Asymptotical distribution free test for the drift of a diffusion process

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Plan of the talk

- Goodness of fit test for the i.i.d observation model
- Diffusion processes
- Weak convergence of the empirical process
- Goodness of fit tests for diffusions
  - Based on $\widehat{F}_T(x)$, the empirical process
  - Based on $V_T(x)$, the scored marked empirical process

Goodness of fit test for the i.i.d observation model

Suppose we observe $X^n = (X_1, \ldots, X_n)$ i.i.d. with distribution function F. To test the two simple hypotheses

$$H_0 : F = F_0$$
$$H_1 : F = F_1,$$

where $F_1 \neq F_0$ means $\sup_{x\in\mathbb{R}} |F_0(x) - F_1(x)| > 0$, we introduce the well known Kolmogorov-Smirnov statistic

$$\Delta_n(X^n) = \sup_x \sqrt{T}|\widehat{F}_n(x) - F_0(x)|$$

where

$$\widehat{F}_n(x) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{\{X_j \leq x\}}$$

is the empirical distribution function.

The test procedure is completely determined by the statistical decision function $\phi_n = \phi_n(X^n)$ where the expected value of $\phi_n(X^n)$ is the probability to reject H_0 having the observation X^n

Fix a number $\alpha \in (0, 1)$ and let us consider $\mathcal{K}_\alpha$ the class of

asymptotic tests of level $1 - \alpha$ or *size* α.

It is defined as

$$\mathcal{K}_\alpha = \left\{ \phi_n : \lim_{n \to +\infty} \mathbf{E}^n_{F_0} \phi_n(X^n) \leq \alpha \right\}$$

The power function of the test based on ϕ_n is the probability of the true decision under H_1, and is given by

$$\beta(\phi_n) = \mathbf{E}^n_{F_1} \phi_n(X^n).$$

A test procedure is consistent if

$$\lim_{n \to +\infty} \mathbf{E}^n_{F_1} \phi_n(X^n) = 1$$

The Kolmogorov-Smirnov procedure is

$$\phi_n(X^n) = \mathbf{1}_{\{\Delta_n > c_\alpha\}}$$

where c_α is defined by

$$\mathbf{P}\left(\sup_{0\leq t\leq 1} |B_0(t)| > c_\alpha\right) = \alpha$$

and B_0 is a Brownian Bridge, i.e. a continuous Gaussian Process with

$$\mathbf{E}B_0(t) = 0, \quad \mathbf{E}B_0(t)B_0(s) = t \wedge s - ts$$

It turns out that the Kolmogorov-Smirnov test is such that

- it belongs to $\mathcal{K}_\alpha$
- it is asymptotical distribution-free
- it is consistent (uniformly) against any alternative $F_1 \neq F_0$.

Diffusion processes

Given a general set-up $(\Omega, \{\mathcal{F}_t\} \subset \mathcal{F}, \mathbf{P})$, let us consider a one dimensional diffusion process

$$\begin{cases} \mathrm{d}X_t = S(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t \\ X_0 = \xi, \end{cases}$$

where $\{W_t : \ t \geq 0\}$ is a Wiener process and $X_0 = \xi$ is a r.v. independent of W_t, $t \geq 0$.

Conditions $\mathcal{ES}$: S is locally bounded, $\sigma^2 > 0$ continuous and for some $A > 0$

$$xS(x) + \sigma(x)^2 \leq A(1 + x^2), \quad x \in \mathbb{R}$$

If assumptions $\mathcal{ES}$ holds, then the stochastic differential equation has a unique weak solution $\{X_t : t \geq 0\}$.

Ergodic diffusions

The scale function and the speed measure of the diffusion solution of the stochastic differential equation are defined as

$$p(x) = \int_0^x \exp\left\{-2\int_0^y \frac{S(v)}{\sigma^2(v)}\,\mathrm{d}v\right\}\mathrm{d}y, \qquad m(\mathrm{d}x) = \frac{1}{\sigma(x)^2 p'(x)}\mathrm{d}x$$

Conditions $\mathcal{RP}$:

$$p(x) = \int_0^x \exp\left\{-2\int_0^y \frac{S(v)}{\sigma^2(v)}\,\mathrm{d}v\right\}\mathrm{d}y \to \pm\infty, \qquad x \to \pm\infty$$

and

$$m(\mathbb{R}) < \infty$$

If assumptions $\mathcal{RP}$ holds the process X_t has the ergodic property: there exists a measure μ such that for every function $g \in L^1(\mu)$

$$\mathbf{P}\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T g(X_t)\mathrm{d}t = \int_{\mathbb{R}} g(z)\mu(dz)\right) = 1$$

μ is called *invariant measure*.

