Nonparametric estimation and testing time homogeneity for Lévy processes

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- Let Ψ be a set, and we define

$$\ell^\infty(\Psi) := \left\{ x : \Psi \to \mathbb{R} \,\middle|\, \sup_{\psi \in \Psi} |x(\psi)| < \infty \right\}$$

- If Ψ is compact w.r.t. a metric ρ, then

$$C_\rho(\Psi) \subset \ell^\infty(\Psi)$$

- If $\Psi = [0, 1]$, then

$$C([0,1]) \subset D([0,1]) \subset \ell^\infty([0,1])$$

Part A:
A uniform CLT for martingales

1 Donsker's theorem

- Donsker (1952): $(\mathbf{R}, \mathfrak{B}(\mathbf{R}), P)$; a probability space.
  $Z_1, Z_2, \ldots$; I.I.D. $\sim P$. $u \in \mathbb{R}$.

$$X_t^{n,u} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \left\{ 1_{(-\infty,u]}(Z_i) - P(-\infty, u] \right\}.$$

- Dudley (1978): $(E, \mathcal{E}, P)$; a probability space.
  $Z_1, Z_2, \ldots$; I.I.D. $\sim P$. $A \in \mathcal{E}_0 \subset \mathcal{E}$.

$$X_t^{n,A} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \{ 1_A(Z_i) - P(A) \}.$$

- Pollard (1982), Ossiander (1987): $(E, \mathcal{E}, P)$; a probability space. $Z_1, Z_2, \ldots$; I.I.D. $\sim P$. $\quad \psi \in \Psi \subset \mathcal{L}^2(P)$.

$$X_t^{n,\psi} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \left\{ \psi(Z_i) - \int_E \psi(z) P(dz) \right\}.$$

- The class Ψ for which CLT holds is called a Donsker class.
- A sufficient condition to be a Donsker class is to satisfy the metric entropy condition with L^2-bracketing.

2 Ossiander's theorem

- For every $\varepsilon \in (0,1]$ choose $N(\varepsilon)$ pairs of elements of $\mathcal{L}^2(P)$, namely, $[l^{\varepsilon,k}, u^{\varepsilon,k}]$, $k = 1, ..., N(\varepsilon)$, such that for every $\psi \in \Psi$ the relation $l^{\varepsilon,k} \le \psi \le u^{\varepsilon,k}$ holds for some k and that

$$(1) \qquad \sqrt{\int_E |u^{\varepsilon,k}(z) - l^{\varepsilon,k}(z)|^2 P(dz)} \le \varepsilon.$$

- Ossiander's theorem says that if this bracketing procedure can be accomplished with

$$\int_0^1 \sqrt{\log N(\varepsilon)}d\varepsilon < \infty, \tag{2}$$

then the sequence of stochastic processes $(t, \psi) \rightsquigarrow X_t^n(\psi)$ defined by

$$X_t^{n,\psi} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \left\{ \psi(Z_i) - \int_E \psi(z) P(dz) \right\}$$

converges weakly in $\ell^\infty([0,1] \times \Psi)$ to a Gaussian process indexed by Ψ.

3 Extension to martingales

- Bae and Levental (1995), N. (1997, 2000, 2007).
- $\{Z_i\}_{i\in\mathbb{N}}$; an arbitrary sequence of E-valued random variables.
- P_i ; the conditional law of Z_i given $\mathcal{F}_{i-1} = \sigma\{Z_1, ..., Z_{i-1}\}$.
- Consider the process $X_t^{n,\psi}$ given by

$$X_t^{n,\psi} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \left\{ \psi(Z_i) - \int_E \psi(z) P_i(dz) \right\}.$$

- Consider the bracketing procedure as above with (1) replaced by

$$\sqrt{\int_E |u^{\varepsilon,k}(z) - l^{\varepsilon,k}(z)|^2 P_i(dz)} \le K_i \varepsilon \qquad \text{almost surely,}$$

  where K_i is a random variable, not depending on ε and k, that is $\mathcal{F}_{i-1}$-measurable.
- Since the left hand side is random in the present case, we have allowed the random coefficient K_i in the right hand side.
- If the entropy condition (2) is satisfied and if the sequence of random variables $\overline{K}^n$ defined by

$$\overline{K}^n = \sqrt{\frac{1}{n}\sum_{i=1}^n |K_i|^2}$$

is bounded in probability, then the uniform tightness of the processes follows from the finite-dimensional convergence and a Lindeberg condition on an envelope function of Ψ.

