ENLARGEMENT OF FILTRATION AND ADDITIONAL INFORMATION IN PRICING MODELS: A BAYESIAN APPROACH

DARIO GASBARRA, ESKO VALKEILA, AND LIOUDMILA VOSTRIKOVA

Abstract. We show how the dynamical Bayesian approach can be used in the initial enlargement of filtrations theory. We use this approach to obtain new proofs and results for Lévy processes. We apply the Bayesian approach to some problems concerning asymmetric information in pricing models, including so-called weak information approach introduced by Baudoin, as well as some other approaches. We give also Bayesian interpretation of utility gain related to asymmetric information.

1. Introduction

The initial enlargement of filtrations is an important topic in the theory of stochastic processes, and it is studied in the fundamental works of Jeulin [20], Jacod [18], Stricker and Yor [23] and Yor [24, 25] and others. Recent interest to this question comes from pricing models in stochastic finance, where the enlargement of filtrations theory is an important tool in modelling of asymmetric information between different agents and the possible additional gain due to this information (see Amendinger et. al. [1], Imkeller et. al. [16] Baudoin [3, 4], Elliot and Jeanblanc [13] and others). For an approach based on anticipating calculus see [21] and others.

The initial enlargement of filtration consists of the following. Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be a filtered probability space with the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying usual conditions and let X be a semimartingale with the $(P,\mathbf{F})$- triplet $T = (B, C, \nu)$ of predictable characteristics of the semimartingale (we refer to [19] and the section 2 for more details on semimartingales). Suppose that we are given a random variable ϑ on $(\Omega, \mathcal{F})$ such that $\sigma(\vartheta) \subsetneq \mathcal{F}_0$. Define now $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\vartheta)$ and then $\mathbf{G} = (\mathcal{G}_t)_{t\geq 0}$ is the initially enlarged filtration. The main problems studied are: is the $(X,\mathbf{F})$ semimartingale still a semimartingale with respect to the filtration $\mathbf{G}$ and if this is true, what is the new triplet $T^\vartheta = (B^\vartheta, C^\vartheta, \nu^\vartheta)$ with respect to $(P,\mathbf{G})$?

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Surprising at the first glance [and very natural at the second glance] the Bayesian approach proposed in the papers by Dzhaparidze et.al. [9, 10] is closely related to the problem of enlargement of filtrations. In the Bayesian approach one of the main concepts is the arithmetic mean measure. This means the following. Suppose that on a probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ with a filtration $\mathbf{F}$ we observe a semimartingale $X = (X_t)_{t\geq 0}$, and the law P^θ of X depends of a parameter $\theta \in \Theta$. Assume that θ is a value of some random variable ϑ, taking values in a measurable polish space $(\Theta, \mathcal{A})$ where $\mathcal{A}$ is σ-algebra on Θ. Denote the law of the random variable ϑ by α. We suppose that for each $\theta \in \Theta$ the measure P^θ is absolutely continuous with respect to P and that the density process z^θ is measurable with respect to $\mathcal{F} \otimes \mathcal{A}$. Then we can introduce on the original space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ the arithmetic mean measure $\bar{P}^\alpha$: for all $B \in \mathcal{F}$

$$\bar{P}^\alpha(B) := \int_\Theta P^\theta(B)\alpha(d\theta) = \int_\Theta \int_B z^\theta dP\alpha(d\theta).$$

One can interpret the measure $\bar{P}^\alpha$ also as a 'randomised experiment'. In [9, 10] it is shown how to compute the predictable characteristics of X with respect to the arithmetic mean measure $\bar{P}^\alpha$ given the characteristics T^θ of X with respect to P^θ.

The Bayesian approach to the initial enlargement of filtration goes as follows. Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be a filtered space with the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying usual conditions with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let X be a semimartingale on this space with the $(P, \mathbf{F})$ -triplet $T = (B, C, \nu)$. We suppose that we have in addition a random variable $\vartheta : (\Omega, \mathcal{F}) \to (\Theta, \mathcal{A})$ with the values in polish space and the prior law α.

We consider next the product space $(\Omega \times \Theta, \mathcal{F} \otimes \mathcal{A}, \mathbb{G}, \mathbb{P})$ with the filtration $\mathbb{G} = (\mathbb{G}_t)_{t\geq 0}$ defined by $\mathbb{G}_t = \mathcal{F}_t \otimes \mathcal{A}$ and $\mathbb{P}$ is the joint law of $(X(\omega), \vartheta(\omega))$. Let $t \in \mathbb{R}^+$ and α^t be the regular a posteriori distribution of the random variable ϑ given the information $\mathcal{F}_t$:

$$\alpha^t(\omega, \theta) := P(\vartheta \in d\theta | \mathcal{F}_t)(\omega).$$

Assume now that $\alpha^t \prec\prec \alpha$. Then, according to the results of Jacod [18] the process $z^\theta = (z_t^\theta)_{t\geq 0}$ where

$$z_t^\theta(\omega) := \frac{d\alpha^t(\omega, \theta)}{d\alpha(\theta)},$$

is a $(P, \mathbf{F})$- martingale with $z_0^\theta = 1$. Define now a measure P^θ by

$$dP_t^\theta := z_t^\theta dP_t,$$

where the sub-script means the restriction of the measure to the sub-sigma-algebra $\mathcal{F}_t$. Then the process X is also a $(P^\theta, \mathbf{F})$ semimartingale. If we know the structure of density martingale z^θ, then using the Itô formula we can write a semimartingale decomposition of it and read the $(P^\theta, \mathbf{F})$ -triplet $T^\theta = (B^\theta, C, \nu^\theta)$. Finally, if T^θ is $\mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$-measurable,

one obtains the $(P, \mathbf{G})$ triplet of the semimartingale X by replacing in T^θ the fixed parameter θ by the random variable ϑ. This method is relatively simple and gives a unifying approach to various concrete models like diffusion processes, counting processes and Lévy processes. It can also be used outside of the semimartingale world. Some applications will be given in the paper [12].
The paper contains two parts. The first one is devoted to the initial enlargement of filtration. We begin with reminding of some basic facts on semimartingale characteristics and Girsanov theorem. Then we apply Bayesian approach to initial enlargement. For somewhat related study see [6]. We continue by giving some examples of initial enlargement with the final value. The Bayesian approach can be developed for the progressive enlargement of filtration. This will be done in a later work. The second part is devoted to so called weak information introduced in Baudoin [3, 4]. We show that the notion of weak information can be interpreted as changing the "true" prior α, the law of the random variable ϑ, to another prior distribution γ for the random variable ϑ. After this the whole analysis can be reduced to the computation of the $\bar{P}^\gamma$ characteristics of the semimartingale X.
Some preliminary results of the Bayesian approach were already obtained in [11]. We extend and generalise the results in many directions: in addition to several examples and new applications, we give a Bayesian interpretation of so-called additional utility of an insider, or of a weak insider and finally gain on false information.

2. Characteristics of a semimartingale

We shall work with a semimartingale X defined on a filtered space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ where $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration. Recall some facts concerning the triplet T of a semimartingale X. Since the triplet T depends on the probability measure P and on the filtration we keep track of the measures and filtrations in what follows. We assume that $\mathbf{F} := \mathbf{F}^X$ is the right-continuous version of natural filtration of the semimartingale X completed with $\mathcal{F}$ sets of probability zero and that $\mathcal{F} = \mathcal{F}^X_\infty$.
Let μ be the jump measure of X, i.e.

$$\int_0^t \int_{|x|>\epsilon} x\mu(ds, dx) := \sum_{s\leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s|>\epsilon\}}.$$

We use the standard notation from [19] and [15]: if $\mu := \mu^X$ is the jump measure of the semimartingale X, then $g \star \mu$ means integral with respect to the jump measure, $g \star \nu$ denotes integral with respect to the $(P, \mathbf{F})$- compensator ν of μ; later $g \cdot U$ is stochastic integral with respect to a local martingale U or Riemann-Stieltjes integral with respect to a bounded variation process U.
Suppose that the semimartingale X has characteristics $T = (B, C, \nu)$ with respect to $(P, \mathbf{F})$. Recall that this means the following (see [19]

for more details and unexplained terminology). Let $l : \mathbb{R} \to \mathbb{R}$ be a truncation function: $l(x) = x$ in the neighbourhood of 0 and l has a compact support. Then one can write the semimartingale X as

$$X = (X - X(l)) + X(l),$$

where $X(l)$ is a jump process, the process with 'big' jumps, defined as

$$X(l)_t := \sum_{s \le t} (\Delta X_s - l(\Delta X_s))$$

with $\Delta X_s = X_s - X_{s-}$.
The process $\tilde{X} = (X - X(l))$ is a special semimartingale with bounded jumps and allows a representation

$$\tilde{X}_t = X_0 + X_t^c + \int_0^t \int_{\mathbb{R}\setminus\{0\}} l(x)(\mu(ds, dx) - \nu(ds, dx)) + B_t(l),$$

where X^c is the continuous local martingale part of X, ν is the $(P, \mathbf{F})$ compensator of μ, $B_t(l)$ is the unique $(P, \mathbf{F})$- predictable locally integrable process such that the process $\tilde{X} - B(l)$ is a $(P, \mathbf{F})$- local martingale. Let C be the continuous process such that the process $(X^c)^2 - C$ is a $(P, \mathbf{F})$ local martingale. Having all this we have defined the *triplet of predictable characteristics* of a semimartingale X as $T = (B(l), C, \nu)$. Later we write B instead of $B(l)$.
Consider the class of real functions $\mathcal{G}$ with the following properties: functions g are bounded, Borel measurable functions on $\mathbb{R}$ vanishing inside of a neighbourhood of 0. Moreover, if η and $\tilde{\eta}$ are measures on $\mathbb{R}$ such that $\eta(\{0\}) = \tilde{\eta}(\{0\}) = 0$, $\eta(|x| > \epsilon) < \infty$ and $\tilde{\eta}(|x| > \epsilon) < \infty$, and if for all $g \in \mathcal{G}$

$$\int_{\mathbb{R}} g(x)\eta(dx) = \int_{\mathbb{R}} g(x)\tilde{\eta}(dx)$$

then $\eta = \tilde{\eta}$.
Recall Theorem II.2.21 from [19, p.80]

Theorem 2.1. *A semimartingale X has the $(P, \mathbf{F})$ triplet $T = (B, C, \nu)$ if and only if*

- *The process $M(l) := X - X(l) - B - X_0$ is a local martingale.*
- *The process*

$$N(l) := M(l)^2 - C^2 - l^2 \star \nu - \sum_{s \le \cdot} (\Delta B_s)^2$$

  is a local martingale.
- *The process $U(l) := g \star (\mu - \nu)$ is a local martingale, where $g \in \mathcal{G}$.*

Assume moreover that we have on $(\Omega, \mathcal{F}, \mathbf{F}, P)$ a family of probability measures P^θ with $\theta \in \Theta$ such that $P_t^\theta \prec\prec P_t$ for all $t \in \mathbb{R}^+$.

Let $\theta \in \Theta$ be fixed. Then X is a $(P^\theta, \mathbf{F})$ semimartingale as well with a triplet $T^\theta = (B^\theta, C^\theta, \nu^\theta)$, and this triplet is related to the triplet $T = (B, C, \nu)$ as follows

$$\begin{aligned} B^\theta &= B + \beta^\theta \cdot C + (Y^\theta - 1)l \star \nu, \\ C^\theta &= C, \\ \nu^\theta &= Y^\theta \cdot \nu, \end{aligned} \tag{2.1}$$

with certain $(P^\theta, \mathbf{F})$-predictable processes $\beta^\theta = (\beta^\theta_t)_{t\geq 0}$ and $Y^\theta = (Y^\theta_t)_{t\geq 0}$ such that P-a.s. for all $t \in \mathbb{R}^+$

$$((\beta^\theta)^2 \cdot C)_t + (|(Y^\theta - 1)l| \star \nu)_t < \infty. \tag{2.2}$$

For more details see [19].
We denote by P^θ_t and P_t the restrictions of the corresponding measures on $\mathcal{F}_t$ and we define density process $z^\theta = (z^\theta_t)_{t\geq 0}$ with

$$z^\theta_t = \frac{dP^\theta_t}{dP_t}.$$

We note that the density process is $(P, \mathbf{F})$- martingale with the property $\inf_{t\in[0,T]} z^\theta_t > 0$ P-a.s. for each $T > 0$, and we define the *stochastic logarithm* m^θ of z^θ by

$$m^\theta := z^\theta / z^\theta_-. \tag{2.3}$$

Then m^θ is a $(P, \mathbf{F})$- local martingale and z^θ is the *stochastic exponential* of m^θ:

$$z^\theta_t = \mathcal{E}(m^\theta)_t.$$

Assume now that X is a $(P, \mathbf{F})$- semimartingale with a triplet $T = (B, C, \nu)$ and that the natural filtration $\mathbf{F}$ of X has the *predictable representation property* : if M is a local martingale with respect to $\mathbf{F}$, then it has a representation:

$$M = M_0 + H \cdot X^c + W \star (\mu - \nu). \tag{2.4}$$

Here the predictable process H belongs to the space L^2_{loc} of locally square-integrable processes with respect to C and the predictable process $W = (W_t(\omega; x))_{t\geq 0}$ belong to $G_{loc}(\mu)$. For information on the space $G_{loc}(\mu)$ see [19, II.1.1,pp. 72-74] and on the predictable representation property see [19, p.185].
By the predictable representation property we have that the local martingale m^θ from (2.3) has the following semimartingale representation

$$m^\theta = \beta^\theta \cdot X^c + \left(Y^\theta - 1 + \frac{\hat{Y}^\theta - \hat{1}}{1 - \hat{1}} \right) \star (\mu - \nu), \tag{2.5}$$

where the processes β^θ and Y^θ are the same as in (2.1) and the "hat" processes are related to the jumps of the compensator ν, namely

$$\hat{1}_t(\omega) := \nu(\omega; \{t\} \times \mathbb{R}_0)$$

and

$$\hat{Y}_t^\theta(\omega) := \int_{\mathbb{R}_0} Y_t^\theta(\omega, x)\nu(\omega, \{t\}, dx).$$

So, to find the triplet T^θ we can read β^θ and Y^θ from (2.5) and use (2.1) .

