,g_k(\cdot)$ satisfy the conditions $\mathscr{B}_1 - \mathscr{B}_2$ instead of $f_1(\cdot),\ldots,f_k(\cdot)$. Then the power of the Rao score test under the local alternative ϑ_u admits the representation*

$$\beta_n(u) = 1 - \int_{\mathbf{\Sigma}_n^{-1/2}\left(\mathbb{C}_\alpha^{(n)} - \mu_n(u)\right)} h_n^{(u)}(y)\,\mathrm{d}y + O(\varepsilon_n^3).$$

Remark 10. *By the Le Cam's Third Lemma the mathematical expectation and covariance matrix of the vector Y_n under the local alternative are given by*

$$\mu_n(u) = u + o(1), \qquad \mathbf{\Sigma}_n = I + o(1).$$

See, for example [43]. Now since $c_{n,\alpha} = \chi^2_{k,\alpha} + o(1)$, then we have the following classical first order approximation of the power

$$\beta_n(u) = 1 - \int_{\|y+u\|<\chi_{k,\alpha}} (2\,\pi)^{-k/2} \exp\left\{-\|y\|^2/2\right\}\,\mathrm{d}y + o(1) =$$
$$= \mathbf{P}\left\{\|\zeta + u\| > \chi_{k,\alpha}\right\} + o(1),$$

where ζ is a k dimensional vector of independent standard Gaussian random variables.

Example (Frequency and amplitude modulation). Let $X^{(n)}$ be a realization of a nonhomogeneous Poisson process with positive intensity function

$$S(\vartheta, x) = \vartheta_1\,\sin(\vartheta_2\,x) + \eta,\ x \in [\,0, n]\,,\ n = 1, 2\ldots.$$

The "dark current" parameter η is known and positive. The pair of amplitude and frequency parameter $\vartheta = (\vartheta_1,\,\vartheta_2)$ is unknown and we want to test the hypotheses

$$\mathscr{H}_0 : (\vartheta_1, \vartheta_2) = (\vartheta_{01}, \vartheta_{02})$$
$$\mathscr{H}_1 : (\vartheta_1, \vartheta_2) \neq (\vartheta_{01}, \vartheta_{02})$$

where $\vartheta_0 = (\vartheta_{01}, \vartheta_{02})$ with is a known point in the parameter space $\Theta = (a_1, b_1)\times(a_2, b_2)$ with $a_1 > 0$, $a_2 > 0$. Here the vector

$$\Delta_n \equiv (\Delta_1, \Delta_2)' = \left(\frac{\partial L(\vartheta, X^{(n)})}{\partial\vartheta_1}, \frac{\partial L(\vartheta, X^{(n)})}{\partial\vartheta_2}\right)'\Bigg|_{\vartheta=\vartheta_0} =$$
$$= \left(\int_0^n \frac{\sin(\vartheta_{02}\,x)}{S(\vartheta_0, x)}\,\pi(\mathrm{d}x),\ \int_0^n \frac{\vartheta_{01}\,x\,\cos(\vartheta_{02}\,x)}{S(\vartheta_0, x)}\,\pi(\mathrm{d}x)\right)'.$$

Let us introduce the normalizing factors

$$\varphi_{1n}^{\text{¡}\,2} = \mathbf{E}_{\vartheta_0}(\Delta_1^2) = \int_0^n \frac{\sin^2(\vartheta_{02}\,x)}{S(\vartheta_0, x)}\,\mathrm{d}x = n\,d_1^{\text{¡}\,2}(1 + o(1)),$$
$$\varphi_{2n}^{\text{¡}\,2} = \mathbf{E}_{\vartheta_0}(\Delta_2^2) = \int_0^n \frac{\vartheta_{01}^2\,x^2\,\cos^2(\vartheta_{02}\,x)}{S(\vartheta_0, x)}\,\mathrm{d}x = n^3\,d_2^{\text{¡}\,2}(1 + o(1)),$$

where

$$d_1^{-2} = \frac{\vartheta_{02}}{2\,\pi}\int_0^{\frac{2\pi}{\vartheta_{02}}} \frac{\sin^2(\vartheta_{02}\,x)}{S(\vartheta_0,x)}\,\mathrm{d}x, \qquad d_2^{-2} = \frac{\vartheta_{02}}{6\,\pi}\int_0^{\frac{2\pi}{\vartheta_{02}}} \frac{\vartheta_{01}^2\,\cos^2(\vartheta_{02}\,x)}{S(\vartheta_0,x)}\,\mathrm{d}x.$$

Observe that the vector Δ_n has two different rates, a classic rate $n^{-1/2}$ (like the i.i.d. case) and a nonclassical rate $n^{-3/2}$. The *Fisher information* matrix $I_n(\vartheta_0)$ is equal to

$$I_n(\vartheta_0) = \begin{pmatrix} n\,d_1^{-2}(1+o(1)) & \sigma(n) \\ \sigma(n) & n^3\,d_2^{-2}(1+o(1)) \end{pmatrix},$$

where

$$\sigma(n) = \mathbf{E}_{\vartheta_0}(\Delta_1\,\Delta_2) = \int_0^n \frac{\vartheta_{01}\,x\,\sin(\vartheta_{02}\,x)\,\cos(\vartheta_{02}\,x)}{S(\vartheta_0,x)}\,\mathrm{d}x = O(n).$$

First we consider the conditions $\mathscr{B}_1 - \mathscr{B}_2$ for the normalized random vector $Y_n = (Y_1,Y_2)' = I_n^{-1/2}(\vartheta_0)\Delta_n$

$$Y_1 = \int_0^n f_1(x)\,\mathrm{d}\pi(x), \qquad Y_2 = \int_0^n f_2(x)\,\mathrm{d}\pi(x).$$

Here the inverse square root matrix

$$I_n^{-1/2}(\vartheta_0) = \begin{pmatrix} d_{1n}\,n^{-1/2} & \gamma_n \\ \gamma_n & d_{2n}\,n^{-3/2} \end{pmatrix},$$

where $\gamma_n = O(n^{-5/2})$, $d_{in} = d_i(1+o(1))$, $i=1,2$, and the functions

$$f_1(x) = d_{1n}\,n^{-1/2}\,\frac{\sin(\vartheta_{02}\,x)}{S(\vartheta_0,x)} + \gamma_n\,\frac{\vartheta_{01}\,x\,\cos(\vartheta_{02}\,x)}{S(\vartheta_0,x)}$$
$$f_2(x) = \gamma_n\,\frac{\sin(\vartheta_{02}\,x)}{S(\vartheta_0,x)} + d_{2n}\,n^{-3/2}\,\frac{\vartheta_{01}\,x\,\cos(\vartheta_{02}\,x)}{S(\vartheta_0,x)}.$$

For the condition $\mathscr{B}_1$ we use the fact that for a function $\phi(\cdot)$ with period τ;

$$\lim_{n\to\infty}\frac{\int_0^n x^k\phi(x)\,\mathrm{d}x}{n^{k+1}} = \frac{\int_0^\tau \phi(x)\,\mathrm{d}x}{\tau\,(k+1)} \qquad k=1,2,\ldots$$

This implies

$$\int_0^n |f_j(x)|^p\;S(\vartheta_0,x)\,\mathrm{d}x \le C\,n^{\frac{2-p}{2}}$$

for $j = 1, 2$ and $p = 3, 4, 5$. Therefore $\mathscr{B}_1$ is satisfied with $\varepsilon_n = n^{-1/2}$. For the condition $\mathscr{B}_2$ we show that the inequality

$$\inf_{2\|\lambda\| \geq c_0\sqrt{n}} \int_0^n \sin^2\left(\lambda_1 f_1(x) + \lambda_2 f_2(x)\right) S(\vartheta_0, x)\,\mathrm{d}x \geq \gamma \ln \sqrt{n}$$

is satisfied for small values of $c_0 > 0$ and any $\gamma > 0$, where $\|\lambda\|^2 = \lambda_1^2 + \lambda_2^2$. Since the intensity function has a positive lower bound we consider only the quantity

$$I_n(\lambda) = \int_0^n \sin^2\left(\lambda_1 f_1(x) + \lambda_2 f_2(x)\right)\,\mathrm{d}x.$$

By the symmetry it is sufficient to consider only the upper half plane $\lambda_2 \geq 0$. In the first quadrant we examine two sets

$$\Lambda_1 = \left\{\lambda :\, 2\|\lambda\| \geq c_0\sqrt{n},\, 0 \leq \lambda_2 \leq \beta\,\lambda_1\right\}$$
$$\Lambda_2 = \left\{\lambda :\, 2\|\lambda\| \geq c_0\sqrt{n},\, \lambda_2 > \beta\,\lambda_1 \geq 0\right\},$$

separately, where $\beta > 0$ is a constant enough small. On the set Λ_1 we have

$$\lambda_1 \geq \frac{c_0\sqrt{n}}{2\sqrt{1+\beta^2}}, \qquad 0 \leq \frac{\lambda_2}{\lambda_1} \leq \beta.$$

Let us introduce the periodic functions

$$h(x) = \frac{\sin(\vartheta_{02}\,x)}{S(\vartheta_0, x)}, \qquad g(x) = \frac{\cos(\vartheta_{02}\,x)}{S(\vartheta_0, x)}$$

with the derivatives (with respect to x)

$$h'(x) = \frac{\vartheta_{02}\eta\,\cos(\vartheta_{02}\,x)}{S(\vartheta_0, x)^2}, \qquad g'(x) = -\vartheta_{02}\frac{\vartheta_{01} + \eta\,\sin(\vartheta_{02}\,x)}{S(\vartheta_0, x)^2}.$$