- Ossiander's theorem is the case of $P_i \equiv P$ and $K_i \equiv 1$.
- We call $\overline{K}^n$ "quadratic modulus". A more precise explanation might be

$$\sup_{\varepsilon\in(0,1]} \max_{1\le k\le N(\varepsilon)} \frac{\sqrt{n^{-1}\sum_{i=1}^{n}\int_E |u^{\varepsilon,k}(z)-l^{\varepsilon,k}(z)|^2 P_i(dz)}}{\varepsilon}.$$

- The main result of Part A, which is omitted, is a uniform CLT for the process

$$X_t^{n,\psi} = W^{n,\psi} * (\mu^n - \nu^n)_t,$$

  where μ^n is an integer-valued random measure on $\mathbb{R}_+ \times E$, ν^n is its predictable compensator, and $W^{n,\psi}$ is a predictable function on $\Omega^n \times \mathbb{R}_+ \times E$ with index parameter $\psi \in \Psi$. Here, Ψ is an arbitrary set.
- In short, we can generalize the definition of the quadratic modulus to the above case.
- In Part B, we will apply it to non-parametric inference for Lévy processes.

Part B:
Non-parametric inference for Lévy processes

0 What is Lévy measure?

1 Model

2 Estimation

3 Testing time-homogeneity

0 What is Lévy measure?

- Let $t \rightsquigarrow Z_t$ is a càdlàg process which is piecewise constant. We denote $\Delta Z_t = Z_t - Z_{t-}$.
- Compound Poisson process:

$$\lambda F(dz), \quad \text{where } \int_{\mathbb{R}} F(dz) = 1.$$

- (Time-homogeneous) Lévy process:

$$\alpha(dz), \quad \text{where } \int_{\mathbb{R}} \alpha(dz) \leq \infty, \ \int_{\mathbb{R}} (z^2 \wedge 1)\alpha(dz) < \infty.$$

- Time-inhomogeneous Lévy process:

$$\alpha(t, dz), \quad \text{where } \int_{\mathbb{R}} \alpha(t, dz) \leq \infty, \ \int_{\mathbb{R}} (z^2 \wedge 1)\alpha(t, dz) < \infty.$$

1 Model

- $t \rightsquigarrow Z_t$; a one-dimensional time-inhomogeneous Lévy process, starting at Z_0, with the Lévy measure

$$L(dt, dz) = dt\alpha(t, dz).$$

- Notice that $\int_{\mathbb{R}} \alpha(t, dz)$ and $\int_{\mathbb{R}} |z|\alpha(t, dz)$ are finite or infinite, and that $\int_{\mathbb{R}}(z^2 \wedge 1)\alpha(t, dz) < \infty$.
- Introducing a non-negative bounded measurable function w on $\mathbb{R}$, we aim to estimate

$$A(t, u) = \int_{[0,t]\times\mathbb{R}} w(z)1_{(-\infty,u]}(z)ds\alpha(s, dz), \quad (t, u) \in [0, 1]\times\mathbb{R}.$$

- We use the weight function given by

$$w(z) = z^2 \wedge 1 \quad \text{or} \quad \frac{z^2}{1 + z^2}. \tag{1}$$

- Since it is not realistic to compute infinitely many sums, for the construction of the estimators we introduce the truncated weight function w_n defined by

$$w_n(z) = w(z)1_{\mathbb{R}\setminus[-c_n,c_n]}(z) \tag{2}$$

  for given sequence of non-negative constants $c_n \downarrow 0$ as $n \to \infty$.
- We emphasize that we *do* need the continuous observation of $t \rightsquigarrow Z_t$ but we do *not* use the jumps such that $|\Delta Z_t| \le c_n$ in the construction of our estimator and test statistics.
- In Section 2, we consider the estimation problem for A based on many observations $Z^1, ..., Z^n$ on the time interval $[0, 1]$, as $n \to \infty$.
- In Section 3, we consider testing hypothesis that the Lévy measure is time homogeneous or not based on an observation Z on the time interval $[0, T]$, as $T \to \infty$.

2 Estimation

- Let $t \rightsquigarrow Z_t^k$, $k = 1, ..., n$ be independent one-dimensional time-inhomogeneous Lévy process, starting at Z_0^k, with the Lévy measure

$$L(dt, dz) = dt\alpha(t, dz).$$

- We assume that we can observe the process $t \rightsquigarrow Z_t^k$ only on a random time interval $[\sigma^k, \tau^k] \subset [0, 1]$.

