Asymptotic properties of Maximum Likelihood Estimator for partially observed fractional diffusion system

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Statistique Asymptotique des Processus Stochastiques VII

OUTLINE

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System (1) has a uniquely defined solution process X which is Gaussian but neither Markovian nor a semimartingale for $H \neq \frac{1}{2}$.

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- $Y^T = X^T = (X_t,\, 0 \leq t \leq T)$ (complete observation problem) ;
- Y^T defined by $dY_t = \mu X_t\,dt + dV_t^H, \quad Y_0 = 0, \quad 0 \leq t \leq T$.
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- Y^T defined by $Y_t = \mu X_t + V_t^H \quad 0 \leq t \leq T$.
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For a fixed value of the parameter ϑ, let $\mathbf{P}_\vartheta^T$ denote the probability measure, induced by (X^T, Y^T) on the function space $\mathcal{C}_{[0,T]} \times \mathcal{C}_{[0,T]}$ and let $\mathcal{F}_t^Y$ be the natural filtration of Y, $\mathcal{F}_t^Y = \sigma\left(Y_s, 0 \leq s \leq t\right)$.

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Let $\mathcal{L}(\vartheta, Y^T)$ be the likelihood, *i.e.* the Radon-Nikodym derivative of $\mathbf{P}_\vartheta^T$, restricted to $\mathcal{F}_T^Y$ with respect to some reference measure on $\mathcal{C}_{[0,T]}$.

THEOREM (PARTIALLY OBSERVED PROBLEM)

The MLE $\hat{\vartheta}_T$ is uniformly on compacts $\mathbb{K} \subset \mathbb{R}_*^+$ consistent, uniformly asymptotically normal

$$\sqrt{T}\left(\hat{\vartheta}_T - \vartheta\right) \overset{law}{\Longrightarrow} \mathcal{N}\left(0, \mathcal{I}(\vartheta)^{-1}\right)$$

where $\mathcal{I}(\vartheta)$ **does not depend on** H:

$$\mathcal{I}(\vartheta) = \frac{1}{2\vartheta} - \frac{2\vartheta}{\alpha(\alpha+\vartheta)} + \frac{\vartheta^2}{2\alpha^3}$$

and $\alpha = \sqrt{\mu^2 + \vartheta^2}$. We have the uniform on $\vartheta \in \mathbb{K}$ convergence of the moments: for any $p > 0$,

$$\lim_{T\to\infty} \mathbf{E}_\vartheta \left|\sqrt{T}\left(\hat{\vartheta}_T - \vartheta\right)\right|^p = \mathbf{E}\left|\mathcal{I}(\vartheta)^{-\frac{1}{2}}\zeta\right|^p \quad \zeta \sim \mathcal{N}(0,1)\,.$$

TRANSFORMATION OF THE OBSERVATION MODEL

Even if fBm are not martingales, there are simple integral transformations which change the fBm to martingales.

Transformation of the observation model

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In particular, defining for $0 < s < t$, $H > \frac{1}{2}$,

$$k_H(t,s) = \kappa_H^{-1} s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H} \,, \quad \kappa_H = 2H\Gamma\left(\frac{3}{2}-H\right)\Gamma\left(\frac{1}{2}+H\right) \,,$$

$$M_t = \int_0^t k_H(t,s) dW_s^H \,,$$

then the process $M = (M_t, t \geq 0)$ is a Gaussian martingale, the *fundamental martingale*.

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then the process $M = (M_t, t \geq 0)$ is a Gaussian martingale, the *fundamental martingale*.

Moreover, the natural filtration of the martingale M coincides with the natural filtration of the fBm W^H.

