Asymptotic properties of parameter estimators of a generalized long memory process

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Abstract

We consider parameter estimators of stationary GARFIMA (Generalised Autoregressive Fractionally Integrated Moving Average) processes which exhibit the long memory property.

Some results about limit distributions of parameter estimates are presented.

A stationary process X_t with long memory, or with long-range dependence (LRD) is characterized by a slow rate of convergence of the covariance function $R(s) = Cov(X_t, X_{t+s}), s = 1, 2, ...$ to zero:

$$\sum_{s=0}^{\infty} |R(s)| = \infty.$$

The first study of processes with long memory was done by Kolmogorov (1941,1961).

Hurst studying empirically the data set of the yearly minimum water level of the Nile River for the years 622-1281 found that it exhibits long-term dependence rather than short-term dependence

(but Thomson (2004) reported that a simple seasonal model has better fit.)

Other references: Mandelbrot (1983), Beran (1994), Robinson (2003) etc.

The (reference) example of LRD processes:

fractional Gaussian noise (**fGn**)

$$f_t^H = B_t^H - B_{t-1}^H,\ t = 0, 1, ...,\ \ (\frac{1}{2} < H < 1),$$

where B_t^H is the fractional Brownian motion (**fBm**) that is a Gaussian process with a mean zero and $E(B_t^H - B_s^H)^2 = |t - s|^{2H}$.

It implies

$$R(s) = Cov(f_t^H, f_{t+s}^H) \sim H(2H - 1)\, s^{2H-2},\ s \to \infty.$$

GARFIMA

= Generalised Autoregressive Fractionally Integrated Moving Average process=
Gegenbauer process.

Hoskings (1981), Grey et al (1989), Chung (1996).

We define the GARFIMA(p, d, η, q) process as a stationary solution of the equation

$$\Psi_p(B)(1 - 2\eta B + B^2)^{d/2}(X_t - m) = \Pi_q(B)e_t\,,\ (*)$$

where: B is a backshift operator (that is $B^k X_t = X_{t-k}$, $k = 1, 2, ...$);
$\Psi_p(z)$ is a stable polynomial of the order $p \geq 0$;
$|\eta| \leq 1$;
$d \in (0, \frac{1}{2})$ when $\eta = \pm 1$, ;
$d \in (0, 1)$ when $|\eta| < 1$;
$\Pi_q(z)$ is a polynomial of the order $q \geq 0$.

The fractional power in the expression $(1 - 2\eta B + B^2)^{d/2}$ is defined by an analytical extension of binomial coefficients through the gamma function:

$$(1 - 2\eta B + B^2)^{-d/2} = \sum_{k=0}^{\infty} C_k^{(d)} z^k \ , \ C_k^{(d)}(\eta) \ = \mathbf{GegenbauerC}[k, d, \eta]$$

The innovation process e_t is a process with uncorrelated values (in particular, a martingale difference with respect to some given filtration $\mathcal{F}_t$ that is $E(e_t|\mathcal{F}_{t-1}) = 0$ and $E(e_t^2) = \sigma^2 < \infty$ a.s.).

Under these assumptions there exists a stationary solution X_t of (*) with $EX_t = m$ and the spectral density is equal

$$f_G(\lambda) = \frac{\sigma^2}{2\pi} \mid \frac{\Pi(e^{-i\lambda})}{\Psi(e^{-i\lambda})} \mid^2 |1 - 2\eta e^{-i\lambda} + e^{-2i\lambda}|^{-2d}, \ 0 < \lambda < \pi.$$

Note that the spectral density of GARFIMA(p,d,η,q) process has a pole at $\lambda = \nu := \arccos\eta$. Also, since

$$R(s) = Const\, s^{2d-1}\cos(s\nu)(1+o(1)),\; s \to \infty,$$

it has the LRD property but in case $|\eta| < 1$

$$|\sum_{s=1}^{\infty} R(s)| < \infty\,.$$

The GARFIMA$(0,d,\eta,0)$ process $X(t)$ has the moving-average representation in terms of Gegenbauer polynomials

$$X(t) = m + \sum_{k=0}^{\infty} C_k^{(d)}(\eta)\, e_k\ ,$$

This representation is easy to use for simulation of trajectories of GARFIMA$(0,d,\eta,0)$ with a proper truncation of infinite series. The trajectories of GARFIMA(p,d,η,q) processes can be obtained by ARMA-transformation of GARFIMA$(0,d,\eta,0)$ processes.

Efficiency of sample mean estimator

One of the striking property of ARFIMA processes is that the variance of the sample mean $\bar{X}_n = \sum_{k=1}^{n} X_k/n$ as an estimator of mean m is rather close to the variance of the best-linear-unbiased estimator (BLUE) μ_{BLUE} even for moderate values of the sample sizes n (see Samarov & Taqqu (1988)).

Theorem. If $0 < |\eta| < 1$, then

$$Var(\mu_{BLUE}) \sim Var(\bar{X}_n) \sim \frac{2\pi f(0)}{n}, \quad n \to \infty.$$

Remark. Samarov & Taqqu (1988)) showed that if $\eta = 1$, then with $\beta = 1 - 2d$

$$Var(\mu_{BLUE}) \sim C(\beta) Var(\bar{X}_n) \sim Const\ n^{-1-2\beta}, n \to \infty$$

where

$$C(\beta) = (2-\beta)\Gamma(3/2-\beta/2)\Gamma(1+2\beta)/\Gamma(1/2+\beta/2) \in (0.981, 1).$$

Though asymptotic efficiency of the sample mean $\bar{X}_n$ in case of GARFIMA processes with $0 < |\eta| < 1$ is equal one as $n \to \infty$, the ratio

$$Var(\mu_{BLUE})/Var(\bar{X}_n)$$

for moderate sample sizes n (about 50 -200) may be essentially less than one when d.