3. Arithmetic mean measure

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ with the right-continuous filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ completed by the $\mathcal{F}$ sets of probability zero and $\mathcal{F} = \mathcal{F}_\infty$. Suppose that we are given with a parametric family of probability measures $(P^\theta)_{\theta\in\Theta}$ where θ belongs to a measurable polish space $(\Theta, \mathcal{A})$.
We make the following assumption

Assumption 1. *For each $\theta \in \Theta$ the probability P^θ is locally absolute continuous with respect to P .*

Then we can define density process: for each $\theta \in \Theta$ and $t \in \mathbb{R}^+$

$$z_t^\theta = \frac{dP_t^\theta}{dP_t}$$

where P_t^θ and P_t are the restrictions of P^θ and P on $\mathcal{F}_t$ respectively. Let us consider measurable with respect to θ versions of density process. Given a probability measure α on $(\Theta, \mathcal{A})$, $t \in \mathbb{R}^+$ and $B \in \mathcal{F}_t$ we can define *the arithmetic mean measure* $\bar{P}_t^\alpha$:

$$\bar{P}_t^\alpha(B) := \int_\Theta P_t^\theta(B)\alpha(d\theta) = \int_{\Theta\times B} z_t^\theta P(d\omega)\alpha(d\theta).$$

Remark 3.1. *In the case of the initial enlargement by a random variable ϑ such that $\alpha = \mathcal{L}(\vartheta|P)$, considered in the section 4., we have $\bar{P}^\alpha = P$. This follows from the fact that in this case P^θ is regular conditional law of X given $\vartheta = \theta$.*

We see that $\bar{P}_t^\alpha$ is absolutely continuous with respect to P_t and that in general, P_t^θ is not absolutely continuous with respect to $\bar{P}_t^\alpha$. For this reason we suppose also that

Assumption 2. *For each $\theta \in \Theta$ the probability P^θ is locally absolute continuous with respect to $\bar{P}^\alpha$.*

Assume now again that X is a $(P, \mathbf{F})$- semimartingale with a triplet $T = (B, C, \nu)$ having representation property. Then X is a $(P^\theta, \mathbf{F})$ semimartingale as well with a triplet $T^\theta = (B^\theta, C^\theta, \nu^\theta)$ where B^θ , C^θ , ν^θ are given in (2.1).
The next theorem is a generalisation of a result by Kolomiets.

Theorem 3.1. *Suppose that the assumptions 1 and 2 hold and X is a $(P, \mathbf{F})$ semimartingale with triplet $T = (B, C, \nu)$. Then, X is also a $(\bar{P}^\alpha, \mathbf{F})$- semimartingale with the triplet $\bar{T} = (\bar{B}, \bar{C}, \bar{\nu})$ defined by*

$$\begin{aligned} \bar{B} &= E_\alpha\{\bar{z}^\theta_- \cdot B^\theta\} = B + E_\alpha\{\bar{z}^\theta_- \beta^\theta \cdot C\} + E_\alpha\{\bar{z}^\theta_-(Y^\theta - 1)l \star \nu\} \\ \bar{C} &= C, \\ \bar{\nu} &= E_\alpha\{\bar{z}^\theta_- Y^\theta \cdot \nu\} \end{aligned} \tag{3.1}$$

where $\bar{z}^\theta$ is the density of P^θ with respect to arithmetic mean measure $\bar{P}^\alpha$.

For the proof see [8, Theorem 3.3].

To interchange the order of integration in (3.1) by using Fubini theorem we introduce the following notation. For each $t \in \mathbb{R}^+$ we define a posteriori measure α^t. To do it for each $B \in \mathcal{A}$ we put

$$\alpha^t(B) := \frac{\int_B z^\theta_t \alpha(d\theta)}{\int_\Theta z^\theta_t \alpha(d\theta)}.$$

Let us define $\alpha^{t-}(d\theta)$ in the following natural way: for each $B \in \mathcal{A}$

$$\alpha^{t-}(B) := \frac{\int_B z^\theta_{t-} \alpha(d\theta)}{\int_\Theta z^\theta_{t-} \alpha(d\theta)}.$$

Assuming that β^θ_t and Y^θ_t are integrable with respect to α^{t-}, we put

$$\bar{\beta}_t = E_{\alpha^{t-}} \beta^\theta_t, \quad \bar{Y}_t = E_{\alpha^{t-}} Y^\theta_t. \tag{3.2}$$

Theorem 3.2. *Suppose that the assumptions 1 and 2 hold and P-a.s. for $t > 0$*

$$(E_{\alpha^{t-}}|\beta^\theta_t| \cdot C)_t + (E_{\alpha^{t-}}|Y^\theta - 1|l \star \nu)_t < \infty. \tag{3.3}$$

Then X is a $(\bar{P}^\alpha, \mathbf{F})$- semimartingale with the triplet $\bar{T} = (\bar{B}, \bar{C}, \bar{\nu})$ defined by

$$\begin{aligned} \bar{B} &= B + \bar{\beta} \cdot C + (\bar{Y} - 1)l \star \nu \\ \bar{C} &= C, \\ \bar{\nu} &= \bar{Y} \cdot \nu \end{aligned} \tag{3.4}$$

where $\bar{\beta}$ and $\bar{Y}$ are given in (3.2).

Proof To prove our result we use classical Fubini theorem. In order to do it, we show that $\bar{B}$ is the process of locally P-integrable variation. In fact, for all $t > 0$

$$\mathrm{Var}(\bar{B})_t \le \mathrm{Var}(B)_t + E_\alpha\{(\bar{z}^\theta_-|\beta^\theta| \cdot C)_t\} + E_\alpha\{(\bar{z}^\theta_-|Y^\theta - 1|l \star \nu)_t\}.$$

Using classical Fubini theorem for positive functions in last two integrals and integration with respect the measure α^{t-} we have: for all $t > 0$

$$\mathrm{Var}(\bar{B})_t \le \mathrm{Var}(B)_t + (E_{\alpha^-}|\beta^\theta| \cdot C)_t + (E_{\alpha^-}|Y^\theta - 1|l \star \nu)_t.$$

We define a localising sequence as follows. For $n \in \mathbb{N}^*$ we put
(3.5)
$$\tau_n = \inf\{t \geq 0 : (E_{\alpha^{t-}}|\beta^\theta_t| \cdot C)_t + (E_{\alpha^{t-}}|Y^\theta - 1|l \star \nu)_t + Var(B)_t > n\}.$$
and notice that τ_n is **F**-stopping time. Moreover, since the jumps of considered processes are bounded by a constant, we can easily verify that for each $n \in \mathbb{N}^*$
$$E_{\bar{P}^\alpha}[((E_{\alpha^{t-}}|\beta^\theta_t|) \cdot C)_{\tau_n} + ((E_{\alpha^{t-}}|Y^\theta - 1|)l \star \nu)_{\tau_n} + Var(B)_{\tau_n}] < n + 3 \max_{x \in \mathbb{R}} l(x),$$
where l is truncation function. Now, we notice that the sequence of **F**- stopping times τ_n is increasing up to infinity due to the condition (3.3). Then, we localise with τ_n and we apply classical Fubini theorem to (3.1) and we have (3.4). □

Remark 3.2. *Theorem 3.2 is a special case of stochastic Fubini theorem. Namely, we know that*
$$z^\theta_t = \mathcal{E}(m^\theta)_t,$$
where
$$m^\theta = \beta^\theta \cdot X^c + \left(Y^\theta - 1 + \frac{\hat{Y}^\theta - \hat{1}}{1 - \hat{1}}\right)$$
Then by Theorem 3.2 we have the following variant of stochastic Fubini theorem
$$\bar{z}_t = \int_\Theta z^\theta_t \alpha(d\theta) = \mathcal{E}(\bar{m})_t$$
with
$$\bar{m} = \bar{\beta} \cdot X^c + \left(\bar{Y} - 1 + \frac{\hat{\bar{Y}} - \hat{1}}{1 - \hat{1}}\right).$$
Some times the verification of the condition (3.3) can be difficult and we can be interested to replace it by another condition expressed in terms of density process. For instance, we can use the following assumption.

Assumption 3. *There exists a localizing sequence of* **F**- *stopping times* τ_n *such that for* $n \geq 1$
$$E\left(\int_\Theta [z^\theta, z^\theta]^{1/2}_{\tau_n} \alpha(d\theta)\right) < \infty$$
where E is the expectation with respect to initial measure measure P.

Theorem 3.3. *Suppose that the assumptions 1,2, 3 hold. Then X is a $(\bar{P}^\alpha, \mathbf{F})$- semimartingale with the triplet $\bar{T} = (\bar{B}, \bar{C}, \bar{\nu})$ defined by (3.4).*

Proof In fact, we have only to show that the assumption 3 implies the local integrability of the variation of $\bar{B}$. Since B is locally integrable with respect to arithmetic mean measure, which follows from the fact that the jumps of B are bounded by a constant, we have only to show that there exists a localizing sequence of stopping times s_n such that for each $n \geq 1$

$$E_{\bar{P}^\alpha}\left((E_{\alpha^-}|\beta^\theta| \cdot C)_{\tau_n} + (E_{\alpha^-}|Y^\theta - 1|l \star \nu)_{\tau_n}\right) < \infty. \tag{3.6}$$

Let for $t \geq 0$

$$\bar{Z}_t = \frac{d\bar{P}_t}{dP_t}.$$

We remark that

$$\bar{Z}_t = \int_\Theta z_t^\theta \alpha(d\theta).$$

Using the fact that $\bar{Z}$ is positive $(P, \mathbf{F})$ martingale and the observation that we are dealing with the predictable positive processes, we obtain:

$$\begin{aligned}
&E_{\bar{P}^\alpha}\left((E_{\alpha^-}|\beta^\theta| \cdot C)_{\tau_n} + (E_{\alpha^-}|Y^\theta - 1|l \star \nu)_{\tau_n}\right)\\
&= E_P\left(\bar{Z}_{\tau_n}(E_{\alpha^-}|\beta^\theta| \cdot C)_{\tau_n} + (E_{\alpha^-}|Y^\theta - 1|l \star \nu)_{\tau_n}\right)\\
&= \int_\Theta E_P\{(z_-^\theta|\beta^\theta| \cdot C)_{\tau_n} + (z_-^\theta|Y^\theta - 1|l \star \nu)_{\tau_n}\}\alpha(d\theta)\\
&= \int_\Theta E_P\{(z_-^\theta|\beta^\theta| \cdot C)_{\tau_n} + (z_-^\vartheta|Y^\theta - 1|l \star \mu^X)_{\tau_n}\}\alpha(d\theta)\\
&= \int_\Theta E_P\{\mathrm{Var}([z^\theta, X(l) - B])_{\tau_n}\}\alpha(d\theta)
\end{aligned}$$

Let

$$\tau_n' = \inf\{t \geq 0 : \sup_{0 \leq s \leq t} |X_s(l) - B_s| > n\}$$

and $s_n = \tau_n' \wedge \tau_n$. By Fefferman inequality, (see [15, Theorem 10.17]) and the fact that $X(l) - B$ is $(P, \mathbf{F})$- local martingale we deduce that

$$E_P \mathrm{Var}([z^\theta, X(l) - B])_{s_n} \leq \| (X(l) - B)^{s_n} \|_{BMO} E_P[z^\theta, z^\theta]_{s_n}^{1/2}.$$

We remark that

$$\| (X(l) - B)^{s_n} \|_{BMO} \leq 2(n + 2\max_x l(x))$$

where l is truncation function. So, after integration with respect to α, we obtain from assumption 3 that (3.6) holds, and, hence, $\bar{B}$ has locally integrable variation with respect to $\bar{P}^\alpha$. □

4. Initial enlargement

4.1. **Triplet and initial enlargement.** We assume that we are given with a semimartingale X on a filtered space $(\Omega, \mathcal{F}, \mathbf{F}, P)$. We suppose that the filtration $\mathbf{F}$ is the right-continuous version of natural filtration $\mathbf{F} = (\mathcal{F}_t^X)_{t \geq 0}$ which is completed by the $\mathcal{F}$ sets of probability zero and $\mathcal{F} = \mathcal{F}_\infty^X$. Let $T = (B, C, \nu)$ be the $(P, \mathbf{F})$-triplet of X. Later, to simplify the notation, we omit the index X in the filtration.

