Second order double Edgeworth expansion in a filtering model based on discrete data

Asymptotical Statistics of Stochastic Processes V,
University of Maine, January 6-8, 2005

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Consider the Gaussian system $H = (X, Y)$ given by

$$dH_t = (a_t + A_t H_t) dt + B_{\alpha, t} dw_t^\alpha, \quad H_0 = h_0.$$ 

Given a constant $y = (y_j)_{j=1}^n$ and a measurable function $g$, we can readily compute the conditional expectation

$$\Pi_{m,n}(y; g) = P[g(X_{t_1}, \ldots, X_{t_m})|(Y_{s_1}, \ldots, Y_{s_n}) = y]$$

as soon as $\mathcal{L}(Y_{s_1}, \ldots, Y_{s_n})$ is nondegenerate, via Monte-Carlo procedure if necessary.
Introduction

Computing $\Pi_{m,n}(y; g)$ becomes difficult if:

- **the Wiener process** $w = (w^\alpha)_{\alpha=1}^{r_w}$ is replaced by a more general Lévy process with jumps;

...then we know $H$ is infinitely divisible, but do not know

$$\mathcal{L}\{X_{t_1}, \ldots, X_{t_m} | (Y_{s_1}, \ldots, Y_{s_n}) = y\}$$

in general.
Introduction

Computing $\Pi_{m,n}(y; g)$ becomes difficult if:

- **the Wiener process** $w = (w^\alpha)_{\alpha=1}^\tau$ is replaced by a more general Lévy process with jumps;

...then we know $H$ is infinitely divisible, but do not know

$$\mathcal{L}\{X_{t_1}, \ldots, X_{t_m} | (Y_{s_1}, \ldots, Y_{s_n}) = y\}$$

in general.

- **moreover the coefficients** $a$, $A$, and $B$ become random;

...then the model is quite divergent.
Introduction

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- the Wiener process $w = (w^\alpha)_{\alpha=1}^{r_w}$ is replaced by a more general Lévy process with jumps;
  
  ...then we know $H$ is infinitely divisible, but do not know
  
  $$\mathcal{L}\{X_{t_1}, \ldots, X_{t_m} | (Y_{s_1}, \ldots, Y_{s_n}) = y\}$$
  
  in general.

- moreover the coefficients $\alpha, A, \text{ and } B$ become random;
  
  ...then the model is quite divergent.

How about “approximately Gaussian” cases for the computation of $\Pi_{m,n}(y; g)$?
Model \((X^\epsilon, Y^\epsilon), \epsilon \in (0, 1]\)

\[
\begin{align*}
\frac{dX_t^\epsilon}{dt} &= \{a(\theta^\epsilon_t) + A_0^X(\theta^\epsilon_t)X^\epsilon_t + A_0^Y(\theta^\epsilon_t)Y^\epsilon_t\}dt \\
&\quad + A_\alpha(\theta^\epsilon_t)dw^\alpha_t + \epsilon \tilde{A}_\beta(\theta^\epsilon_{t-})dL_t^\beta, \\
X_0^\epsilon &= x_0 \text{ (constant)},
\end{align*}
\]

\[
\begin{align*}
\frac{dY_t^\epsilon}{dt} &= \{b(\theta^\epsilon_t) + B_0^X(\theta^\epsilon_t)X^\epsilon_t + B_0^Y(\theta^\epsilon_t)Y^\epsilon_t\}dt \\
&\quad + B_\alpha(\theta^\epsilon_t)dw^\alpha_t + \epsilon \tilde{B}_\beta(\theta^\epsilon_{t-})dL_t^\beta, \\
Y_0^\epsilon &= y_0 \text{ (constant)}.
\end{align*}
\]