- A typical example is that

$$Z_0^k > 0, \quad \sigma^k = 0, \quad \tau^k = \inf\{t \in [0,1] : Z_t^k \leq 0\}.$$

- We can also consider the right censored case

$$\sigma^k = 0, \quad \tau^k = \inf\{t \in [0,1] : Z_t^k \leq 0\} \wedge C^k$$

where C^k is a stopping time which represents the consoring.

- However, our setting is more general; actually what we assume is just that there exists a measurable function y on $[0,1]$ which is bounded away from 0 such that

$$\sup_{t\in[0,1]} \left| \frac{1}{n} \sum_{k=1}^{n} 1_{]]\sigma^k,\tau^k]]}(\cdot,t) - y(t) \right| \to 0 \quad \text{in probabiilty, as } n \to \infty.$$

- We introduce the integer-valued random measure N^k given by

$$N^k(\omega; dt, dz) = \sum_{t \in (\sigma^k(\omega), \tau^k(\omega)]} 1_{\{\Delta Z_t^k(\omega) \neq 0\}} \varepsilon_{(t, \Delta Z_t^k(\omega))}(dt, dz)$$

where ε_a denotes the Dirac measure at point a.

- Then, the predictable compensator of N^k is given by

$$1_{]]\sigma^k, \tau^k]]}(\omega, t) dt \alpha(t, dz).$$

- Summing up with respect to k, we have the integer-valued random measure

$$\mu^n(\omega; dt, dz) = \sum_{k=1}^{n} N^k(\omega; dt, dz).$$

- The predictable compensator of μ^n is given by

$$\nu^n(\omega; dt, dz) = Y_t^n(\omega) dt \alpha(t, dz)$$

where

$$Y_t^n(\omega) = \sum_{k=1}^{n} 1_{]]\sigma^k, \tau^k]]}(\omega, t). \tag{3}$$

- We propose the generalized Nelson-Aalen estimator $\widehat{A}^n$ given by

$$(t,u) \rightsquigarrow \widehat{A}^n(t,u) = \int_{[0,t]\times\mathbb{R}} w_n(z) 1_{(-\infty,u]}(z) Y_s^{n-} \mu^n(ds,dz),$$

where

$$Y_s^{n-} = \begin{cases} 1/Y_s^n, & \text{if } \; Y_s^n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

- **Theorem 1** *Suppose that there exists a measurable function $y(t)$ on $[0,1]$, which is bounded and bounded away from zero, such that*

$$(4)\quad \sup_{t\in[0,1]}\left|\frac{Y_t^n}{n} - y(t)\right| \longrightarrow 0 \quad \textit{in probability, as } n\to\infty.$$

Further suppose that the sequence $\{c_n\}$ satisfies

$$\sqrt{n}\int_{[0,1]\times\mathbb{R}} w(z)1_{[-c_n,c_n]}(z)dt\alpha(t,dz) \to 0 \quad \textit{as } n\to\infty.$$

Then it holds that $\sqrt{n}(\widehat{A}^n - A)$ converges weakly in $\ell^\infty([0,1]\times\mathbb{R})$, as $n\to\infty$, to the zero-mean Gaussian process G with the covariance

$$E(G(t_1,u_1)G(t_2,u_2)) = \int_{[0,t_1\wedge t_2]\times(-\infty,u_1\wedge u_2]} \frac{|w(z)|^2}{y(s)} ds\alpha(s,dz).$$

Furthermore, almost all paths of the process $(t, u) \rightsquigarrow G(t, u)$ *are continuous with respect to the pseudo-metric* d_G *given by*

$$d_G((t_1, u_1), (t_2, u_2)) = \sqrt{E|G(t_1, u_1) - G(t_2, u_2)|^2}.$$

- **Remark.** The assumption (4) has been standard in the context of survival analysis; see (8.4.1) of Anderson *et al.* (1993).
- **Remark.** Our result extends the preceding work by Basawa and Brockwell (1982).