Suppose that we have also a random variable ϑ with values in measurable polish space $(\Theta, \mathcal{A})$. Define now the initially enlarged filtration $\mathbf{G} = (\mathcal{G}_t)_{t\geq 0}$ by

$$\mathcal{G}_t := \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(\vartheta)).$$

Then we complete the filtration with $\mathcal{G}_\infty$ sets of P-probability zero. Our problem is to find the semimartingale decomposition of X with respect to the enlarged filtration $\mathbf{G}$.

Let α be the distribution of the random variable ϑ, i.e. $P(\vartheta \in d\theta) = \alpha(d\theta)$. Let for $t \in \mathbb{R}^+$ α^t be its regular conditional distribution with respect to the sigma-field $\mathcal{F}_t$. Following Bayesian terminology we say that α is the *a priori distribution* and α^t is the *a posteriori distribution* with respect to the information $\mathcal{F}_t$, of the random variable ϑ.

We make the following standing assumption

Assumption 4. *The posterior distributions α^t and the prior distribution α satisfy: for each $t \in [0, T]$ and P-a.s.*

$$\alpha^t \prec\prec \alpha \tag{4.1}$$

We stop to discuss the right-continuity of the filtration $\mathbf{G}$: in Amendinger [2, Proposition 3.3] it is shown that under the assumption $\alpha^t \sim \alpha$ we have that $\mathcal{G}_t = F_t \vee \sigma(\vartheta)$. But if one checks the proof of this result in [2] it can be seen that in fact it is sufficient to assume only assumption 4. So under assumption 4 we can take $\mathcal{G}_t = F_t \vee \sigma(\vartheta)$.

We consider next the product space $(\Omega \times \Theta, \mathcal{F} \otimes \mathcal{A}, \mathbb{G}, \mathbb{P})$ with the filtration $\mathbb{G} = (\mathbb{G}_t)_{t\geq 0}$ defined by

$$\mathbb{G}_t = \bigcap_{s>t} (\mathcal{F}_s \otimes \mathcal{A}) \tag{4.2}$$

and $\mathbb{P}$ joint law of $(\omega, \vartheta(\omega))$. Again, under assumption 4 we can take $\mathbb{G}_t = F_t \otimes \mathcal{A}$.

Denote the optional and predictable sigma-fields on $(\Omega \times \mathbb{R}^+)$ with respect to $\mathbf{F}$ by $\mathcal{O}(\mathbf{F})$ and $\mathcal{P}(\mathbf{F})$. With the filtration $\mathbb{G}$ we have that

$$\mathcal{P}(\mathbb{G}) = \mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$$

and

$$\mathcal{O}(\mathbf{F}) \otimes \mathcal{A} \subset \mathcal{O}(\mathbb{G}).$$

The following result is due to Jacod.

Lemma 4.1. *Under assumption 4 there exists a strictly positive $\mathcal{O}(\mathbb{G})$-measurable function $(\omega, t, \theta) \mapsto z_t^\theta(\omega)$, such that:*

1. *For each $\theta \in \Theta$, z^θ is a $(P, \mathbf{F})$- martingale.*
2. *For each $t \in \mathbb{R}^+$, the measure $z_t^\theta \alpha(d\theta)$ is a version of the regular conditional distribution $\alpha^t(d\theta)$ so that $P_t \times \alpha$-a.s.*

$$\frac{d\alpha^t}{d\alpha}(\theta) = z_t^\theta. \tag{4.3}$$

The proof of this lemma is in [18, Lemme 1.8., p.18-19].
For each $\theta \in \Theta$ define also a measure P^θ as follows

$$dP^\theta_t := z^\theta_t dP_t. \tag{4.4}$$

The measure P^θ is absolutely continuous with respect to the measure P, and so X is a $(P^\theta, \mathbf{F})$- semimartingale, too. Hence it has a $(P^\theta, \mathbf{F})$-triplet $T^\theta = (B^\theta, C, \nu^\theta)$.
Next we indicate how one can use the prior and posterior distribution to obtain the semimartingale decomposition of a $(P, \mathbf{F})$- semimartingale with respect to the filtration $\mathbf{G}$.

(1) We are given a semimartingale X with $(P, \mathbf{F})$- triplet $T = (B, C, \nu)$, where the natural filtration $\mathbf{F}$ has the representation property, random variable ϑ, prior $\alpha(d\theta) = P(\vartheta \in d\theta)$ and posterior $\alpha^t(d\theta) = P(\vartheta \in d\theta | \mathcal{F}_t)$.
(2) Compute $\frac{d\alpha^t}{d\alpha}(\theta)$ with the Ito formula as $\mathcal{E}(m^\theta)$ and read β^θ and Y^θ from the representation (2.5), use (2.1) to obtain T^θ .
(3) If T^θ is $\mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$ - measurable, replace θ by ϑ in T^θ to obtain the triplet of X with respect to $(P, \mathbf{G})$.

In the following theorem we give the link between Girsanov theorems and enlargement of filtrations.

Theorem 4.1. *Assume that the process X is a cadlag $(P, \mathbf{F})$- semimartingale with triplet $T = (B, C, \nu)$ and we have the martingale representation property with respect to natural filtration $\mathbf{F}$. Let ϑ be a random variable such that the assumption (4.1) is satisfied. Suppose also that $L^1(\Omega, \mathcal{F}, P)$ is separable and the condition (3.3) holds.*
Then, if we consider cadlag versions, the following conditions are equivalent:

(a) *The process X is a $(P^\theta, \mathbf{F})$- semimartingale with the triplet $T^\theta = (B^\theta, C, \nu^\theta)$ on the space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ for α almost all θ and the application $T' : (\omega, t, \theta) \to T^\theta_t(\omega)$ is $\mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$- measurable.*
(b) *The process X is a $(\mathbb{P}, \mathbb{G})$- semimartingale with the triplet $T' : (\omega, t, \theta) \to T^\theta_t(\omega)$ on product space $(\Omega \times \Theta, \mathcal{F} \otimes \mathcal{A}, \mathbb{G}, \mathbb{P})$ where $\mathbb{P}$ is the joint law of $(\omega, \vartheta(\omega)$,*
(c) *The process X is a $(P, \mathbf{G})$- semimartingale on the space $(\Omega, \mathcal{F}, P)$ with the triplet $T^\vartheta = (B^\vartheta, C, \nu^\vartheta)$.*

Remark 4.1. *It should be noticed that separability condition will be used only in the direction:*

$$c) \Rightarrow b) \Rightarrow a).$$

To prove the theorem we need some lemmas concerning the transformation of triplets, stopping times and martingales.

Lemma 4.2. *The application* $X : (\omega, t, \theta) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *is* $\mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$*-measurable if and only if* $X^\vartheta : (\omega, t, \vartheta(\omega)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *is* $\mathcal{P}(\mathbf{G})$*-measurable.*

Proof It is sufficient to establish the property on semi-algebras generating the corresponding σ-algebras. Let now $a, b, c \in \mathbb{R}, a < b, A \in \mathcal{F}_a, B \in \mathcal{A}$ and

$$X(\omega, t, \theta) = c\mathbf{1}_{(a,b]}(t)\mathbf{1}_A(\omega)\mathbf{1}_B(\theta). \tag{4.5}$$

Then X is an element of semi-algebra generating $\mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$ and

$$X^\vartheta(\omega, t, \vartheta(\omega)) = c\mathbf{1}_{(a,b]}(t)\mathbf{1}_A(\omega)\mathbf{1}_B(\vartheta(\omega)) = c\mathbf{1}_{(a,b]}(t)\mathbf{1}_{A\cap\vartheta^{-1}(B)}(\omega). \tag{4.6}$$

Since the set $A \cap \vartheta^{-1}(B)$ belongs to $\mathcal{F}_a \vee \sigma(\vartheta)$, it belongs also to $\mathcal{G}_a$, and the function X^ϑ defined by (4.6) is an element of $\mathcal{P}(\mathbf{G})$.
Inversely, let $a, b, c \in \mathbb{R}, a < b, C \in \mathcal{G}_{a-}$, then

$$X^\vartheta(\omega, t, \vartheta(\omega)) = c\mathbf{1}_{(a,b]}(t)\mathbf{1}_C(\omega) \tag{4.7}$$

is an element of semi-algebra generating $\mathcal{P}(\mathbf{G})$. Since $\mathcal{G}_{a-} = \bigvee_{s<a}(\mathcal{F}_s \vee \sigma(\vartheta))$ it is sufficient to consider the elements of generating algebra, namely $\bigcup_{s<a}(\mathcal{F}_s \vee \sigma(\vartheta))$. In turn, if $C \in \bigcup_{s<a}(\mathcal{F}_s \vee \sigma(\vartheta))$, then there exists $s < a$ such that $C \in \mathcal{F}_s \vee \sigma(\vartheta)$. Next, the sigma-algebra $\mathcal{F}_s \vee \sigma(\vartheta)$ is generated by the sets $A \cap \vartheta^{-1}(B)$ with $A \in \mathcal{F}_s$ and $B \in \mathcal{A}$. So, we have to consider only the elements X^ϑ of the form (4.7) with $C = A \cap \vartheta^{-1}(B)$. But the corresponding application X is (4.5) and it is $\mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$-measurable. □

Lemma 4.3. *Let for each* $\theta \in \Theta$ *the process* $(X_t^\theta)_{t\geq 0}$ *be* $\mathbf{F}$*-adapted cadlag process. Let* $L > 0$ *and*

$$\tau_L^\theta = \inf\{s \geq 0 : X_s^\theta(\omega) > L\}. \tag{4.8}$$

If the application $X : (\omega, t, \theta) \to X_t^\theta$ *is* $\mathcal{O}(\mathbb{G})$ *then*

$$\tau_L^\vartheta = \inf\{s \geq 0 : X_s^{\vartheta(\omega)}(\omega) > L\}$$

is $\mathbf{G}$*-stopping time.*

Proof Let $t \in \mathbb{R}^+$. Then

$$\{(\omega, \theta) : \tau_L^\theta > t\} = \{(\omega, \theta) : \sup_{s\leq t} X_s^\theta \leq L\} \in \mathbb{G}_t$$

where $\mathbb{G}_t$ is defined by (4.2). It means that for all $u > t$

$$\{(\omega, \theta) : \tau_L^\theta > t\} \in \mathcal{F}_u \otimes \mathcal{A}.$$

Since $\mathcal{F}_u \otimes \mathcal{A}$ is generated by semi-algebra of the sets of the form $A \times B$ with $A \in \mathcal{F}_u$ and $B \in \mathcal{A}$, we can restrict ourselves to this special type of sets. But

$$\{\omega : (\omega, \vartheta(\omega)) \in A \times B\} \in \mathcal{F}_u \vee \sigma(\vartheta)$$

and, hence, for $u > t$

$$\{\omega : \tau_L^\vartheta > t\} \in \mathcal{F}_u \vee \sigma(\vartheta).$$

Then, τ_L^ϑ is $\mathbf{G}$-stopping time. □

Lemma 4.4. *Let $\theta \in \Theta$ and $(M_t^\theta)_{t\geq 0}$ be $\mathbf{F}$-adapted cadlag process. Let M be the application $(t,\omega,\theta) \to M_t^\theta(\omega)$. Suppose that $L^1(\Omega,\mathcal{F},P)$ is separable. Then the following conditions are equivalent:*

a) *M^θ is $(P^\theta,\mathbf{F})$-martingale for α-almost all θ and M is $\mathcal{O}(\mathbb{G})$-measurable process,*
b) *M is $(\mathbb{P},\mathbb{G})$-martingale,*
c) *M^ϑ is $(P,\mathbf{G})$-martingale.*

Proof We show that

$$a) \overset{(i)}{\Rightarrow} c) \overset{(ii)}{\Rightarrow} b) \overset{(iii)}{\Rightarrow} a).$$

