Second order double Edgeworth expansion in a filtering model based on discrete data

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Introduction

Consider the Gaussian system $H = (X, Y)$ given by

$$dH_t = (a_t + A_t H_t)dt + B_{\alpha,t} dw_t^{\alpha}, \qquad H_0 = h_0.$$

Given a constant $y = (y_j)_{j=1}^n$ and a measurable function g, we can readily compute the *conditional expectation*

$$\Pi_{m,n}(y; g) = P[g(X_{t_1}, \ldots, X_{t_m})|(Y_{s_1}, \ldots, Y_{s_n}) = y]$$

as soon as $\mathscr{L}(Y_{s_1}, \ldots, Y_{s_n})$ is nondegenerate, via Monte-Carlo procedure if necessary.

Introduction

Computing $\boldsymbol{\Pi}_{m,n}(y;g)$ becomes difficult if:

- *the Wiener process $w = (w^\alpha)_{\alpha=1}^{r_w}$ is replaced by a more general Lévy process with jumps*;
  ...then we know H is inifinitely divisible, but do not know

$$\mathscr{L}\{X_{t_1},\ldots,X_{t_m}|(Y_{s_1},\ldots,Y_{s_n})=y\}$$

in general.

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  in general.
- *moreover the coefficients $\boldsymbol{a}$, $\boldsymbol{A}$, and $\boldsymbol{B}$ become random*;
  ...then the model is quite diveregent.

How about "*approximately Gaussian*" cases for the computation of $\mathbf{\Pi}_{m,n}(\boldsymbol{y};\boldsymbol{g})$?

Model $(X^\epsilon, Y^\epsilon), \epsilon \in (0,1]$

$$\begin{cases} dX_t^\epsilon &= \{a(\theta_t^\epsilon) + A_0^X(\theta_t^\epsilon)X_t^\epsilon + A_0^Y(\theta_t^\epsilon)Y_t^\epsilon\}dt \\ & \qquad +A_\alpha(\theta_t^\epsilon)dw_t^\alpha + \epsilon\tilde{A}_\beta(\theta_{t-}^\epsilon)dL_t^\beta, \\ X_0^\epsilon &= x_0 \quad \text{(constant)}, \end{cases} \tag{1}$$

$$\begin{cases} dY_t^\epsilon &= \{b(\theta_t^\epsilon) + B_0^X(\theta_t^\epsilon)X_t^\epsilon + B_0^Y(\theta_t^\epsilon)Y_t^\epsilon\}dt \\ & \qquad +B_\alpha(\theta_t^\epsilon)dw_t^\alpha + \epsilon\tilde{B}_\beta(\theta_{t-}^\epsilon)dL_t^\beta, \\ Y_0^\epsilon &= y_0 \quad \text{(constant)}. \end{cases} \tag{2}$$

- $w = (w^\alpha)_{\alpha=1}^{r_w}$ is a standard Wiener process;
- $L = (L^\beta)_{\beta=1}^{r_L}$ is a pure-jump Lévy process with mean zero;
- θ^ϵ is an adapted càdlàg process *independent of* (w, L).

Model $(X^\epsilon, Y^\epsilon), \epsilon \in (0,1]$

$$\begin{cases} dX_t^\epsilon & = \{a(\theta_t^\epsilon) + A_0^X(\theta_t^\epsilon)X_t^\epsilon + A_0^Y(\theta_t^\epsilon)Y_t^\epsilon\}dt \\ & \qquad + A_\alpha(\theta_t^\epsilon)dw_t^\alpha + \epsilon \tilde{A}_\beta(\theta_{t-}^\epsilon)dL_t^\beta, \\ X_0^\epsilon & = x_0 \quad (\text{constant}), \end{cases} \tag{1}$$

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- We suppose that $\theta^0 = \lim_{\epsilon \to 0} \theta^\epsilon$ *exists (in a suitable topology) and is non-random.*

Model (X^ϵ, Y^ϵ), $\epsilon \in (0,1]$

$$\begin{pmatrix} dX_t^\epsilon \\ dY_t^\epsilon \end{pmatrix} = \left\{ C_0(\theta_t^\epsilon) + C_0^*(\theta_t^\epsilon) \begin{pmatrix} X_t^\epsilon \\ Y_t^\epsilon \end{pmatrix} \right\} dt$$
$$+ C_w(\theta_t^\epsilon) dw_t + \epsilon C_L(\theta_{t-}^\epsilon) dL_t,$$
$$\begin{pmatrix} X_0^\epsilon \\ Y_0^\epsilon \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

- (X^ϵ, Y^ϵ) is a small perturbation of the Gaussian system (X^0, Y^0), with the small parameter $\epsilon \in (0,1]$ expressing the magnitude of non-Gaussianity of (X^ϵ, Y^ϵ).

