Hypotheses Testing: Poisson versus Self-Exciting and Self-Correcting

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Statement of the problem

Model:
Stationary (counting) point process

\[ X_t = f(X_t); \quad t > 0 \]

The simplest case — Poisson process with a constant intensity \( S > 0 \), i.e., the increments of \( X \) on the disjoint intervals are independent and distributed according to Poisson law

\[
P(X_t - X_s = k) = S^k \frac{e^{-S}}{k!} \quad \text{for} \quad 0 \leq s < t; \quad k = 0, 1, \ldots
\]

Observations:
Sequence \( X_T = f(t_1; t_2; \ldots \) of events of \( X \) on \([0; T]\).

Purpose:
Test if \( X \) is a Poisson process (with known intensity).

Alternatives:
Self-exciting and self-correcting processes.

Asymptotics:
\( T \to \infty \).
Model: Stationary (counting) point process $X = \{X_t, \ t \geq 0\}$. 

Observations: Sequence $X_T = \{t_1, t_2, \ldots\}$ of events of $X$ on $[0; T]$. 

Purpose: Test if $X$ is a Poisson process (with known intensity). 

Alternatives: Self-exciting and self-correcting processes. 

Asymptotics: $T \to 1$. 

Model: Stationary (counting) point process \( X = \{ X_t, \ t \geq 0 \} \).

The simplest case — Poisson process with a constant intensity \( S > 0 \), i.e., the increments of \( X \) on the disjoint intervals are independent and distributed according to Poisson law

\[
P\{ X_t - X_s = k \} = \frac{S^k (t - s)^k}{k!} e^{-S(t-s)}, \quad 0 \leq s < t, \quad k = 0, 1, \ldots
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Purpose: Test if \( X \) is a Poisson process (with known intensity).

Alternatives: Self-exciting and self-correcting processes.

Asymptotics: \( T \to \infty \).
Recalls

The process $X$ admits a unique (Doob-Meyer) decomposition $X_t = A_t + M_t$ where $M_t = f_{M_t}^t; F_t > 0$ is a martingale and $A_t = f_{A_t}^t; F_t > 0$ is a predictable increasing process.

$A_t = Z_t^0 S(v;d) dv$ where $S = f_{S_t}(t;d)$ is called the intensity function.
The process $X$ admits a unique (Doob-Meyer) decomposition

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Recalls

The process \( X \) admits a unique (Doob-Meyer) decomposition

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where \( M = \{ M_t, \mathcal{F}_t, t \geq 0 \} \) is a martingale and \( A = \{ A_t, \mathcal{F}_t, t \geq 0 \} \) is a predictable increasing process.

\[
A_t = \int_0^t S(v, \omega) \, dv
\]

where \( S = \{ S(t, \omega), \mathcal{F}_t, t \geq 0 \} \) is called intensity function.
Self-exciting processes


Defined by intensity function of the form

\[ S(t;X) = S_0 + \int_{t-s}^{t} g(s) \, ds \]

where \( S > 0 \) and the function \( g(t) > 0 \) satisfies the condition

\[ \int_{0}^{1} g(t) \, dt < 1 \]

The rate of the process is

\[ \lambda = S_0 + \int_{0}^{1} g(t) \, dt \]

\[ E[X_t] = t \]
Self-exciting processes

Self-exciting processes


Defined by intensity function of the form

$$S(t, X) = S + \int_{-\infty}^{t} g(t - s) \, dX_s$$

where $S > 0$ and the function $g(\cdot) \geq 0$ satisfies the condition

$$\rho = \int_{0}^{\infty} g(t) \, dt < 1.$$
Self-exciting processes


Defined by intensity function of the form

\[ S(t, X) = S + \int_{-\infty}^{t} g(t - s) \, dX_s \]

where \( S > 0 \) and the function \( g(\cdot) \geq 0 \) satisfies the condition

\[ \rho = \int_{0}^{\infty} g(t) \, dt < 1. \]

The rate of the process is \( \mu = \frac{S}{1 - \rho} \), \( \mathbb{E}X_t = \mu \, t \).
Example