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Then $\zeta = (\zeta_t, t \geq 0) = \begin{pmatrix} Z_t \\ \int_0^t s^{2H-1} dZ_s \end{pmatrix}$ is the solution of

$$d\zeta_t = -\vartheta \lambda \mathbf{A}(t) \zeta_t d\langle M \rangle_t + b(t) dM_t, \quad \zeta_0 = 0 \,,$$

with

$$\mathbf{A}(t) = \begin{pmatrix} t^{2H-1} & 1 \\ t^{4H-2} & t^{2H-1} \end{pmatrix} \quad \text{and} \quad b(t) = \begin{pmatrix} 1 \\ t^{2H-1} \end{pmatrix} .$$

SKETCH OF THE PROOF

We have

$$\mathcal{L}(\vartheta, X^T) = \frac{d\mathbf{P}_\vartheta}{d\mathbf{P}_0}\left(\zeta^T\right)$$
$$= \exp\left(-\vartheta\lambda\int_0^T (\mathbf{A}\zeta_s)^*\,\mathbf{B}^+ d\zeta_s - \frac{\vartheta^2\lambda^2}{2}\int_0^T (\mathbf{A}\zeta_s)^*\,\mathbf{B}^+\mathbf{A}\zeta_s d\langle M\rangle_s\right)$$

where $\mathbf{B}^+ = b\,(b^* b)^{-2}\, b^*$ and, by derivating w.r.t. ϑ

$$\sqrt{T}\left(\hat{\vartheta}_T - \vartheta\right) = -\frac{\frac{1}{\sqrt{T}}\int_0^T \lambda l(s)^*\zeta_s dM_s}{\frac{1}{T}\int_0^T \lambda^2\zeta_s^* l(s) l(s)^*\zeta_s d\langle M\rangle_s} \quad l(t) = \begin{pmatrix} t^{2H-1} \\ 1 \end{pmatrix}.$$

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and the explicit expression of the Laplace transform

$$L_T\left(a,\vartheta\right) = \mathbf{E}_\vartheta \exp\left\{-a\lambda^2\int_0^T \zeta_s^* l(s)l(s)^*\zeta_s\, d\langle M\rangle_s\right\}$$

that implies that

$$\lim_{T\to\infty} L_T\left(\frac{a}{T},\vartheta\right)=\exp\left\{-a\frac{1}{2\vartheta}\right\},$$

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$$\sqrt{T}\left(\hat{\vartheta}_T - \vartheta\right) = -\frac{\frac{1}{\sqrt{T}}\int_0^T \lambda l(s)^* \zeta_s dM_s}{\frac{1}{T}\int_0^T \lambda^2 \zeta_s^* l(s) l(s)^* \zeta_s d\langle M\rangle_s} \quad l(t) = \begin{pmatrix} t^{2H-1} \\ 1 \end{pmatrix}.$$

we have the convergence of the following integrals:

$$\frac{1}{T}\int_0^T \lambda^2 \zeta_s^* l(s) l(s)^* \zeta_s d\langle M\rangle_s \longrightarrow \frac{1}{2\vartheta} \quad \text{a.s.},$$

$$\frac{1}{\sqrt{T}}\int_0^T \lambda l(s)^* \zeta_s dM_s \stackrel{law}{\Longrightarrow} \mathcal{N}\left(0, \frac{1}{2\vartheta}\right).$$

Transformation of the observation model

Let us introduce $Z^O = \left(Z^O_t, t \geq 0\right)$ the *fundamental semimartingale* associated to Y, namely

$$Z^O_t = \int_0^t k_H(t,s) dY_s \,.$$

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Let us introduce $Z^O = \left(Z_t^O, t \geq 0\right)$ the *fundamental semimartingale* associated to Y, namely

$$Z_t^O = \int_0^t k_H(t,s) dY_s \,.$$

It is governed by the dynamic

$$dZ_t^O = \mu\lambda l(t)^* \zeta_t d\langle N\rangle_t + dN_t \,, \quad Z_0^O = 0 \,,$$

where, for recall, $\zeta = (\zeta_t, t \geq 0)$ is the solution of:

$$d\zeta_t = -\vartheta\lambda \mathbf{A}(t)\zeta_t d\langle M\rangle_t + b(t) dM_t, \quad \zeta_0 = 0 \,,$$

$$l(t) = \begin{pmatrix} t^{2H-1} \\ 1 \end{pmatrix} , \quad \mathbf{A}(t) = \begin{pmatrix} t^{2H-1} & 1 \\ t^{4H-2} & t^{2H-1} \end{pmatrix} \quad \text{and} \quad b(t) = \begin{pmatrix} 1 \\ t^{2H-1} \end{pmatrix} .$$