(i): Let E be the expectation with respect to P and $\mathbb{E}$ be the expectation with respect to $\mathbb{P}$ which is the joint law of $(\omega,\vartheta(\omega))$. For each $s<t, A\in\mathcal{F}_s, B\in\mathcal{A}$ we have

$$E(\mathbf{1}_A(\omega)\mathbf{1}_B(\vartheta(\omega))(M_t^\vartheta - M_s^\vartheta)) = \mathbb{E}(\mathbf{1}_A(\omega)\mathbf{1}_B(\theta)(M_t^\theta - M_s^\theta)).$$

Let E_α be the expectation with respect to α and E_θ is the expectation with respect to P^θ. Then by Fubini theorem and conditioning we obtain

$$\mathbb{E}(\mathbf{1}_A(\omega)\mathbf{1}_B(\theta)(M_t^\theta - M_s^\theta)) = E_\alpha[\mathbf{1}_B(\theta)E_\theta(\mathbf{1}_A(\omega)E_\theta(M_t^\theta - M_s^\theta|\mathcal{F}_s))] = 0$$

since M^θ is a martingale α-a.s. with respect to $(P^\theta,\mathbf{F})$. Hence, P-a.s.

$$E(M_t^\vartheta - M_s^\vartheta|\mathcal{F}_s\vee\sigma(\vartheta)) = 0$$

Since M^ϑ is cadlag, using corollary 2.4 of [22] ,p.59, we have

$$E(M_t^\vartheta - M_s^\vartheta|\mathcal{G}_s) = \lim_{u\downarrow s} E(M_t^\vartheta - M_s^\vartheta|\mathcal{F}_u\vee\sigma(\vartheta)) = 0$$

which gives c).
(ii): If M^ϑ is $(P,\mathbf{G})$-martingale, then for each $t\in\mathbb{Q}^+$ M_t^ϑ is $\mathcal{G}_t = \bigcap_{s>t}(\mathcal{F}_s\vee\sigma(\vartheta))$-measurable and it can be written in the form $M_t^\vartheta(\omega) = M(\omega,t,\vartheta(\omega))$ (P-a.s.) where M is measurable with respect to $\mathbb{G}_t = \bigcap_{s>t}(\mathcal{F}_t\otimes\mathcal{A})$. Taking right-continuous version having left-hand limits we obtain the application $M : (\omega,t,\theta) \to (\mathbb{R},\mathcal{B}(\mathbb{R}))$ which is $\mathcal{O}(\mathbb{G})$. For all $s<t$ and $A\in\mathcal{F}_s,\ B\in\mathcal{A}$ we have:

$$\mathbb{E}(\mathbf{1}_A(\omega)\mathbf{1}_B(\theta)(M(\omega,t,\theta) - M(\omega,s,\theta)) = E(\mathbf{1}_A(\omega)\mathbf{1}_B(\vartheta(\omega))(M_t^\vartheta - M_s^\vartheta)) = 0$$

which means that $\mathbb{P}$-a.s.

$$\mathbb{E}(M(\omega,t,\theta) - M(\omega,s,\theta)|\mathcal{F}_s\otimes\mathcal{A}) = 0$$

and we have b) in the same way as c) before, since M is cadlag.
(iii): If we have b), then for each (ω,t,θ) we have $M_t^\theta = M(\omega,t,\theta)$. For $A\in\mathcal{F}_s$ and $B\in\mathcal{A}$ we obtain by Fubini theorem

$$0 = \mathbb{E}(\mathbf{1}_A(\omega)\mathbf{1}_B(\theta)(M(\omega,t,\theta) - M(\omega,s,\theta))) = E_\alpha(\mathbf{1}_B(\theta)E_\theta(\mathbf{1}_A(\omega)(M_t^\theta - M_s^\theta))).$$

Hence, for each $s<t$ and α - a.s.

$$E_\theta(\mathbf{1}_A(M_t^\theta - M_s^\theta)) = 0.$$

The measurability problem which may occur here is that α-a.s. set can depend on A and s. Since $L^1(\Omega, \mathcal{F}, P)$ is separable, we obtain that α-a.s. for all s and all $\mathcal{F}_s$-measurable bounded functions g_s

$$E_\theta(g_s(M_t^\theta - M_s^\theta)) = 0$$

and, hence,

$$E_\theta(M_t^\theta - M_s^\theta | \mathcal{F}_s) = 0$$

which gives $a)$.

□

Proof We show that $a), b), c)$ are equivalent. With the notation of Theorem 2.1, the processes $M^\theta(l)$, $N^\theta(l)$ and $U^\theta(l)$ are $(P, \mathbf{F})$ local martingales. Since the semimartingale $\tilde{X}$ has bounded jumps, all these local martingales are also locally bounded, i.e. for each θ there exists a localizing sequence τ_L^θ such that the stopped processes are bounded. By Lemma 4.3 the replacing θ by ϑ in stopping times gives $\tau_L^\vartheta(\omega)$ which is $(P, \mathbf{G})$-stopping times. Moreover, the application $\tau_L : (\omega, t, \theta) \to \tau_L^\theta$ is $(\mathbb{P}, \mathbb{G})$-stopping time.
Next, by Lemma 4.2 the replacing of θ by ϑ in T^θ which supposed to be $\mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$-measurable, gives T^ϑ which is $\mathcal{P}(\mathbf{G})$-measurable. Moreover, the application $T' : (\omega, t, \theta) \to T^\theta$ is $\mathcal{P}(\mathbb{G})$-measurable.
Finally, the claim follows now from the Lemma 4.4 which guaranties the conservation of martingale properties in the case of replacing θ by the variable ϑ and in the case of replacing of the initial space by product space. □

In the considered case when P^θ is the conditional law of semimartingale X given $\vartheta = \theta$, one can rewrite the assumption 3 in terms of so-called decoupling measure Q as in [14]. Let us suppose that the density process $z = (z^\theta)_{\theta \in \Theta}$ is $\mathcal{O}(\mathbf{F}) \otimes \mathcal{A}$ measurable. Then we can replace θ by ϑ to obtain z^ϑ. We denote by P_t and Q_t the restrictions of the measures P and Q to $\mathcal{G}_t$ where $\mathbf{G} = (\mathcal{G}_t)_{t \geq 0}$ is enlarged with the initial value ϑ filtration . If for all $t > 0$, $z_t^\vartheta > 0$ P-a.s. , we can define Q by

$$dQ_t = (z_t^\vartheta)^{-1} dP_t.$$

The decoupling measure has the following property: $(Q, \mathbf{G})$- triplet of X is the same as the $(P, \mathbf{F})$- triplet of X and $\mathcal{L}(\vartheta | Q) = \mathcal{L}(\vartheta | P)$. We can also use an another definition of a decoupling measure Q, namely as a solution of the following martingale problem, if it exists and is unique: the $(Q, \mathbf{G})$- triplet of X is the same as the $(P, \mathbf{F})$- triplet of X and $\mathcal{L}(\vartheta | Q) = \mathcal{L}(\vartheta | P)$.

Remark 4.2. *If for all $t > 0$, $z_t^\vartheta > 0$ P-a.s., the assumption 3 is equivalent to the assumption:*

(4.9) $$E_Q[z^\vartheta, z^\vartheta]_{\tau_n}^{1/2} < \infty$$

for some localizing sequence of $\mathbf{F}$*- stopping times* τ_n*. We note that* $[z^\vartheta, z^\vartheta]^{1/2}$ *is* $(Q, \mathbf{G})$*- locally integrable (see* [19, Corollary I.4.55]*). Here we require the existence of a localizing sequence of* $\mathbf{F}$*-stopping times.*

Theorem 4.2. *Under the settings of Theorem (4.1), assume a) and (4.9) hold. Then* X *is a* $(P, \mathbf{G})$*- semimartingale with the triplet* $T^\vartheta = (B^\vartheta, C, \nu^\vartheta)$*.*

Proof Using the proof of Theorem (4.1) we conclude that it remains to prove that B^ϑ is of locally integrable variation with recpect to P. Since B^ϑ is obtained from B^θ by replacing θ by ϑ, we have: for $t > 0$

$$\mathrm{Var}(B^\vartheta)_t \leq \mathrm{Var}(B)_t + (|\beta^\vartheta| \cdot C)_t + (|Y^\vartheta - 1|l \star \nu)_t.$$

Since B is locally integrable with respect to P, the question of local integrability of B^ϑ is reduced to the existence of localizing sequence of $\mathbf{F}$ stopping times τ_n such that for each $n \geq 1$

$$E_P\left(|\beta^\vartheta| \cdot C)_\tau + |Y^\vartheta - 1|l \star \nu\right)_{\tau_n} < \infty. \tag{4.10}$$

We have:

$$\begin{aligned}
&E_P\left((|\beta^\vartheta| \cdot C)_{\tau_n} + (|Y^\vartheta - 1|l \star \nu)_{\tau_n}\right) \\
&= E_Q\{z^\vartheta_\tau\left((|\beta^\vartheta| \cdot C)_{\tau_n} + (|Y^\vartheta - 1|l \star \nu)_{\tau_n}\right)\} \\
&= E_Q\{(z^\vartheta_- |\beta^\vartheta| \cdot C)_{\tau_n} + (z^\vartheta_- |Y^\vartheta - 1|l \star \nu)_{\tau_n}\} \\
&= E_Q\{(z^\vartheta_- |\beta^\vartheta| \cdot C)_{\tau_n} + (z^\vartheta_- |Y^\vartheta - 1|l \star \mu^X)_{\tau_n}\} \\
&= E_Q \mathrm{Var}([z^\vartheta, X(l) - B])_{\tau_n}
\end{aligned}$$

By Fefferman inequality, (see [15, Theorem 10.17]) and the fact that $X(l) - B$ is both $(Q, \mathbf{G})$- and $(P, \mathbf{F})$- local martingale we deduce that

$$E_Q \mathrm{Var}([z^\vartheta, X(l) - B])_{\tau_n} \leq \| (X(l) - B)^{\tau_n} \|_{BMO} \, E_Q[z^\vartheta, z^\vartheta]^{1/2}_{\tau_n}.$$

From proposition 2.38 in [17] it follows easily that the $(P, \mathbf{F})$-local martingale $(X(l) - B)$ is $(P, \mathbf{F})$-locally in BMO since it has bounded jumps, and by assumption (4.9) there is a localizing sequence of $\mathbf{F}$-stopping times τ_n tending to infinity which makes the last expression finite. Hence, the inequality (4.10) holds and B^ϑ has locally integrable variation with respect to P. □

Remark 4.3. *Assumption 4.9 can be expressed in term of information. More precicely,*

$$E_Q([z^\vartheta, z^\vartheta]^{1/2}_\tau) \leq C(1 + E_Q(z^\vartheta_\tau log z^\vartheta_\tau))$$

The boundness of this information was used in [10] *to verify stochastic Fubini theorem.*

4.2. **Initial enlargement and gaussian martingales.** Let us first consider a classical example of initial enlargement of filtration. Here X is a continuous Gaussian martingale with respect to the measure P starting from zero and such that there exists $\lim_{t\to\infty} X_t = X_\infty$.
Let $\vartheta = X_\infty$. We denote by $\langle X\rangle$ the predictable quadratic variation of X and we put $\langle X\rangle_{t,\infty} := \langle X\rangle_\infty - \langle X\rangle_t$.
The prior distribution $\alpha(d\theta) := P(\vartheta \in d\theta)$ is a $\mathcal{N}(0, \langle X\rangle_\infty)$ and the posterior distribution α^t of ϑ given $\mathcal{F}_t$ is $\mathcal{N}(X_t, \langle X\rangle_{t,\infty})$.
Assume $\langle X\rangle_{t,\infty} > 0$ for all $t \in \mathbb{R}^+$, then α^t is equivalent to α, so the assumption (4.1) is valid.
From the Ito formula with the function $f(x,y) = x^2/y$ applied to the first term in exponential we have:

$$\begin{aligned} \frac{d\alpha^t}{d\alpha}(\theta) &= \frac{\sqrt{\langle X\rangle_\infty}}{\sqrt{\langle X\rangle_{t,\infty}}}\exp\Big\{-\frac{(\theta - X_t)^2}{2\langle X\rangle_{t,\infty}} + \frac{\theta^2}{2\langle X\rangle_\infty}\Big\} \\ &= \exp\Big\{\int_0^t \beta_s^\theta dX_s - \frac{1}{2}\int_0^t \left(\beta_s^\theta\right)^2 d\langle X\rangle_s\Big\}, \end{aligned}$$

where

$$\beta_s^\theta := \frac{\theta - X_s}{\langle X\rangle_{s,\infty}}.$$

Since β^θ is predictable process for each $\theta \in \Theta$, continuous in θ uniformly in $t \in [0,T]$ for each $T > 0$, the application $(\omega, t, \theta) \to \beta_t^\theta$ is $\mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$-measurable. By Theorem 4.1 we can now conclude that the process

$$X_t - \int_0^t \frac{(X_\infty - X_s)}{\langle X\rangle_{s,\infty}} d\langle X\rangle_s$$

is a $(P, \mathbf{G})$- Gaussian martingale with the bracket $\langle X\rangle$.
We give some special cases of the above results.