We write

$$\lambda_1 f_1(x) + \lambda_2 f_2(x) = \left(\frac{\lambda_1}{\sqrt{n}} + \lambda_2\,\gamma_n\right) h(x) + \left(\lambda_1\,\gamma_n + \frac{\lambda_2}{n^{3/2}}\right) x\,g(x).$$

Let the period $\tau = \frac{2\pi}{\vartheta_{02}}$ and

$$a_k = k\,\tau, \qquad b_k = k\,\tau + \delta, \quad k = 0, 1, 2, \ldots,$$

where $\delta > 0$ is a small fixed number. We have

$$\begin{aligned} I_n(\lambda) \geq \int_0^n \sin^2 (\lambda_1 f_1(x) + \lambda_2 f_2(x)) \, \mathrm{d}x \geq \\ \geq \sum_{k=0}^{\left[\frac{n}{\tau}\right]} \int_{a_k}^{b_k} \sin^2 (\lambda_1 f_1(x) + \lambda_2 f_2(x)) \, \mathrm{d}x, \end{aligned}$$

where $[\cdot]$ denotes the integer part. The change of variable $u = \lambda_1 f_1(x) + \lambda_2 f_2(x)$ gives

$$\int_{a_k}^{b_k} \sin^2 (\lambda_1 f_1(x) + \lambda_2 f_2(x)) \, \mathrm{d}x = \int_{u_{1k}}^{u_{2k}} \frac{\sin^2 (u)}{\lambda_1 f_1'(x) + \lambda_2 f_2'(x)} \, \mathrm{d}u,$$

with obvious notations. Below we will see that the denominator of the integrand is positive. Let us first consider the bounds of the integral. The lower limit of the integral

$$\begin{aligned} u_{1k} = \lambda_1 f_1(a_k) + \lambda_2 f_2(a_k) = (\lambda_1 \gamma_n + \lambda_2 n^{-3/2}) \, a_k \, g(0) = \\ = \frac{\lambda_1}{n^{3/2}} \left(n^{3/2} \gamma_n + \frac{\lambda_2}{\lambda_1} \right) a_k \, g(0). \end{aligned}$$

Hence on the set Λ_1,

$$|u_{1k}| \leq C_1 \frac{\lambda_1}{\sqrt{n}} \left(\frac{C_2}{n} + \beta \right) \tag{4.3}$$

for some positive constants C_1 and C_2. The upper limit

$$\begin{aligned} u_{2k} = \lambda_1 f_1(b_k) + \lambda_2 f_2(b_k) = (\lambda_1 n^{-1/2} + \lambda_2 \gamma_n) \, h(\delta) + \\ + (\lambda_1 \gamma_n + \lambda_2 n^{-3/2}) \, b_k \, g(\delta) \equiv \frac{\lambda_1}{\sqrt{n}} \{ h(\delta) + s_n + t_n \} \end{aligned}$$

where the terms

$$s_n = \frac{\lambda_2}{\lambda_1} \gamma_n \sqrt{n} \, h(\delta), \qquad t_n = (\gamma_n \sqrt{n} + \frac{\lambda_2}{n \, \lambda_1}) \, b_k \, g(\delta).$$

Hence on Λ_1

$$|s_n| \leq C_3 \, n^{-2}, \qquad |t_n| \leq \frac{C_4}{n} + C_5 \, \beta,$$

from which we get

$$u_{2k} \geq \frac{\lambda_1}{\sqrt{n}} \{ h(\delta) - |s_n| - |t_n| \} \geq C' \frac{\lambda_1}{\sqrt{n}} (1 - \beta) \geq C \frac{\lambda_1}{\sqrt{n}}, \tag{4.4}$$

where we used the fact that for $\beta > 0$ enough small the quantity $1 - \beta > 0$. Now we consider the denominator of the integrand which can be written as

$$
\begin{aligned}
J_n &\equiv \lambda_1 f_1^0(x) + \lambda_2 f_2^0(x) = \\
&= \frac{\lambda_1}{\sqrt{n}} \left\{ h^0(x) + \frac{\lambda_2}{\lambda_1} \gamma_n \sqrt{n}\, h^0(x) + \left(\gamma_n \sqrt{n} + \frac{\lambda_2}{n\,\lambda_1} \right) (g(x) + x\ g^0(x)) \right\} = \\
&= \frac{\lambda_1}{\sqrt{n}} \left\{ h^0(x) + r_n \right\},
\end{aligned}
$$

with obvious notation. Remark that $h^0(x)$ has a positive lower bound on $[a_k, b_k]$ and $|r_n| \le C_1 \left(C_2\, n^{-1} + \beta \right)$. Hence on the set Λ_1

$$
J_n \ge \frac{\lambda_1}{\sqrt{n}} (1 - C_3\, \beta) \ge C_4 \frac{\lambda_1}{\sqrt{n}},
$$

where $C_4 > 0$. This inequality shows that the quantity $J_n \ge C_5 > 0$, because on Λ_1 we have $\lambda_1 \ge C\sqrt{n}$. On the other hand we obtain the upper bound

$$
J_n = \frac{\lambda_1}{\sqrt{n}} \left\{ h^0(x) + r_n \right\} \le C \frac{\lambda_1}{\sqrt{n}}.
$$

Now we can write

$$
\begin{aligned}
I_n(\lambda) &\ge \sum_{k=0}^{\left[\frac{n}{\tau}\right]} \int_{u_{1k}}^{u_{2k}} \frac{\sin^2(u)}{\lambda_1 f_1^0(x) + \lambda_2 f_2^0(x)}\, \mathrm{d}u \ge \frac{C\sqrt{n}}{\lambda_1} \sum_{k=0}^{\left[\frac{n}{\tau}\right]} \int_{u_{1k}}^{u_{2k}} \sin^2\, \mathrm{d}u = \\
&= \frac{C\sqrt{n}}{2\,\lambda_1} \sum_{k=0}^{\left[\frac{n}{\tau}\right]} \left\{ u_{2k} \left(1 - \frac{\sin(2\,u_{2k})}{2\,u_{2k}} \right) - u_{1k} \left(1 - \frac{\sin(2\,u_{1k})}{2\,u_{1k}} \right) \right\} \equiv \\
&\equiv \sum_{k=0}^{\left[\frac{n}{\tau}\right]} (d_{2k} - d_{1k}),
\end{aligned}
$$

with obvious notations. Therefore

$$
\inf_{\lambda \in \Lambda_1} I_n(\lambda) \ge C \sum_{k=0}^{\left[\frac{n}{\tau}\right]} \left(\inf_{\lambda \in \Lambda_1} d_{2k} - \sup_{\lambda \in \Lambda_1} d_{1k} \right).
$$

From (4.4) the first term admits the lower bound

$$
\inf_{\lambda \in \Lambda_1} d_{2k} \ge C_1 \left(1 - \sup_{\lambda \in \Lambda_1} \frac{\sin(2\,u_{2k})}{2\,u_{2k}} \right) \ge C_2 > 0,
$$

because on the set Λ_1 we have $u_{2k} > C > 0$. This implies

$$0 < \sup_{\lambda \in \Lambda_1} \frac{\sin(2\, u_{2k})}{2\, u_{2k}} < C' < 1.$$

Also by (4.3) the second term in the summation can be written as

$$\sup_{\lambda \in \Lambda_1} d_{1k} \leq C \sup_{\lambda \in \Lambda_1} \frac{\sqrt{n}}{\lambda_1} \, |u_{1k}| \leq C_1\, n^{-1} + C_2\, \beta,$$

for some constants C_1, C_2. These estimates allow us to write

$$\inf_{\lambda \in \Lambda_1} d_{2k} - \sup_{\lambda \in \Lambda_1} d_{1k} \geq C' - C_1\, n^{-1} - C_2\, \beta \geq C_3 > 0.$$

Therefore

$$\inf_{\lambda \in \Lambda_1} I_n(\lambda) \geq C \sum_{k=0}^{\left[\frac{n}{\tau}\right]} \left(\inf_{\lambda \in \Lambda_1} d_{2k} - \sup_{\lambda \in \Lambda_1} d_{1k} \right) \geq C'' n \geq \gamma \ln \sqrt{n},$$

as desired. Now we consider the set Λ_2 on which we have

$$\lambda_2 \geq \frac{c_0 \sqrt{n}}{2\sqrt{1+\beta^{-2}}}, \qquad 0 \leq \frac{\lambda_1}{\lambda_2} \leq \frac{1}{\beta}.$$

Letting

$$c_k = \frac{3\pi}{2\, \vartheta_{02}} + k\, \tau, \qquad d_k = c_k + \delta,$$

we can write

$$I_n(\lambda) \geq \sum_{k=\left[\frac{n}{2\tau}\right]}^{\left[\frac{n}{\tau}\right]} \int_{c_k}^{d_k} \sin^2\left(\lambda_1\, f_1(x) + \lambda_2\, f_2(x)\right)\, \mathrm{d}x \geq$$

$$\geq \sum_{k=\left[\frac{n}{2\tau}\right]}^{\left[\frac{n}{\tau}\right]} \int_{\nu_{1k}}^{\nu_{2k}} \frac{\sin^2(u)}{\lambda_1\, f_1'(x) + \lambda_2\, f_2'(x)}\, \mathrm{d}u.$$

Observe that on the interval $[c_k, c_k + \delta]$ with $\delta > 0$ enough small the function $g'(x)$ has a positive lower bound, which follows easily from the fact that $\eta > \vartheta_{01}$. Indeed since the intensity function is supposed to be positive then $S(\vartheta_0, \frac{3\pi}{2\vartheta_{02}}) = \eta - \vartheta_{01} > 0$.