Let \( g(t) = \alpha e^{-\gamma t} \), where \( \alpha > 0, \gamma > 0 \) and \( \alpha/\gamma < 1 \). Then the point process \( X \) with intensity function

\[
S(t, X) = S + \alpha \sum_{t_i \leq t} e^{-\gamma (t-t_i)}
\]

is self-exciting with the rate

\[
\mu = \frac{S \gamma}{\gamma - \alpha}.
\]
One-sided parametric alternative

We assume that the observed process is either Poisson with constant intensity $S$ or is a self-exciting process with intensity function $S(t; X) = S + Z_{1h(t; s)} dX_s; > 0$:

The function $h(t; s)$ is supposed to be known and $h(t; s) = f(t; s) > 0$.

To have contiguous alternatives we consider $T = u = \frac{p}{T}$ and test $H_0: u = 0$ against $H_1: u > 0$:
One-sided parametric alternative

We assume that the observed process is either Poisson with constant intensity $S_*$ or is self-exciting process with intensity function

$$S(t, X) = S_* + \theta \int_{-\infty}^{t} h(t - s) \, dX_s, \quad \theta > 0.$$
We assume that the observed process is either Poisson with constant intensity $S_*$ or is self-exciting process with intensity function

$$S(t, X) = S_* + \theta \int_{-\infty}^{t} h(t - s) \, dX_s, \quad \theta > 0.$$ 

The function $h(\cdot)$ is supposed to be known and

$$h(\cdot) \in \mathcal{L}_+ (\mathbb{R}_+) = \left\{ f(\cdot) \geq 0 : \int_{0}^{\infty} f(t) \, dt < \infty \right\}.$$
One-sided parametric alternative

We assume that the observed process is either Poisson with constant intensity $S_*$ or is self-exciting process with intensity function

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$$h(\cdot) \in \mathcal{L}_+(\mathbb{R}_+) = \left\{ f(\cdot) \geq 0 : \int_0^{\infty} f(t) \, dt < \infty \right\}.$$ 

To have contiguous alternatives we consider $\theta = \theta_T = u / \sqrt{T}$ and test

$$\mathcal{H}_0 : \quad u = 0 \quad \text{against} \quad \mathcal{H}_1 : \quad u > 0.$$
LAUMP tests

Randomised test: $T = X_T$ is the probability to accept $H_1$.

Size of the test: $(T) = E_0 T_X T_X$.

Power function of the test: $T_u; T_u = E_u T_X T_X$; $u > 0$.

For $\mu(0; 1)$ we denote $K = n_T$: $\lim_{T \to 1} (T) = o$.

Definition. A test $T$ is called locally asymptotically uniformly most powerful (LAUMP) in the class $K$ if for any other test $T_2 K$ and any constant $K > 0$ we have $\lim_{T \to 1} \inf_0 u K T_u T_u = 0$: $T_u > 0$.
Randomised test: $\phi_T = \phi_T \left( X^T \right)$ is the probability to accept $\mathcal{H}_1$. 
LAUMP tests

Randomised test: $\phi_T = \phi_T (X^T)$ is the probability to accept $\mathcal{H}_1$.

Size of the test: $\alpha(\phi_T) = E_0 \phi_T (X^T)$. 

Randomised test: $\phi_T = \phi_T \left( X^T \right)$ is the probability to accept $\mathcal{H}_1$.

Size of the test: $\alpha(\phi_T) = \mathbb{E}_0 \phi_T \left( X^T \right)$.

Power function of the test: $\beta_T \left( u, \phi_T \right) = \mathbb{E}_u \phi_T \left( X^T \right), \quad u \geq 0$. 
Randomised test: $\phi_T = \phi_T (X^T)$ is the probability to accept $\mathcal{H}_1$.

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Power function of the test: $\beta_T (u, \phi_T) = \mathbb{E}_u \phi_T (X^T)$, $u \geq 0$.

For $\varepsilon \in (0, 1)$ we denote $\mathcal{H}_\varepsilon = \left\{ \phi_T : \lim_{T \to \infty} \alpha(\phi_T) = \varepsilon \right\}$.
LAUMP tests

Randomised test: $\phi_T = \phi_T \left( X^T \right)$ is the probability to accept $\mathcal{H}_1$.