- Let Y be a Brownian motion and put $X_t = \int_0^t a_s dY_s$, where a is deterministic square-integrable function on $\mathbb{R}^+$. If $a_s := I_{(0,T]}(s)$, then we have: $\vartheta = Y_T$, $\langle X\rangle_{t,\infty} = T - t$ for $t \le T$ and $\beta_s^\theta = \dfrac{\theta - Y_s}{T - s}$ and hence we have the classical representation of the Brownian bridge:

$$Y_t = \int_0^t \frac{Y_T - Y_s}{T - s} ds + Y_t^{\mathbf{G}},$$

  where $Y^{\mathbf{G}}$ is a Brownian motion with respect to $\mathbf{G}$.
- In the previous case take $a = I_{(0,T+\eta]}$. We obtain the case of *final value distorted by a small noise* example from [1].
- Assume that Y is a fractional Brownian motion and let $X_t := E[Y_T | F_t^Y]$ be the prediction martingale. This example and related will be studied in detail in [12].

4.3. **Initial enlargement in the Poisson filtration.** Assume that X is a Poisson process with intensity 1 on $(\Omega, \mathcal{F}, \mathbf{F}, P)$ stopped in time T and let $\vartheta = X_T$. Here prior distribution α is $Poisson(T)$ and posterior distribution

$$\alpha^t(\theta) = \begin{cases} e^{T-t}\frac{(T-t)^{\theta - X_t}}{(\theta - X_t)!} & \text{if } \theta \geq X_t, \\ 0 & \text{if } \theta < X_t. \end{cases} \tag{4.11}$$

Next, for all $t \in [0, T[$ we have $\alpha^t \prec\prec \alpha$ and

$$\frac{d\alpha^t}{d\alpha}(\theta) = e^{-t}\frac{(T-t)^{\theta - X_t}}{T^\theta} I_{\{\theta \geq X_t\}} \frac{\theta!}{(\theta - X_t)!}.$$

We put $Y_s^\theta := \dfrac{\theta - X_{s-}}{T-s}$ and we remark that Y^θ predictable process with $0 \leq Y_s^\theta < \infty$ for all $s \in [0, T]$ – this follows from the fact that $\Delta X_T = 0$ $\mathbb{P}$ - a.s. Since

$$\frac{d\alpha^t}{d\alpha}(\theta) = \exp\{\int_0^t (Y_s^\theta - 1)ds\} \prod_{s \leq t} \left(Y_s^\theta\right)^{\Delta X_s},$$

we obtain that with respect to the filtration $\mathbf{G}$ the standard Poisson process has the semimartingale representation: for $t < T$

$$X_t = n_t + \int_0^t \frac{X_T - X_{s-}}{T-s} ds,$$

where $n = (n_t)_{t \geq 0}$ is a $(P, \mathbf{G})$- martingale.

4.4. **Lévy processes: initial enlargement with the final value.** Let X be a Lévy process. Then for each $\lambda \in \mathbb{R}$ the characteristic function of X_t is

$$Ee^{i\lambda X_t} = e^{-t\psi(\lambda)}$$

where ψ is characteristic exponent given by

$$\psi(\lambda) = ia\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} \left(1 - e^{i\lambda x} + i\lambda x I_{\{|x|<1\}}\right) \pi(dx)$$

with π a measure on $\mathbb{R}$ verifying $\int_{\mathbb{R}}(1 \wedge x^2)\pi(dx) < \infty$. The $(P, \mathbf{F})$-triplet of X is $T = (aI, \sigma^2 I, Leb \otimes \pi)$, where $I_t = t$.
We consider again stopped in T process and we take $\vartheta := X_T$. The process X is a time-homogeneous Markov process with independent increments and hence

$$\alpha^t(d\theta) = P(X_T \in d\theta | X_t) = P(X_{T-t} + x \in d\theta)|_{x = X_t}.$$

To be able to continue we assume that the law of the random variable X_s has a density $f(s, y)$ with respect to fixed dominating measure η, i.e. for $B \in \mathcal{B}(\mathbb{R})$

$$P(X_s \in B) = \int_B f(s, y)\eta(dy).$$

Moreover, we assume that $f \in C_b^{1,2}(\mathbb{R}^+ \times U)$ where U is an open set belonging to $\mathbb{R}$.
Since for $t \in [0, T[\ \alpha^t \prec\prec \alpha$, we can write that η- a.s.

$$\frac{d\alpha^t}{d\alpha}(\theta) = \frac{f(T-t, \theta - X_t)}{f(T, \theta)}. \tag{4.12}$$

Use Itô formula to obtain

$$\begin{aligned} f(T-t, \theta - X_t) &= f(T, \theta) - \int_0^t \frac{\partial f}{\partial s}(T-s, \theta - X_{s-}) ds \\ &\quad - \int_0^t \frac{\partial f}{\partial x}(T-s, \theta - X_{s-}) dX_s \\ &\quad + \frac{1}{2}\sigma^2 \int_0^t \frac{\partial^2 f}{\partial x^2}(T-s, \theta - X_{s-}) ds \\ &\quad + \sum_{s \le t} \left(\Delta f(T-s, \theta - X_s) + \frac{\partial f}{\partial x}(T-s, \theta - X_{s-}) \Delta X_s \right). \end{aligned} \tag{4.13}$$

We know that the expression in (4.12) is a $(P, \mathbf{F})$- martingale for each θ. So, we can identify the continuous martingale part on the right hand side of (4.13) and then the continuous martingale part of (4.12) as

$$- \int_0^t \frac{\frac{\partial f}{\partial x}(T-s, \theta - X_{s-})}{f(T, \theta)} dX_s^c. \tag{4.14}$$

Recall that $z_t^\theta = \frac{d\alpha^t}{d\alpha}(\theta)$. According to the Girsanov theorem the term β^θ in the equation (2.1) is obtained as (for more details on this kind of computations see [19, Lemma III.3.31])

$$\begin{aligned} \beta_t^\theta &= \frac{d < z^\theta, X^c >_t}{z_{t-}^\theta d < X^c, X^c >_t} = \frac{-\frac{\partial f}{\partial x}(T-t, \theta - X_{t-})}{f(T-t, \theta - X_{t-})} \\ &= -\frac{\partial}{\partial x} \log f(T-t, x)|_{x = \theta - X_{t-}}. \end{aligned} \tag{4.15}$$

Consider next the pure jump martingale in (4.12): we have that

$$\Delta f(T-t, \theta - X_t) = f(T-t, \theta - X_t) - f(T-t, \theta - X_{t-})$$

and so

$$\frac{\Delta z_t^\theta}{z_{t-}^\theta} = \frac{f(T-t, \theta - X_t)}{f(T-t, \theta - X_{t-})} - 1,$$

from this we obtain (for more details, see [19, p. 175]) that the P^θ compensator ν^θ of μ^X is

$$\nu^\theta(dt, du) = \frac{f(T-t, \theta - (X_{t-} + u))}{f(T-t, \theta - X_{t-})} \pi(du) dt. \tag{4.16}$$

Moreover, since the expression on the right-hand side of (4.12) is a martingale, the function $f(t,u)$ satisfies the following integro-differential equation, which might be called a Kolmogorov backward integro-differential equation:

$$\begin{aligned}\frac{\partial f}{\partial t}(T-t,\theta-x) &= \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(T-t,\theta-x) - a\frac{\partial f}{\partial x}(T-t,\theta-x)\\ &\quad + \int_{\mathbb{R}}\Big(f\left(T-t,\theta-(x+y)\right)\\ &\quad - f\left(T-t,\theta-x\right) + \frac{\partial f}{\partial x}\left(T-t,\theta-x\right)y\Big)\pi(dy) \end{aligned}\tag{4.17}$$

with the boundary condition $f(T,\theta-x) = \delta_{\{0\}}(\theta-x)$.

4.4.1. *Example: Brownian motion.* We look again the Brownian case, as in subsection 4.2, but now using the Lévy processes approach. Since the triplet of X is $T=(0,\sigma^2 I,0)$, the equation (4.17) is reduced to:

$$\frac{\partial f}{\partial t}(T-t,\theta-x) = \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(T-t,\theta-x)$$

with boundary condition $f(T,\theta-x)=\delta_{\{0\}}(\theta-x)$.
It is well-known that the solution is

$$f(T-t,\theta-x) = \frac{1}{\sqrt{2\pi(T-t)}}\exp\{-\frac{(\theta-x)^2}{2(T-t)}\}$$

and so $\beta^\theta = \dfrac{\theta-X_s}{T-s}$ and a new drift is $B_t^\theta = \displaystyle\int_0^t \frac{\theta-X_s}{T-s}ds$.

4.4.2. *Example: Gamma process.* Let X be a Gamma process. This means that the $(P,\mathbf{F})$ triplet of X is $T=(\frac{a}{b}t, 0, \frac{a}{u}e^{-bu}dudt)$. We know also that the density $f(t,x)=P(X_t\in dx)$ is $f(t,x)=\frac{b^{at}}{\Gamma(at)}x^{at-1}e^{-bx}$ with some parameters $a,b>0$ (see [5, p.73]). In particular we have that $X_t-\frac{a}{b}t$ is a $(P,\mathbf{F})$ martingale.
Put again $\vartheta = X_T$ and we have from (4.16) that the (P^θ,F) compensator is

$$\nu^\theta(dx,dt) = (1-\frac{x}{\theta-X_{t-}})^{a(T-t)-1}\frac{a}{x}dxdt.$$

Hence, $(P^\theta,\mathbf{F})$ drift of the process X is

$$\int_0^t\int_0^{\theta-X_{t-}} x(1-\frac{x}{\theta-X_{s-}})^{a(T-s)-1}\frac{a}{x}dxdt = \int_0^t\frac{\theta-X_{s-}}{T-s}ds,$$

and this means that the process $X_t-\frac{a}{b}t-\int_0^t\frac{\theta-X_{s-}}{T-s}ds$ is a $(P^\theta,\mathbf{F})$-martingale.

4.4.3. *Example: Poisson process.* We look again at the Poisson case, as in subsection 4.3. We indicate briefly how one can use the approach described in 4.4, where we know only the triplet of the process X. So, let X be a Poisson process with intensity λ.
Put again $\vartheta = X_T$. Put $p(t,k) := P(X_t = k)$ and we assume that for $k \geq 0$ the functions $p(\cdot, k) \in C^1(\mathbb{R}^+)$.
We know (see (4.12)) that

$$\frac{d\alpha^t}{d\alpha}(\theta) = \frac{p(T-t, \theta - X_t)}{p(T,\theta)}.$$

We start with the trivial identity, which is the analog of Ito-formula here:

$$p(T-t, \theta - X_t) = p(T,\theta) - \int_0^t p_t(T-s, \theta - X_{s-})ds + \sum_{s \leq t} \Delta p(T-s, \theta - X_s). \tag{4.18}$$

Using the fact that $\Delta X_t \in \{0,1\}$, we have the following identity

$$\Delta p(T-s, \theta - X_s) = (p(T-s, \theta - (X_{s-}+1)) - p(T-s, \theta - X_{s-}))\Delta X_s;$$

and since the right-hand side of (4.18) is a $(P, \mathbf{F})$- martingale, we obtain that the functions $p(t,k)$ satisfy the following system of differential equations

$$p_t(T-s,k) = \lambda(p(T-s,k) - p(T-s,k+1)); \tag{4.19}$$

and, hence,

$$p(T-s,k) = e^{-\lambda(T-s)}\frac{(\lambda(T-s))^k}{k!}$$

is the solution of the system (4.19) with boundary condition $p(T, \theta - x) = \delta_{\{0\}}(\theta - x)$. It remains to note that

$$p(T-s,k) - p(T-s,k+1) = p(T-s,k)(\frac{k+1}{\lambda(T-s)} - 1) \tag{4.20}$$

and we can conclude that with respect to the measure P^θ the process X has intensity $\frac{\theta - X_{s-}}{T-s}$. This means that the process $X_t - \int_0^t \frac{\theta - X_{s-}}{T-s}ds$ is a $(P^\theta, \mathbf{F})$- martingale.

5. Progressive enlargement

5.1. **Progressive enlargement with random time.** We assume that we are given with a semimartingale X on a filtered space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ with the right-continuous version of natural filtration $\mathbf{F} = (\mathcal{F}_t^X)_{t \geq 0}$ completed by the $\mathcal{F}$ sets of probability zero and $\mathcal{F} = \mathcal{F}_\infty^X$. Let $T = (B, C, \nu)$ be the $(P, \mathbf{F})$-triplet of X. Later, to simplify the notation, we omit the index X in the filtration.