Furthermore on the interval $[c_k, c_k + \delta]$ the functions $g(\cdot)$ and $h^{0}(\cdot)$ are nonnegative and also $0 \le \frac{\lambda_1}{\lambda_2} \le \beta^{\text{¡}\,1}$. Hence we obtain

$$\frac{C_1\,\lambda_2}{\sqrt{n}} \le \lambda_1\, f_1^{0}(x) + \lambda_2\, f_2^{0}(x) \le \frac{C_2\,\lambda_2}{\sqrt{n}},$$

where C_1 and C_2 are two positive constants. Therefore the denominator of the integrand has a positive lower bound on the set Λ_2 and

$$I_n(\lambda) \ge \frac{C\,\sqrt{n}}{\lambda_2} \sum_{k=\left[\frac{n}{2\tau}\right]}^{\left[\frac{n}{\tau}\right]} \int_{\nu_{1k}}^{\nu_{2k}} \sin^2(u)\;\mathrm{d}u.$$

The difference of limits of the integral admits the estimation

$$\begin{aligned} \nu_{2k} - \nu_{1k} = &\left(\frac{\lambda_1}{\sqrt{n}} + \lambda_2\,\gamma_n\right)\left[h\left(\frac{3\,\pi}{2\,\vartheta_{02}} + \delta\right) - h\left(\frac{3\,\pi}{2\,\vartheta_{02}}\right)\right] + \\ &+ \left(\lambda_1\,\gamma_n + \frac{\lambda_2}{n^{3/2}}\right) d_k\, g\left(\frac{3\,\pi}{2\,\vartheta_{02}} + \delta\right) \ge C_1\left(\frac{\lambda_1}{\sqrt{n}} + \lambda_2\,\gamma_n\right) + \frac{C_2\,\lambda_2}{\sqrt{n}} \ge \\ &\ge C_1\,\gamma_n\,\lambda_2 + \frac{C_2\,\lambda_2}{\sqrt{n}} \ge \frac{C\,\lambda_2}{\sqrt{n}} \end{aligned} \tag{4.5}$$

for some positive constant C. Now we consider two following cases

$$\frac{c_0\,\sqrt{n}}{2\sqrt{1 + \beta^{\text{¡}\,2}}} \le \lambda_2 \le L\,\sqrt{n}, \qquad \lambda_2 > L\,\sqrt{n},$$

separately, where $L > 0$ is a constant enough large. Note that by (4.5) the inequality $\nu_{2k} - \nu_{1k} \ge C^{0} > 0$ holds on the set Λ_2 which implies

$$\int_{\nu_{1k}}^{\nu_{2k}} \sin^2(u)\;\mathrm{d}u \ge C$$

for some constant $C > 0$. Therefore in the first case

$$I_n(\lambda) \ge \frac{\sqrt{n}}{\lambda_2} \sum_{k=\left[\frac{n}{2\tau}\right]}^{\left[\frac{n}{\tau}\right]} C \ge C^{0} n.$$

In the second case we estimate the integral as follows

$$\begin{aligned} &\int_{\nu_{1k}}^{\nu_{2k}} \sin^2(u)\;\mathrm{d}u = \frac{1}{2}\left(\nu_{2k} - \nu_{1k} - \frac{\sin(2\,\nu_{2k})}{2} + \frac{\sin(2\,\nu_{1k})}{2}\right) \ge \\ &\ge \frac{1}{2}\left(\nu_{2k} - \nu_{1k} - 1\right) \ge \frac{1}{2}\left(\frac{C\,\lambda_2}{\sqrt{n}} - 1\right) \ge \frac{C\,\lambda_2}{2\,\sqrt{n}} \Bigg(\text{15Tf16.1.95Tf2Tf4.230TD[(k)]TJ F3c911.95T} \end{aligned}$$

where we used the fact that $L > 0$ can be chosen enough large. Therefore in this case

$$I_n(\lambda) \geq \frac{C\sqrt{n}}{\lambda_2} \sum_{k=\left[\frac{n}{2\tau}\right]}^{\left[\frac{n}{\tau}\right]} \int_{\nu_{1k}}^{\nu_{2k}} \sin^2(u)\ \mathrm{d}u \geq \sum_{k=\left[\frac{n}{2\tau}\right]}^{\left[\frac{n}{\tau}\right]} C_1 \geq C_2\, n.$$

These estimates allow us to write

$$\inf_{\lambda \in \Lambda_2} I_n(\lambda) \geq C' n \geq \gamma \ln \sqrt{n}.$$

The second quadrant can be treated similarly and we omit the details. Therefore the conditions $\mathscr{B}_1 - \mathscr{B}_2$ are satisfied. Hence we have the representation

$$\mathbf{P}_{\vartheta_0}^{(n)}\{Y_n \in \mathbb{C}\} = \int_{\mathbb{C}} h_n(y_1, y_2)\, \mathrm{d}y_1\, \mathrm{d}y_2 + O(n^{-\frac{3}{2}}). \tag{4.6}$$

Here the density

$$\begin{aligned} h_n(y_1, y_2) &= (2\pi)^{-1} \exp\left\{-\frac{y_1^2 + y_2^2}{2}\right\} \{1 + Q_n(3,0)\, H_3(y_1) + \\ &+ Q_n(2,1)\, H_2(y_1)\, H_1(y_2) + Q_n(1,2)\, H_1(y_1)\, H_2(y_2) + Q_n(0,3)\, H_3(y_2) + \\ &+ R_n(4,0)\, H_4(y_1) + R_n(3,1)\, H_3(y_1)\, H_1(y_2) + R_n(2,2)\, H_2(y_1)\, H_2(y_2) + \\ &+ R_n(1,3)\, H_1(y_1)\, H_3(y_2) + R_n(0,4)\, H_4(y_2) + \frac{1}{2}\, Q_n^2(3,0)\, H_6(y_1) + \\ &+ \frac{1}{2}\left(2\, Q_n(3,0)\, Q_n(2,1) + Q_n^2(2,1)\right) H_4(y_1) H_2(y_2) + \\ &+ \frac{1}{2}\left(2\, Q_n(0,3)\, Q_n(1,2) + Q_n^2(1,2)\right) H_2(y_1) H_4(y_2) + \frac{1}{2}\, Q_n^2(0,3)\, H_6(y_2)\Big\}, \end{aligned}$$

with the coefficients

$$\frac{1}{\nu_1!\, \nu_2!} \int_0^n f_1^{\nu_1}(x)\, f_2^{\nu_2}(x)\, S(\vartheta_0, x)\, \mathrm{d}x = \begin{cases} Q_n(\nu_1, \nu_2), & \text{if } \nu_1 + \nu_2 = 3, \\ R_n(\nu_1, \nu_2), & \text{if } \nu_1 + \nu_2 = 4. \end{cases}$$

Therefore the equation

$$\mathbf{P}_{\vartheta_0}^{(n)}\left\{\Delta_n' \, I_n^{-1}(\vartheta_0) \Delta_n > c_{n,\alpha}\right\} = \alpha + O(n^{-3/2})$$

is satisfied with

$$\begin{aligned} c_{n,\alpha} &= -2\, \ln\alpha - 4\, a_n\, \ln\alpha + 8\, b_n\, \ln^2\alpha - 16\, c_n\, \ln^3\alpha + O(n^{-2}) = \\ &= (1 + 2\, a_n)\, \chi_{2,\alpha}^2 + 2\, b_n\, \chi_{2,\alpha}^4 + 2\, c_n\, \chi_{2,\alpha}^6 + O(n^{-2}), \end{aligned}$$

where

$$\begin{aligned}
a_n &= \frac{3}{2}\left[R_n(0,4) + R_n(4,0)\right] + \frac{1}{2} R_n(2,2) - \frac{15}{4}\left[Q_n^2(3,0) + Q_n^2(0,3)\right] - \frac{3}{4}\, d_n \\
b_n &= \frac{-3}{8}\left[R_n(0,4) + R_n(4,0)\right] - \frac{1}{8} R_n(2,2) + \frac{5}{18}\left[Q_n^2(3,0) + Q_n^2(0,3)\right] + \frac{3}{4}\, d_n \\
c_n &= \frac{-5}{32}\left[Q_n^2(3,0) + Q_n^2(0,3)\right] - \frac{5}{32}\, d_n,
\end{aligned} \tag{4.7}$$

and

$$d_n = 2\, Q_n(3,0)\, Q_n(2,1) + 2\, Q_n(0,3)\, Q_n(1,2) + Q_n^2(2,1) + Q_n^2(1,2).$$

Now we consider the power of the Rao score test at *local alternative* of the form

$$\vartheta_u = (\vartheta_{01} + \frac{u_1}{\sqrt{n}},\ \vartheta_{02} + \frac{u_2}{n^{3/2}})$$

where $u = (u_1, u_2)' \neq 0$. Note that the components of ϑ_u converge to the corresponding components of the null hypothesis $\vartheta = \vartheta_0$ with two different rates $n^{-1/2}$ and $n^{-3/2}$. The power is equal to

$$\beta_n(u) = \mathbf{P}_{\vartheta_u}^{(n)}\left\{\Delta_n'\, I_n^{-1}(\vartheta_0)\Delta_n > c_{n,\alpha}\right\} = \mathbf{P}_{\vartheta_u}^{(n)}\left\{Y_n' Y_n > c_{n,\alpha}\right\}.$$