- \(w = (w^\alpha)_{\alpha=1}^r\) is a standard Wiener process;
- \(L = (L^\beta)_{\beta=1}^r\) is a pure-jump Lévy process with mean zero;
- \(\theta^\epsilon\) is an adapted càdlàg process independent of \((w, L)\).
**Model** \((X^\epsilon, Y^\epsilon), \epsilon \in (0, 1]\)

\[
\begin{align*}
    dX_t^\epsilon &= \{ a(\theta_t^\epsilon) + A_0^X(\theta_t^\epsilon)X_t^\epsilon + A_0^Y(\theta_t^\epsilon)Y_t^\epsilon \}dt \\
    &\quad + A_\alpha(\theta_t^\epsilon)dw_t^\alpha + \epsilon \tilde{A}_\beta(\theta_{t-}^\epsilon)dL_t^\beta, \\
    X_0^\epsilon &= x_0 \quad \text{(constant)},
\end{align*}
\tag{1}
\]

\[
\begin{align*}
    dY_t^\epsilon &= \{ b(\theta_t^\epsilon) + B_0^X(\theta_t^\epsilon)X_t^\epsilon + B_0^Y(\theta_t^\epsilon)Y_t^\epsilon \}dt \\
    &\quad + B_\alpha(\theta_t^\epsilon)dw_t^\alpha + \epsilon \tilde{B}_\beta(\theta_{t-}^\epsilon)dL_t^\beta, \\
    Y_0^\epsilon &= y_0 \quad \text{(constant)}.
\end{align*}
\tag{2}
\]

- We suppose that \(\theta^0 = \lim_{\epsilon \to 0} \theta^\epsilon \) exists (in a suitable topology) and is non-random.
**Model** \((X^\epsilon, Y^\epsilon), \epsilon \in (0, 1]\)

\[
\begin{align*}
\left( dX_t^\epsilon \right) &= \left\{ C_0(\theta_t^\epsilon) + C_0^*(\theta_t^\epsilon) \begin{pmatrix} X_t^\epsilon \\ Y_t^\epsilon \end{pmatrix} \right\} dt \\
&+ C_w(\theta_t^\epsilon) dw_t + \epsilon C_L(\theta_t^\epsilon) dL_t,
\end{align*}
\]

\[
\begin{pmatrix} X_0^\epsilon \\ Y_0^\epsilon \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.
\]

- \((X^\epsilon, Y^\epsilon)\) is a small perturbation of the Gaussian system \((X^0, Y^0)\), with the small parameter \(\epsilon \in (0, 1]\) expressing the magnitude of non-Gaussianity of \((X^\epsilon, Y^\epsilon)\).
Our aim here is to approximate the conditional expectation

$$
\Pi^\epsilon(y; g) = \mathbb{P}[g(X^\epsilon_{t_1}, \ldots, X^\epsilon_{t_m})|(Y^\epsilon_{s_1}, \ldots, Y^\epsilon_{s_n}) = y] \tag{3}
$$

for $\epsilon \to 0$ under suitable conditions,

given

- $0 < t_1 < t_2 < \cdots < t_m$ and $0 < s_1 < s_2 < \cdots < s_n$;
- $y = (y_j)_{j=1}^n$;
- $g : \mathbb{R}^m \to \mathbb{R}$. 

Second order double Edgeworth expansion in a filtering model based on discrete data – p.5/17
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$$\Pi^\epsilon(y; g) = P[g(X^\epsilon_{t_1}, \ldots, X^\epsilon_{t_m})|(Y^\epsilon_{s_1}, \ldots, Y^\epsilon_{s_n}) = y]$$  \hspace{1cm} (3)$$

for $$\epsilon \to 0$$ under suitable conditions,

given

- $$0 < t_1 < t_2 < \cdots < t_m$$ and $$0 < s_1 < s_2 < \cdots < s_n$$;
- $$y = (y_j)_{j=1}^n$$;
- $$g : \mathbb{R}^m \rightarrow \mathbb{R}$$.