Moreover the invariant measure μ has a density given by

$$f(y) = \frac{1}{G(S)} \frac{1}{\sigma(y)^2} \exp\left\{ 2 \int_0^y \frac{S(v)}{\sigma(v)^2} dv \right\}$$

and the invariant distribution function is

$$F(x) = \int_{-\infty}^{x} \frac{1}{G(S)} \frac{1}{\sigma(y)^2} \exp\left\{ 2 \int_0^y \frac{S(v)}{\sigma(v)^2} dv \right\} dy$$

Here we have introduced the notation

$$G(S) = \int_{-\infty}^{\infty} \frac{1}{\sigma(x)^2} \exp\left\{ 2 \int_0^x \frac{S(v)}{\sigma^2(v)} \, \mathrm{d}v \right\} \, \mathrm{d}x \, (= m(\mathbb{R}))$$

Later on we suppose that X_0 has F as distribution. So we deal with processes $\{X_t : t \geq 0\}$, strictly stationary.

Local time for diffusions

The local time for a diffusion is defined via the Tanaka-Meyer formula. For every $x \in \mathbb{R}$ and $t \geq 0$ we have

$$L_t^X(x) = |X_t - x| - |X_0 - x| - \int_0^t \mathrm{sgn}(X_s - x)\mathrm{d}X_s$$

The process $\{L_t^X(x) : x \in \mathbb{R}, t \geq 0\}$, is countinuous and increasing in t and continuous and nonnegative in x.

The main theorem for local time is the *occupation time* formula. For a diffusion process it is: let φ be a measurable function, then with probability one,

$$\int_0^t \varphi(X_s)\sigma(X_s)^2 ds = \int_{\mathbb{R}} L_t^X(x)\varphi(x)dx.$$

The last formula becomes

$$\int_0^t \varphi(X_s)ds = \int_{\mathbb{R}} L_t^X(x)\frac{\varphi(x)}{\sigma(x)^2}dx.$$

Equivalence of experiments

Suppose we observe different diffusion processes $\{X_t : 0 \leq t \leq T\}$ with drift coefficients S_1, S_2 and $S_0 = 0$ and initial value X_0^1, X_0^2 and X_0^0.

$\mathcal{EM}$. *The functions S_1, S_2 and σ satisfy condition $\mathcal{ES}$ and the densities (with respect to the Lebesgue measure) of the corresponding initial values X_0^1, X_0^2 and X_0^0 have the same support (if the initial value is nonrandom, then we suppose that it takes the same value for all processes).*

If condition $\mathcal{EM}$ holds true, all the measures P_S^T induced by the process $\{X_t, : 0 \leq t \leq T\}$ in the space $\mathcal{C}_T$ (the space of all the continuos function on $[0, T]$ with uniform metric and Borel σ-algebra $\mathcal{B}(\mathcal{C}_T)$) for different S are equivalent.

Moreover if conditions $\mathcal{ES}$, $\mathcal{RP}$ and $\mathcal{EM}$ hold then the corresponding Radon-Nikodym derivative or likelihood ratio,

$$L(S, S_1, X^T) = \frac{dP_S^T}{dP_{S_1}^T}(X^T)$$

is given by

$$L(S, S_1, X^T) = \frac{G(S_1)}{G(S)} \exp\left\{2\int_0^{X_0} \frac{S(v) - S_1(v)}{\sigma(v)^2} dv + \int_0^T \frac{S(X_t) - S_1(X_t)}{\sigma(X_t)^2} dX_t\right\} \cdot$$
$$\cdot \exp\left\{-\frac{1}{2}\int_0^T \frac{S(X_t)^2 - S_1(X_t)^2}{\sigma(X_t)^2} dt\right\}.$$

In the following we will denote by $\mathbf{E}_S^T$ the mathematical expectation with respect to the measure P_S^T.

Statistical framework

Suppose we observe the process

$$X^T = \{X_t : 0 \leq t \leq T\}$$

solution of the following s.d.e.

$$\begin{cases} \mathrm{d}X_t = S(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t \\ X_0 = \xi, \end{cases}$$

where σ is known and S is unknown.

