SUBSAMPLING FOR NON-STATIONARY PROCESSES

Dominique Dehay [a] and Jacek Leśkow [b]

[a] University of Rennes 2, France
dominique.dehay@uhb.fr

[b] Nowy Sacz Graduate School of Business, Nowy Sącz, Poland

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I- Almost periodically correlated (APC) processes

I-1 - Examples

▷ $X(t) = X_1(t)\cos t + X_2(t)\cos(\pi t), \qquad t \in \mathbb{R} \qquad$ where $\{X_i(t) : t \in \mathbb{R}\}$, $i = 1, 2$ uncorrelated and stationary processes.

▷ $X(t) = \sum_{m=1}^{M} A_m X_m(t - d_m) + N(t), \qquad t \in \mathbb{R} \qquad$ where

– M : number of users;

– A_m and d_m : real parameters (amplitude and time-delay);

– X_m : real-valued, zero-mean, APC, uncorrelated on one another and on $N := \{N(t) : t \in \mathbb{R}\}$;

– noise N : real-valued, zero-mean and APC.

▷ $X(t) = Z(t)\cos t \quad$ and $\quad Y(t) = Z(t)\cos t + Z(t-1)\cos(\pi t)$ where Z is an Ornstein Uhlenbeck process.

I-2 - Background

▷ *Definition .* $\{X(t) \,:\, t \in \mathbb{R}\}$ **almost periodically correlated** (APC) (Gladyshev, 1963) :

- $\mathrm{E}\{X(t)\} = 0, \quad X(t) \in \mathbb{R},$
- $B(t,\tau) = \mathrm{E}\{X(t+\tau)X(t)\}$ almost periodic in t for any $\tau \in \mathbb{R}$.

▷ Then (Hurd, 1991)

$$B(t,\tau) \sim \sum_{\lambda \in \Lambda} a(\lambda,\tau) e^{i\lambda t}$$

where the **spectral covariance** is

$$a(\lambda,\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T B(t,\tau) e^{-i\lambda t} dt,$$

and the **set of frequencies** $\Lambda = \Big\{\lambda \in \mathbb{R} : a(\lambda,\tau) \neq 0\Big\}$ is countable.

Example 1

▷ $X(t) = X_1(t)\cos t + X_2(t)\cos(\pi t), \qquad t \in \mathbb{R}, \qquad$ where $\{X_i(t) : t \in \mathbb{R}\}$, $i = 1, 2$, independent, zero-mean and stationary with continuous covariance function $r_i(\,\cdot\,)$.

▷ Then, X is an APC process, and

$$\Lambda = \{-2\pi, -2, 0, 2, 2\pi\}$$

$$\begin{aligned} a(0,\tau) &= \frac{1}{2}\, r_1(\tau)\cos\tau + \frac{1}{2}\, r_2(\tau)\cos(\pi\tau) \\ a(2,\tau) = \overline{a(-2,\tau)} &= \frac{1}{4}\, r_1(\tau)\,\mathrm{e}^{i\tau} \\ a(2\pi,\tau) = \overline{a(-2\pi,\tau)} &= \frac{1}{4}\, r_2(\tau)\,\mathrm{e}^{i\pi\tau}. \end{aligned}$$

Example 2 (Gardner, 1991)(Gardner, Napolitano, 2006)

▷ $X(t) = \sum_{m=1}^{M} A_m \, \mathrm{e}^{i\phi_m} X_m(t - d_m) + N(t),$ where

– user signals $X_m := \{X_m(t) : t \in \mathbb{R}\}$: real-valued, zero-mean, APC, uncorrelated on one another, and on $N := \{N(t) : t \in \mathbb{R}\}$;

– the noise N is also real-valued zero-mean UAPC.

▷ Then X is an APC process, and

$$a(\lambda, \tau) = \sum_{m=1}^{M} A_m^2 \, \mathrm{e}^{-i2\lambda d_m} \, a_m(\lambda, \tau) + a_N(\lambda, \tau).$$

Example 3

▷ $X(t) = Z(t)\cos t, \qquad t \in \mathbb{R},$
where Z is an Ornstein Uhlenbeck process.