The mean and the covariance matrix of the vector Y_n under ϑ_u are given by

$$\begin{aligned}
&\mu_n(u) \equiv (\mu_1(u), \mu_2(u))' = \mathbf{E}_{\vartheta_u} Y_n = \\
&= \left(\int_0^n f_1(x)(S(\vartheta_u, x) - S(\vartheta_0, x))\,\mathrm{d}x,\ \int_0^n f_2(x)(S(\vartheta_u, x) - S(\vartheta_0, x))\,\mathrm{d}x\right)'
\end{aligned}$$

and

$$\mathbf{\Sigma}_n = \begin{pmatrix} \int_0^n f_1^2(x)\, S(\vartheta_u, x)\,\mathrm{d}x & \int_0^n f_1(x)\, f_2(x)\, S(\vartheta_u, x)\,\mathrm{d}x \\ \int_0^n f_1(x)\, f_2(x)\, S(\vartheta_u, x)\,\mathrm{d}x & \int_0^n f_2^2(x)\, S(\vartheta_u, x)\,\mathrm{d}x \end{pmatrix}.$$

The Taylor expansion

$$S(\vartheta_u, x) = S(\vartheta_0, x) + \frac{u_1}{\sqrt{n}} \sin\left(\vartheta_{02}^{(n)}\, x\right) + \frac{u_2}{n^{3/2}}\, \vartheta_{01}^{(n)}\, x\, \cos\left(\vartheta_{02}^{(n)}\, x\right), \tag{4.8}$$

where $(\vartheta_{01}^{(n)}, \vartheta_{02}^{(n)})$ (depending on x) is an intermediate point, implies that the elements of the main diagonal have the form $1 + O(n^{-1/2})$ and the covariance term is of order

$O(n^{-1/2})$. Therefore we obtain

$$\mathbf{\Sigma}_n^{-1/2} = \begin{pmatrix} p_n & q_n \\ q_n & s_n \end{pmatrix},$$

where $p_n = 1 + O(n^{-1/2})$, $q_n = O(n^{-1/2})$ and $s_n = 1 + O(n^{-1/2})$. Now we consider the normalized random vector

$$\begin{aligned} W_n = (W_1, W_2)' &= \mathbf{\Sigma}_n^{-1/2} \left(Y_n - \mu_n(u)\right) = \\ &= \left(\int_0^n g_1(x)\, \pi_u(\mathrm{d}x),\ \int_0^n g_2(x)\, \pi_u(\mathrm{d}x)\right)' \end{aligned}$$

where $\pi_u(\mathrm{d}x) = X^{(n)}(\mathrm{d}x) - S(\vartheta_u, x)\,\mathrm{d}x$ and the functions

$$\begin{aligned} g_1(x) &= p_n\, f_1(x) + q_n\, f_2(x) \\ g_2(x) &= q_n\, f_1(x) + s_n\, f_2(x). \end{aligned}$$

Therefore by the same way as for $f_1(\cdot)$ and $f_2(\cdot)$ it can be shown that the conditions $\mathscr{B}_1 - \mathscr{B}_2$ are satisfied for $g_1(\cdot)$ and $g_2(\cdot)$. Hence we obtain the following representation

$$\mathbf{P}_{\vartheta_u}^{(n)} \left\{W_n \in \mathbb{C}\right\} = \int_{\mathbb{C}} h_n^{(u)}(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 + O(n^{-\frac{3}{2}}).$$

The density $h_n^{(u)}(y_1, y_2)$ has the same form as $h_n(y_1, y_2)$ with the corresponding coefficients (depending on u) given by

$$\int_0^n g_1^{\nu_1}(x)\, g_2^{\nu_2}(x) S(\vartheta_u, x)\,\mathrm{d}x = \begin{cases} Q_n^{(u)}(\nu_1, \nu_2), & \text{if } \nu_1 + \nu_2 = 3, \\ R_n^{(u)}(\nu_1, \nu_2), & \text{if } \nu_1 + \nu_2 = 4. \end{cases}$$

Therefore the power of the Rao score test at the local alternative ϑ_u admits the representation

$$\begin{aligned} \beta_n(u) = 1 - \mathbf{P}_{\vartheta_u}^{(n)} \left\{Y_n \in \mathbb{C}_\alpha^{(n)}\right\} &= 1 - \mathbf{P}_{\vartheta_u}^{(n)} \left\{W_n \in \mathbf{\Sigma}_n^{-1/2} \left(\mathbb{C}_\alpha^{(n)} - \mu_n(u)\right)\right\} = \\ &= 1 - \int_{\mathbf{\Sigma}_n^{-1/2}\left(\mathbb{C}_\alpha^{(n)} - \mu_n(u)\right)} h_n^{(u)}(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 + O(n^{-\frac{3}{2}}). \end{aligned}$$

The convex set

$$\boldsymbol{\Sigma}_n^{-1/2}\left(\mathbb{C}_\alpha^{(n)} - \mu_n(u)\right) = \left\{\boldsymbol{\Sigma}_n^{-1/2}\left(x - \mu_n(u)\right),\ x \in \mathbb{C}_\alpha^{(n)}\right\}$$

is the interior of the ellipse

$$y'\boldsymbol{\Sigma}_n\, y + 2\,\mu_n'(u)\,\boldsymbol{\Sigma}_n^{1/2}\, y + \mu_n'(u)\,\mu_n(u) - c_{n,\alpha} = 0$$

which can be written as

$$A\, y_1^2 + B\, y_2^2 + 2\, C\, y_1\, y_2 + 2\, D\, y_1 + 2\, E\, y_2 + F = 0,$$

where

$$\begin{aligned}
&A = \frac{q_n^2 + s_n^2}{\left(p_n\, s_n - q_n^2\right)^2}, \quad B = \frac{p_n^2 + q_n^2}{\left(p_n\, s_n - q_n^2\right)^2}, \quad C = \frac{-\, q_n\,(p_n + s_n)}{\left(p_n\, s_n - q_n^2\right)^2}\\
&D = \frac{s_n\,\mu_1(u) - q_n\,\mu_2(u)}{p_n\, s_n - q_n^2}, \quad E = \frac{p_n\,\mu_2(u) - q_n\,\mu_1(u)}{p_n\, s_n - q_n^2},\\
&F = \mu_n'(u)\,\mu_n(u) - c_{n,\alpha} = \mu_1(u)^2 + \mu_2(u)^2 - c_{n,\alpha}.
\end{aligned}$$

Remark 11. *The matrix $\boldsymbol{\Sigma}^{-1/2}$ can be written as $\boldsymbol{\Sigma}^{-1/2} = I + \mathbf{O}(n^{-1/2})$, where I is 2×2 identity matrix and $\mathbf{O}(n^{-1/2})$ denotes a matrix with the elements of the order $O(n^{-1/2})$. Therefore*

$$\boldsymbol{\Sigma}_n^{-1/2}\left(\mathbb{C}_\alpha^{(n)} - \mu_n(u)\right) = \mathbb{C}_\alpha^{(n)} - \mu_n(u) + \mathbf{O}(n^{-1/2})$$

and hence we obtain the representation

$$\begin{aligned}
\beta_n(u) &= 1 - (2\,\pi)^{-1} \int_{\mathbb{C}_\alpha^{(n)} - \mu_n(u)} \exp\left\{-\frac{y_1^2 + y_2^2}{2}\right\}\, \mathrm{d}y_1\, \mathrm{d}y_2 + O(n^{-1/2}) = \\
&= \mathbf{P}\left\{\|(\zeta_1, \zeta_2)' + \mu_n(u)\| > c_{n,\alpha}\right\} + O(n^{-1/2}),
\end{aligned}$$

where ζ_1, ζ_2 are two independent standard Gaussian random variables. Furthermore since

$$\mu_n(u) = \left(\frac{u_1}{d_1}, \frac{u_2}{d_2}\right)' + o(1),$$

then we have

$$\beta_n(u) = \mathbf{P}\left\{\left\|(\zeta_1, \zeta_2)' + \left(\frac{u_1}{d_1}, \frac{u_2}{d_2}\right)'\right\| > \chi_{2,\alpha}\right\} + o(1).$$