- We can prove the asymptotic efficiency of the proposed estimator, following the general theory developed in Chapter 3.11 of van der Vaart and Wellner (1996).
- Denote

  $$L^p = L^p([0,1]\times\mathbb{R}, \mathbf{B}([0,1])\otimes\mathbf{B}(\mathbb{R}), \frac{1}{y(t)}dt\alpha(t,dz), \quad \forall p \geq 1.$$

- We set $\mathbb{H} = L^2$ and $H = L^2 \cap L^\infty$.
- We consider a family $\mathbf{P}^n = \{P^n_h : h \in H\}$ of probability measures on $(\Omega, \mathcal{F})$ indexed by H given as follows: under the probablity measure P^n_h, the Lévy measure of Z^k's is given by

$$dt\alpha^n_h(t,dz) = \left(1 + \frac{h(t,z)}{2\sqrt{n}y(t)}\right)^2 dt\alpha(t,dz).$$

- Since $H = L^2 \cap L^\infty \subset L^1$, under some conditions on the filtration, the log-likelihood ratio is given by

$$\log \frac{dP_h^n|\mathcal{F}_1^n}{dP_0^n|\mathcal{F}_1^n} = 2\left(\log\left|1+\frac{h}{2\sqrt{ny}}\right|\right) * \mu_1^n - \left(\left|1+\frac{h}{2\sqrt{ny}}\right|^2 - 1\right) * \nu_1^{n,0}$$

- We can show that this model is *locally asymptotically normal.*

- Next we define the sequence of unknown parameter $A^n : H \to \ell^\infty([0,1] \times \mathbb{R})$ by

$$A^n(h)(t,u) = \int_{[0,t]\times\mathbb{R}} w(z)1_{(-\infty,u]}(z)ds\alpha_h^n(s,dz).$$

- This sequence A^n is *differentiable* with rate $\sqrt{n}$ and that its derivative $\dot{A} : H \to \ell^\infty([0,1] \times \mathbb{R})$ is given by

$$\dot{A}(h)(t,u) = \langle f, h \rangle_{\mathbb{H}}, \quad \text{where} \quad f(s,z) = 1_{[0,t]}(s)w(z)1_{(-\infty,u]}(z).$$

- We can show that the estimator $\widehat{A}^n$ is regular.
- Also, the limit appearing in Theorem 1 achieves the bound of asymptotic efficiency in $\ell^\infty([0,1] \times \mathbb{R})$.
- We can conclude that our estimator is asymptotically efficipent in the sense of the convolution theorem. The asymptotic efficiency in the sense of the asymptotic minimax theorem also holds for a certain choice of loss function.

- That is, for any bounded continuous subconvex loss function $\ell : \ell^{\infty}([0,1] \times \Psi) \to [0, \infty)$, it holds that

$$\sup_{I \subset H} \limsup_{n \to \infty} \sup_{h \in I} E_h^n \ell \left(\sqrt{n} (\widehat{A}^n - A^n(h)) \right) = E\ell(G),$$

and that, for any estimator T^n,

$$\sup_{I \subset H} \liminf_{n \to \infty} \sup_{h \in I} E_h^n \ell \left(\sqrt{n} (T^n - A^n(h)) \right) \geq E\ell(G),$$

where the supremum with respect to $I \subset H$ is taken over all finite subsets.

3 Testing time-homogeneity

- Our starting point is the time-inhomogeneous case, that is, the Lévy measure L of the Lévy process $t \rightsquigarrow Z_t$ is given by

$$L(dt, dz) = dt\alpha(t, dz).$$

- We assume that the process Z is observable on $[0, T]$, and consider the asymptotics $T \to \infty$.
- We wish to test

  H_0: the Lévy measure is time-homogeneous, that is, $\alpha(t, dz) = \alpha_0(dz)$, v.s.,

  H_1: there exists $\theta_0 \in (0, 1)$ such that

$$\alpha(t, dz) = \begin{cases} \alpha_0(dz) & \text{for } t \in [0, \theta_0 T], \\ \alpha_1(dz) & \text{for } t \in (\theta_0 T, T], \end{cases}$$

  where $\alpha_0 \neq \alpha_1$.

- Let us consider the class $\Psi = [0,1] \times \Xi$, where Ξ is a class of measurable function on $\mathbb{R}$ with an envelope function $\bar{\xi}$ such that

$$\int_{\mathbb{R}} |\bar{\xi}(z)| \vee |\bar{\xi}(z)|^2 \alpha_0(dz).$$

- We define the semi-norm $||\cdot||_{L^2(\alpha_0)}$ by

$$||\xi||_{L^2(\alpha_0)} = \sqrt{\int_{\mathbb{R}} |\xi(z)|^2 \alpha_0(dz)}.$$

- We put

$$A^T(\theta,\xi) = \frac{1}{T}\int_{[0,T]\times\mathbb{R}} k_\theta(tT^{-1})\xi(z)dt\alpha(t,dz), \quad \theta \in [0,1],\ \xi \in \Xi,$$