Aim

Our aim here is to approximate the conditional expectation

$$\Pi^{\epsilon}(y;g) = P[g(X^{\epsilon}_{t_1},\ldots,X^{\epsilon}_{t_m})|(Y^{\epsilon}_{s_1},\ldots,Y^{\epsilon}_{s_n}) = y] \quad (3)$$

for $\epsilon \longrightarrow 0$ under suitable conditions,

given

- $0 < t_1 < t_2 < \cdots < t_m$ and $0 < s_1 < s_2 < \cdots < s_n$;
- $y = (y_j)_{j=1}^{n}$;
- $g : \mathbf{R}^m \longrightarrow \mathbf{R}$.

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We will apply *the double Edgeworth expansion* (*DEE* for short) developed in Yoshida (2003, SPA).

Claim

Our claim is the following:

If the model (X^ϵ, Y^ϵ) *is sufficiently smooth in* ϵ *and if* L_1 *admits moments of any order, then we have the expansion*

$$\Pi^\epsilon(y;g) \sim c_0(y;g) + \epsilon c_1(y;g) + \epsilon^2 c_2(y;g) + \ldots \qquad (4)$$

as $\epsilon \to 0$, *with specified coefficients* $\{c_i(y;g)\}_{i\geq 0}$.

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as $\epsilon \longrightarrow 0$, with specified coefficients $\{c_i(y;g)\}_{i\geq 0}$.

Main merits of our result are:

- no structural restriction of θ^ϵ is imposed;
- given the forms of coefficients and θ^ϵ, a computer automatically evaluates $\Pi^\epsilon(y;g)$,

Double Edgeworth expansion (rough explanation!)

Let Z^ϵ and F^ϵ be random vectors admitting *smooth stochastic expansions*

$$\begin{aligned} Z^\epsilon &\sim \zeta_0 + \epsilon\zeta_1 + \epsilon^2\zeta_2 + \cdots , \\ F^\epsilon &\sim f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots . \end{aligned}$$

Under some *regularity conditions*, the DEE is given in the form

$$P[g(Z^\epsilon)|F^\epsilon = y] \sim c_0(y;g) + \epsilon c_1(y;g) + \epsilon^2 c_2(y;g) + \ldots \quad (5)$$

as $\epsilon \longrightarrow 0$, with specified coefficients $\{c_i(y;g)\}_{i\geq 0}$.

Double Edgeworth expansion (rough explanation!)

If densities of $\mathscr{L}(Z^\epsilon, F^\epsilon)$ and $\mathscr{L}(F^\epsilon)$ fulfill

$$\begin{aligned} p_\epsilon(z,y) &\sim p_0(z,y)\{1 + p_1(z,y)\epsilon + p_2(z,y)\epsilon^2 + \ldots\}, \\ q_\epsilon(y) &\sim q_0(y)\{1 + q_1(y)\epsilon + q_2(y)\epsilon^2 + \ldots\}, \end{aligned}$$

then the conditional density of $\mathscr{L}(Z^\epsilon | F^\epsilon = y)$ fulfills

$$\begin{aligned} r_\epsilon(z|y) &= \frac{p_\epsilon(z,y)}{q_\epsilon(y)} \\ &\sim \frac{p_0(z,y)}{q_0(y)} \Big[1 + (p_1(z,y) - q_1(y))\epsilon \\ &\qquad + \{p_2(z,y) - q_2(y) \\ &\qquad - q_1(y)(p_1(z,y) - q_1(y))\}\epsilon^2 + \ldots \Big] \end{aligned}$$

Formula up to the second order

We are concerned here with up to the second order $c_1(y;g)$: the general formulae for $c_0(y;g)$ and $c_1(y;g)$ are given by

$$
\begin{aligned}
c_0(y;g) &= P[g(\zeta_0)|f_0 = y], \\
c_1(y;g) &= \left(p^{f_0}(y)\right)^{-1} \Big[\int_{\mathbf{R}^m} g(z)(-\partial_z)\cdot\{\chi_{\zeta_1}(z,y)\}\,dz \\
&\quad + \int_{\mathbf{R}^m} g(z)(-\partial_y)\cdot\{\chi_{f_1}(z,y)\}\,dz \\
&\quad - P[g(\zeta_0)|f_0 = y](-\partial_y)\cdot\left\{P[f_1|f_0 = y]p^{f_0}(y)\right\} \Big],
\end{aligned}
$$