Size of the test: $\alpha(\phi_T) = \mathbb{E}_0 \phi_T \left( X^T \right)$.

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For $\varepsilon \in (0, 1)$ we denote $\mathcal{K}_\varepsilon = \left\{ \phi_T : \lim_{T \to \infty} \alpha(\phi_T) = \varepsilon \right\}$.

**Definition.** A test $\phi^*_T$ is called locally asymptotically uniformly most powerful (LAUMP) in the class $\mathcal{K}_\varepsilon$ if for any other test $\phi_T \in \mathcal{K}_\varepsilon$ and any constant $K > 0$ we have

$$\lim_{T \to \infty} \inf_{0 \leq u \leq K} \left[ \beta_T \left( u, \phi^*_T \right) - \beta_T \left( u, \phi_T \right) \right] \geq 0.$$
We consider

$$
\Delta_T (X^T) = \frac{1}{S_* \sqrt{T I^*_h}} \int_0^T \int_0^{t^-} h(t - s) \, dX_s \, [dX_t - S_\ast \, dt]
$$

\[
= \frac{1}{S_* \sqrt{T I^*_h}} \sum_{0 \leq t_j \leq T} \sum_{t_i < t_j} h(t_j - t_i) - \frac{1}{\sqrt{T I^*_h}} \sum_{0 \leq t_j \leq T} \int_0^{T-t_j} h(v) \, dv
\]

where

$$
I^*_h = \int_0^\infty h(t)^2 \, dt + S_\ast \left( \int_0^\infty h(t) \, dt \right)^2.
$$
First main result

**Theorem.** Let $h(\cdot) \in \mathcal{L}^1_+ (\mathbb{R}_+)$ and bounded. Then the test

$$
\hat{\phi}_T (X^T) = 1 \{ \Delta_T (X^T) > z_\varepsilon \}
$$

is LAUMP in the class $\mathcal{K}_\varepsilon$, and for any $u > 0$

$$
\beta_T \left( u, \hat{\phi}_T \right) \longrightarrow \beta (u) = \mathbb{P} \left\{ \zeta > z_\varepsilon - u \sqrt{I_h^*} \right\}.
$$

Here and in the sequel we denote $\zeta \sim \mathcal{N} (0, 1)$, and $z_\varepsilon$ is the $1 - \varepsilon$ quantile of $\zeta$, i.e., $\mathbb{P} \{ \zeta > z_\varepsilon \} = \varepsilon$. 
Key elements of the proof

The likelihood ratio is LAN under hypothesis $\mathcal{H}_0$, because

$$Z_T(u) = \exp \left\{ \int_0^T \ln \left( 1 + \frac{u}{S_* \sqrt{T}} \int_0^{t-} h(t-s) \, dX_s \right) \, dX_t - \frac{u}{\sqrt{T}} \int_0^T \int_0^t h(t-s) \, dX_s \, dt \right\} =$$

$$= \exp \left\{ u \sqrt{I^*_h} \Delta_T(X^T) - \frac{u^2}{2} I^*_h + r_T(u, X^T) \right\}$$

and $\mathcal{L}_0 \{ \Delta_T(X^T) \} \implies N(0,1)$. 
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and $\mathcal{L}_0 \{ \Delta_T (X^T) \} \implies \mathcal{N} (0, 1)$.

Power of the test: the third lemma of Le Cam.
Key elements of the proof

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$$= \exp \left\{ u \sqrt{I_h^*} \Delta_T (X^T) - \frac{u^2}{2} I_h^* + r_T(u, X^T) \right\}$$

and $\mathcal{L}_0 \{ \Delta_T (X^T) \} \implies \mathcal{N} (0, 1)$.

Power of the test: the third lemma of Le Cam.