For a fixed function σ let us introduce the class

$$\mathcal{S}_\sigma = \{S : \text{conditions } \mathcal{ES}, \mathcal{RP}, \mathcal{EM} \text{ are fulfilled}\}$$

We can be interested in the estimation of $F_S(x)$, $f_S(x)$ or $S(x)$ for every x in $\mathbb{R}$.

Empirical distribution function

A natural estimator for the stationary distribution function is the empirical distribution function defined as

$$\widehat{F}_T(x) = \frac{1}{T}\int_0^T \mathbf{1}_{\{X_t \le x\}} dt.$$

The estimator $\widehat{F}_T(x)$ is an unbiased estimators and it is uniformly consistent by the Glivenko-Cantelli theorem:

Theorem (Kutoyants, 1997)

Let $S \in \mathcal{S}_\sigma$. Then

$$E_S(\widehat{F}_T(x)) = F_S(x), \quad \text{for all} \quad x \in \mathbb{R}$$

and

$$P_S\left(\lim_{T\to+\infty} \sup_{x\in\mathbb{R}} |\widehat{F}_T(x)) - F_S(x)| = 0\right) = 1$$

Weak convergence of the empirical process

Let us introduce the empirical process

$$\eta_T(x) = \sqrt{T}(\hat{F}_T(x)) - F_S(x))$$

Let us introduce the covariance function

$$R_S(x,y) = 4\int_{-\infty}^{+\infty} \frac{(F_S(v\wedge x) - F_S(v)F_S(x))(F_S(v\wedge y) - F_S(v)F_S(y))}{\sigma(v)^2 f_S(v)}\mathrm{d}v.$$

The next result give us the asymptotic normality of the empirical distribution function.

Theorem If $R_S(x^*,x^*) < +\infty$ then the sequence of random variables

$$\sqrt{T}(\hat{F}_T(x^*)) - F_S(x^*))$$

weakly converges to a gaussian random variable $\mathcal{N}(0, R_S(x^*,x^*))$

The limit process

For a fixed S, we denote by $\{\eta_T^S(x):\ x \in \mathbb{R}\}$ the family of processes defined by

$$\eta_T^S(x) = \sqrt{T}\Big(\hat{F}_T(x) - F_S(x)\Big) = \frac{1}{\sqrt{T}}\int_0^T \left(\mathbf{1}_{\{X_t^S < x\}} - F_S(x)\right) dt$$

Let us consider $\mathcal{C}_0(\mathbb{R})$ the space of all the continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ vanishing at infinity, with $||\varphi|| = \sup_{x\in\mathbb{R}} |\varphi(x)|$. We denote $\mathcal{B}_0$ the corresponding Borel σ-algebra.

On $(\mathcal{C}_0(\mathbb{R}), \mathcal{B}_0)$ we define the Gaussian process

$$\eta_S = \{\eta_S(x):\ x \in R\}$$

with mean 0 for every $x \in \mathbb{R}$ and with covariance function $R_S(x, y)$.

We denote with $\mathbf{P}_S$ the law on $(\mathcal{C}_0(\mathbb{R}), \mathcal{B}_0)$ of the process η_S.

Let us introduce the following function

$$H_{x,S}(y) = 2\int_0^y \frac{F_S(v\wedge x) - F_S(v)F_S(x)}{\sigma(v)^2 f_S(v)}\mathrm{d}v,$$

and let us introduce the class $\mathcal{S}_\sigma^* \subset \mathcal{S}_\sigma$ such that for every S_* in $\mathcal{S}_\sigma^*$ there exist a $\delta > 0$, and a vicinity

$$V_\delta = \left\{ S \in \mathcal{S}_\sigma^* : \sup_{x\in\mathbb{R}} |S_*(x) - S(x)| < \delta, \right\},$$

such that $\sup_{S\in V_\delta} G(S) < +\infty$.

Let us introduce the following conditions:

Condition $\mathcal{Q}_1$. The function $S_* \in \mathcal{S}_\sigma^*$ and for some $\delta > 0$

$$\sup_{S\in V_\delta}\sup_{x\in\mathbb{R}} 4E_S\left(\frac{F_S(\xi\wedge x)\Big(1 - F_S(\xi\vee x)\Big)}{\sigma(\xi)f_S(\xi)}\right)^2 < +\infty$$

Condition $\mathcal{Q}_2$:

$$\sup_{S\in V_\delta}\sup_{x\in\mathbb{R}} E_S H_{x,S}(\xi)^2 < +\infty.$$

Condition $\mathcal{Q}_3$: let $R > 0$, for $x, y \in [-R, R]$

$$\sup_{S \in V_\delta} E_S |H_{x,S}(\xi) - H_{y,S}(\xi)|^2 < P(R)|x-y|^2$$

where $P(R)$ is a polynomial function depending only on R.