▷ Then

$$\begin{aligned} \mathrm{E}\{X(s)X(s+\tau)\} &= \mathrm{e}^{-|\tau|}\cos s\,\cos(s+\tau) \\ &= a(-2,\tau)\,\mathrm{e}^{-i2s} + a(0,\tau) + a(2,\tau)\,\mathrm{e}^{i2s}, \end{aligned}$$

and

$$\Lambda = \{-2, 0, 2\},$$

$$a(0,\tau) = \frac{1}{2}\,\mathrm{e}^{-|\tau|}\cos\tau, \qquad a(2,\tau) = \overline{a(-2,\tau)} = \frac{1}{4}\,\mathrm{e}^{-|\tau|}\,\mathrm{e}^{i\tau}.$$

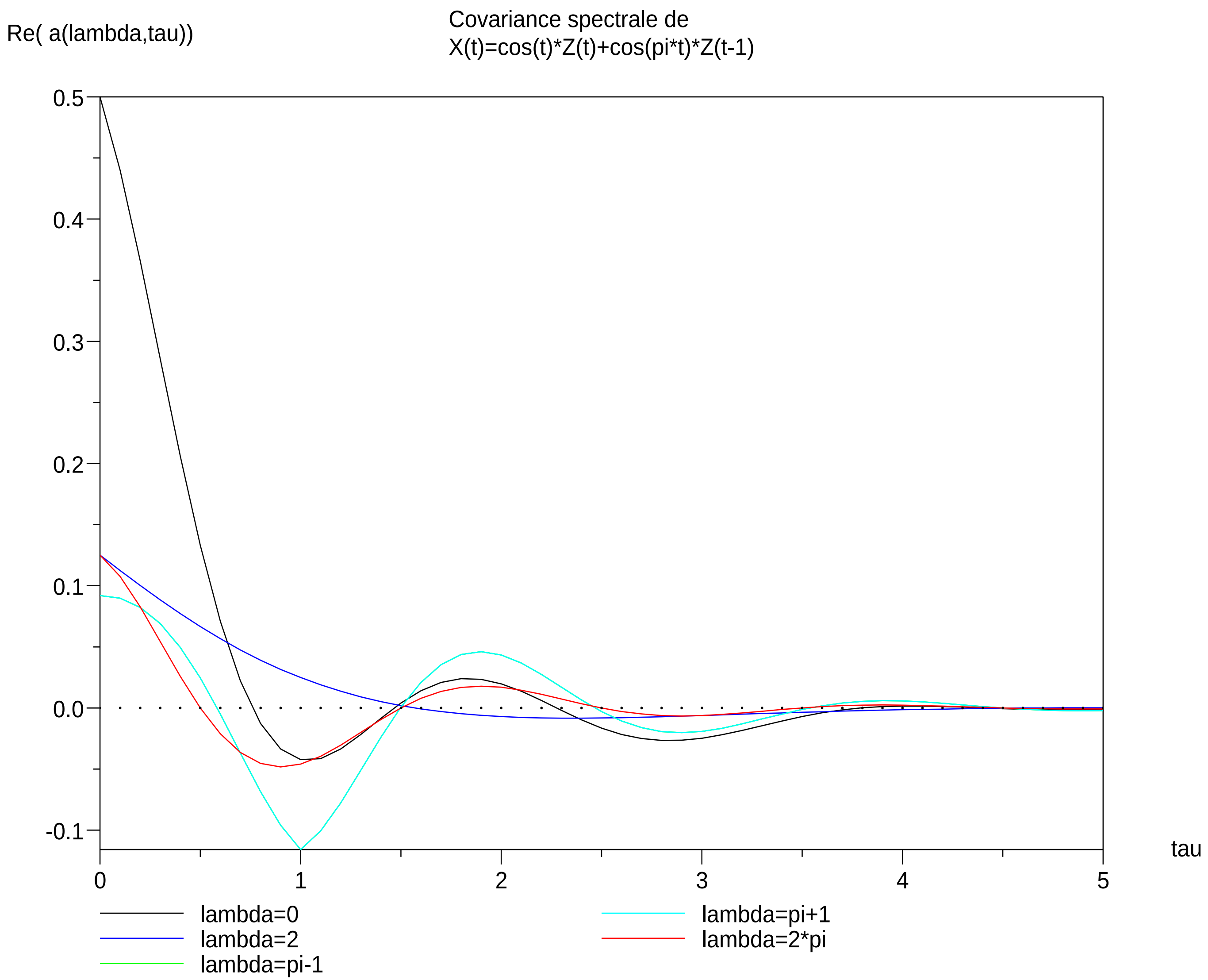
X is APC. More precisely it is *periodically correlated* (PC) : the function $s \mapsto \mathrm{E}\{X(s)X(s+\tau)\}$ is periodic for any $\tau \in \mathbb{R}$.