LAUMP: our test is asymptotically as good as N-P test for any simple alternative.
Simulations

We take $S = 1$ and $h(t) = \frac{1}{2}e^{-\frac{t}{2}}$, i.e.,

$$S(t; X_t) = 1 + \mu_2 p T X_t i t e^{(t - t_i)} = \frac{1}{2}.$$
Simulations

We take \( S_* = 1 \) and \( h(t) = \frac{1}{2} e^{-t/2} \), i.e.,

\[
S(t, X) = 1 + \frac{u}{2\sqrt{T}} \sum_{t_i \leq t} e^{-(t-t_i)/2}.
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Simulations

We take \( S_* = 1 \) and \( h(t) = \frac{1}{2} e^{-t/2} \), i.e.,

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S(t, X) = 1 + \frac{u}{2\sqrt{T}} \sum_{t_i \leq t} e^{-(t-t_i)/2}.
\]

The test \( \hat{\phi}_T^\varepsilon = \hat{\phi}_T(X^T) = 1 \{ \Delta_T(X^T) > z_\varepsilon \} \) is LAUMP in \( \mathcal{K}_\varepsilon \).
Simulations

We take $S_\ast = 1$ and $h(t) = \frac{1}{2} e^{-t^2/2}$, i.e.,

$$S(t, X) = 1 + \frac{u}{2\sqrt{T}} \sum_{t_i \leq t} e^{-(t-t_i)/2}.$$

The test $\hat{\phi}_T^{\varepsilon} = \hat{\phi}_T(X^T) = \mathbb{1}\{\Delta_T(X^T) > z_{\varepsilon}\}$ is LAUMP in $\mathcal{H}_{\varepsilon}$.

The test $\tilde{\phi}_T^{\varepsilon} = \tilde{\phi}_T(X^T) = \mathbb{1}\{\Delta_T(X^T) > z\}$, where the threshold $z = z(\varepsilon, T)$ is chosen so that this test is of exact size $\varepsilon$, is also LAUMP.
Size of the test $\phi_T^{0.05}$

Size of the test $\phi_{0.05}^T$

Obtained by simulating $10^7$ Poisson trajectories
Power function of the test $\hat{\phi}_{T}^{0.05}$

![Graph showing power function for different values of T (100, 300, 1000) and a Gaussian distribution.](image)
Power function of the test $\hat{\phi}_{T}^{0.05}$

Obtained by simulating $10^6$ self-exciting trajectories (for each $u$)
Threshold for the test $\tilde{\phi}_T^{0.05}$

Threshold for the test $\tilde{\phi}_T^{0.05}$

\[ z \simeq 1.78 \text{ for } T = 100, \quad 1.74 \text{ for } 300, \quad 1.70 \text{ for } 1000, \quad \text{against } z_{0.05} \simeq 1.64 \]
Threshold for the test $\tilde{\phi}_T^{0.05}$ (zoom)
Threshold for the test $\tilde{\phi}_T^{0.05}$ (zoom)

Obtained by simulating $10^7$ Poisson trajectories
We assume that the observed process is either Poisson with constant intensity $S$ or is self-exciting process with intensity function $S(t; X) = S + 1 \int_{\supp u(t)} dX$.

The function $u(t)$ is now unknown but belongs to $U_r$ with $U_r = \int_0^1 u(t) dt = r$ where $\supp u(t)$ is bounded and $C_b$ is the set of nonnegative functions bounded by a same constant $b$. 

Asymptotical Statistics of Stochastic Processes (S.A.P.S.) V - p.16/33
We assume that the observed process is either Poisson with constant intensity $S_*$ or is self-exciting process with intensity function

$$S(t, X) = S_* + \frac{1}{\sqrt{T}} \int_{-\infty}^{t} u(t-s) \, dX_s.$$
We assume that the observed process is either Poisson with constant intensity $S_*$ or is self-exciting process with intensity function

$$S(t, X) = S_* + \frac{1}{\sqrt{T}} \int_{-\infty}^{t} u(t - s) \, dX_s.$$  

The function $u(\cdot)$ is now unknown but belongs to $\bigcup_{r>0} \mathcal{U}_r$ with

$$\mathcal{U}_r = \left\{ u(\cdot) \in C^b_+ : \int_{0}^{\infty} u(t) \, dt = r, \, \text{supp} \, u(\cdot) \, \text{is bounded} \right\}$$

where $C^b_+$ is the set of nonnegative functions bounded by a same constant $b$.  