where $\cdot$ denotes the divergence and

$$\chi_\tau(z,y) := P[\tau|(\zeta_0, f_0) = (z,y)]p^{\zeta_0,f_0}(z,y).$$

Our setup revisited

Recall our model:

$$\begin{cases} dX_t^\epsilon &= \{a(\theta_t^\epsilon) + A_0^X(\theta_t^\epsilon)X_t^\epsilon + A_0^Y(\theta_t^\epsilon)Y_t^\epsilon\}dt \\ & \qquad + A_\alpha(\theta_t^\epsilon)dw_t^\alpha + \epsilon\tilde{A}_\beta(\theta_{t-}^\epsilon)dL_t^\beta, \\ X_0^\epsilon &= x_0 \quad \text{(constant)}, \end{cases} \tag{1}$$

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We shall apply the DEE (4) with

$$\begin{aligned} Z^\epsilon &= (X_{t_1}^\epsilon, X_{t_2}^\epsilon, \ldots, X_{t_m}^\epsilon)^\top, \\ F^\epsilon &= (Y_{s_1}^\epsilon, Y_{s_2}^\epsilon, \ldots, Y_{s_n}^\epsilon)^\top. \end{aligned}$$

Our setup revisited

Our technical assumptions is:

[A1] *A Malliavin operator (Bichteler et al., 1987) exists on the underlying stochastic basis, and* $\boldsymbol{Z}^{\epsilon} = (\boldsymbol{X}^{\epsilon}_{t_1}, \boldsymbol{X}^{\epsilon}_{t_2}, \ldots, \boldsymbol{X}^{\epsilon}_{t_m})^{\top}$ *and* $\boldsymbol{F}^{\epsilon} = (\boldsymbol{Y}^{\epsilon}_{s_1}, \boldsymbol{Y}^{\epsilon}_{s_2}, \ldots, \boldsymbol{Y}^{\epsilon}_{s_n})^{\top}$ *admit smooth stochastic expansions*;

[A2] *the determinants of the Malliavin covariance matrix of* $(\boldsymbol{Z}^{\epsilon}, \boldsymbol{F}^{\epsilon})$*, say* $\boldsymbol{\Delta}_{\epsilon}$*, fulfills* $\limsup_{\epsilon \to 0} P[\boldsymbol{\Delta}^{-p}] < \infty$ *for every* $p > 1$.

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Under the assumptions, we have

$$\zeta_0 = (\boldsymbol{X}^0_{t_1}, \ldots, \boldsymbol{X}^0_{t_m})^\top, \quad \zeta_1 = (\boldsymbol{X}^{[1]}_{t_1}, \ldots, \boldsymbol{X}^{[1]}_{t_m})^\top,$$
$$f_0 = (\boldsymbol{Y}^0_{s_1}, \ldots, \boldsymbol{Y}^0_{s_n})^\top, \quad f_1 = (\boldsymbol{Y}^{[1]}_{s_1}, \ldots, \boldsymbol{Y}^{[1]}_{s_n})^\top,$$

where $\boldsymbol{X}^{[1]} := (\partial_\epsilon)_0 \boldsymbol{X}^\epsilon$ and $\boldsymbol{Y}^{[1]} := (\partial_\epsilon)_0 \boldsymbol{Y}^\epsilon$.

The first order $c_0(x;g)$

$$\begin{pmatrix} dX_t^0 \\ dY_t^0 \end{pmatrix} = \left\{ C_0(\theta_t^0) + C_0^*(\theta_t^0) \begin{pmatrix} X_t^0 \\ Y_t^0 \end{pmatrix} \right\} dt + C_w(\theta_t^0) dw_t,$$
$$\begin{pmatrix} X_0^0 \\ Y_0^0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

The first order coefficient $c_0(x;g)$ is given by

$$c_0(y;g) = P[g(\zeta_0)|f_0 = y], \tag{7}$$

Since $\mathscr{L}(\zeta_0, f_0)$ hence $\mathscr{L}(\zeta_0|f_0 = y)$ is Gaussian, there is nothing to be done.