Moreover for $x > L > 0$

$$\sup_{S \in V_\delta} E_S |H_{x,S}(\xi)|^2 \leq C e^{-\alpha|x|}$$

where $C > 0$ and $\alpha > 0$ are constant.

Condition $\mathcal{U}_1$: The convergence

$$\lim_{T \to +\infty} \frac{4}{T} \int_0^T \left(\frac{F_S(X_t \wedge x) - F_S(x) F_S(X_t)}{\sigma(X_t) f_S(X_t)} \right)^2 dt = R_S(x, x)$$

is uniformly in P_S probability on $S \in V_\delta$.

The main result regarding the weak convergence of the empirical process uniformly in the space $\mathcal{C}_0(\mathbb{R})$ is given by the following

Theorem (N. 1998)

Let conditions $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ and $\mathcal{U}_1$ hold. Then the empirical process $\{\eta_T^S(x) : x \in \mathbb{R}\}$, weakly converge in $\mathcal{C}_0(\mathbb{R})$ to the process $\{\eta_S(x) : \ x \in \mathbb{R}\}$ uniformly on V_δ.

Remark: condition $\mathcal{Q}_1$ assures the weak convergence of finite dimensional laws. Conditions $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are used to prove the tightness of the family of process. Condition $\mathcal{U}_1$ assures the uniformity of the result.

Remark: To prove the convergence of $\eta_T(x) = \sqrt{T}(\hat{F}_T(x)) - F_S(x))$ to $\eta_S = \{\eta_S(x) : \ x \in R\}$ for a fixed S all the conditions can be relaxed avoiding to require uniformity with respect to $S \in V_\delta$.

Example

Let us suppose that the coefficients S and σ satisfied the following

Condition $\mathbf{C}$:

a) There exists a constant $L > 0$ such that for every $|v| > L$

$$S(v)\,\mathrm{sgn}(v) < -\gamma$$

for some $\gamma > 0$.

b) There exist two constants k_1 and k_2 such that

$$0 < k_1 < \sigma(x) < k_2 < +\infty$$

for all $x \in \mathbb{R}$.

If condition $\mathbf{C}$ holds true then condition $\mathcal{Q}_1$ to $\mathcal{Q}_3$ and $\mathcal{U}_1$ are satisfied. So we have a class of non parametric diffusion model for which the empirical process converge weakly.

Remark. Regarding the weak convergence of the empirical process η_T Van der Vaart and Van Zanten (2005) have established the following general result. (We state their general result in our set-up)

Theorem

(i) The process η_T weak converges to the process η in $\ell^\infty(J)$ for every compact $J \subset \mathbb{R}$ if and only if $\int_{\mathbb{R}} F^2(1-F)^2 dp < \infty$, where p is the scale function of the diffusion. Here $\ell^\infty(J)$ denotes the space of the continuous function $g : J \to \mathbb{R}$ endowed with the sup norm.

(ii) We have the convergence in $\ell^\infty(\mathbb{R})$ if and only if the limit process η lies in $C_0(\mathbb{R})$.

Remark 1. The condition $\int_{\mathbb{R}} F^2(1-F)^2 dp$ is equivalent to the existence of the covariance function of the process η, that is $R_S(x,x) < \infty$ for every $x \in \mathbb{R}$.

Remark 2. The result is based on $\frac{1}{t}\sup_{x\in I} l_t^X(x) = O_{\mathbf{P}}(1)$, where $l_t^X(x) = L_t^X(x)p'(x)$ is the diffusion local time for X.