Suppose that we have also a random variable ϑ with values in measurable polish space $(\Theta, \mathcal{A})$. Define now the initially enlarged filtration $\mathbf{G} = (\mathcal{G}_t)_{t\geq 0}$ by

$$\mathcal{G}_t := \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(\vartheta)).$$

Consider a counting process $N = (N_t)_{t\geq 0}$ defined by

$$N_t = \mathbf{1}_{\{\vartheta \leq t\}}$$

and let $\mathbf{F}^N = (\mathcal{F}_t^N)_{t\geq 0}$ be the right-continuous version of the filtration generated by N. Put

$$\mathcal{H}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \mathcal{F}_s^N)$$

and we complete the filtration with $\mathcal{H}_\infty$ sets of probability zero. Then, $\mathbf{H} = (\mathcal{H}_t)_{t\geq 0}$ is progressively enlarged filtration with the random process N or with random time ϑ.
As it was shown in 3., under the conditions of theorem 4.1 X is $(P, \mathbf{G})$-semimartingale. Since the filtration $\mathbf{H}$ is smaler then $\mathbf{G}$, the result of Stricker (see [**?**, p.53, Theorem 4]) on filtration shrinkage says that X will be $(P, \mathbf{H})$-semimartingale. So, $(P, \mathbf{H})$-triplet of X exists. But to find it by projection we have to avoid some technical difficulties related with the local integrability of the variation of drift part. Namely, this localisation can be done with **G**- stopping times but not always with **H**-stopping times. To assure the existence of **H**-stopping times for localising procedure we assume that for each $t > 0$ and P-a.s.

$$(\tilde{\beta} \cdot C)_t + (\tilde{Y} \star \nu)_t < \infty \tag{5.1}$$

where

$$\tilde{\beta}_t = E_{\alpha^{t-}} |\beta_t^\theta|, \quad \tilde{Y}_t = E_{\alpha^{t-}} |(Y_t^\theta - 1)l|. \tag{5.2}$$

and $E_{\alpha^{t-}}$ stays for the expectation with respect to a posreriori measure α^{t-}.

Proposition 5.1. *Assume the conditions of theorem 3.1 and (5.1). Then the $(P, \mathbf{H})$ triplet $T^N = (B^N, C^N, \nu^N)$ of X is simply the $(P, \mathbf{H})$-dual predictable projection of the $(P, \mathbf{G})$-triplet $T^\vartheta = (B^\vartheta, C^\vartheta, \nu^\vartheta)$ of X* .

Proof We begin with the decomposition of the semimartingale $X = \tilde{X} + X(l)$ where $\tilde{X} = X - X(l)$ and

$$X(l)_t := \sum_{s\leq t} (\Delta X_s - l(\Delta X_s))$$

with $\Delta X_s = X_s - X_{s-}$ and l truncation function. Since $\tilde{X}$ is special semimartingale it can be decomposed on martingale and drift parts both with respect to the filtration $\mathbf{G}$ and with respect to the filtration $\mathbf{H}$:

$$\tilde{X} = M^{\mathbf{G}} + B^{\mathbf{G}}, \quad \tilde{X} = M^{\mathbf{H}} + B^{\mathbf{H}}.$$

To show that $B^{\mathbf{H}}$ is dual predictable projection of $B^{\mathbf{G}}$ we have to find an increasing sequence of $\mathbf{H}$-stopping times τ_n, $\tau_n \uparrow \infty$ such that for each bounded $(P, \mathbf{H})$-predictable process K

$$E(K \cdot B^H)_{\tau_n} = E(K \cdot B^H)_{\tau_n} \tag{5.3}$$

where E stands for the expectation with respect to P.
For $n \in \mathbb{N}^*$ we put

$$\tau_n = \inf\{t \geq 0 : (\tilde{\beta} \cdot C)_t + (\tilde{Y} \star \nu)_t + Var(B)_t > n\} \tag{5.4}$$

where $\tilde{\beta}$ and $\tilde{Y}$ are given by (5.2). Since τ_n is $(P, \mathbf{F})$-stopping time and the filtration $\mathbf{H}$ is bigger then $\mathbf{F}$, it is also $(P, \mathbf{H})$-stopping time. Moreover, we can easily verify by Fubini theorem that for each n

$$E(Var(B^{\mathbf{G}})_{\tau_n}) \leq E[(|\beta^{\vartheta}| \cdot C)_{\tau_n} + Var(B)_{\tau_n} + (|(Y^{\vartheta} - 1)l| \star \nu)_{\tau_n}] =$$
$$E[(\tilde{\beta} \cdot C)_{\tau_n} + Var(B)_{\tau_n} + (\tilde{Y} \star \nu)_{\tau_n}] < n + 3 \max_{x \in \mathbb{R}} l(x) < \infty.$$

Without loss of generality we can suppose that $M^{\mathbf{H}}$ and $M^{\mathbf{G}}$ are martingales and that $\tilde{X}$ is integrable for each $t > 0$. In fact, to satisfy these conditions it is sufficient to make an additional localising. Then for each $(P, \mathbf{H})$-predictable bounded process K we have:

$$E(K \cdot M^H)_{\tau_n \wedge t} + E(K \cdot B^H)_{\tau_n \wedge t} = E(K \cdot M^G)_{\tau_n \wedge t} + E(K \cdot B^G)_{\tau_n \wedge t}.$$

Since M^G and M^H are martingales we obtain (5.3) letting $t \to \infty$.
By [**?**, Theorem 4.6.1, p. 191] we have that $C^{\mathbf{H}} = C^{\mathbf{G}} =: C$.
Consider next the compensators $\nu^{\mathbf{H}}$ and $\nu^{\mathbf{G}}$. From [19, II.1.16] it follows that the jump measure $\mu(\omega, dt, dx)$ of X is both $\tilde{\mathcal{P}}(\mathbf{G})$ σ-finite and $\tilde{\mathcal{P}}(\mathbf{H})$ σ-finite, and its respective $(P, \mathbf{H})$- and $(P, \mathbf{G})$- dual predictable projections $\nu^{\mathbf{H}}(\omega, dt, dx)$ and $\nu^{\mathbf{G}}(\omega, dt, dx)$ do exist.
What it remains to show is that $\nu^{\mathbf{G}}$ is the $(P, \mathbf{G})$-dual predictable projection of $\nu^{\mathbf{H}}$. But for every non-negative $\tilde{\mathcal{P}}(\mathbf{G})$-measurable bounded integrand $W(\omega, t, x)$ we have

$$E(W \star \nu^{\mathbf{H}})_{\tau_n} = E(W \star \mu)_{\tau_n} = E(W \star \nu^{\mathbf{G}})_{\tau_n}$$

which means that $(\nu^{\mathbf{H}})$ is $(P, \mathbf{H})$ dual predictable projection of $\nu^{\mathbf{G}}$. □
It remains now to find the formulas for dual predictable projection of $(P, \mathbf{G})$- triplet of X. For this we note that under the conditions of theorem 3.1 the $(P, \mathbf{G})$-tiplet is given by:

$$\nu^{\mathbf{G}} = Y^{\vartheta} \cdot \nu, \quad B^{\mathbf{G}} = B + \beta^{\vartheta} \cdot C + (Y^{\vartheta} - 1)l \star \nu$$

where $T = (B, C, \nu)$ is a triplet of X under P. We suppose that for each $\mathbf{F}$-predictable bounded stopping time τ

$$E[\beta^{\vartheta}_{\tau} \mathbf{1}_{\{\tau < \infty\}}] < \infty, \tag{5.5}$$

$$E[\int_{\mathbb{R}\setminus\{0\}} |(Y^{\vartheta}_{\tau}(\omega,x)-1)l(x)|K(\omega,\tau,dx)\mathbf{1}_{\{\tau<\infty\}}] < \infty, \tag{5.6}$$

$$E[\int_{\mathbb{R}\setminus\{0\}} Y^{\vartheta}_{\tau}(\omega,x)K(\omega,\tau,dx)\mathbf{1}_{\{\tau<\infty\}}] < \infty. \tag{5.7}$$

Where $K(\omega,\tau,dx)$ is a transition kernel appearing in desintegration formula for compensator ν:

$$d\nu_t(\omega,x) = dA_t(\omega)K(\omega,t,dx).$$

Proposition 5.2. *Assume the conditions of theorem 3.1 and the conditions (5.1), (5.5), (5.6), (5.7). Then $(P,\mathbf{H})$-triplet of X is given by:*

$$\nu^{\mathbf{H}} = Y^{\mathbf{H}}\cdot\nu,\quad B^{\mathbf{H}} = B + \beta^{\mathbf{H}}\cdot C + (Y^{\mathbf{H}}-1)l\star\nu.$$

Here $\beta^{\mathbf{H}}$ and $Y^{\mathbf{H}}$ are left-continuous versions of the conditional expectations with respect to P:

$$\beta^{\mathbf{H}}_s(\omega) = E(\beta^{\vartheta}_s(\omega)|\mathcal{H}_{s-}),\quad Y^{\mathbf{H}}_s(\omega,x) = E(Y^{\vartheta}_s(\omega,x)|\mathcal{H}_{s-}). \tag{5.8}$$

Proof Let $n\in\mathbb{N}^*$ and τ_n is defined by (5.4). Since B is $(P,\mathbf{H})$-predictable we have:

$$B^{\mathbf{H}}_{\tau_n\wedge t} = B_{\tau_n\wedge t} + (\beta^{\vartheta}\cdot C)^p_{\tau_n\wedge t} + ((Y^{\vartheta}-1)l\star\nu)^p_{\tau_n\wedge t}$$

where p stands for $(P,\mathbf{H})$-predictable projection.
Without additional restrictions we can take only positive bounded $(P,\mathbf{H})$-predictable process U. Then for all $\mathbf{F}$-predictable stopping time τ we have:

$$E[U_\tau\beta^{\vartheta}_\tau\mathbf{1}_{\{\tau<\infty\}}] = E[U_\tau E[\beta^{\vartheta}_\tau|\mathcal{H}_{\tau-}]\mathbf{1}_{\{\tau<\infty\}}]$$

since U_τ and $\mathbf{1}_{\{\tau<\infty\}}$ is $\mathcal{H}_{\tau-}$-measurable. Then performing time change with the process C as it was indicated in [22], p.177, we obtain

$$E[(U\beta^{\vartheta}\cdot C)_{\tau_n\wedge t}] = E[(UE(\beta^{\vartheta}|\mathcal{H}_-)\cdot C)_{\tau_n\wedge t}].$$

The same procedure can be used to find the predictable projection for the term contaning the integral with respect to the compensator. Namely, for each bounded $\tilde{\mathcal{P}}(\mathbf{H})$-predictable process $W(\cdot,\omega,x)$ we have:

$$E[(\int_{\mathbb{R}^*} W(\cdot,\omega,x)(Y^{\vartheta}_{\cdot}(\omega,x)-1)l(x)K(\omega,\cdot,dx)\cdot A)_{\tau_n\wedge t} =$$

$$E[(E(\int_{\mathbb{R}^*} W(\cdot,\omega,x)(Y^{\vartheta}_{\cdot}(\omega,x)-1)l(x)K(\omega,\cdot,dx)|\mathcal{H}_-)\cdot A)_{\tau_n\wedge t}$$

It remains to show that for each $s\in\mathbb{R}^+$ and P-a.s.

$$E(\int_{\mathbb{R}^*} W(s,\omega,x)(Y^{\vartheta}_s(\omega,x)-1)l(x)K(\omega,s,dx)|\mathcal{H}_{s-}) =$$

$$\int_{\mathbb{R}^*} W(s,\omega,x)E((Y^{\vartheta}_s(\omega,x)-1)|\mathcal{H}_{s-})l(x)K(\omega,s,dx)$$

Let $(E, \mathcal{E})$ be measurable space of jumps. Since the functions W and K are measurable with respect to $\mathcal{P}(\mathbf{H}) \times \mathcal{E}$ and Y^ϑ is measurable with respect to $\mathcal{P}(\mathbf{G}) \times \mathcal{E}$, they can be approximated by linear combinations of the functions of the type $v(\omega, s)h(x)$ and $u(\omega, s)g(x)$ where v and u are $\mathcal{P}(\mathbf{H})$- and $\mathcal{P}(\mathbf{G})$-measurables and h, g are $\mathcal{E}$-mesurables. Since $\mathcal{E}$ is generated by countable algebra $\mathcal{A}$ we have only to consider as functions of x the indicator functions of the sets belonging to $\mathcal{A}$. Finally, with these reductions we have P-a.s.:

$$E(\int_{\mathbb{R}^*} v(\omega, s)u(\omega, s)\mathbf{1}_B(x)K(\omega, s, dx)|\mathcal{H}_{s-}) =$$

$$E(v(\omega, s)u(\omega, s)K(\omega, s, B)|\mathcal{H}_{s-}) = v(\omega, s)E(u(\omega, s)|\mathcal{H}_{s-})K(\omega, s, B) =$$

$$\int_{\mathbb{R}^*} v(\omega, s)E(u(\omega, s)|\mathcal{H}_{s-})\mathbf{1}_B(x)K(\omega, s, dx).$$