4.3 Edgeworth expansion

In this section for simplicity in the notations we don't write the parameter ϑ. Let $X^{(n)}$ be a nonhomogeneous Poisson process with intensity function (with respect to the Lebesgue measure) $S(x)$, $x \in \mathbb{A}_n$, where $\mathbb{A}_n$ is a subset of d dimensional Euclidean space $\mathbb{R}^d$. The distribution of the random element $X^{(n)}$ is denoted by $\mathbf{P}^{(n)}$ and the mathematical expectation with respect to $\mathbf{P}^{(n)}$ by $\mathbf{E}$. Let $f_1, f_2, \ldots, f_k$ be k real valued Borel measurable functions (depending on n) defined on $\mathbb{A}_n$ and introduce k dimensional stochastic integral

$$Y_n = \left(\int_{\mathbb{A}_n} f_1(x)\, \pi(\mathrm{d}x), \; \ldots, \; \int_{\mathbb{A}_n} f_k(x)\, \pi(\mathrm{d}x) \right)',$$

where $\pi(\mathrm{d}x) = X^{(n)}(\mathrm{d}x) - S(x)\,\mathrm{d}x$ is the centered Poisson process and prime means the transposition. Here the stochastic integrals

$$\int_{\mathbb{A}_n} f_\nu(x)\, \pi(\mathrm{d}x) = \sum_{x_i \in \mathbb{A}_n} f_\nu(x_i), \qquad \nu = 1, \ldots, k$$

where $\{x_i\}$ are the *events* (random points) of the Poisson process. The random vector Y_n has mean $\mathbf{E}(Y_n) = 0$ and covariance matrix

$$\mathbf{\Sigma}_n = [\, \sigma_{ij}(n) \,], \qquad i, j = 1, \ldots, k$$

where

$$\sigma_{ij}(n) = \mathbf{E} \left\{ \int_{\mathbb{A}_n} f_i(x)\, \pi(\mathrm{d}x) \int_{\mathbb{A}_n} f_j(x)\, \pi(\mathrm{d}x) \right\} = \int_{\mathbb{A}_n} f_i(x)\, f_j(x)\, S(x)\, \mathrm{d}x.$$

For proof see [27], page 18. We suppose that $\mathbf{\Sigma}_n$ is nonsingular. Because otherwise Y_n has a degenerate distribution and we can obtain a random vector with the same distribution whose support is a strict subspace of $\mathbb{R}^k$. In this section we obtain an Edgeworth type expansion for the distribution function of the k dimensional stochastic integral Y_n in terms of the powers of a small parameter $\varepsilon_n \to 0$. Our aim is to give an expansion useful for the Rao score test and we do not seek the general case. For simplicity of the exposition we consider only two terms (of orders $O(\varepsilon_n)$ and $O(\varepsilon_n^2)$)

after the Gaussian term. Let us introduce the vector notations

$$\mathbf{f}_n(x) = (f_1(x), \dots, f_k(x))', \quad \lambda = (\lambda_1, \dots, \lambda_k)' \in \mathbb{R}^k$$

and the inner product $\langle \lambda, \mathbf{f}_n(x) \rangle = \sum_{\nu=1}^{k} \lambda_\nu f_\nu(x)$. The Euclidian norm in $\mathbb{R}^k$ is denoted by $\|\cdot\|$ and we use the usual notation $|\cdot|$ for (absolute value of) a real or complex number. Below we need the Hermit polynomials

$$H_0(t) = 1,\ H_1(t) = t,\ H_2(t) = t^2 - 1,\ H_3(t) = t^3 - 3t,\ H_4(t) = t^4 - 6t^2 + 3,$$
$$H_5(t) = t^5 - 10t^3 + 15t,\ H_6(t) = t^6 - 15t^4 + 45t^2 - 15.$$

First assume that $\mathbf{\Sigma}_n = I$, where I denotes the $k \times k$ identity matrix. The expansion for two terms after the Gaussian term is obtained under the following conditions:

$\mathscr{B}_1$. *There exists a sequence of real numbers* $\varepsilon_n \to 0$, *as* $n \to \infty$ *and constants* $C_p > 0,\ p = 3, 4, 5$ *such that*

$$\int_{\mathbb{A}_n} \left(|f_1(x)|^p + \dots + |f_k(x)|^p\right) S(x)\,\mathrm{d}x \le C_p\,\varepsilon_n^{p-2}.$$

$\mathscr{B}_2$. *There exists* $\gamma \ge \frac{3+3k}{2}$ *such that the inequality*

$$\inf_{2\|\lambda\| \le c_0 \varepsilon_n^{-1}} \int_{\mathbb{A}_n} \sin^2\left(\langle \lambda, \mathbf{f}_n(x)\rangle\right) S(x)\,\mathrm{d}x \ge \gamma \ln \varepsilon_n^{-1}$$

holds for small values of $c_0 > 0$.

Theorem 29. *Let* $\mathbf{\Sigma}_n = I$ *and the conditions* $\mathscr{B}_1 - \mathscr{B}_2$ *be satisfied. Then uniformly over all convex subsets* $\mathbb{C}$ *of* $\mathbb{R}^k$ *with the Lebesgue measure less than to a given* $\eta > 0$ *we have*

$$\mathbf{P}^{(n)}\left\{Y_n \in \mathbb{C}\right\} = \int_{\mathbb{C}} h_n(y)\,\mathrm{d}y + O(\varepsilon_n^3), \tag{4.9}$$

where for $y = (y_1, \dots, y_k)' \in \mathbb{R}^k$ *the density*

$$h_n(y) = \frac{e^{\frac{-\|y\|^2}{2}}}{\sqrt{(2\pi)^k}} \Bigg\{ 1 + \sum_{\nu_1 + \cdots + \nu_k = 3} Q_n(\nu_1, \dots, \nu_k) H_{\nu_1}(y_1) \dots H_{\nu_k}(y_k) +$$
$$+ \sum_{\nu_1 + \cdots + \nu_k = 4} R_n(\nu_1, \dots, \nu_k) H_{\nu_1}(y_1) \dots H_{\nu_k}(y_k) +$$
$$+ \frac{1}{2} \sum_{\sum \nu_j = 3} \sum_{\sum \nu'_j = 3} Q_n(\nu_1, \dots, \nu_k)\, Q_n(\nu'_1, \dots, \nu'_k) H_{\nu_1 + \nu'_1}(y_1) \dots H_{\nu_k + \nu'_k}(y_k) \Bigg\}.$$

The summations are extended over all nonnegative integers that sum to 3 or 4 and

$$\frac{1}{\nu_1! \dots \nu_k!} \int_{\mathbb{A}_n} f_1^{\nu_1}(x) \dots f_k^{\nu_k}(x)\, S(x)\, \mathrm{d}x = \begin{cases} Q_n(\nu_1, \cdots, \nu_k), & \text{if } \sum \nu_j = 3, \\ R_n(\nu_1, \cdots, \nu_k), & \text{if } \sum \nu_j = 4. \end{cases}$$

Note that the coefficients $Q_n(\nu_1, \dots, \nu_k)$ and $R_n(\nu_1, \dots, \nu_k)$ are dominated by ε_n and ε_n^2, respectively. Indeed by $\mathscr{B}_1$ with $p = 3$,

$$\begin{aligned} |Q_n(\nu_1, \dots, \nu_k)| &\le C_1 \int_{\mathbb{A}_n} \{|f_1(x)| + \dots + |f_k(x)|\}^3 \; S(x)\, \mathrm{d}x \le \\ &\le C_2 \int_{\mathbb{A}_n} \left\{|f_1(x)|^3 + \dots + |f_k(x)|^3\right\} \, S(x)\, \mathrm{d}x \le C_3\, \varepsilon_n, \end{aligned}$$

and similarly $|R_n| \le C\, \varepsilon_n^2$. Here we used the inequality

$$\left| \sum_{i=1}^{m} a_i \right|^s \le m^{s-1} \sum_{i=1}^{m} |a_i|^s, \quad s \ge 1. \tag{4.10}$$

For arbitrary nonsingular covariance matrix $\mathbf{\Sigma}_n$, we use the transformation

$$\mathbf{\Sigma}_n^{-1/2}\, Y_n = \left(\int_{\mathbb{A}_n} g_1(x)\, \pi(\mathrm{d}x),\ \dots,\ \int_{\mathbb{A}_n} g_k(x)\, \pi(\mathrm{d}x) \right)'$$

where the matrix $\mathbf{\Sigma}_n^{-1/2}$ is the square root of $\mathbf{\Sigma}_n^{-1}$. Then uniformly over all convex subsets $\mathbb{C} \subseteq \mathbb{R}^k$

$$\mathbf{P}^{(n)} \{Y_n \in \mathbb{C}\} = \mathbf{P}^{(n)} \left\{\mathbf{\Sigma}_n^{-1/2} Y_n \in \mathbf{\Sigma}_n^{-1/2} \mathbb{C}\right\} = \int_{\mathbf{\Sigma}_n^{-1/2}\mathbb{C}} h_n(y)\, \mathrm{d}y + O(\varepsilon_n^3),$$

where $\mathbf{\Sigma}_n^{-1/2}\mathbb{C} = \left\{\mathbf{\Sigma}_n^{-1/2} y,\ y \in \mathbb{C}\right\}$ is a convex set and the coefficients of $h_n(y)$ are given by

$$\frac{1}{\nu_1! \dots \nu_k!} \int_{\mathbb{A}_n} g_1^{\nu_1}(x) \dots g_k^{\nu_k}(x)\, S(x)\, \mathrm{d}x.$$

Proof. The proof is a modification of of the proof of the Theorem 3.5 in [27], page 135. The characteristic function of Y_n is given by

$$\phi_n(\lambda) = \mathbf{E}\, e^{i\langle \lambda, Y_n \rangle} = \exp\left\{ \int_{\mathbb{A}_n} \left[e^{i\langle \lambda, \mathbf{f}_n(x) \rangle} - 1 - i\langle \lambda, \mathbf{f}_n(x) \rangle \right] \, S(x)\, \mathrm{d}x \right\},$$

which by using the formula

$$e^{iu} = 1 + iu - \frac{u^2}{2!} + \frac{(iu)^3}{3!} + \frac{u^4}{4!} + \frac{(iu)^5}{5!} \int_0^1 e^{iut}\,(1-t)^4\,\mathrm{d}t,$$

can be written as

$$\begin{aligned}\phi_n(\lambda) &= \exp\left\{-\frac{\|\lambda\|^2}{2}\right\}\\ &\exp\left\{\frac{i^3}{3!}\int_{\mathbb{A}_n}\langle\lambda,\mathbf{f}_n(x)\rangle^3\,S(x)\,\mathrm{d}x + \frac{1}{4!}\int_{\mathbb{A}_n}\langle\lambda,\mathbf{f}_n(x)\rangle^4\,S(x)\,\mathrm{d}x + r_n(\lambda)\right\}.\end{aligned}$$