Example 4

- ▷ $Y(t) = Z(t)\cos t + Z(t-1)\cos(\pi t), \forall t \in \mathbb{R}$.

- ▷ Then the process Y is APC with

$$\Lambda = \Big\{-2\pi, -1-\pi, 1-\pi, -2, 0, 2, \pi-1, \pi+1, 2\pi\Big\}.$$



Real parts of the spectral covariances
for $X(t) = Z(t)\ \cos t + Z(t-1)\ \cos(\pi t)$.

I-3 - Continuous time estimation of $a(\lambda,\tau)$

$\triangleright$ *Aim* : estimation of $a(\lambda,\tau)$, and find $\Lambda = \big\{\lambda \in \mathbb{R} : a(\lambda,\tau) \neq 0\big\}$.

$\triangleright$ *Observation* : sample $\{X(t)\,;\, t \in [0,T]\}$ with $T \longrightarrow \infty$.

$\triangleright$ *Estimator of* $a(\lambda,\tau)$ (Hurd Leskow, 1992) :

$$\widehat{a}_T(\lambda,\tau) = \begin{cases} \dfrac{1}{T}\displaystyle\int_{|\tau|}^{T-|\tau|} X(t+\tau)X(t)\, e^{-i\lambda t}\, dt, & -T/2 \leq \tau \leq T/2, \\ 0, & \text{otherwise.} \end{cases}$$

$\triangleright$ $\widehat{a}_T(\lambda,\tau)$ *is a consistent estimator of* $a(\lambda,\tau)$ *:*

If $\sup_t \mathrm{E}\{|X(t)|^{4+\delta}\} < \infty$, for $\delta > 0$, and $\alpha_X(t) \longrightarrow 0$ as $t \to \infty$,

then $\widehat{a}_T(\lambda,\tau) \xrightarrow{L^2} a(\lambda,\tau)$.

$\triangleright$ $\widehat{a}_T(\lambda,\tau)$ *is asymptotically normal :*

If $\sup_t \mathrm{E}\{|X(t)|^{4+\delta}\} < \infty$, $\displaystyle\int_0^T \alpha_X(u)^{\delta/(4+\delta)}\,du < \infty$, for $\delta > 0$,

$\displaystyle\sum_{\lambda\in\Lambda} 1/\lambda^2 < \infty$, and for any u,

$v \mapsto \mathrm{cov}\{X(u+v+\tau)X(u+v), X(v+\tau)X(v)\}$ is almost periodic,

then $\sqrt{T}\{\widehat{a}_T(\lambda,\tau) - a(\lambda,\tau)\} \xrightarrow{\mathcal{L}} \mathcal{N}\big(0, V_\infty(\lambda,\tau)\big)$.