The fact that $\mathcal{E}$ is generated by countable algebra permit us to approximate in L^1 sense each $\mathcal{E}$-measurable bounded fonction by at most countable set of linear combinations of indicator functions with the sets from $\mathcal{A}$. So, for each $s \in \mathbb{R}^+$ and $K(\omega, s, \cdot)$-a.a. x we will have (5.8). Similar consideration with $\nu^{\mathbf{G}}$ as previously permit to obtain that $\nu^{\mathbf{H}} = Y^{\mathbf{H}} \cdot \nu$ with $Y^{\mathbf{H}}$ given by (5.8). $\square$

Let us give the Bayesian interpretation of the conditional expectation with respect to $\mathcal{H}_-$. First we remark that $\mathcal{H}_{s-} = \mathcal{F}_{s-} \vee \mathcal{F}^N_{s-}$ and that we can only consider the sets belonging to semi-algebras generating $\mathcal{F}_{s-}$ and $\mathcal{F}^N_{s-}$. Let $u < s$ and $A \in \mathcal{F}_u$, then for $\beta^{\mathbf{G}} = \beta^{\mathbf{G}}(\omega, \vartheta)$ we have by Fubini theorem:

$$E(\mathbf{1}_A \mathbf{1}_{[0,u]\cup[s,\infty[}\beta^{\mathbf{G}}_s) =$$

$$\mathbb{E}[\mathbf{1}_A \mathbf{1}_{[0,u]\cup[s,\infty[}(\theta)\beta^{\mathbf{G}}_s(\omega, \theta)] = E[\mathbf{1}_A E_{\alpha^{s-}}(\mathbf{1}_{[0,u]\cup[s,\infty[}(\theta)\beta^{\mathbf{G}}_s(\omega, \theta))]$$

where $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$, the joint law of (ω, ϑ), and α^{s-} is posterior distribution of ϑ with respect to $\mathcal{F}_{s-}$.
For $u < s$ we put

$$Q([0, u]) = E_{\alpha^{s-}}(\mathbf{1}_{[0,u]\cup[s,\infty[}(\theta)\beta^{\mathbf{G}}_s(\omega, \theta)).$$

Then by (5.1), Q is a σ-finite positive measure which is absolutely continuous with respect to $\alpha^{s-}(\cdot \cap [0, s[) + \delta_{\{s\}}\alpha^{s-}([s, \infty[)$ and $\beta^{\mathbf{H}}_s(\omega, \theta)$ is nothing else as the Radon-Nikodym derivative of Q with respect to above mentionned measure. Explicite calculation of Radon-Nikodym derivative gives

$$\beta^{\mathbf{H}}_s(\omega, \theta) = \mathbf{1}_{\{\theta < s\}}\beta^\theta_s(\omega) + \mathbf{1}_{\{\theta \geq s\}} \frac{\int_s^\infty \beta^\theta_s(\omega)\alpha^{s-}(d\theta)}{\alpha^{s-}([s, \infty))} \tag{5.9}$$

Similar calculation can be made for $Y^{\mathbf{H}}$ too. We have

$$Y_s^{\mathbf{H}}(\omega,\theta,x) = \mathbf{1}_{\{\theta<s\}} Y_s^{\theta}(\omega,x) + \mathbf{1}_{\{\theta\geq s\}} \frac{\int_s^\infty Y_s^\theta(\omega,x)\alpha^{s-}(d\theta)}{\alpha^{s-}([s,\infty))}$$

If the application $T : (\omega,t,\theta) \to T_t^\theta(\omega)$ is $\mathcal{P}(\mathbf{F})\otimes\mathcal{A}$-measurable we replace θ by ϑ to obtain that

$$\beta_s^{\mathbf{H}}(\omega) = \mathbf{1}_{\{\vartheta<s\}}\beta_s^{\vartheta}(\omega) + \mathbf{1}_{\{\vartheta\geq s\}} \frac{\int_s^\infty \beta_s^\theta(\omega)\alpha^{s-}(d\theta)}{\alpha^{s-}([s,\infty))} \tag{5.10}$$

and analogously

$$Y_s^{\mathbf{H}}(\omega,x) = \mathbf{1}_{\{\vartheta<s\}} Y_s^{\vartheta}(\omega,x) + \mathbf{1}_{\{\vartheta\geq s\}} \frac{\int_s^\infty Y_s^\theta(\omega,x)\alpha^{s-}(d\theta)}{\alpha^{s-}([s,\infty))} \tag{5.11}$$

Theorem 5.1. *Assume that the process X is a $(P,\mathbf{F})$-semimartingale with triplet $T = (B,C,\nu)$, that we have the martingale representation property with respect to natural filtration $\mathbf{F}$ and $L^1(\Omega,\mathcal{F},P)$ is separable. Suppose the application $T : (\omega,t,x) \to T_t^\theta(\omega)$ is $\mathcal{P}(\mathbf{F})\otimes\mathcal{A}$-measurable and that the conditions (5.1), (5.5), (5.6), (5.7) hold. Then X is $(P,\mathbf{H})$-semimartingale with the triplet $T^{\mathbf{H}} = (B^{\mathbf{H}},C,\nu^{\mathbf{H}})$ given by*

$$\nu^{\mathbf{H}} = Y^{\mathbf{H}}\cdot\nu,$$

$$B^{\mathbf{H}} = B + \beta^{\mathbf{H}}\cdot C + (Y^{\mathbf{H}}-1)l\star\nu$$

where $\beta^{\mathbf{H}}$ and $Y^{\mathbf{H}}$ are defined in (5.10),(5.11).

Proof The result follows from the propositions 5.1 and 5.2 and performed above calculus. □

5.2. **Progressive enlargement in brownian filtration.** Let X be a brownian motion with respect to the measure P starting from zero and stoped in T and $\vartheta = X_T$.
The prior distribution $\alpha(d\theta) := P(\vartheta\in d\theta)$ is a $\mathcal{N}(0,T)$ and the posterior distribution α^t of ϑ given $\mathcal{F}_t$ is $\mathcal{N}(X_t, T-t)$.
As it was shown in 2.,

$$\beta_t^\theta = \frac{\theta - X_t}{T-t}.$$

Hence, by straightforward calculations we have

$$\beta_t^{\mathbf{H}} = \mathbf{1}_{\{X_T<t\}}\frac{(X_T-X_t)}{(T-t)} + \mathbf{1}_{\{X_T\geq t\}}\frac{1}{\sqrt{2\pi(T-t)}}\frac{\exp[-(t-X_t)/(T-t)]}{[1-\Phi((t-X_t)/(T-t))]}$$

where $\Phi(\cdot)$ is the distribution function of standard normal law.

6. Weak information

In this and in the next sections we discuss briefly some other applications of the Bayesian viewpoint related with the enlargement and arithmetic mean measure.

6.1. **Weak insider information.**

The notion of weak information in mathematical finance was introduced by Baudoin [3, 4]. Before we discuss briefly this notion, recall our basic setup. We have a semimartingale X on a filtered space $(\Omega, \mathcal{F}, \mathbf{F}^X, P)$ with the right-continuous version of natural filtration $\mathbf{F} = (\mathcal{F}_t^X)_{t\geq 0}$ completed by the $\mathcal{F}$ sets of probability zero and $\mathcal{F} = \mathcal{F}_\infty^X$. We assume the predictable representation property for $\mathbf{F}^X$ and we denote by $T = (B, C, \nu)$ the $(P, \mathbf{F})$-triplet of X. Later, to simplify the notation, we omit the index X in the filtration. Let ϑ be a F_T- measurable random variable with the values in measurable polish space $(\Theta, \mathcal{A})$. Let $\alpha := \mathcal{L}(\vartheta|P)$, $\alpha^t(d\theta) := P(\vartheta \in d\theta|\mathcal{F}_t)$, assume that we have (4.1), and define z_t^θ by (4.3) and finally put $dP_t^\theta = z_t^\theta dP_t$. Recall that in this case the arithmetic mean measure is

$$\bar{P}_t^\alpha(B) := \int_\Theta P_t^\theta(B)\alpha(d\theta) = P(B).$$

In particular, the $(P, \mathbf{F})$- triplet of the semimartingale X does not change under the arithmetic mean measure $\bar{P}^\alpha$ (see Remark 3.1).

Consider three types of agents in a pricing model, where by the stock price is given by the semimartingale X: ordinary agents, strong insiders and weak insiders. We do not want to go in too detailed description of the pricing model, but we define the three types by giving the information and the (historical) probability of the agent.

- *ordinary agents* For the ordinary agent the information is given by $\mathbf{F}$, the probability is P and he uses the triplet $T = (B, C, \nu)$ to build his stratergy.
- *strong insiders* For the strong insider the information is given in the pair (X, ϑ), and we can model this by initial enlargement of filtration. By using Theorem 4.1 we see that one possibility to model strong insider is to change the probability P to P^θ, and the strong insider works with filtration $\mathbf{F}$ and with the new triplet T^θ.

Let us now describe the notion of *weak insider* in more detail. Let γ be the probability distribution on $(\Theta, \mathcal{A})$. Following [3, p. 112] we assume that $\gamma \prec\prec \alpha$. Then it is easy to see that $\bar{P}^\gamma \prec\prec \bar{P}^\alpha = P$, where

$$\bar{P}_t^\gamma(B) = \int_{\Theta\times B} z_t^\theta \gamma(d\theta) dP,$$

and the measure $\bar{P}^\gamma$ is the arithmetic mean measure with respect to the prior distribution γ; in [3] the corresponding measure on $(\Omega, \mathbf{F}, F)$ is called the *minimal probability associated with the conditioning* (T, ϑ, γ).

Hence we can model the weak insiders as follows:

- *weak insiders* For the weak insider the information is given by the filtration $\mathbf{F}$, but he changes the probability measure P to the measure $\bar{P}^\gamma$ and he works with the triplet $\bar{T}^\gamma = (\bar{B}^\gamma, C, \bar{\nu}^\gamma)$.

Assume that we have

$$\gamma^t \prec\prec \gamma$$

and we have assumption 3 with respect to the measure $P \otimes \gamma$.
We can now use Theorem 3.1 to compute the new triplet with respect to the measure $\bar{P}^\gamma$ and we obtain:

$$\begin{aligned} \bar{B}^\gamma &= B + \bar{\beta}^\gamma \cdot C + (\bar{Y}^\gamma - 1)l \star \nu, \\ \bar{C}^\gamma &= C, \\ \bar{\nu}^\gamma &= \bar{Y}^\gamma \cdot \nu, \end{aligned} \tag{6.1}$$

where the predictable local characteristics $\bar{\beta}^\gamma$ and $\bar{Y}^\gamma$ are given by

$$\bar{\beta}^\gamma_t = E_{\gamma^{t-}} \beta^\theta_t, \quad \bar{Y}^\gamma_t = E_{\gamma^{t-}} Y^\theta_t \tag{6.2}$$

with γ^t and γ^{t-} a posteriori distributions under γ. Recall that γ^t is defined by : for each $A \in \mathcal{A}$

$$\gamma^t(A) := \frac{\int_A z^\theta_t \gamma(d\theta)}{\int_\Theta z^\theta_t \gamma(d\theta)}$$

and γ^{t-} is given by the same formula with replacing z^θ_t by z^θ_{t-}.
Define now $\bar{m}^\gamma$ as

$$\bar{m}^\gamma = \bar{\beta}^\gamma \cdot X^c + \left(\bar{Y}^\gamma - 1 + \frac{\hat{\bar{Y}}^\gamma - \hat{1}}{1 - \hat{1}} \right) \star (\mu - \nu),$$

then we have that

$$\frac{d\bar{P}^\gamma_t}{dP_t} = \mathcal{E}(\bar{m}^\gamma)_t.$$

By definition of $\bar{P}^\gamma_t$ and γ^t we have also that

$$\frac{d\gamma^t}{d\gamma}(\theta) \;=\; \frac{dP^\theta_t}{d\bar{P}^\gamma_t} = \frac{dP^\theta_t}{dP_t}\frac{dP_t}{d\bar{P}^\gamma_t} = z^\theta_t \frac{1}{\mathcal{E}(\bar{m}^\gamma)_t}.$$

In comparison with $\dfrac{d\alpha^t}{d\alpha}(\theta)$ which is equal to z^θ_t ($P_t \times \alpha$ -a.s.), it means that

$$\frac{d\gamma^t}{d\gamma}(\theta) = \frac{d\alpha^t}{d\alpha}(\theta)\frac{1}{\mathcal{E}(\bar{m}^\gamma)_t}.$$