Application: Hypotheses testing for diffusion models

Suppose that we observe the process $\{X_t : 0 \leq t \leq T\}$, solution of the stochastic differential equation

$$\mathrm{d}X_t = S(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$$

and we wish to test the two simple hypothesis

$$H_0 : S = S_0$$
$$H_1 : S = S_1$$

where $S_1 \neq S_0$ means

$$\sup_x |S_1(x) - S_0(x)| > 0.$$

The test procedure is completely determined by the statistical decision function

$$\phi_T = \phi_T(X^T)$$

The expected value of $\phi_T(X^T)$ is the probability to reject H_0 having the observation $X^T = \{X_t : 0 \leq t \leq T\}$

Historical remarks on testing problems for diffusions

– Parametric framework

- Lin'kov, (1981). Simple hypothesis on $S(\theta, x)$ for ergodic diffusion
- Kutoyants, (2004). Composite hypothesis on $S(\theta, x)$ for ergodic diffusion
- Iacus and Kutoyants, (2001). Test on some functionals for dynamical systems with small noise (semiparametric)

– Nonparametric framework

- Kutoyants, (2004). Goodness of fit test for ergodic diffusion based on Kolmogorov-Smirnov statistics

Goodness of fit test based on $\widehat{F}_T(x)$

If $\sup_x |S_1(x) - S_0(x)| > 0$, then we have

$$\sup_x |F_{S_1}(x) - F_{S_0}(x)| > 0$$

Then to test

$$\begin{aligned} H_0 &: S = S_0 \\ H_1 &: S = S_1 \end{aligned}$$

we can introduce the statistic

$$\Delta_T(X^T) = \sup_x \sqrt{T} |\widehat{F}_T(x) - F_{S_0}(x)|$$

and the decision function

$$\hat{\phi}_T(X^T) = \mathbf{1}_{\{\Delta_T(x) > c_\alpha\}}$$

where c_α is the solution of the equation

$$\mathbf{P}\left(\sup_x |\eta_{S_0}(x)| > c_\alpha\right) = \alpha$$

Goodness of fit test based on $\widehat{F}_T(x)$

We have the following result

Theorem (Kutoyants 2004)

Let $\sigma = 1$ and condition $\mathbf{C}$ be satisfied for S. Then the test based on

$$\widehat{\phi}_T(X^T) = \mathbf{1}_{\{\Delta_T(X^T) > c_\alpha\}}$$

belongs to $\mathcal{K}_\alpha$ and it is consistent.

Remarks: The fact that $\phi_T(X^T) \in \mathcal{K}_\alpha$ follows from the weak convergence of the process $\eta_{S_0}^T$ to the gaussian process η_{S_0} and the continuous mapping theorem.

Due to the structure of the covariance function of η_{S_0} the Kolmogorov-Smirnov statistics is not asymptotically distribution free.

Goodness of fit test based on $V_T(x)$

To test the two simple hypothesis

$$H_0 : S = S_0$$
$$H_1 : S = S_1$$

we introduce the score marked empirical process

$$\begin{aligned} V_T(x) &= \frac{1}{\sqrt{T}} \int_0^T \mathbf{1}_{(-\infty,x]}(X_t) \frac{1}{\sigma(X_t)} (\mathrm{d}X_t - S_0(X_t)\mathrm{d}t) \\ &= \frac{1}{\sqrt{T}} \int_0^T \mathbf{1}_{(-\infty,x]}(X_t) \mathrm{d}W_t. \end{aligned}$$

We will consider the following statistical decision function

$$\phi_T^* = \mathbf{1}_{\{\sup_{x\in\mathbb{R}} |V_T(x)| > c_\alpha\}}$$

where the *critical value* c_α is defined by

$$\mathbf{P}\left(\sup_{0\le t\le 1} |B(t)| > c_\alpha \right) = \alpha$$

Here B denote a standard Brownian motion on the positive real line.

The weak convergence of $V_T(x)$

We introduce in the space $(C_B(\mathbb{R}), \mathcal{B})$ the Gaussian process $\{\Gamma(x) : x \in \mathbb{R}\}$ such that $\mathbf{E}(\Gamma(x)) = 0$ and covariance function given by

$$\mathbf{E}(\Gamma(x)\Gamma(y)) = \int_{-\infty}^{x \wedge y} f_{S_0}(z)\mathrm{d}z = F_{S_0}(x \wedge y)$$

Theorem (N. and Nishiyama 2006) Let $S \in \mathcal{S}_\sigma$. Then the family of stochastic process $\{V_T(x) : x \in \mathbb{R}\}$ weackly converges on the space $(C_B(\mathbb{R}), \mathcal{B})$ to the Gaussian process $\{\Gamma(x) : x \in \mathbb{R}\}$.