TRANSFORMATION OF THE OBSERVATION MODEL

Classical Girsanov theorem and the general filtering theorem give the following likelihood **explicit** function

$$\mathcal{L}_T(\vartheta, Z^{O,T}) = \exp\left\{\mu\lambda \int_0^T I^* \pi_t(\zeta) dZ_t^0 - \frac{\mu^2\lambda^2}{2} \int_0^T \pi_t(\zeta) II^* \pi_t(\zeta)^* d\langle N\rangle_t\right\}$$

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where the conditional expectation $\pi_t(\zeta) = \mathbf{E}_\vartheta\left(\zeta_t | \mathcal{F}_t^Y\right)$ satisfies:

$$d\pi_t(\zeta) = \left(-\vartheta\lambda\mathbf{A} - \mu^2\lambda^2\gamma_{\zeta,\zeta} ll^*\right)\pi_t(\zeta) d\langle N\rangle_t + \mu\lambda\gamma_{\zeta\zeta} l dZ_t^O, \quad \pi_0(\zeta) = 0\,.$$

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and the filtering error $\gamma_{\zeta,\zeta}(t) = \mathbf{E}_\vartheta\left(\zeta_t - \pi_t(\zeta)\right)^*\left(\zeta_t - \pi_t(\zeta)\right)$ is the solution of the Ricatti equation: $\gamma_{\zeta\zeta}(0) = 0$ and

$$d\gamma_{\zeta\zeta}(t) = \left(-\vartheta\lambda\left(\mathbf{A}\gamma_{\zeta\zeta} + \gamma_{\zeta\zeta}\mathbf{A}^*\right) + bb^* - \mu^2\lambda^2\gamma_{\zeta\zeta}ll^*\gamma_{\zeta\zeta}\right)d\langle N\rangle_t\,.$$

Transformation of the observation model

Conditional expectation dynamic can be rewritten in the equivalent form

$$d\pi_t(\zeta) = -\vartheta\lambda \mathbf{A}\pi_t(\zeta) d\langle N\rangle_t + \mu\lambda\gamma_{\zeta\zeta} l d\nu_t$$

where the innovation process $(\nu_t, t \geq 0)$ is defined by:

$$d\nu_t = dZ_t^O - \mu\lambda l(t)^* \pi_t(\zeta) d\langle N\rangle_t \,, \quad \nu_0 = 0 \,.$$

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Then, the likelihood ratio $\mathcal{Z}_T(\vartheta_1, \vartheta_2, Z^{O,T}) =$

$$\begin{aligned} &= \frac{\mathcal{L}_T(\vartheta_2, \zeta^{O,T})}{\mathcal{L}_T(\vartheta_1, \zeta^{O,T})} = \frac{d\mathbf{P}^T_{\vartheta_2}}{d\mathbf{P}^T_{\vartheta_1}} / \mathcal{F}^Y_T \,, \\ &= \exp\left\{ \mu\lambda \int_0^T I^* \delta_{\vartheta_1,\vartheta_2} d\nu_t^{\vartheta_1} - \frac{\mu^2\lambda^2}{2} \int_0^T \delta^*_{\vartheta_1,\vartheta_2} I I^* \delta_{\vartheta_1,\vartheta_2} d\langle N\rangle_t \right\} \end{aligned}$$

where $\delta_{\vartheta_1,\vartheta_2}(t)$ is the difference $\pi_t^{\vartheta_2}(\zeta) - \pi_t^{\vartheta_1}(\zeta)$.