The reminder term $r_n(\lambda)$ by the Cauchy-Schwartz inequality and $\mathscr{B}_1$ with $p=5$

$$\begin{aligned}|r_n(\lambda)| &\le C\int_{\mathbb{A}_n}|\langle\lambda,\mathbf{f}_n(x)\rangle|^5\;S(x)\,\mathrm{d}x \le\\ &\le C\,\|\lambda\|^5\int_{\mathbb{A}_n}\sum_{\nu=1}^{k}|f_\nu(x)|^5\;S(x)\,\mathrm{d}x \le C\,\|\lambda\|^5\varepsilon_n^3,\end{aligned}$$

for some constant $C>0$ and all $\lambda\in\mathbb{R}^k$. We define $\gamma_{3,n},\gamma_{4,n}$ and $\gamma_{5,n}$ (depending on λ) by

$$\begin{aligned}\gamma_{3,n} &= \frac{i^3\varepsilon_n^{\text{¡}\,1}}{3!}\int_{\mathbb{A}_n}\langle\lambda,\mathbf{f}_n(x)\rangle^3\,S(x)\,\mathrm{d}x\\ \gamma_{4,n} &= \frac{\varepsilon_n^{\text{¡}\,2}}{4!}\int_{\mathbb{A}_n}\langle\lambda,\mathbf{f}_n(x)\rangle^4\,S(x)\,\mathrm{d}x\\ \gamma_{5,n} &= \varepsilon_n^{\text{¡}\,3}r_n(\lambda),\end{aligned}$$

which by $\mathscr{B}_1$ satisfy the inequalities

$$|\gamma_{j,n}| \le C^{\text{¤}}\,\|\,\lambda\,\|^j, \qquad j=3,4,5,$$

for some constant $C^{\text{¤}}>0$. Now we can write

$$\phi_n(\lambda) = e^{\text{¡}\,\frac{\|\lambda\|^2}{2}}\exp\left\{\gamma_{3,n}\,\varepsilon_n + \gamma_{4,n}\,\varepsilon_n^2 + \gamma_{5,n}\,\varepsilon_n^3\right\}.$$

For the function

$$g(z) = \exp\left\{\gamma_{3,n}\,z + \gamma_{4,n}\,z^2 + \gamma_{5,n}\,z^3\right\}$$

by the Taylor formula we have

$$g(z) = 1 + z\,\gamma_{3,n} + \frac{z^2}{2!}\left(2\,\gamma_{4,n} + \gamma_{3,n}^2\right) + \frac{z^3}{3!}\,g^{(3)}(\tilde z),$$

where $\tilde{z}$ is an intermediate point. Letting $z = \varepsilon_n$ yields

$$
\begin{aligned}
\phi_n(\lambda) = e^{\mathsf{i}\ \frac{\|\lambda\|^2}{2}} g(\varepsilon_n) = e^{\mathsf{i}\ \frac{\|\lambda\|^2}{2}} \left\{ 1 + \varepsilon_n\, \gamma_{3,n} + \frac{\varepsilon_n^2}{2!} \left(2\, \gamma_{4,n} + \gamma_{3,n}^2 \right) \right\} + \\
+ e^{\mathsf{i}\ \frac{\|\lambda\|^2}{2}} \frac{\varepsilon_n^3}{3!}\ g^{(3)}(\tilde{\varepsilon}_n)
\end{aligned}
$$

where $|\tilde{\varepsilon}_n| \le \varepsilon_n$. It is easy to see that for all λ in the domain $\|\lambda\| < c_0\, \varepsilon_n^{\mathsf{i}\,1}$ we have the estimate

$$
\left| g^{(3)}(\tilde{\varepsilon}_n) \right| \le C\ |q\left(\|\lambda\|\right)|\ \exp\left\{ C^{¤} \left(c_0 + c_0^2 + c_0^3\right) \|\lambda\|^2 \right\}
$$

for some constant $C > 0$ and $q(\cdot)$ is a polynomial (not depending on n) of order 9 of $\|\lambda\|$. Therefore the reminder term

$$
\left| e^{\mathsf{i}\ \frac{\|\lambda\|^2}{2}} \frac{\varepsilon_n^3}{3!}\ g^{(3)}(\tilde{\varepsilon}_n) \right| \le C^{0} \varepsilon_n^3\ |q\left(\|\lambda\|\right)|\ \exp\left\{ \left(C^{¤} \left(c_0 + c_0^2 + c_0^3\right) - \frac{1}{2} \right) \|\lambda\|^2 \right\}. \tag{4.11}
$$

The exponent $\alpha = C^{¤} \left(c_0 + c_0^2 + c_0^3\right) - \frac{1}{2}$ is negative for $c_0 > 0$ enough small (see $\mathscr{B}_2$). The characteristic function $\phi_n(\lambda)$ we write as

$$
\phi_n(\lambda) = \Psi_n(\lambda) + e^{\mathsf{i}\ \frac{\|\lambda\|^2}{2}} \frac{\varepsilon_n^3}{3!}\ g^{(3)}(\tilde{\varepsilon}_n)
$$

where

$$
\begin{aligned}
\Psi_n(\lambda) &= e^{\mathsf{i}\ \frac{\|\lambda\|^2}{2}} \left\{ 1 + \varepsilon_n\, \gamma_{3,n} + \frac{\varepsilon_n^2}{2!} \left(2\, \gamma_{4,n} + \gamma_{3,n}^2 \right) \right\} = \\
&= e^{\mathsf{i}\ \frac{\|\lambda\|^2}{2}} \left\{ 1 + \frac{i^3}{3!} \int_{\mathbb{A}_n} \langle \lambda, \mathbf{f}_n(x) \rangle^3\, S(x)\, \mathrm{d}x + \right. \\
&\left. + \frac{1}{4!} \int_{\mathbb{A}_n} \langle \lambda, \mathbf{f}_n(x) \rangle^4\, S(x)\, \mathrm{d}x - \frac{1}{72} \left(\int_{\mathbb{A}_n} \langle \lambda, \mathbf{f}_n(x) \rangle^3\, S(x)\, \mathrm{d}x \right)^2 \right\} = \\
&= e^{\mathsf{i}\ \frac{\|\lambda\|^2}{2}} \left\{ 1 + \sum_{\nu_1 + ¢¢¢ \nu_k = 3} Q_n(\nu_1, \dots, \nu_k) (i\lambda_1)^{\nu_1} \dots (i\lambda_k)^{\nu_k} + \right. \\
&+ \sum_{\nu_1 + ¢¢¢ \nu_k = 4} R_n(\nu_1, \dots, \nu_k) (i\lambda_1)^{\nu_1} \dots (i\lambda_k)^{\nu_k} + \\
&\left. + \frac{1}{2} \sum_{\sum \nu_j = 3} \sum_{\sum \nu_j' = 3} Q_n(\nu_1, \dots, \nu_k)\, Q_n(\nu_1^{0}, \dots, \nu_k^{0}) (i\lambda_1)^{\nu_1 + \nu_1'} \dots (i\lambda_k)^{\nu_k + \nu_k'} \right\}.
\end{aligned}
$$

The inverse Fourier transform of $\Psi_n(\lambda)$ is equal to

$$h_n(y) = \frac{e^{\frac{-\|y\|^2}{2}}}{\sqrt{(2\pi)^k}} \left\{ 1 + \sum_{\nu_1+\cdots+\nu_k=3} Q_n(\nu_1,\dots,\nu_k)\, H_{\nu_1}(y_1)\dots H_{\nu_k}(y_k) + \right.$$
$$+ \sum_{\nu_1+\cdots+\nu_k=4} R_n(\nu_1,\dots,\nu_k)\, H_{\nu_1}(y_1)\dots H_{\nu_k}(y_k) +$$
$$\left. + \frac{1}{2} \sum_{\sum \nu_j=3} \sum_{\sum \nu_j'=3} Q_n(\nu_1,\dots,\nu_k)\, Q_n(\nu_1',\dots,\nu_k') H_{\nu_1+\nu_1'}(y_1)\dots H_{\nu_k+\nu_k'}(y_k) \right\}.$$

Remind that the inverse Fourier transform of the function $f(t) = (it)^\nu\, e^{-\frac{t^2}{2}}$ is equal to $\hat{f}(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}\, H_\nu(x)$. Let us define the measures

$$\mu_n(\mathbb{E}) = \mathbf{P}^{(n)}\left\{Y_n \in \mathbb{E}\right\}, \qquad \nu_n(\mathbb{E}) = \int_{\mathbb{E}} h_n(y)\, \mathrm{d}y$$

on the measurable space $\left(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)\right)$ of Borel sets. The measure ν_n is a finite signed measure with the density $h_n(y)$ with respect to the Lebesgue measure. Let $\mathbb{C}$ be a convex set in $\mathbb{R}^k$. Our goal is to estimate the difference

$$\mathbf{P}^{(n)}\left\{Y_n \in \mathbb{C}\right\} - \int_{\mathbb{C}} h_n(y)\, \mathrm{d}y = \mu_n(\mathbb{C}) - \nu_n(\mathbb{C}).$$

We follow [4], page 17. Fix $\varepsilon > 0$ and for real Borel measurable function $F(\cdot)$ on $\mathbb{R}^k$ define

$$M_F(y;\varepsilon) = \sup_{z \in B(y,\varepsilon)} F(z), \qquad m_F(y;\varepsilon) = \inf_{z \in B(y,\varepsilon)} F(z)$$
$$B(y,\varepsilon) = \left\{ z \in \mathbb{R}^k \ :\ \|z-y\| < \varepsilon \right\}.$$

Lemma 3. *For any $\varepsilon > 0$ there exists a probability measure K on measurable space $\left(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)\right)$ with*

$$K\left(B(0,\varepsilon)\right) = 1$$

such that its characteristic function $\hat{K}$ satisfies the inequality

$$\left|\hat{K}(\lambda)\right| \le C \exp\left\{-\|\varepsilon\,\lambda\|^{1/2}\right\}, \qquad \lambda \in \mathbb{R}^k$$

where the constant C doesn't depend on ε.