where

$$k_\theta(t) = (1-\theta)1_{[0,\theta]}(t) - \theta 1_{(\theta,1]}(t).$$

- Our idea comes from the fact that, under H_0, it holds that $A^T(\theta, \xi) = 0$ for all $(\theta, \xi) \in [0, 1] \times \Xi$.
- Under either H_0 or H_1, a natural estimator for A^T is

$$\widehat{A}^T(\theta, \xi) = \frac{1}{T} \int_{[0,T]\times\mathbb{R}} 1_{\mathbb{R}\setminus[-c_T, c_T]}(z) k_\theta(tT^{-1}) \xi(z) \mu(dt, dz),$$

where

$$\mu(\cdot; dt, dz) = \sum_s 1_{\{\Delta Z_s(\cdot)\neq 0\}} \varepsilon_{(s, \Delta Z_s(\cdot))}(dt, dz),$$

and where $\{c_T\}$ is a sequence of non-negative constant such that $c_T \to 0$.

- **Theorem 2** *Assume H_0. Suppose that*

$$(5) \qquad \int_0^1 \sqrt{\log N_{[\ ]}(\Xi, ||\cdot||_{L^2(\alpha_0)}, \varepsilon)}d\varepsilon < \infty.$$

Let $\{c_T\}$ be any sequence of non-negative constants such that $c_T \to 0$. Then, it holds that $\sqrt{T}\widehat{A}^T$ converges weakly in $\ell^\infty([0,1]\times\Xi)$ to the zero-mean Gaussian process $(\theta,\xi) \rightsquigarrow G(\theta,\xi)$ with the covariance

$$EG(\theta,\xi)G(\theta',\xi') = (\theta \wedge \theta' - \theta\theta') \int_{\mathbb{R}} \xi(z)\xi'(z)\alpha_0(dz).$$

- Based on this weak convergence result, by the continuous mapping theorem, we can derive the asymptotic behavior of the test statistics

$$(6) \qquad \sup_{(\theta,\xi)\in[0,1]\times\Xi} |\sqrt{T}\widehat{A}^T(\theta,\xi)|.$$

- We can choose any class Ξ which satisfies (5). The most elementary examples are as follows.
- **Corollary 3** *Choose* w *as in (1). Set*

$$\begin{aligned}\Xi_1 &= \{w(z)1_{(-\infty,u]}(z) : u \in \mathbb{R}\},\\ \Xi_2 &= \{w(z)1_{(v,u]}(z) : -\infty < v \le u < \infty\},\end{aligned}$$

and define

$$\widehat{S}_i^T = \frac{\sup_{(\theta,\xi)\in[0,1]\times\Xi_i} |\sqrt{T}\widehat{A}^T(\theta,\xi)|}{\sqrt{4\widehat{A}^T(1/2,w^2)}}, \quad i = 1,2.$$

Assume H_0*, and suppose that* $u \rightsquigarrow \int_{(-\infty,u]} |w(z)|^2\alpha_0(dz)$ *is continuous and that* $\int_{\mathbb{R}} |w(z)|^2\alpha_0(dz) > 0$*. Then,* $\widehat{S}_i^T$

converges weakly in $\mathbb{R}$ *to* S_i*, where*

$$S_1 = \sup_{t,s\in[0,1]} |B_t^\circ B_s|,$$
$$S_2 = \sup_{t,s,s'\in[0,1]} |B_t^\circ (B_s - B_{s'})|,$$

and where $t \rightsquigarrow B_t^\circ$ *is a standard Brownian bridge and* $t \rightsquigarrow B_t$ *is a standard Brownian motion, which are independent.*

- Obviously, we can use more general classes Ξ like

$$\Xi_{2p} = \{w(z)1_{(v_1,u_1]\cup\cdots\cup(v_p,u_p]}(z) : -\infty < v_1 \le u_1 \le \cdots \le v_p \le u_p < \infty\}$$

and

$$\Xi_{2p}^{\pm} = \{w(z)(1_{(v_1,u_1]\cup\cdots\cup(v_p,u_p]}(z) - 1_{\mathbb{R}\setminus(v_1,u_1]\cup\cdots\cup(v_p,u_p]}(z)) : -\infty < v_1 \le u_1 \le \cdots \le v_p \le u_p < \infty\},$$

where p is a positive integer.