Asymptotical Statistics of Stochastic Processes (S.A.P.S.) V – p.16/33
We consider the following hypotheses testing problem

\[ \mathcal{H}_0 : \ u(\cdot) \equiv 0 \quad \text{against} \quad \mathcal{H}_1 : \ u(\cdot) \in \mathcal{U}_r, \quad r > 0. \]
We consider the following hypotheses testing problem

\[ \mathcal{H}_0 : \ u(\cdot) \equiv 0 \quad \text{against} \quad \mathcal{H}_1 : \ u(\cdot) \in \mathcal{U}_r, \ r > 0. \]

The power function \( \beta_T(u, \phi_T) = \mathbb{E}_u \phi_T(X^T) \) of a test \( \phi_T \) depends now on the function \( u = u(\cdot) \in \mathcal{U}_r \) with some \( r > 0 \).
We consider the following hypotheses testing problem

\[ H_0 : u(\cdot) \equiv 0 \quad \text{against} \quad H_1 : u(\cdot) \in \mathcal{U}_r, \quad r > 0. \]

The power function \( \beta_T (u, \phi_T) = \mathbb{E}_u \phi_T (X^T) \) of a test \( \phi_T \) depends now on the function \( u = u(\cdot) \in \mathcal{U}_r \) with some \( r > 0 \).

We seek to maximize the minimal power of test on the class \( \mathcal{U}_r \).
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The power function \( \beta_T (u, \phi_T) = \mathbb{E}_u \phi_T (X^T) \) of a test \( \phi_T \) depends now on the function \( u = u (\cdot) \in \mathcal{U}_r \) with some \( r > 0 \).

We seek to maximize the minimal power of test on the class \( \mathcal{U}_r \).

However, for any test \( \phi_T \in \mathcal{H} \varepsilon \) we have

\[
\lim_{T \to \infty} \inf_{u(\cdot) \in \mathcal{U}_r} \beta_T (u, \phi_T) = \varepsilon.
\]
Hence we introduce the set

\[ \mathcal{U}_{r,N} = \{ u(\cdot) \in \mathcal{U}_r : \text{supp } u(\cdot) \subset [0, N] \} \]

and denote

\[ B_T (r, N, \phi_T) = \inf_{u(\cdot) \in \mathcal{U}_{r,N}} \beta_T (u, \phi_T). \]
Hence we introduce the set

\[ \mathcal{U}_{r,N} = \{ u(\cdot) \in \mathcal{U}_r : \text{supp} u(\cdot) \subset [0, N] \} \]

and denote

\[ B_T(r, N, \phi_T) = \inf_{u(\cdot) \in \mathcal{U}_{r,N}} \beta_T(u, \phi_T). \]

**Definition.** A test \( \phi_T^*(\cdot) \) is called LAUMP in the class \( \mathcal{K}_\xi \) if for any other test \( \phi_T(\cdot) \in \mathcal{K}_\xi \) and any \( K > 0 \) we have

\[ \lim_{N \to \infty} \lim_{T \to \infty} \inf_{0 \leq r \leq K} [B_T(r, N, \phi_T^*) - B_T(r, N, \phi_T)] \geq 0. \]
Second main result

We consider the test

\[ \hat{\phi}_T (X^T) = 1\{\Delta_T(X^T) > z_{\varepsilon}\} \quad \text{where} \quad \Delta_T (X^T) = \frac{X_T - S_{*T}}{\sqrt{S_{*T}}} . \]
Second main result

We consider the test

\[ \hat{\phi}_T(X^T) = 1\{\Delta_T(X^T) > z_\epsilon\} \]

where

\[ \Delta_T(X^T) = \frac{X_T - S_*T}{\sqrt{S_*T}}. \]

**Theorem.** The test \( \hat{\phi}_T \) is LAUMP in the class \( \mathcal{K}_\epsilon \), and for any \( u(\cdot) \in \mathcal{U}_r \)

\[ \beta_T(u, \hat{\phi}_T) \longrightarrow \hat{\beta}(u) = \mathbb{P}\left\{\zeta > z_\epsilon - r \sqrt{S_*}\right\}. \]
For any simple alternative $u(\cdot)$ the likelihood ratio can be written as

$$Z_T(u(\cdot)) = \exp \left\{ \Delta_T(u, X^T) - \frac{1}{2} I(u) + r_T(u, X^T) \right\}$$

with $I(u) = \int_0^\infty u(t)^2 \ dt + S_\ast r^2$. 
Key elements of the proof

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\[ Z_T(u(\cdot)) = \exp \left\{ \Delta_T(u, X^T) - \frac{1}{2} I(u) + r_T(u, X^T) \right\} \]

with $I(u) = \int_0^\infty u(t)^2 \, dt + S_\ast r^2$.