Example: Brownian motion. Let X be a Brownian motion stopped in T and suppose that the Brownian filtration $\mathbf{F}$ is enlarged by $\vartheta = X_T$. In this example $T = (0, I, 0)$ and $\beta^\theta = \dfrac{\theta - X_t}{T - t}$. Consider the example of *final value distorted with a noise.* We suppose that the weak insider knows in advance the value y of random variable $Y = X_T + \epsilon$, where ϵ is independent of X_T and has $\mathcal{N}(0, \eta^2)$ as law. The prior of the insider with weak information is $\gamma = P(X_T|Y)$, which by theorem of normal correlation is $\mathcal{N}(m, \sigma^2)$ with $\sigma^2 = (T^{-1} + \eta^{-2})^{-1}$ and $m = Y\sigma^2/\eta^2$.
For $t < T$ the posterior distribution is $\gamma^t := P(X_T|Y, X_t)$, which by the theorem of normal correlation is $\mathcal{N}(m_t, \sigma_t^2)$ with $\sigma_t^2 = ((T-t)^{-1} + \eta^{-2})^{-1}$ and $m_t = (Y\eta^{-2} + X_t(T-t)^{-1})\sigma_t^2$.
According to previous results on the triplets a new drift of X under the insider measure is given by

$$\bar{B}_t^\gamma = \int_0^t \frac{E_{\gamma^s}\vartheta - X_s}{T - s} ds. \tag{6.3}$$

Since

$$E_{\gamma^s}\vartheta = \frac{Y(T-s) + X_s\eta^{-2}}{T - s + \eta^{-2}}$$

we have after simplification that

$$\bar{B}_t^\gamma = \int_0^t \frac{Y - X_s}{T - s + \eta^2} ds.$$

Remark 6.1. *One can analyse the increasing information along the same lines. By this we mean that the insider obtains dynamically more and more precise information about the random variable ϑ. A model of this type is the following: in addition to the price process X the insider observes the process Y, where*

$$Y_t = \vartheta + \epsilon_t,$$

where ϵ is a semimartingale, independent of the random variable ϑ such that $\epsilon_t \to 0$ P- a.s., when $t \to T$. This kind of models are analysed in [7].

7. Additional expected logarithmic utility of an insider

7.1. **Introduction.** We consider the pricing model with two assets, the stock (risky asset) and the bond (riskless asset). We assume as in [1] that the process X has the dynamics

$$dX_t = \mu_t d\langle M\rangle_t + dM_t \tag{7.1}$$

here μ is a predictable process, and M is a continuous Gaussian martingale with a deterministic bracket $\langle M\rangle$. The bond B has dynamics

$dB_t = rB_tdt$ and we assume that the interest rate r is equal to zero, so that $B_t = 1$ for all t.
We assume that the stock price S has the dynamics

$$dS_t = S_t dX_t.$$

We assume that if we have fixed the investment strategy π we have the dynamics

$$dV_t^\pi = \pi_t V_t^\pi dX_t.$$

Then it can be shown that with respect to logarithmic utility, the average optimal strategy π^o of an ordinary investor is to take $\pi^o := \mu$. Note that here the average optimal strategy is computed with respect to the measure P.

7.2. **Additional expected utility of strong insiders.** Now consider a strong insider who knows the final value of the stock. We assume that it is the same as the insider knows the final value of the martingale M_T. Put again $\vartheta = M_T$.
Then he can model the dynamics of X as

$$dX_t = (\mu_t + \beta_t^\theta)d\langle M\rangle_t + dM_t^\theta. \tag{7.2}$$

Here M^θ is a continuous **G**-martingale with

$$M_t^\theta = M_t - \int_0^t \beta_s^\theta d\langle M\rangle_s$$

and

$$\beta_t^\theta = \frac{\theta - M_t}{\langle M\rangle_{t,T}}$$

where $\langle M\rangle_{t,T} = \langle M\rangle_T - \langle M\rangle_t$. Again the optimal expected investment strategy of an insider agent for the logarithmic utility is $\pi^i = \mu + \beta^\theta$. Note that the expectation is computed with respect to the measure $\mathbb{P}$ which is the joint law of $(M, \vartheta(\omega))$. The log-value of the optimal strategy for the ordinary investor is

$$\log V_t^{\pi^o} = \log V_0 + \int_0^t \mu_s dM_s + \frac{1}{2}\int_0^t \mu_s^2 d\langle M\rangle_s. \tag{7.3}$$

Similarly, the log-value of the optimal strategy for the insider investor is

$$\log V_t^{\pi^i} = \log V_0 + \int_0^t (\mu_s + \beta_s^\theta)dM_s^\theta + \frac{1}{2}\int_0^t (\beta_s^\theta + \mu_s)^2 d\langle M\rangle_s. \tag{7.4}$$

To calculate the expectation $\mathbb{E}$ with respect to $\mathbb{P}$ we need the following lemma.

Lemma 7.1. *Let for each $\theta \in \Theta$ $u^\theta = (u^\theta_t)_{t\geq 0}$ be a positive* **F***-adapted cadlag process. Suppose that the application $u : (\omega, t, \theta) \to u^\theta_t(\omega)$ is $\mathcal{O}(\mathbb{G})$- measurable with $\mathbb{G}$ defined by (4.2). Then*

$$\mathbb{E}\int_0^t u^\theta_s d\langle M\rangle_s = E\int_0^t \bar{u}^\alpha_s d\langle M\rangle_s \tag{7.5}$$

where E is the expectation with respect to P and $\bar{u}^\alpha_s$ is the posterior mean of u^θ_s namely

$$\bar{u}^\alpha_s = E_{\alpha^{s-}} u^\theta_s$$

Proof Recall first the following fact. Assume that $y = (y_t)_{t\geq 0}$ is a positive uniformly integrable $(P, \mathbf{F})$-martingale and D is a predictable increasing process, with $D_0 = 0$. Then by [15, Theorem 5.16, p. 144 and Remark 5.3, p. 137]

$$E\,(y_t D_t) = E\int_0^t ({}^pY)_s dD_s = \mathbb{E}\int_0^t Y_{s-} dD_s. \tag{7.6}$$

Since z^θ is the conditional density of the law of X given $\vartheta = \theta$ with respect to P, we have using (7.6) and ordinary Fubini theorem that

$$\begin{aligned}
\mathbb{E}(\int_0^t u^\theta_s d\langle M\rangle_s) &= E(\int_\Theta z^\theta_t \int_0^t u^\theta_s d\langle M\rangle_s d\alpha) = \int_\Theta E\left(z^\theta_t \int_0^t u^\theta_s d\langle M\rangle_s\right) d\alpha \\
&= (\int_\Theta E\int_0^t z^\theta_{s-} u^\theta_s d\langle M\rangle_s d\alpha) = E\int_0^t \left(\int_\Theta z^\theta_{s-} u^\theta_s d\alpha\right) d\langle M\rangle_s \\
&= E\int_0^t \bar{u}^\alpha_s d\langle M\rangle_s.
\end{aligned}$$

This proves (7.5). □

Let us now compute the expected utility from the insider point of view. This means that we take the expectation of (7.4) with respect to the insider measure $\mathbb{P}$ which is the joint law of (ω, ϑ). In the computation we use the fact that the martingale M has a drift $\int_0^\cdot \beta^\theta_s d\langle M\rangle_s$ with respect to the insider measure. We obtain:

$$\begin{aligned}
\mathbb{E}(\log V^{\pi^i}_t - \log V^{\pi^o}_t) &= \frac{1}{2}\mathbb{E}\int_0^t (\mu_s + \beta^\theta_s)^2 d\langle M\rangle_s \\
&\quad -\frac{1}{2}\mathbb{E}\int_0^t \mu^2_s d\langle M\rangle_s - \mathbb{E}\int_0^t \mu_s dM_s \\
&= \frac{1}{2}\mathbb{E}\int_0^t (\beta^\theta_s)^2 d\langle M\rangle_s \\
&= \frac{1}{2}E\int_0^t \bar{v}^\alpha_s(\beta) d\langle M\rangle_s
\end{aligned}$$

where $\bar{v}_s^\alpha(\beta)$ is the posterior variance of the process β_s^θ. Next we give Bayesian interpretation of this result. Note first that the Kullback-Leibler information in the prior with respect to posterior is

$$I(\alpha|\alpha^\tau) := E_{\alpha^\tau} \log \frac{d\alpha^\tau}{d\alpha}(\theta).$$

In our case we have:

$$\mathbb{E}(\log V_t^{\pi^i} - \log V_t^{\pi^o}) = EI(\alpha|\alpha^t).$$

For more information on this kind of computations we refer to [10].
We compute next the difference of the expected gain from the ordinary agent point of view. This has the interpretation that the ordinary agent has excess to the insider information, but he thinks that this is false. We model this by the measure $P \otimes \alpha$ – this means that the ordinary agent does not change his triplet. So the expected utility gain has to be calculated using the measure $P \otimes \alpha$. With a similar computation to the previous one we obtain that

$$E_{P\otimes\alpha}(\log V_t^{\pi^o} - \log V_t^{\pi^i}) = \frac{1}{2} E_{P\otimes\alpha} \int_0^t (\beta_s^\theta)^2 d\langle M\rangle_s.$$

The Kullback-Leibler information in the posterior α^τ with respect to the prior α is define by

$$I(\alpha^\tau|\alpha) := E_\alpha \log \frac{d\alpha}{d\alpha^\tau}.$$

For our model we can conclude that

$$E_{P\otimes\alpha}(\log V_t^{\pi^o} - \log V_t^{\pi^i}) = EI(\alpha^t|\alpha).$$

Note that the differences of the expected gains are in both cases positive – this reflects the fact the investors act optimally according to their own model.

7.3. **Additional expected logarithmic utility of weak insider.** Assume that γ and α are two different equivalent priors for the parameter ϑ; we can define the arithmetic mean measures $\bar{P}^\gamma$ and $\bar{P}^\alpha$; we can compute the $(\mathbf{F}, \bar{P}^\gamma)$ and $(\mathbf{F}, \bar{P}^\alpha)$ triplets of the semimartingale X by (3.1). Note that here we do not assume that α is the marginal law of the parameter ϑ.
Denote the optimal strategies based on the weak information for the prior γ and α by $\pi^{w,\gamma}$ and $\pi^{w,\alpha}$ respectively.
Then, with a familiar computation

$$E_{\bar{P}^\gamma}(\log V_t^{w,\gamma} - \log V_t^{w,\alpha}) = \frac{1}{2} E_{\bar{P}^\gamma}\left(\int_0^t (\overline{\beta}_s^\gamma - \overline{\beta}_s^\alpha)^2 d\langle M\rangle_s\right) \tag{7.7}$$

where

$$\bar{\beta}_s^\gamma = E_{\gamma^{s-}}\beta_s^\theta, \quad \bar{\beta}_s^\alpha = E_{\alpha^{s-}}\beta_s^\theta.$$

We remark that the right-hand side of (7.7) is nothing else as Kullback-Leibler information of $\bar{P}^\alpha$ in $\bar{P}^\gamma$ and, hence,

$$E_{\bar{P}^\gamma}\left(\log V_t^{w,\gamma} - \log V_t^{w,\alpha}\right) = I(\bar{P}^\alpha|\bar{P}^\gamma)_t.$$

Note that

$$\begin{aligned}
&0 \le I(\bar{P}^\alpha|\bar{P}^\gamma)_t = E_{\bar{P}^\gamma}\log\Big(\frac{d\bar{P}_t^\gamma}{d\bar{P}_t^\alpha}\Big) = \\
&= \int\limits_\Theta \int\limits_\Omega \Big\{\log\Big(\frac{dP_t^\theta}{d\bar{P}_t^\alpha}\Big) - \log\Big(\frac{dP_t^\theta}{d\bar{P}_t^\gamma}\Big)\Big\} P_t^\theta(d\omega)\gamma(d\theta) \\
&= E_\gamma\big\{I(P_t^\theta|\bar{P}_t^\alpha) - I(P_t^\theta|\bar{P}_t^\gamma)\big\} = E_{\bar{P}_t^\gamma}\big\{I(\alpha|\alpha^t) - I(\gamma|\gamma^t)\big\}
\end{aligned}$$

In particular this means that

$$E_{\bar{P}_t^\gamma} I(\gamma|\gamma^t) = \inf_\alpha E_{\bar{P}_t^\gamma} I(\alpha|\alpha^t)$$

where infinum is taken over all measures α which are equivalent to γ. The interpretation is that if one believes in his own prior γ, he expects to get less information from the data than any other person using the same model with a "wrong" prior.

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Department of Mathematics and Statistics, P. O. Box 4, FIN-00014 University of Helsinki, Finland
E-mail address: `Dario.Gasbarra@helsinki.fi`

Institute of Mathematics, P.O. Box 1100, FIN-02015 Helsinki University of Technology, FINLAND
E-mail address: `Esko.Valkeila@tkk.fi`

Departement de Mathematiques, Universite d'Angers, France
E-mail address: `vostrik@univ-angers.fr`