We will apply \textit{the double Edgeworth expansion} (DEE for short) developed in Yoshida (2003, SPA).
Claim

Our claim is the following:

If the model \((X^\epsilon, Y^\epsilon)\) is sufficiently smooth in \(\epsilon\) and if \(L_1\) admits moments of any order, then we have the expansion

\[
\Pi^\epsilon(y; g) \sim c_0(y; g) + \epsilon c_1(y; g) + \epsilon^2 c_2(y; g) + \ldots
\]

as \(\epsilon \to 0\), with specified coefficients \(\{c_i(y; g)\}_{i \geq 0}\).
Claim

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*If the model \((X^\epsilon, Y^\epsilon)\) is sufficiently smooth in \(\epsilon\) and if \(L_1\) admits moments of any order, then we have the expansion*

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\]

*as \(\epsilon \to 0\), with specified coefficients \(\{c_i(y; g)\}_{i \geq 0}\).*

Main merits of our result are:

- no structural restriction of \(\theta^\epsilon\) is imposed;
- given the forms of coefficients and \(\theta^\epsilon\), a computer automatically evaluates \(\Pi^\epsilon(y; g)\).
Double Edgeworth expansion (rough explanation!)

Let $Z^\epsilon$ and $F^\epsilon$ be random vectors admitting smooth stochastic expansions

\[
Z^\epsilon \sim \zeta_0 + \epsilon \zeta_1 + \epsilon^2 \zeta_2 + \cdots ,
\]
\[
F^\epsilon \sim f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots .
\]

Under some regularity conditions, the DEE is given in the form

\[
P[g(Z^\epsilon)|F^\epsilon = y] \sim c_0(y; g) + \epsilon c_1(y; g) + \epsilon^2 c_2(y; g) + \cdots \quad (5)
\]

as $\epsilon \to 0$, with specified coefficients $\{c_i(y; g)\}_{i \geq 0}$. 

Double Edgeworth expansion (rough explanation!)

If densities of $\mathcal{L}(Z^\epsilon, F^\epsilon)$ and $\mathcal{L}(F^\epsilon)$ fulfill

$$p_\epsilon(z, y) \sim p_0(z, y)\{1 + p_1(z, y)\epsilon + p_2(z, y)\epsilon^2 + \ldots \},$$
$$q_\epsilon(y) \sim q_0(y)\{1 + q_1(y)\epsilon + q_2(y)\epsilon^2 + \ldots \},$$

then the conditional density of $\mathcal{L}(Z^\epsilon|F^\epsilon = y)$ fulfills

$$r_\epsilon(z|y) = \frac{p_\epsilon(z, y)}{q_\epsilon(y)}$$
$$\sim \frac{p_0(z, y)}{q_0(y)} \left[ 1 + (p_1(z, y) - q_1(y))\epsilon \\
+ \{p_2(z, y) - q_2(y) \\
- q_1(y)(p_1(z, y) - q_1(y)) \} \epsilon^2 + \ldots \right]$$
We are concerned here with up to the second order $c_1(y; g)$: the general formulae for $c_0(y; g)$ and $c_1(y; g)$ are given by

\[
c_0(y; g) = P[g(\zeta_0)|f_0 = y],
\]

\[
c_1(y; g) = \left( p^{f_0}(y) \right)^{-1} \left[ \int_{\mathbb{R}^m} g(z)(-\partial_z) \cdot \{\chi_{\zeta_1}(z, y)\} \, dz \right.
\]

\[
+ \int_{\mathbb{R}^m} g(z)(-\partial_y) \cdot \{\chi_{f_1}(z, y)\} \, dz
\]

\[
- P[g(\zeta_0)|f_0 = y](-\partial_y) \cdot \left\{ P[f_1|f_0 = y]p^{f_0}(y) \right\},
\]

where $\cdot$ denotes the divergence and

\[
\chi_{\tau}(z, y) := P[\tau|(\zeta_0, f_0) = (z, y)]p^{\zeta_0, f_0}(z, y).
\]
Our setup revisited