Proof. See [3], page 87.

Lemma 4. *Let μ be a finite measure and ν be a finite signed measure on $\left(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)\right)$. Then for any real valued Borel measurable function $F(\cdot)$ on $\mathbb{R}^k$ we have the estimate*

$$\left| \int_{\mathbb{R}^k} F(\cdot)\ \mathrm{d}(\mu - \nu) \right| \leq \tau_1(F;\varepsilon) + \tau_2(F;\varepsilon)$$

where

$$\begin{aligned}
\tau_1(F;\varepsilon) &= \max\left\{ \int_{\mathbb{R}^k} M_F(\cdot\,;\varepsilon)\ \mathrm{d}(\mu-\nu) \star K,\ -\int_{\mathbb{R}^k} m_F(\cdot\,;\varepsilon)\ \mathrm{d}(\mu-\nu)\star K \right\}, \\
\tau_2(F;\varepsilon) &= \max\left\{ \int_{\mathbb{R}^k} \left(M_F(\cdot\,;\varepsilon) - F(\cdot)\right)\ \mathrm{d}\nu^+,\ \int_{\mathbb{R}^k} \left(F(\cdot) - m_F(\cdot\,;\varepsilon)\right)\ \mathrm{d}\nu^+ \right\}.
\end{aligned}$$

Here ν^+ is the positive part of $\nu = \nu^+ - \nu^{\text{¡}}$ in the Hahn-Jordan decomposition and $\star$ means convolution.

Proof. See [3], Lemma 11.1, page 93.

We continue the proof of Theorem 29. Let

$$F(y) = \begin{cases} 1, & \text{if } \ y \in \mathbb{C} \\ 0, & \text{if } \ y \notin \mathbb{C}, \end{cases}$$

where $\mathbb{C}$ is an arbitrary convex set. The Lemma 4 for the indicator function $F(\cdot)$ and the measures μ_n and ν_n gives

$$\begin{aligned}
\left| \mathbf{P}^{(n)}\left\{Y_n \in \mathbb{C}\right\} - \int_{\mathbb{C}} h_n(y)\ \mathrm{d}y \right| &= \left|\mu_n(\mathbb{C}) - \nu_n(\mathbb{C})\right| \leq \\
&\leq \tau_1(F;\varepsilon) + \tau_2(F;\varepsilon).
\end{aligned} \tag{4.12}$$

Observe that since the Fourier transform $\left(\phi_n(\lambda) - \Psi_n(\lambda)\right)\hat{K}(\lambda)$ of $(\mu_n - \nu_n)\star K$ is absolutely integrable on $\mathbb{R}^k$ then the signed measure $(\mu_n - \nu_n)\star K$ has a density $p_n(y)$. Now we have

$$\begin{aligned}
\tau_1(F;\varepsilon) &= \max\left\{(\mu_n - \nu_n)\star K(\mathbb{C}^\varepsilon),\ -(\mu_n-\nu_n)\star K(\mathbb{C}^{\text{¡}\,\varepsilon})\right\} = \\
&= \max\left\{ \int_{\mathbb{C}^\varepsilon} p_n(y)\ \mathrm{d}y,\ -\int_{\mathbb{C}^{-\varepsilon}} p_n(y)\ \mathrm{d}y \right\} \leq \int_{\mathbb{C}^\varepsilon} |p_n(y)|\ \mathrm{d}y,
\end{aligned} \tag{4.13}$$

where the sets

$$\mathbb{C}^{\varepsilon} = \bigcup_{y \in \mathbb{C}} B(y,\varepsilon), \qquad \mathbb{C}^{-\varepsilon} = \{y\,:\, B(y,\varepsilon) \subseteq \mathbb{C}\}\,.$$

The Fourier inversion formula yields

$$\begin{aligned} |p_n(y)| &\le C \int_{\mathbb{R}^k} \left|(\phi_n(\lambda) - \Psi_n(\lambda))\ \hat{K}(\lambda)\right|\,\mathrm{d}\lambda = \\ &= C\left(\int_{\|\lambda\| < c_0\,\varepsilon_n^{-1}} + \int_{\|\lambda\| \ge c_0\,\varepsilon_n^{-1}}\right) \equiv C\,(I_1 + I_2) \end{aligned}$$

for some constant $C > 0$ and all $y \in \mathbb{R}^k$. The first integral by (4.11) admits the estimation

$$\begin{aligned} I_1 &\le C' \varepsilon_n^3 \int_{\|\lambda\| < c_0\,\varepsilon_n^{-1}} |q\,(\|\lambda\|)|\,\exp\left\{\left(C^{*}\,(c_0 + c_0^2 + c_0^3) - \frac{1}{2}\right)\,\|\lambda\|^2\right\}\mathrm{d}\lambda \le \\ &\le C''\varepsilon_n^3. \end{aligned}$$

For I_2 we write

$$\begin{aligned} I_2 &= \int_{\|\lambda\| \ge c_0\,\varepsilon_n^{-1}} \left|(\phi_n(\lambda) - \Psi_n(\lambda))\ \hat{K}(\lambda)\right|\,\mathrm{d}\lambda \le \\ &\le \int_{\|\lambda\| \ge c_0\,\varepsilon_n^{-1}} |\phi_n(\lambda)|\left|\hat{K}(\lambda)\right|\,\mathrm{d}\lambda + \int_{\|\lambda\| \ge c_0\,\varepsilon_n^{-1}} |\Psi_n(\lambda)|\,\mathrm{d}\lambda \equiv I_2(1) + I_2(2), \end{aligned}$$

with obvious notations. Since $\varepsilon_n^{-1} \to \infty$ then the last integral converges exponentially to zero which implies

$$I_2(2) \le C\varepsilon_n^3.$$

For the first integral by $\mathscr{B}_2$ and the Lemma 3 we have

$$\begin{aligned} I_2(1) &= \int_{\|\lambda\| \ge c_0\,\varepsilon_n^{-1}} \exp\left\{-2\int_{\mathbb{A}_n} \sin^2\left(\langle\frac{\lambda}{2}, \mathbf{f}_n(x)\rangle\right) S(x)\,\mathrm{d}x\right\}\left|\hat{K}(\lambda)\right|\,\mathrm{d}\lambda = \\ &= \int_{\|\lambda\| \ge 2\,c_0\,\varepsilon_n^{-1}} \exp\left\{-2\int_{\mathbb{A}_n} \sin^2\left(\lambda, \mathbf{f}_n(x)\rangle\right) S(x)\,\mathrm{d}x\right\}\left|\hat{K}(\lambda)\right|\,\mathrm{d}\lambda \le \\ &\le \varepsilon_n^{2\gamma}\int_{2\,\|\lambda\| \ge c_0\,\varepsilon_n^{-1}} \left|\hat{K}(\lambda)\right|\,\mathrm{d}\lambda \le C_1\varepsilon_n^{2\gamma}\int_{2\,\|\lambda\| \ge c_0\,\varepsilon_n^{-1}} e^{-\|\varepsilon\lambda\|^{1/2}}\,\mathrm{d}\lambda \le \\ &\le C_2\,\varepsilon_n^{2\gamma}\,\varepsilon^{-k}\int_{\sqrt{\frac{c_0\,\varepsilon\,\varepsilon_n^{-1}}{2}}}^{\infty} u^{2k-1}e^{-u}\,\mathrm{d}u. \end{aligned}$$

Now by letting $\varepsilon = \varepsilon_n^3$ we get

$$I_2(1) \leq C_3\, \varepsilon_n^{2\gamma}\, \varepsilon^{-k} \int_0^1 u^{2k-1} e^{-u}\, \mathrm{d}u \leq C\, \varepsilon_n^{2\gamma - 3k} \leq C\, \varepsilon_n^3.$$

These estimates allow us to write $I_2 \leq C\, \varepsilon_n^3$ and consequently

$$|p_n(y)| \leq C\, \varepsilon_n^3, \qquad y \in \mathbb{R}^k \tag{4.14}$$

for some constant $C > 0$. Turning to (4.13) we can write

$$\tau_1(F;\varepsilon) \leq \int_{\mathbb{C}^\varepsilon} |p_n(y)|\, \mathrm{d}y \leq C\, \varepsilon_n^3. \tag{4.15}$$