The second order $\epsilon c_2(x; g)$

Recall the formula:

$$c_1(y; g) = \Big(p^{f_0}(y)\Big)^{-1} \Big[\int_{\mathbf{R}^m} g(x)(-\partial_x) \cdot \{\chi_{\zeta_1}(x, y)\}\, dx + \int_{\mathbf{R}^m} g(x)(-\partial_y) \cdot \{\chi_{f_1}(x, y)\}\, dx - P[g(\zeta_0)|f_0 = y](-\partial_y) \cdot \Big\{ P[f_1|f_0 = y] p^{f_0}(y) \Big\} \Big],$$

with

$$\chi_\tau(x, y) := P[\tau|(\zeta_0, f_0) = (x, y)] p^{\zeta_0, f_0}(x, y).$$

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$$
\begin{aligned}
c_1(y;g) = \left(p^{f_0}(y)\right)^{-1} \Bigg[\int_{\mathbf{R}^m} g(x)(-\partial_x) \cdot \{\chi_{\zeta_1}(x,y)\} dx \\
+ \int_{\mathbf{R}^m} g(x)(-\partial_y) \cdot \{\chi_{f_1}(x,y)\} dx \\
- c_0(y;g)(-\partial_y) \cdot \left\{ P[f_1 | f_0 = y] p^{f_0}(y) \right\} \Bigg].
\end{aligned}
$$

We should begin with computing

$$
\begin{aligned}
\lambda_j^X(x,y) &= P[X_{t_j}^{[1]} | (\zeta_0, f_0) = (x,y)] \quad (j = 1,2,\ldots,m), \\
\lambda_k^Y(x,y) &= P[Y_{s_k}^{[1]} | (\zeta_0, f_0) = (x,y)] \quad (k = 1,2,\ldots,n), \\
\gamma_k^Y(y) &= P[Y_{s_k}^{[1]} | f_0 = y] \quad (k = 1,2,\ldots,n).
\end{aligned}
$$

The second order $\epsilon c_2(x;g)$

$$c_1(y;g) = \left(p^{f_0}(y)\right)^{-1}\Bigg[\int_{\mathbf{R}^m} g(x)(-\partial_x)\cdot\{\chi_{\zeta_1}(x,y)\}dx$$

$$+\int_{\mathbf{R}^m} g(x)(-\partial_y)\cdot\{\chi_{f_1}(x,y)\}dx$$

$$-\, c_0(y;g)(-\partial_y)\cdot\Big\{P[f_1|f_0=y]p^{f_0}(y)\Big\}\Bigg].$$

Using the $\theta^{[1]}$-conditional Gaussianity of $(X^{[1]}_{t_i}, Y^{[1]}_{s_j})$ as well as disintegration argument, we have, e.g.,

$$\lambda^X_j(x,y) = P^{\theta^{[1]}}[\mu^{[1],X}_j]+P^{\theta^{[1]}}[\Xi^X_{j:1}](x-\mu^{0,1})+P^{\theta^{[1]}}[\Xi^X_{j:2}](y-\mu^{0,2}),$$

and so on.

The second order $\epsilon c_2(x; g)$

$$c_1(y;g) = \sum_{j=1}^{m} V_j^{(1)}(y;g) + \sum_{k=1}^{n} V_k^{(2)}(y;g) + c_0(y;g)\left\{ \sum_{j=1}^{m} V_j^{(3)}(y) + \sum_{k=1}^{n} V_k^{(4)}(y) \right\}, \tag{6}$$

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$$\begin{aligned}
V_j^{(1)}(y;g) &= S_j^{(1)}(y)U_1(y;g) + \sigma_{j;(1)}^{-1} U_2(y;g) P^{\theta^{[1]}}[\Xi_{j:1}^X], \\
V_k^{(2)}(y;g) &= S_k^{(2)}(y)U_1(y;g) + \sigma_{m+k;(1)}^{-1} U_2(y;g) P^{\theta^{[1]}}[\Xi_{k:1}^Y], \\
V_j^{(3)}(y) &= P^{\theta^{[1]}}[\mu_j^{[1],X}]\sigma_{j:(2)}^{-1}(y-\mu^{0,2}) \\
&\quad + \sigma_{j:(2)}^{-1}(y-\mu^{0,(2)}) P^{\theta^{[1]}}[\Xi_{j:2}^X](y-\mu^{0,2}) - P^{\theta^{[1]}}[\xi_{j:j}^X], \\
V_k^{(4)}(y) &= P^{\theta^{[1]}}[\mu_k^{[1],Y}]\sigma_{m+k:(2)}^{-1}(y-\mu^{0,2}) + \sigma_{m+k:(2)}^{-1}(y-\mu^{0,2}) P^{\theta^{[1]}}[\Xi_{k:2}^Y](y-\mu^{0,} \\
&\quad - P^{\theta^{[1]}}[\xi_{k:m+k}^Y] - \sigma_k^{(22),-1}(y-\mu^{0,2})\Big\{ P^{\theta^{[1]}}[\mu_k^{[1],Y}] + P^{\theta^{[1]}}[\psi_k^Y](y-\mu \\
&\quad + P^{\theta^{[1]}}[\psi_{k:k}^Y]
\end{aligned}$$