IBRAGIMOV-KHASMINSKII PROGRAM

From Ibragimov-Khasminskii, it is sufficient to check the three following conditions: (here $\mathcal{Z}_T(u, Z^{O,T}) = \mathcal{Z}_T(\vartheta, \vartheta + \frac{u}{\sqrt{T}}, Z^{O,T})$)

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$$\mathcal{Z}_T(u, Z^{O,T}) \overset{law}{\Longrightarrow} \underbrace{\exp\left\{ u.\eta - \frac{u^2}{2}\mathcal{I}(\vartheta) \right\}}_{Z(u)} \text{ with } \quad \eta \sim \mathcal{N}\left(0, \mathcal{I}(\vartheta)\right) ,$$

(A.2) for some $C, \chi > 0$: for all u such that $\vartheta + \frac{u}{\sqrt{T}} \in \mathbf{K} \subset \mathbb{R}_*^+$,

$$\mathbf{E}_\vartheta \sqrt{\mathcal{Z}_T(u, Z^{O,T})} \leq C \exp\left(-\chi u^2\right) ,$$

(A.3) there exists $C > 0$ such that, for all $|u_1| < R$ and $|u_2| < R$,

$$\mathbf{E}_\vartheta \left(\sqrt{\mathcal{Z}_T(u_1, Z^{O,T})} - \sqrt{\mathcal{Z}_T(u_2, Z^{O,T})} \right)^2 \leq C|u_1 - u_2|^2 .$$

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Let $L_T(a,\vartheta_1,\vartheta_2)$ be the Laplace transform of the integral of the quadratic form of the difference $\delta_{\vartheta_1,\vartheta_2}(t)=\pi_t^{\vartheta_2}(\zeta)-\pi_t^{\vartheta_1}(\zeta)$:

$$L_T(a,\vartheta_1,\vartheta_2)=\mathbf{E}_{\vartheta_1}\exp\left\{-a\frac{\mu^2\lambda^2}{2}\int_0^T\delta^*_{\vartheta_1,\vartheta_2}ll^*\delta_{\vartheta_1,\vartheta_2}d\langle N\rangle_t\right\}.$$

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Let us introduce the following condition (L): There exists $a_0<0$ such that for all $a>a_0$, $\forall u_1,u_2\in\mathbb{R}$,

$$\lim_{T\to\infty}L_T(a,\vartheta+\frac{u_1}{\sqrt{T}},\vartheta+\frac{u_2}{\sqrt{T}})=\exp\left(-a\frac{(u_2-u_1)^2}{2}\mathcal{I}(\vartheta)\right),$$

and for all T, $L_T(a,\vartheta+\frac{u_1}{\sqrt{T}},\vartheta+\frac{u_2}{\sqrt{T}})\leq C\exp\left(-a\chi\left(u_1-u_2\right)^2\right)$.

Let $L_T(a, \vartheta_1, \vartheta_2)$ be the Laplace transform of the integral of the quadratic form of the difference $\delta_{\vartheta_1,\vartheta_2}(t) = \pi_t^{\vartheta_2}(\zeta) - \pi_t^{\vartheta_1}(\zeta)$:

$$L_T(a, \vartheta_1, \vartheta_2) = \mathbf{E}_{\vartheta_1} \exp\left\{ -a\frac{\mu^2\lambda^2}{2} \int_0^T \delta^*_{\vartheta_1,\vartheta_2} ll^* \delta_{\vartheta_1,\vartheta_2} d\langle N \rangle_t \right\} .$$

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$$\lim_{T\to\infty} L_T(a, \vartheta + \frac{u_1}{\sqrt{T}}, \vartheta + \frac{u_2}{\sqrt{T}}) = \exp\left(-a\frac{(u_2 - u_1)^2}{2} \mathcal{I}(\vartheta) \right) ,$$

and for all T, $L_T(a, \vartheta + \frac{u_1}{\sqrt{T}}, \vartheta + \frac{u_2}{\sqrt{T}}) \leq C \exp\left(-a\chi\,(u_1 - u_2)^2 \right)$.

Proposition

Suppose condition (L) is satisfied. Then properties (A.1.–A.3) hold.