It remains to estimate

$$\tau_2(F;\varepsilon) = \max\left\{ \int_{\mathbb{R}^k} \left(M_F(\cdot\,;\varepsilon) - F(\cdot)\right)\, \mathrm{d}\nu_n^+,\; \int_{\mathbb{R}^k} \left(F(\cdot) - m_F(\cdot\,;\varepsilon)\right)\, \mathrm{d}\nu_n^+ \right\}.$$

It can be shown that for the indicator function $F(\cdot)$,

$$M_F(y\,;\varepsilon) - F(y) = \begin{cases} 1, & \text{if } \; y \in \mathbb{C}^\varepsilon - \mathbb{C} \\ 0, & \text{if } \; y \notin \mathbb{C}^\varepsilon - \mathbb{C}, \end{cases}$$

and

$$F(y) - m_F(y\,;\varepsilon) = \begin{cases} 1, & \text{if } \; y \in \mathbb{C} - \mathbb{C}^{-\varepsilon} \\ 0, & \text{if } \; y \notin \mathbb{C} - \mathbb{C}^{-\varepsilon}. \end{cases}$$

Therefore we get

$$\begin{aligned} \tau_2(F;\varepsilon) &\leq \max\left\{ \nu_n^+\left(\mathbb{C}^\varepsilon - \mathbb{C}\right),\, \nu_n^+\left(\mathbb{C} - \mathbb{C}^{-\varepsilon}\right) \right\} \leq \\ &\leq \nu_n^+\left(\mathbb{C}^\varepsilon - \mathbb{C}\right) + \nu_n^+\left(\mathbb{C} - \mathbb{C}^{-\varepsilon}\right) = \nu_n^+\left(\mathbb{C}^\varepsilon - \mathbb{C}^{-\varepsilon}\right) \leq \\ &\leq \int_{\mathbb{C}^\varepsilon - \mathbb{C}^{-\varepsilon}} |h_n(y)|\; \mathrm{d}y. \end{aligned}$$

To estimate the last integral we use the following inequality

$$\begin{aligned} \int_{\mathbb{C}^\varepsilon - \mathbb{C}^{-\rho}} (2\pi)^{-k/2}\, \|y\|^s \exp\left\{ -\frac{\|y\|^2}{2} \right\} \mathrm{d}y \leq \\ \leq 2^{s-1}\left(2s + k - 1\right) \frac{\Gamma\left(k + s - 1/2\right)}{\Gamma\left(k/2\right)} (\varepsilon + \rho) \end{aligned}$$

which is true for any convex set $\mathbb{C}$, $s \geq 0$, $k > 1$ and every pair of positive numbers ε, ρ. For a proof see [3], Corollary 3.2., page 24. It is easy to check that for $k = 1$ there exists a constant $C > 0$ for all convex sets $\mathbb{C}$ such that

$$\int_{\mathbb{C}^{\varepsilon} - \mathbb{C}^{-\varepsilon}} |y|^s \, e^{- y^2/2} \, \mathrm{d}y \leq C \, \varepsilon.$$

Indeed in this case a convex set is a subinterval of the real line and

$$\int_b^{b+\varepsilon} |y|^s \, e^{- y^2/2} \, \mathrm{d}y \leq \varepsilon \sup_{y \in \mathbb{R}} |y|^s \, e^{- y^2/2} \leq C \, \varepsilon$$

uniformly in b. Therefore for every $s \geq 0$, $k \geq 1$ and any convex set $\mathbb{C}$

$$\int_{\mathbb{C}^{\varepsilon} - \mathbb{C}^{-\varepsilon}} \|y\|^s \exp\left\{ -\frac{\|y\|^2}{2} \right\} \mathrm{d}y \leq C \, \varepsilon,$$

where the constant C doesn't depend on $\mathbb{C}$. By considering the structure of $h_n(\cdot)$ and the last inequality we can write

$$\tau_2(F; \varepsilon) \leq \int_{\mathbb{C}^{\varepsilon} - \mathbb{C}^{-\varepsilon}} |h_n(y)| \, \mathrm{d}y \leq C \, \varepsilon = C \, \varepsilon_n^3$$

uniformly for any convex set $\mathbb{C}$. Finally from 4.12, 4.18 and the last inequality we get

$$\left| \mathbf{P}^{(n)} \{ Y_n \in \mathbb{C} \} - \int_{\mathbb{C}} h_n(y) \, \mathrm{d}y \right| \leq \tau_1(F; \varepsilon) + \tau_2(F; \varepsilon) \leq C \, \varepsilon_n^3$$

which completes the proof of Theorem.

Let $2 < r < 3$ be an arbitrary given number and introduce the following more weak condition instead of $\mathscr{B}_2$,

$\mathscr{B}_2^0$. *There exists* $\gamma \geq \frac{3+k\,r}{2}$ *such that the inequality*

$$\inf_{2\|\lambda\| \geq c_0 \varepsilon_n^{-1}} \int_{\mathbb{A}_n} \sin^2\left(\langle \lambda, \mathbf{f}(x) \rangle \right) S(x) \, \mathrm{d}x \geq \gamma \ln \varepsilon_n^{-1}$$

holds for small values of $c_0 > 0$.

Remark 12. *Let the conditions* $\mathscr{B}_1 - \mathscr{B}_2^0$ *be fulfilled. Then uniformly over all convex sets* $\mathbb{C} \subseteq \mathbb{R}^k$

$$\mathbf{P}^{(n)} \{ Y_n \in \mathbb{C} \} = \int_{\mathbb{C}} h_n(y) \, \mathrm{d}y + O(\varepsilon_n^r). \tag{4.16}$$

Proof. The only difference is in the estimations of $I_2(1)$ and $\tau_1(F;\varepsilon)$. By $\mathscr{B}_2^0$ and the Lemma 3 we have

$$I_2(1) \le C_2\, \varepsilon_n^{2\gamma}\, \varepsilon^{-k} \int_{\sqrt{\frac{c_0\,\varepsilon\,\varepsilon_n^{-1}}{2}}}^{1} u^{2k-1} e^{-u}\, \mathrm{d}u,$$

which by letting $\varepsilon = \varepsilon_n^r$ leads to

$$I_2(1) \le C\, \varepsilon_n^{2\gamma - kr} \le C\, \varepsilon_n^3.$$

This estimation allows us to write $I_2 \le C\, \varepsilon_n^3$ and consequently

$$|p_n(y)| \le C\, \varepsilon_n^3, \qquad y \in \mathbb{R}^k \tag{4.17}$$

for some constant $C > 0$. Letting $\eta = \frac{3-r}{k}$, the inequality (4.13) gives

$$\tau_1(F;\varepsilon) \le \int_{\mathbb{R}^k} |p_n(y)|\, \mathrm{d}y = \int_{\|y\| < \varepsilon_n^{-\eta}} |p_n(y)|\, \mathrm{d}y + \int_{\|y\| \ge \varepsilon_n^{-\eta}} |p_n(y)|\, \mathrm{d}y \equiv I_3 + I_4, \tag{4.18}$$

uniformly over all convex sets $\mathbb{C}$, because the right hand side doesn't depend on $\mathbb{C}$. The term I_3 by (4.17) is dominated by

$$I_3 \le C\ \varepsilon_n^{3-k\eta} = C\ \varepsilon_n^r.$$

For the second integral we write

$$\begin{aligned} I_4 = &\int_{B(0,\varepsilon)\setminus\left\{\|y\| \ge \varepsilon_n^{-\eta}\right\}} |\,\mathrm{d}\,(\mu_n - \nu_n) \star K| \le \int_{B(0,\varepsilon)\setminus\left\{\|y\| \ge \varepsilon_n^{-\eta}\right\}} |\mathrm{d}\,\mu_n \star K| + \\ &+ \int_{B(0,\varepsilon)\setminus\left\{\|y\| \ge \varepsilon_n^{-\eta}\right\}} |\mathrm{d}\,\nu_n \star K| \le \mu_n \left\{\|y\| \ge \varepsilon_n^{-\eta} - \varepsilon\right\} + \\ &+ \int_{\left\{\|y\| \ge \varepsilon_n^{-\eta} - \varepsilon\right\}} |h_n(y)|\ \mathrm{d}y \equiv I_4(1) + I_4(2), \end{aligned}$$

with obvious notations. The first term by the corollary 10, Chapter 1 admits the estimation

$$I_4(1) = \mathbf{P}^{(n)} \left\{\|Y_n\| \ge \varepsilon_n^{-\eta} - \varepsilon\right\} = O(\varepsilon_n^r).$$

By considering the structure of $h_n(y)$ and the fact that $\varepsilon_n^{-\eta} - \varepsilon \to \infty$ as $n \to \infty$,

$$I_4(2) = \int_{\left\{\|y\| \ge \varepsilon_n^{-\eta} - \varepsilon\right\}} |h_n(y)|\ \mathrm{d}y \le C\, \varepsilon_n^3,$$

because the integral converges exponentially to zero. Hence $I_4 \leq C\,\varepsilon_n^r$ and we get the estimation

$$\tau_1(F;\varepsilon) \leq I_3 + I_4 \leq C\ \varepsilon_n^r$$

for some constant $C > 0$ and uniformly for all convex sets $\mathbb{C}$, because I_3 and I_4 don't depend on $\mathbb{C}$.

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