\[ \inf_{u(\cdot) \in \mathcal{U}_{r,N}} I(u) \text{ is attained on } u^*(t) = \frac{r}{N} \mathbb{1}_{\{0 \leq t \leq N\}} \] which corresponds to the least favorable for detection self-exciting process in $\mathcal{U}_{r,N}$. 
Key elements of the proof

For any simple alternative \( u (\cdot) \) the likelihood ratio can be written as

\[
Z_T (u(\cdot)) = \exp \left\{ \Delta_T (u, X^T) - \frac{1}{2} I(u) + r_T (u, X^T) \right\}
\]

with \( I(u) = \int_0^\infty u(t)^2 \, dt + S_* r^2 \).

\[
\inf_{u(\cdot) \in \mathcal{U}_{r,N}} I(u) \quad \text{is attained on } u^*(t) = \frac{r}{N} \mathbf{1}_{\{0 \leq t \leq N\}} \quad \text{which corresponds to the least favorable for detection self-exciting process in } \mathcal{U}_{r,N}.
\]

Power of the test: the third lemma of Le Cam.
Key elements of the proof

For any simple alternative \( u(\cdot) \) the likelihood ratio can be written as

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Z_T(u(\cdot)) = \exp \left\{ \Delta_T(u, X^T) - \frac{1}{2} I(u) + r_T(u, X^T) \right\}
\]

with \( I(u) = \int_0^\infty u(t)^2 \, dt + S^* \, r^2 \).

\[
\inf_{u(\cdot) \in \mathcal{U}_{r,N}} I(u) \text{ is attained on } u^*(t) = \frac{r}{N} \mathbb{1}_{\{0 \leq t \leq N\}} \text{ which corresponds to the least favorable for detection self-exciting process in } \mathcal{U}_{r,N}.
\]

Power of the test: the third lemma of Le Cam.

LAUMP: our test is “asymptoticaly” as good as N-P test for the least favorable alternative.
Simulations

We take $S = 1$ and $u(t) = rNf(tN)g$, i.e., $S(t;X) = 1 + rNpT_Xt_i < t^f_t + T_g$.

This choice of $u(t)$ allows us to compare the power function of our LAUMP test $^T_XT = f_XT > z$ with the asymptotic power of Neyman-Pearson test for the least favorable alternative.
Simulations

We take $S_* = 1$ and $u(t) = \frac{r}{N} \mathbb{1}_{\{0 \leq t \leq N\}}$, i.e.,

$$S(t, X) = 1 + \frac{r}{N \sqrt{T}} \sum_{t_i < t} \mathbb{1}_{\{t - t_i \leq N\}}.$$
We take $S_* = 1$ and $u(t) = \frac{r}{N} \mathbb{1}_{0 \leq t \leq N}$, i.e.,

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This choice of $u(\cdot)$ allows us to compare the power function of our LAUMP test

$$\hat{\phi}_T^\epsilon (X^T) = \mathbb{1}_{X_T > z_\epsilon \sqrt{T} + T}$$

with the asymptotic power of Neyman-Pearson test for the least favorable alternative.
Power function of $\hat{\phi}_T^{0.05}$, $u(t) = \frac{r}{5} \mathbb{1}_{\{0 \leq t \leq 5\}}$
Power function of $\hat{\phi}_T^{0.05}$, $u(t) = \frac{r}{5} 1\{0 \leq t \leq 5\}$

Obtained by simulating $10^6$ self-exciting trajectories (for each $r$)
Power function of $\hat{\phi}_T^{0.05}$, $u(t) = \frac{r}{50} \mathbb{1}_{\{0 \leq t \leq 50\}}$
Power function of $\hat{\phi}_{T}^{0.05}$, $u(t) = \frac{r}{50} 1\{0 \leq t \leq 50\}$