The second order $\epsilon c_2(x;g)$

$$c_1(y;g) = \sum_{j=1}^{m} V_j^{(1)}(y;g) + \sum_{k=1}^{n} V_k^{(2)}(y;g) + c_0(y;g)\left\{ \sum_{j=1}^{m} V_j^{(3)}(y) + \sum_{k=1}^{n} V_k^{(4)}(y) \right\}, \tag{6}$$

$$\begin{aligned}
S_j^{(1)}(y) &= P^{\theta^{[1]}}[\mu_j^{[1],X}]\sigma_{j:(1)}^{-1} + \sigma_{j:(2)}^{-1}(y-\mu^{0,2})P^{\theta^{[1]}}[\Xi_{j:1}^X] + P^{\theta^{[1]}}[\Xi_{j:2}^X](y-\mu^{0,2})\sigma \\
S_k^{(2)}(y) &= P^{\theta^{[1]}}[\mu_k^{[1],Y}]\sigma_{m+k:(1)}^{-1} + \sigma_{m+k:(2)}^{-1}(y-\mu^{0,2})P^{\theta^{[1]}}[\Xi_{k:1}^Y] \\
&\quad + P^{\theta^{[1]}}[\Xi_{k:2}^Y](y-\mu^{0,2})\sigma_{m+k:(1)}^{-1}, \\
U_1(y;g) &= \int_{\mathbf{R}^m} g(x)(x-\mu^{0,1})p^{\zeta_0|f_0}(x|y)dx, \\
U_2(y;g) &= \int_{\mathbf{R}^m} g(x)(x-\mu^{0,1})(x-\mu^{0,(1)})^\top p^{\zeta_0|f_0}(x|y)dx.
\end{aligned}$$

Simulation 1

Let $g(x) = x$, $m = 1$, $n = 2$ with $t_1 = s_2 = 1$ and $s_1 = 0.5$;

$$\begin{cases} dX_t^\epsilon &= 1.3\theta_t^\epsilon dt + 0.5 dw_t^1 + dw_t^2, \\ X_0^\epsilon &= 0, \end{cases}$$

$$\begin{cases} dY_t^\epsilon &= -0.5\theta_t^\epsilon dt + dw_t^1 + 0.8 dw_t^2, \\ Y_0^\epsilon &= 0. \end{cases}$$

Further $\epsilon = 0.2$, $y_{0.5} = 0$ and $y_1 = 0.05$, and

$$\theta_t^\epsilon = \epsilon(t + 2w_t^*).$$

♠ This is a *two*-dimensional conditioning version of the numerical example in Yoshida (2003, Example 2).

Simulation result 1

Results of Example 1 with 10^7 Monte Carlo trials.

	Example 1
Theoretical value	**0.205018**
Actually counted trials $(/10^7)$	4875
Consumed time	2h23m56s
Monte Carlo	**0.204624**
DEE 1st order	0.039634
DEE 2nd order	0.169918
DEE 1st+2nd	**0.209552**

Simulation 2

Let $g(x) = x$, $m = 1$, $n = 1$ with $t_1 = s_1 = 1$;

$$\begin{cases} dX_t^\epsilon &= 2\theta_t^\epsilon dt + 3dw_t^1 + 2dw_t^2, \\ X_0^\epsilon &= 0, \end{cases}$$

$$\begin{cases} dY_t^\epsilon &= 1.5\theta_t^\epsilon dt + 0.7dw_t^1 + 0.8dw_t^2, \\ Y_0^\epsilon &= 0. \end{cases}$$

Further $\epsilon = 0.1$, $y_1 = 0.1$, and $\theta_t^\epsilon = \epsilon L_t^*$, where

$$\mathscr{L}(L_1) = NIG(5, 1, 1, 0),$$

a normal inverse Gaussian Lévy motion.

Simulation result 2

Results of Example 2 with 10^6 Monte Carlo trials, where "Difference rate"=(DEE-MC)/MC.

	Example 2
Actually counted trials ($/10^6$)	18935
Consumed time	20m34s
Monte Carlo	**0.298084**
DEE 1st order	0.327434
DEE 2nd order	-0.0297154
DEE 1st+2nd	**0.297718**
Difference rate	-0.123%

Concluding remarks

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- Truncation functional;
- Multidimensional setting;
- The coefficients of the model may depend on (t, ϵ).
- Estimation of the parameters should be made;