Recall our model:

\[
\begin{align*}
    dX_t^\varepsilon &= \{a(\theta_t^\varepsilon) + A_X^0(\theta_t^\varepsilon)X_t^\varepsilon + A_Y^0(\theta_t^\varepsilon)Y_t^\varepsilon\}dt \\
    &\quad + A_\alpha(\theta_t^\varepsilon)dw_\alpha + \epsilon\tilde{A}_\beta(\theta_t^-)dL_\beta, \\
    X_0^\varepsilon &= x_0 \text{ (constant)},
\end{align*}
\]

\[
\begin{align*}
    dY_t^\varepsilon &= \{b(\theta_t^\varepsilon) + B_X^0(\theta_t^\varepsilon)X_t^\varepsilon + B_Y^0(\theta_t^\varepsilon)Y_t^\varepsilon\}dt \\
    &\quad + B_\alpha(\theta_t^\varepsilon)dw_\alpha + \epsilon\tilde{B}_\beta(\theta_t^-)dL_\beta, \\
    Y_0^\varepsilon &= y_0 \text{ (constant)}.
\end{align*}
\]

We shall apply the DEE (4) with

\[
\begin{align*}
    Z^\varepsilon &= (X_{t_1}^\varepsilon, X_{t_2}^\varepsilon, \ldots, X_{t_m}^\varepsilon)^\top, \\
    F^\varepsilon &= (Y_{s_1}^\varepsilon, Y_{s_2}^\varepsilon, \ldots, Y_{s_n}^\varepsilon)^\top.
\end{align*}
\]
Our setup revisited

Our technical assumptions is:

[A1] A Malliavin operator (Bichteler et al., 1987) exists on the underlying stochastic basis, and $Z^\varepsilon = (X_{t_1}^\varepsilon, X_{t_2}^\varepsilon, \ldots, X_{t_m}^\varepsilon) ^\top$ and $F^\varepsilon = (Y_{s_1}^\varepsilon, Y_{s_2}^\varepsilon, \ldots, Y_{s_n}^\varepsilon) ^\top$ admit smooth stochastic expansions;

[A2] the determinants of the Malliavin covariance matrix of $(Z^\varepsilon, F^\varepsilon)$, say $\Delta_\varepsilon$, fulfills $\limsup_{\varepsilon \to 0} P[\Delta^{-p}] < \infty$ for every $p > 1$. 
Our setup revisited

Our technical assumptions is:

[A1] A Malliavin operator (Bichteler et al., 1987) exists on the underlying stochastic basis, and $Z^\varepsilon = (X_{t_1}^\varepsilon, X_{t_2}^\varepsilon, \ldots, X_{t_m}^\varepsilon)^\top$ and $F^\varepsilon = (Y_{s_1}^\varepsilon, Y_{s_2}^\varepsilon, \ldots, Y_{s_n}^\varepsilon)^\top$ admit smooth stochastic expansions;

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Under the assumptions, we have

\[
\zeta_0 = (X_{t_1}^0, \ldots, X_{t_m}^0)^\top, \quad \zeta_1 = (X_{t_1}^{[1]}, \ldots, X_{t_m}^{[1]})^\top, \\
f_0 = (Y_{s_1}^0, \ldots, Y_{s_n}^0)^\top, \quad f_1 = (Y_{s_1}^{[1]}, \ldots, Y_{s_n}^{[1]})^\top,
\]

where $X^{[1]} := (\partial_\varepsilon)_0 X^\varepsilon$ and $Y^{[1]} := (\partial_\varepsilon)_0 Y^\varepsilon$. 
The first order $c_0(x; g)$

\[
\begin{align*}
\left(\begin{array}{c}
\frac{dX^0_t}{dt} \\
\frac{dY^0_t}{dt}
\end{array}\right) &= \left\{ C_0(\theta^0_t) + C^*_0(\theta^0_t)\left(\begin{array}{c}
X^0_t \\
Y^0_t
\end{array}\right)\right\}dt + C_w(\theta^0_t)dw_t, \\
\left(\begin{array}{c}
X^0_0 \\
Y^0_0
\end{array}\right) &= \left(\begin{array}{c}
x_0 \\
y_0
\end{array}\right).
\end{align*}
\]