Good convergence is obtained if $1 \ll N \ll T$
Self-correcting processes


Defined by intensity function of the form

\[ S(t; \lambda t) = a \lambda t \]

where \( a > 0 \) and the function \( S \) satisfies the following conditions:

1. \( 0 < S(x) < 1 \) for any \( x > 0 \),
2. there exists some \( c > 0 \) such that \( S(x) > c \) for any \( x > 0 \),
3. \( \lim_{x \to \infty} S(x) = 1 \), and
4. \( \lim_{x \to \infty} S(x) < 1 \).
Self-correcting processes

Self-correcting processes


Defined by intensity function of the form

$$S(t, X_t) = \Psi(at - X_t)$$

where $a > 0$ and the function $\Psi(\cdot)$ satisfies the following conditions:

- $0 \leq \Psi(x) < \infty$ for any $x \in \mathbb{R}$,
- there exists some $c > 0$ such that $\Psi(x) \geq c$ for any $x > 0$,
- $\lim_{x \to \infty} \Psi(x) > 1$, and $\lim_{x \to -\infty} \Psi(x) < 1$. 
Let $b > 0$, $c > 0$ and

$$S(t, X_t) = \exp \left\{ a + b (t - cX_t) \right\}.$$ 

Then the point process with such intensity function is self-correcting.
One-sided parametric alternative

We assume that the observed process is either Poisson with constant intensity $S$ or is self-exciting process with intensity function $S(t; X_t) = S_t\left[\frac{S_t}{X_t}\right]; > 0$.

The function $(\cdot)$ is supposed to be known, positive, continuously differentiable at $0$, and such that $(0) = 1$ and $\_\_ (0) > 0$.

To have contiguous alternatives we consider $T = u S_0 T$ and test $H_0: u = 0$ against $H_1: u > 0$. 

We assume that the observed process is either Poisson with constant intensity $S_*$ or is self-exciting process with intensity function

$$S(t, X_t) = S_* \psi (\theta [S_* t - X_t]), \quad \theta > 0.$$
We assume that the observed process is either Poisson with constant intensity $S_*$ or is self-exciting process with intensity function

$$S(t, X_t) = S_* \psi(\theta [S_* t - X_t]), \quad \theta > 0.$$ 

The function $\psi(\cdot)$ is supposed to be known, positive, continuously differentiable at 0, and such that $\psi(0) = 1$ and $\psi'(0) > 0$. 
One-sided parametric alternative

We assume that the observed process is either Poisson with constant intensity $S_*$ or is self-exciting process with intensity function

$$S(t, X_t) = S_\ast \psi \left( \theta [S_\ast t - X_t] \right), \quad \theta > 0.$$ 

The function $\psi (\cdot)$ is supposed to be known, positive, continuously differentiable at 0, and such that $\psi (0) = 1$ and $\dot{\psi} (0) > 0$.

To have contiguous alternatives we consider $\theta = \theta_T = \frac{u}{S_\ast \dot{\psi} (0) T}$ and test

$$\mathcal{H}_0 : \quad u = 0 \quad \text{against} \quad \mathcal{H}_1 : \quad u > 0.$$
Test statistics

We consider

$$\Delta_T (X^T) = \frac{1}{S_* T} \int_0^T (S_* t - X_t) \ [dX_t - S_* \ dt] =$$

$$= \frac{X_T - (X_T - S_* T)^2}{2S_* T}.$$
Theorem. The test

\[ \hat{\phi}_T (X^T) = 1_{\{\Delta_T(X^T) > c_\varepsilon\}} \]

belongs to the class \( \mathcal{K}_\varepsilon \) and for any \( u > 0 \) its power function

\[ \beta_T(u, \hat{\phi}_T) \rightarrow \beta(u, \hat{\phi}) = P \left\{ |\zeta| \leq h(u) \ z_{1-\varepsilon}^{1/2} \right\}. \]

Here

\[ c_\varepsilon = \frac{1 - z_{1-\varepsilon}^2}{2} \quad \text{and} \quad h(u) = \sqrt{\frac{2u}{1 - e^{-2u}}}. \]
Key elements of the proof

The likelihood ratio is LAQ under hypothesis $\mathcal{H}_0$, because

$$Z_T(u) = \exp \left\{ u \Delta_T(X^T) - \frac{u^2}{2} J_T(X^T) + r_T(u, X^T) \right\}$$

where $J_T(X^T) = \frac{1}{S_* T^2} \int_0^T (S_* t - X_t)^2 \, dt$. 