In particular

$$T\mathrm{var}\{Re(\widehat{a}(\lambda,\tau))\} \longrightarrow V_\infty^{(1,1)}(\lambda,\tau),$$

with

$$V_\infty^{(1,1)}(\lambda,\tau) \;:=\; \frac{1}{2}\int_{\mathbb{R}} \Big\{ \Big(B_c(2\lambda, u+\tau, \tau, u) + B_c(0, u+\tau, \tau, u) \Big) \cos(\lambda u) + \\ + B_s(2\lambda, u+\tau, \tau, u) \sin(\lambda u) \Big\}\, du,$$

$$B_c\big(\lambda, u, v, w\big) \;:=\; \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{cov}\Big\{ X(s)X(s+u), X(s+v)X(s+w) \Big\} \cos(\lambda s)\, ds,$$

$$B_s\big(\lambda, u, v, w\big) \;:=\; \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{cov}\Big\{ X(s)X(s+u), X(s+v)X(s+w) \Big\} \sin(\lambda s)\, ds.$$

$\triangleright$ The asymptotic covariance of the statistic $\sqrt{T}\{\widehat{a}_T(\lambda,\tau) - a(\lambda,\tau)\}$ is quite difficult to handle. For this purpose we will consider the subsampling version of the estimator $\widehat{a}_T(\lambda,\tau)$.

$\triangleright$ We finish this section by providing the definition of α-mixing process, quite useful in subsequent considerations.

Definition . The process X is called α-mixing if $\alpha_X(s) \to 0$ as $s \to \infty$, where

$$\alpha_X(s) = \sup\left\{\left|\mathrm{P}(A \cap B) - \mathrm{P}(A)\mathrm{P}(B)\right| : t \in \mathbb{R}, A \in \mathcal{F}_{-\infty}^{t}(X), B \in \mathcal{F}_{t+s}^{\infty}(X)\right\}$$

and $\mathcal{F}_{t_1}^{t_2}(X) = \sigma\left\{X(t)\,;\, t_1 \leq t \leq t_2\right\}$.

II - Subsampling estimators for stochastic processes

II-1 - Subsampling method (Politis, Romano and Wolf, 1999)

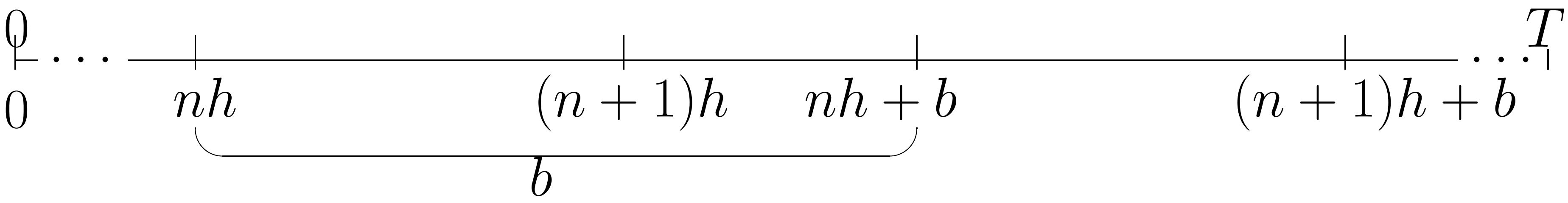
▷ for $h > 0$, $b > 0$ and $n \geq 0$, let $Y_{b,n} := \{X(u) : u \in [nh; nh + b]\}$.

▷ b is the *subsampling size*. $b = b_T \to \infty$ as $T \to \infty$, $0 < b \leq T$.

▷ h is the *overlap factor*.

▷ minimal overlap when $h = b$.

▷ maximal overlap when $h \to 0$.



Definition . The **subsampling version** $\widetilde{a}_{b,n}$ of the estimator $\widehat{a}_T(\lambda,\tau)$ generated by the data $Y_{b,t}$ is defined as

$$\widetilde{a}_{b,n} := \widetilde{a}_{b,n}(\lambda,\tau) = \widehat{a}_b(Y_{b,n}) = \widehat{a}_b\{X(nh+u) : 0 \le u \le b\}$$

$$= \begin{cases} \dfrac{1}{b}\displaystyle\int_{|\tau|}^{b-|\tau|} X(nh+u+\tau)X(nh+u)\,\mathrm{e}^{-i\lambda u}\,du, & \text{for } |\tau| \le b/2 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition . $\displaystyle\lim_{b\to\infty} \mathrm{E}\{\widetilde{a}_{b,n}\} = a(\lambda,\tau)\,\mathrm{e}^{i\lambda nh}.$