The first order coefficient $c_0(x; g)$ is given by

\[
c_0(y; g) = P[g(\zeta_0)|f_0 = y],
\]

Since $L(\zeta_0, f_0)$ hence $L(\zeta_0|f_0 = y)$ is Gaussian, there is nothing to be done.
**The second order** $\epsilon c_2(x; g)$

Recall the formula:

$$c_1(y; g) = \left(p^{f_0}(y)\right)^{-1} \left[ \int_{\mathbb{R}^m} g(x)(-\partial_x) \cdot \{\chi_{\zeta_1}(x, y)\} \, dx \right]$$

$$+ \int_{\mathbb{R}^m} g(x)(-\partial_y) \cdot \{\chi_{f_1}(x, y)\} \, dx$$

$$- P[g(\zeta_0)|f_0 = y](-\partial_y) \cdot \left\{ P[f_1|f_0 = y]p^{f_0}(y) \right\},$$

with

$$\chi_\tau(x, y) := P[\tau|(\zeta_0, f_0) = (x, y)]p^{\zeta_0, f_0}(x, y).$$
The second order $\epsilon c_2(x; g)$

$$c_1(y; g) = \left(p^{f_0(y)}\right)^{-1} \left[ \int_{\mathbb{R}^m} g(x)(-\partial_x) \cdot \{\chi_{\zeta_1}(x, y)\} dx ight]$$

$$+ \int_{\mathbb{R}^m} g(x)(-\partial_y) \cdot \{\chi_{f_1}(x, y)\} dx$$

$$- c_0(y; g)(-\partial_y) \cdot \left\{ P[f_1|f_0 = y]p^{f_0(y)} \right\}.$$

We should begin with computing

$$\lambda^X_j(x, y) = P[X_t^{[1]}|\zeta_0, f_0 = (x, y)] \quad (j = 1, 2, \ldots, m),$$

$$\lambda^Y_k(x, y) = P[Y_s^{[1]}|\zeta_0, f_0 = (x, y)] \quad (k = 1, 2, \ldots, n),$$

$$\gamma^Y_k(y) = P[Y_s^{[1]}|f_0 = y] \quad (k = 1, 2, \ldots, n).$$
The second order $\epsilon c_2(x; g)$

\[
c_1(y; g) = \left( p^{f_0}(y) \right)^{-1} \left[ \int_{\mathbb{R}^m} g(x)(-\partial_x) \cdot \{ \chi_{\zeta_1}(x, y) \} dx \right.
\]
\[
+ \left. \int_{\mathbb{R}^m} g(x)(-\partial_y) \cdot \{ \chi_{f_1}(x, y) \} dx \right.
\]
\[
- c_0(y; g)(-\partial_y) \cdot \left\{ P[f_1|f_0 = y]p^{f_0}(y) \right\} .
\]

Using the $\theta^{[1]}$-conditional Gaussianity of $(X^{[1]}_{t_i}, Y^{[1]}_{s_j})$ as well as disintegration argument, we have, e.g.,

\[
\lambda^X_j(x, y) = P^{\theta^{[1]}}[\mu^{[1], X}_j] + P^{\theta^{[1]}}[\Xi^{X}_{j:1}](x - \mu^{0, 1}) + P^{\theta^{[1]}}[\Xi^{X}_{j:2}](y - \mu^{0, 2}),
\]

and so on.
The second order $\epsilon c_2(x; g)$

$$c_1(y; g) = \sum_{j=1}^{m} V_j^{(1)}(y; g) + \sum_{k=1}^{n} V_k^{(2)}(y; g) + c_0(y; g) \left\{ \sum_{j=1}^{m} V_j^{(3)}(y) + \sum_{k=1}^{n} V_k^{(4)}(y) \right\},$$