Table 1. **Efficiency of the sample mean**

d	ν	**n**=50	**n**=100	**n**=200
0.9	0.95π	0.7235	0.5231	0.8154
0.9	π/60	0.2975	0.6111	0.5591

Functional Limit Theorems for estimators of parameters GARFIMA processes

Here we consider asymptotic behavior of estimators of the parameters (d, η) of GARFIMA processes.

We considered as Chung (1996) the Conditional Sum Squares (CSS) estimators which have the same efficiency as the MLE estimators (but less computer intensive to calculate).

We assume below that the innovation process e_t is a martingale-difference with $\sup_t E(|e_t|^{2+\beta} \mid \mathcal{F}_{t-1}) < \infty$.

CSS estimators maximize the following score function

$$-\frac{n}{2}\log \sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n} \hat{e}_i^2, \;\; with \quad \hat{e}_i = (X_i - m) - \sum_{j=1}^{i-1} b_j (X_{i-j} - m).$$

Theorem (**). *Let* $(\mu_n, \widehat{d}_n, \widehat{\eta}_n)$ *be the CSS-estimators of the parameters* (μ, d, η). *Then*

$$(\mu_{[nt]} - \mu)\sqrt{n} \xrightarrow{w} \sqrt{2\pi f(0)}\, W_t, \quad (\widehat{d}_{[nt]} - d)\sqrt{n} \xrightarrow{w} \varkappa W_t,$$

where $\varkappa^{-2} = \frac{\pi^2}{6} - \frac{\pi\nu}{2} + \frac{\nu^2}{2}$, W_t *is a standard BM.*

If $0 < \nu < \pi$, $d > 0$ *then*

$$(\widehat{\eta}_{[nt]} - \eta)\, n \xrightarrow{w} \frac{2\sin(\nu)}{d} Y_0(t), \quad Y_0(t) = \frac{\int_0^t [W_s^{(1)} dW_s^{(2)} - W_s^{(2)} dW_s^{(1)}]}{\int_0^t [(W_s^{(1)})^2 + (W_s^{(2)})^2] ds}$$

where $(W_s^{(1)}, W_s^{(2)})$ *is a standard two-dimensional BM.*

If $\nu = 0$ *or* π , $d > 0$ *then*

$$(\widehat{\eta}_{[nt]} - \eta)\, n^2 \xrightarrow{w} \frac{1}{d} Y_1(t), \quad Y_1(t) = \frac{\int_0^t (\int_0^s W_u du) dW_s}{\int_0^t (\int_0^s W_u du)^2 ds}.$$

The following theorem presents the distribution of $Y_0(1)$ in terms of integrals of the modified Bessel function of the second order $K_0(z)$.

Theorem. For $x > 0$

$$P\{|Y_0(1)| > x\} = \frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^k \int\limits_{x(2k+1)}^{\infty} \mathbf{BesselK}[0, u] du$$

$$P\{\sup_{0 \le s \le 1} |Y_0(s)| > x\} = \frac{8}{\pi} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{k+m} \int\limits_{x(2k+1)(2m+1)}^{\infty} \mathbf{BesselK}[0, u] du$$

Table 2. Percentiles of $\mathrm{P}\{|\mathbf{Y}_0(\mathbf{1})| > \mathbf{x}\} = \alpha$

α	.010	.020	.050	.100	.200
Monte-Carlo (Chung)	4.238	3.596	2.777	2.171	1.570
Numerical (Theorem (**))	4.233	3.6041	2.7844	2.1748	1.571

References

Beran, J. (1994) *Statistics for long-memory processes.* Chapman & Hall. New York, 1994.

Chung, Ching-Fan (1996) *A generalized fractionally integrated autoregressive moving-average process.* J. Time Ser. Anal. 17, no. 2, pp 111–140.

Davydov (1970) *The invariance principle for stationary processes.* (Russian) Teor. Verojatnost. i Primenen. 15 1970 pp 498–509.

Granger, C.W.J. & Joyeux, R. (1980) *An introduction to long-range time series models and fractional differencing.* J. Time Ser. Anal., v.1, pp 15-31.

Gray, H.L., Zhang, N. & Woodward, W.A. (1989) *On generalized fractional processes.* J. Time Ser. Anal., v.10 , pp 233-257.

Hosking, J.R.M. (1981) *Fractional differencing.* Biometrika, 1981, v.68 , pp 165-176.

Kolmogorov, (1941) *A.N. Local structure of turbulence in fluid for very large Reynolds numbers. Transl. in Turbulence*. S. K. Friedlander & L.Topper (eds.), Interscience Publishers, New York, pp 151-155, 1961.

Robinson, P. (2003) *Time series with long memory.* Edited by Peter M. Robinson. Advanced Texts in Econometrics. Oxford University Press, Oxford.

Samarov, A. and Taqqu, M.S. (1988) *On the efficiency of the sample mean in long-memory noise.* J. Time Ser. Anal., pp 191-200.

Thomson, D. (2004) *Un test pour les processus a longue memoire.* Abstract of Annual Meetings of Stat.Soc. of Canada.