- Next, let us consider the asymptotic behavior of the test statistics under the alternative H_1.
- **Theorem 4** *Assume H_1. Suppose that*

$$\int_0^1 \sqrt{\log N_{[\,]}(\Xi, ||\cdot||_{L^2(\alpha_0)}, \varepsilon)}d\varepsilon + \int_0^1 \sqrt{\log N_{[\,]}(\Xi, ||\cdot||_{L^2(\alpha_1)}, \varepsilon)}d\varepsilon < \infty.$$

Let $\{c_T\}$ be a sequence of non-negative constants such that

$$\sqrt{T}\int_{\mathbb{R}} \bar{\xi}(z)1_{[-c_T,c_T]}(z)(\alpha_0(dz)+\alpha_1(dz)) \to 0 \quad \text{as } T \to \infty.$$

Then, it holds that $\sqrt{T}(\widehat{A}^T - A^*)$ converges weakly in $\ell^\infty([0,1]\times \Xi)$ to a zero-mean Gaussian process G^*, where

$$A^*(\theta,\xi) = \begin{cases} \theta(1-\theta_0)\int_{\mathbb{R}} \xi(z)(\alpha_0(dz)-\alpha_1(dz)), & \theta \in [0,\theta_0], \\ (1-\theta)\theta_0\int_{\mathbb{R}} \xi(z)(\alpha_0(dz)-\alpha_1(dz)), & \theta \in (\theta_0,1], \end{cases}$$

and the covariance of G^* *is given by*

$$
EG^*(\theta,\xi)G^*(\theta',\xi') = \begin{cases} (\theta\wedge\theta' - \theta\theta')\int_{\mathbb{R}}\xi(z)\xi'(z)\alpha_0(dz) \\ +\theta\theta'(1-\theta_0)\int_{\mathbb{R}}\xi(z)\xi'(z)(\alpha_1(dz)-\alpha_0(dz)), \\ \quad for\ \theta,\theta'\in[0,\theta_0], \\ \\ (\theta\wedge\theta' - \theta\theta')\int_{\mathbb{R}}\xi(z)\xi'(z)((1-\theta_0)\alpha_0(dz)+\theta_0\alpha_1(dz)), \\ \quad for\ \theta\in[0,\theta_0],\theta'\in[\theta_0,1], \\ \\ (\theta\wedge\theta' - \theta\theta')\int_{\mathbb{R}}\xi(z)\xi'(z)\alpha_1(dz) \\ +(1-\theta)(1-\theta')\theta_0\int_{\mathbb{R}}\xi(z)\xi'(z)(\alpha_0(dz)-\alpha_1(dz)), \\ \quad for\ \theta,\theta'\in[\theta_0,1]. \end{cases}
$$

- Notice that the above theorem formally includes the case of the null hypothesis H_0; that is, the assumption $\alpha_0 \neq \alpha_1$ is not essential.
- In general, we have

$$\sqrt{T}\widehat{A}^T \approx \sqrt{T}A^* + G^* \quad \text{for large } T.$$

- When $\alpha_0 = \alpha_1$, the deterministic process A^* is identically zero, and the Gaussian part becomes $G^* = G$ where G is from Theorem 2.
- When $\alpha_0 \neq \alpha_1$, the value $|A^*(\theta, \xi)|$ becomes strictly positive.
- So our test statistics (6) is reasonable.

- The power of the test depends on the choice of Ξ, which is important in practice. For the sake of explanation, we shall consider the integrable case $\alpha(dz) = \lambda F(dz)$ where $\lambda > 0$ and $F(\mathbb{R}) = 1$.
- When we treat the model

  $$\alpha_0(dz) = \lambda_0 F(dz) \quad \text{and} \quad \alpha_1(dz) = \lambda_1 F(dz) \quad \text{with} \quad \lambda_0 \neq \lambda_1,$$

  the one element class $\Xi = \{1_{\mathbb{R}}(z)\}$ is trivially enough.
- On the other hand, when the model is

  $$\alpha_0(dz) = \lambda F_0(dz) \quad \text{and} \quad \alpha_1(dz) = \lambda F_1(dz) \quad \text{with} \quad F_0 \neq F_1,$$

  the choice of the class Ξ is serious; if F_0 and F_1 are different in a complex way, then the class Ξ_{2p} or $\Xi_{2p}^{\pm}$ with a big integer p would be good.
- The selection of p depends on the complexity of the model which is considered.