(6)
The second order $\epsilon c_2(x; g)$

\[ c_1(y; g) = \sum_{j=1}^{m} V_j^{(1)}(y; g) + \sum_{k=1}^{n} V_k^{(2)}(y; g) + c_0(y; g) \left\{ \sum_{j=1}^{m} V_j^{(3)}(y) + \sum_{k=1}^{n} V_k^{(4)}(y) \right\} , \]

\[ V_j^{(1)}(y; g) = S_j^{(1)}(y) U_1(y; g) + \sigma_{j;1}^{-1} U_2(y; g) P^{\theta[1]} [\Xi_j] , \]

\[ V_k^{(2)}(y; g) = S_k^{(2)}(y) U_1(y; g) + \sigma_{m+k;1}^{-1} U_2(y; g) P^{\theta[1]} [\Xi_k] , \]

\[ V_j^{(3)}(y) = P^{\theta[1]} [\mu_j^{[1]}, X] \sigma_{j;2}^{-1} (y - \mu_0^2) \]

\[ + \sigma_{j;2}^{-1} (y - \mu_0^2) P^{\theta[1]} [\Xi_j] (y - \mu_0^2) - P^{\theta[1]} [\xi_j] , \]

\[ V_k^{(4)}(y) = P^{\theta[1]} [\mu_k^{[1]}, Y] \sigma_{m+k;2}^{-1} (y - \mu_0^2) + \sigma_{m+k;2}^{-1} (y - \mu_0^2) P^{\theta[1]} [\Xi_k] (y - \mu_0^2) \]

\[ - P^{\theta[1]} [\xi_k] - \sigma_{m+k;2}^{-1} (y - \mu_0^2) \left\{ P^{\theta[1]} [\mu_k^{[1]}, Y] + P^{\theta[1]} [\psi_k] (y - \mu_0^2) \right\} + P^{\theta[1]} [\psi_k] (y - \mu_0^2) , \]
The second order $\epsilon c_2(x; g)$

$$c_1(y; g) = \sum_{j=1}^{m} V_j^{(1)}(y; g) + \sum_{k=1}^{n} V_k^{(2)}(y; g) + c_0(y; g) \left\{ \sum_{j=1}^{m} V_j^{(3)}(y) + \sum_{k=1}^{n} V_k^{(4)}(y) \right\},$$

(6)

$$S_j^{(1)}(y) = P^{[1]} \left[ \mu_j^{[1], X} \sigma_{j:1}^{-1} + \sigma_{j:2}^{-1} (y - \mu_{0,2}^j) P^{[1]} [\Xi_j^{X:1}] + P^{[1]} [\Xi_j^{X:2}] (y - \mu_{0,2}^j) \right],$$

$$S_k^{(2)}(y) = P^{[1]} \left[ \mu_k^{[1], Y} \sigma_{m+k:1}^{-1} + \sigma_{m+k:2}^{-1} (y - \mu_{0,2}^k) P^{[1]} [\Xi_k^{Y:1}] + P^{[1]} [\Xi_k^{Y:2}] (y - \mu_{0,2}^k) \sigma_{m+k:1}^{-1} \right],$$

$$U_1(y; g) = \int_{\mathbb{R}^m} g(x) (x - \mu_{0,1}^j) P_{\xi_0} f_0(x|y) dx,$$

$$U_2(y; g) = \int_{\mathbb{R}^m} g(x) (x - \mu_{0,1}^j) (x - \mu_{0,1}^{(1)})^T P_{\xi_0} f_0(x|y) dx.$$
**Simulation 1**