Remarks: we have proved the result for a more general class of function that the indicator function. The convergence of the finite dimensional laws follows from the central limit theorem for stochastic integrals.The tightness follows from the theorem 2.3 of Nishiyama, 1999, applying the cited result of Van der Vart and Van Zanten (2005) on the local time for diffusions: $\frac{1}{t} \sup_{x \in I} l_t^X(x) = O_{\mathbf{P}}(1)$.

The limit process $\{\Gamma(x) : x \in \mathbb{R}\}$ admits the following representation in distribution

$$\Gamma(x) = B(F_{S_0}(x))$$

where B is a standard Brownian Motion.

The weak convergence, the representation $\Gamma(x) = B(F_{S_0}(x))$ and the continuous mapping theorem yield the following relations in distribution

$$\sup_{x\in\mathbb{R}} |V_T(x)| \to \sup_{x\in\mathbb{R}} |\Gamma(x)| = \sup_{0\le t\le 1} |B(t)|$$

where is finite, and $\to$ denote the weak convergence. So if we consider the following statistical decision function

$$\phi_T^* = \mathbf{1}_{\{\sup_{x\in\mathbb{R}} |V_T(x)|>c_\alpha\}}$$

where the *critical value* c_α is defined by

$$\mathbf{P}\left(\sup_{0\le t\le 1} |B(t)| > c_\alpha\right) = \alpha$$

we have proved that $\phi_T^* \in \mathcal{K}_\alpha$ and that the test is asymptotically distribution free

Consistency of the test based on $V_T(x)$

Let us introduce the following condition.

$\mathcal{A}$: For some $x \in \mathbb{R}$ it holds

$$\int_{-\infty}^{+\infty} \mathbf{1}_{(-\infty,x]}(y)\frac{1}{\sigma(y)}(S_0(y) - S_1(y))f_{S_1}(y)\mathrm{d}y \neq 0$$

Theorem (N-Nishiyama, 2006) Let S_0 and S_1 belong to $\mathcal{S}_\sigma$ and the condition $\mathcal{A}$ be satisfied. Then the test based on the statistical decision function

$$\phi_T^* = \mathbf{1}_{\{\sup_{x\in\mathbb{R}} |V_T(x)|>c_\alpha\}}$$

where the *critical value* c_α is defined by $\mathbf{P}\left(\sup_{0\leq t\leq 1} |B(t)| > c_\alpha\right) = \alpha$, is consistent.

To prove the consistence it is enough to show that, under H_1

$$\mathbf{P}\left(\lim_{T\to+\infty} \sup_{x\in\mathbb{R}} |V_T(x)| = +\infty\right) = 1.$$

We can write

$$\sup_{x\in\mathbb{R}} |V_T(x)| \geq \sqrt{T} \sup_{x\in\mathbb{R}} |A_T(x)| - \sup_{x\in\mathbb{R}} |V_T^1(x)|$$

Where, under H_1 the process

$$V_T^1(x) = \frac{1}{\sqrt{T}} \int_0^T \mathbf{1}_{(-\infty,x]}(X_t) \frac{1}{\sigma(X_t)} (\mathrm{d}X_t - S_1(X_t)\mathrm{d}t)$$

weakly converge to the corresponding Gaussian process so the limit process if tight. Moreover

$$A_T(x) = \frac{1}{T} \int_0^T \mathbf{1}_{(-\infty,x]}(X_t) \frac{1}{\sigma(X_t)} (S_0(X_t) - S_1(X_t))\mathrm{d}t$$

converge a.s. uniformly in x to

$$A(x) = \int_{-\infty}^{+\infty} \mathbf{1}_{(-\infty,x]}(y) \frac{1}{\sigma(y)} (S_0(y) - S_1(y)) f_{S_1}(y)\mathrm{d}y.$$

If condition $\mathcal{A}$ is satisfied we have

$$\lim_{T\to+\infty} \sqrt{T} \sup_{x\in\mathbb{R}} |A_T(x)| = +\infty \quad \text{a.s.}$$

and the test is consistent.

On-going works

Goodness of fit test - contiguous alternatives

We consider the problem of testing $H_0 : S = S_0$ *versus* $H_1 : S = S_0 + \frac{h}{\varphi(T)}$, and we want to study the asymptotic properties of likelihood ratio

$$L(S_0, S, X^T) = \frac{dP_S^T}{dP_{S_0}^T}(X^T)$$

under H_1.

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