Definition . **modified subsampling version** of the estimator $\widehat{a}_T(\lambda,\tau)$:

$$\widehat{a}_{b,n} := \widetilde{a}_{b,n}\,\mathrm{e}^{-i\lambda nh}$$

$$= \begin{cases} \dfrac{1}{b}\displaystyle\int_{nh+|\tau|}^{nh+b-|\tau|} X(u+\tau)X(u)\,\mathrm{e}^{-i\lambda u}\,du, & \text{for } |\tau| \le b/2 \\ 0 & \text{otherwise.} \end{cases}$$

II-2 - Empirical process

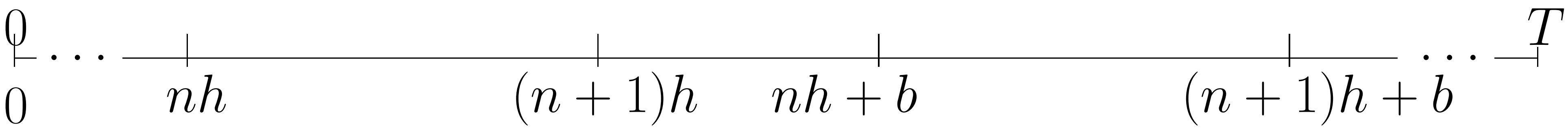
▷ $a(\lambda,\tau)$ and $\widehat{a}_n$ and $\widehat{a}_{b,n}$ are complex-valued, so we can see them as two-dimensional real-valued : real part and imaginary part.

▷ Partial order in $\mathbb{C} \sim \mathbb{R}^2$ defined by : $x \preceq y \Leftrightarrow x_1 \leq y_1$ and $x_2 \leq y_2$, for $x = x_1 + ix_2$ and $y = y_1 + iy_2$.

Definition . The **empirical process** $L_{T,b}(x)$ induced by the sub-sampling procedure and the sample $\{X(t) : t \in [0,T]\}$ is defined as

$$L_{T,b}(x) = \frac{1}{q}\sum_{n=0}^{q-1} \mathbf{1}\Big\{\sqrt{b}\{\widehat{a}_{b,n} - \widehat{a}_T\} \preceq x\Big\},$$

where $q = \lfloor \frac{T-b}{h} \rfloor + 1$ is the number of $[nh\,,\,nh+b]$ in $[0,T]$.



II-3 - Subsampling consistency

Assume that

▷ $\sup_t \mathrm{E}\left\{|X(t)|^{4+\delta}\right\} < \infty, \quad \int_0^T \alpha_X(u)^{\delta/(4+\delta)}\,du < \infty, \quad$ for $\delta > 0$,

▷ $\sum_{\lambda\in\Lambda} 1/\lambda^2 < \infty$,

▷ $v \mapsto \mathrm{cov}\Big\{X(u+v+\tau)X(u+v), X(v+\tau)X(v)\Big\}$ is almost periodic,

▷ $b/T \to 0$, h constant or $h \uparrow$, with $h/T \to a$, $0 \le a \le 1$, as $T \to \infty$.

▷ Then $\widehat{a}_T(\lambda,\tau)$ is asymptotically normal.
Denote the limit distribution by $J := \mathcal{N}\big(0, V_\infty\big)$.

Lemma . $\sqrt{T}\{\widehat{a}_T - a(\lambda,\tau)\} \xrightarrow{\mathcal{L}} J$ and $\sqrt{b}\{\widehat{a}_{b,n} - a(\lambda,\tau)\} \xrightarrow{\mathcal{L}} J$, as $T \to \infty$ and $n \to \infty$.

Proof . Bias, variance, APC property, central limit theorem for strong mixing non-stationary processes.