Let \( g(x) = x \), \( m = 1 \), \( n = 2 \) with \( t_1 = s_2 = 1 \) and \( s_1 = 0.5 \);

\[
\begin{cases}
    dX_t^\varepsilon = 1.3\theta_t^\varepsilon dt + 0.5dw_t^1 + dw_t^2, \\
    X_0^\varepsilon = 0,
\end{cases}
\]

\[
\begin{cases}
    dY_t^\varepsilon = -0.5\theta_t^\varepsilon dt + dw_t^1 + 0.8dw_t^2, \\
    Y_0^\varepsilon = 0.
\end{cases}
\]

Further \( \varepsilon = 0.2 \), \( y_{0.5} = 0 \) and \( y_1 = 0.05 \), and

\[
\theta_t^\varepsilon = \varepsilon(t + 2w_t^*).
\]

♠ This is a *two*-dimensional conditioning version of the numerical example in Yoshida (2003, Example 2).
Simulation result 1

Results of Example 1 with $10^7$ Monte Carlo trials.

<table>
<thead>
<tr>
<th></th>
<th>Example 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theoretical value</td>
<td>0.205018</td>
</tr>
<tr>
<td>Actually counted trials ($/10^7$)</td>
<td>4875</td>
</tr>
<tr>
<td>Consumed time</td>
<td>2h23m56s</td>
</tr>
<tr>
<td>Monte Carlo</td>
<td>0.204624</td>
</tr>
<tr>
<td>DEE 1st order</td>
<td>0.039634</td>
</tr>
<tr>
<td>DEE 2nd order</td>
<td>0.169918</td>
</tr>
<tr>
<td>DEE 1st+2nd</td>
<td>0.209552</td>
</tr>
</tbody>
</table>
**Simulation 2**

Let \( g(x) = x, \ m = 1, \ n = 1 \) with \( t_1 = s_1 = 1; \)

\[
\begin{cases}
    dX_t^\epsilon = 2\theta_t^\epsilon dt + 3dw_t^1 + 2dw_t^2, \\
    X_0^\epsilon = 0,
\end{cases}
\]

\[
\begin{cases}
    dY_t^\epsilon = 1.5\theta_t^\epsilon dt + 0.7dw_t^1 + 0.8dw_t^2, \\
    Y_0^\epsilon = 0.
\end{cases}
\]

Further \( \epsilon = 0.1, \ y_1 = 0.1, \) and \( \theta_t^\epsilon = \epsilon L_t^*, \) where

\[
\mathcal{L}(L_1) = NIG(5, 1, 1, 0),
\]

a normal inverse Gaussian Lévy motion.
## Simulation result 2

Results of Example 2 with $10^6$ Monte Carlo trials, where “Difference rate” = $(\text{DEE-MC})/\text{MC}$.

<table>
<thead>
<tr>
<th></th>
<th>Example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actually counted trials ($/10^6$)</td>
<td>18935</td>
</tr>
<tr>
<td>Consumed time</td>
<td>20m34s</td>
</tr>
<tr>
<td>Monte Carlo</td>
<td>0.298084</td>
</tr>
<tr>
<td>DEE 1st order</td>
<td>0.327434</td>
</tr>
<tr>
<td>DEE 2nd order</td>
<td>-0.0297154</td>
</tr>
<tr>
<td>DEE 1st+2nd</td>
<td>0.297718</td>
</tr>
<tr>
<td>Difference rate</td>
<td>-0.123%</td>
</tr>
</tbody>
</table>
Concluding remarks

We have provided a practical scheme for computing intractable conditional expectations in a class of perturbed Gaussian systems.

- Truncation functional;
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- Multidimensional setting;
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- Truncation functional;
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- The coefficients of the model may depend on \( (t, \epsilon) \).
Concluding remarks

We have provided a practical scheme for computing intractable conditional expectations in a class of perturbed Gaussian systems.

- Truncation functional;
- Multidimensional setting;
- The coefficients of the model may depend on \((t, \epsilon)\).
- Estimation of the parameters should be made;