Power of the test: the third lemma of Le Cam.
Key elements of the proof

The likelihood ratio is LAQ under hypothesis $\mathcal{H}_0$, because

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where $J_T(X^T) = \frac{1}{S_* T^2} \int_0^T (S_* t - X_t)^2 \, dt$.

$$\mathcal{L}_0\left\{ \Delta_T(X^T), J_T(X^T) \right\} \implies \left( \frac{1 - W(1)^2}{2}, \int_0^1 W(s)^2 \, ds \right)$$

where $\{ W(s), 0 \leq s \leq 1 \}$ is standard Wiener process.
Key elements of the proof

The likelihood ratio is LAQ under hypothesis $\mathcal{H}_0$, because

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where $\{ W(s), 0 \leq s \leq 1 \}$ is standard Wiener process.

Power of the test: the third lemma of Le Cam.
Simulations

We take $S = 1$ and $(t) = e^t$, i.e., $S(t; X_t) = e^t X_t$.

Simulations allow us to compare the power function of our test $^T_X T \geq c$ with its asymptotic power, as well as with the Neyman-Pearson envelope (simulated for $T = 1000$).

Asymptotical Statistics of Stochastic Processes (S.A.P.S.) V – p.30/33
We take $S_*=1$ and $\psi(t) = e^t$, i.e.,

$$S(t, X_t) = e^{t-X_t}.$$
Simulations

We take $S_* = 1$ and $\psi(t) = e^t$, i.e.,

$$S(t, X_t) = e^{t - X_t}.$$ 

Simulations allows us to compare the power function of our test

$$\hat{\phi}_T (X^T) = \mathbb{1}_{\{\Delta_T(X^T) > c_\varepsilon\}}$$

with its asymptotic power, as well as with the Neyman-Pearson envelope (simulated for $T = 1000$).
Power function of $\hat{\phi}_T^{0.05}$

![Graph showing power function and N-P envelope for different T values](image)

- **N-P envelope**
- **Limiting**
- $T=1000$
- $T=300$
- $T=100$

Obtained by simulating $10^6$ self-correcting trajectories (for each $u$).

Asymptotical Statistics of Stochastic Processes (S.A.P.S.) V – p.31/33
Power function of $\phi_{T}^{0.05}$

Obtained by simulating $10^6$ self-correcting trajectories (for each $u$)

Asymptotical Statistics of Stochastic Processes (S.A.P.S.) V – p.31/33
Remark

Note that these problems of hypotheses testing are similar to the corresponding problems of hypotheses testing for diffusion processes:

\[ dX_t = \theta_T \psi(X_t) \, dt + dW_t, \quad X(0) = 0, \quad 0 \leq t \leq T \]

where we put \( \theta_T = u \varphi_T \) and test two hypotheses

\[ \mathcal{H}_0 : \quad u = 0 \quad \text{against} \quad \mathcal{H}_1 : \quad u > 0, \]

i.e., Wiener process (\( \mathcal{H}_0 \)) against, say, ergodic diffusion (\( \mathcal{H}_1 \)). Then the statistic

\[ \Delta^*_T (X^T) = \varphi_T \int_0^T \psi(X_t) \, dX_t \]

has the similar to \( \Delta_T (X^T) \) asymptotic behavior.
Remark (continuation)

For example, if \( \psi(x) = -x \) (Ornstein-Uhlenbeck process under alternative), then the test

\[
\phi^* (X^T) = \mathbb{1}\{\Delta^*_T(X^T) > c_\varepsilon\}
\]

with

\[
\Delta^*_T(X^T) = -\frac{1}{T} \int_0^T X_t \, dX_t \quad \text{and} \quad c_\varepsilon = \frac{1 - z^2_{1-\varepsilon}}{2}
\]

has the same asymptotic properties as \( \hat{\psi}(X^T) \).