Let $$U_{T,b}(x) := \frac{1}{q}\sum_{n=0}^{q-1} \mathbf{1}\left\{\sqrt{b}\{\widehat{a}_{b,n} - a(\lambda,\tau)\} \preceq x\right\}.$$

Lemma . $U_{T,b}(x) \xrightarrow{L^2} J(x)$ at each continuity point of $J(x)$ as $T \to \infty$.

Proof . Bias, variance, strong mixing inequality for bounded functions.

Theorem

▷ The subsampling is consistent, that is

$$L_{T,b}(x) := \frac{1}{q}\sum_{n=0}^{q-1} \mathbf{1}\{\sqrt{b}\{\widehat{a}_{b,n} - \widehat{a}_T\} \preceq x\} \xrightarrow{\text{Prob}} J(x)$$

at each point of continuity $x \in \mathbb{R}^2$ of $J(x)$ as $T \to \infty$.

Theorem

$\triangleright \quad L^*_{T,b}(x) := \frac{1}{q}\sum_{n=0}^{q-1} \mathbf{1}\Big\{\sqrt{b}\{|\widehat{a}_{b,n}|^2 - |\widehat{a}_T|^2\} \leq x\Big\} \xrightarrow{\text{Prob}} F(x)$

at each point of continuity $x \in \mathbb{R}$ of $F(x)$ as $n \to \infty$, where

$\sqrt{T}\Big\{|\widehat{a}_T|^2 - |a(\lambda,\tau)|^2\Big\} \xrightarrow{\mathcal{L}} F$ as $T \to \infty$. That is

$F = \mathcal{N}(0,\sigma^2)$, where $\sigma^2 = DV_\infty D^\top,\ D = 2\Big(\mathrm{Re}(a(\lambda,\tau))\,,\ \mathrm{Im}(a(\lambda,\tau))\Big)$.

$\triangleright$ For any $\epsilon > 0$, $\qquad \sup_{|x|\geq\epsilon} |L^*_{T,b}(x) - F(x)| \xrightarrow{\text{Prob}} 0.$

Recall that

$$V_\infty^{(1,1)} := \frac{1}{2}\int_{\mathbb{R}} \Big\{\Big(B_c(2\lambda, u+\tau,\tau,u) + B_c(0,u+\tau,\tau,u)\Big)\cos(\lambda u) + \\ + B_s(2\lambda,u+\tau,\tau,u)\sin(\lambda u)\Big\}\, du,$$

II-5 - Confidence interval

$\triangleright$ *Corollary* The one-sided subsampling confidence interval for $|a(\lambda,\tau)|^2$ is consistent for any $\alpha \in [0,1]$

$$\mathrm{P}\Big[\sqrt{T}\big\{|\widehat{a}_T|^2 - |a(\lambda,\tau)|^2\big\} \leq c_{T,b}(1-\alpha)\Big] \longrightarrow 1-\alpha, \quad as \quad T \to \infty,$$

where $c_{T,b}(1-\alpha) := \inf\big\{x : L^*_{T,b}(x) \geq 1-\alpha\big\}$, $\alpha \in [0,1)$.

$\triangleright$ *Testing problem :*

$$\begin{cases} H_0 : a(\lambda,\tau) = 0 \\ \\ H_1 : a(\lambda,\tau) \neq 0 \end{cases}$$

II-5 - Choice of b, open problem

▷ b small $\longrightarrow \widehat{a}_{b,n}$ is a poor estimator

▷ $b \sim T$ small $\longrightarrow \widehat{a}_{b,n} \sim \widehat{a}_T$

▷ First approach : "under good hypotheses"

$$\mathrm{E}\left\{\frac{b}{q}\sum_{n=0}^{q-1}\Big(Re(\widehat{a}_{b,n}) - Re(\widehat{a}_T)\Big)^2\right\} - V_\infty^{(1,1)} = O\left(\frac{1}{b}\right) + O\left(\frac{b}{T}\right)$$

as $T, b \to \infty$, $b < T$.

Take $b \sim \sqrt{T}$. Is it the good choice ?

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