Proof of Proposition

Actually, (A.1) is a direct consequence of (L). Indeed, for $u_1 = 0$ and $u_2 = u$, we have:

$$\lim_{T\to\infty} L_T(a,\vartheta,\vartheta + \frac{u}{\sqrt{T}}) = \exp\left(-a\frac{u^2}{2}\mathcal{I}(\vartheta)\right) .$$

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It gives the convergence of the following integrals:

$$\frac{\mu^2\lambda^2}{2}\int_0^T \delta^*_{\vartheta,u,T} ll^* \delta_{\vartheta,u,T} d\langle N\rangle_t \longrightarrow \frac{u^2}{2}\mathcal{I}\left(\vartheta\right) \quad \text{a.s.}$$

and

$$\mu\lambda\int_0^T l^*\delta_{\vartheta,u,T} d\nu_t \overset{law}{\Longrightarrow} \mathcal{N}\left(0, u^2\mathcal{I}\left(\vartheta\right)\right) ,$$

which achieves the proof of (A.1).

PROOF OF PROPOSITION

The condition (A.2) holds thanks to: $\mathbf{E}_\vartheta \sqrt{\mathcal{Z}_T(u)} =$

$$
\begin{aligned}
&= \mathbf{E}_\vartheta \exp\left(\frac{\mu\lambda}{2}\int_0^T l^*\delta_{\vartheta,u,T}d\nu_t^\vartheta - \frac{\mu^2\lambda^2}{4}\int_0^T \delta^*_{\vartheta,u,T} ll^*\delta_{\vartheta,u,T} d\langle N\rangle_t\right) \\
&= \mathbf{E}_\vartheta \exp\left(\frac{\mu\lambda}{2}\int_0^T l^*\delta_{\vartheta,u,T}d\nu_t^\vartheta - \frac{\mu^2\lambda^2 p}{8}\int_0^T \delta^*_{\vartheta,u,T} ll^*\delta_{\vartheta,u,T} d\langle N\rangle_t\right) \times \\
&\quad \times \exp\left(\frac{\mu^2\lambda^2}{4}\left(\frac{p}{2}-1\right)\int_0^T \delta^*_{\vartheta,u,T} ll^*\delta_{\vartheta,u,T} d\langle N\rangle_t\right) \\
&\overset{(a)}{\leq} \left(\mathbf{E}_\vartheta \exp\left(\frac{\mu\lambda p}{2}\int_0^T l^*\delta_{\vartheta,u,T}d\nu_t^\vartheta - \frac{\mu^2\lambda^2 p^2}{8}\int_0^T \delta^*_{\vartheta,u,T} ll^*\delta_{\vartheta,u,T} d\langle N\rangle_t\right)\right)^{\frac{1}{p}} \\
&\quad \times \left(\mathbf{E}_\vartheta \exp\left(\frac{\mu^2\lambda^2 q}{4}\left(\frac{p}{2}-1\right)\int_0^T \delta^*_{\vartheta,u,T} ll^*\delta_{\vartheta,u,T} d\langle N\rangle_t\right)\right)^{\frac{1}{q}}
\end{aligned}
$$

Proof of Proposition

To prove (A.3), let us note that

$$\begin{aligned}
\mathbf{E}_{\vartheta}\left(\sqrt{\mathcal{Z}_T(u_1)}-\sqrt{\mathcal{Z}_T(u_2)}\right)^2 &= 2\left(1-\mathbf{E}_{\vartheta}\mathcal{Z}_T(u_1)\sqrt{\frac{\mathcal{Z}_T(u_2)}{\mathcal{Z}_T(u_1)}}\right)\\
&= 2\left(1-\mathbf{E}_{\vartheta_1}\sqrt{\mathcal{Z}_T(\vartheta_1,\vartheta_2)}\right)\\
&\leq 2\left(1-\exp\left(-\chi\left(u_2-u_1\right)^2\right)\right)\\
&\leq C|u_1-u_2|^2 .
\end{aligned}$$

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&= 2\left(1 - \mathbf{E}_{\vartheta_1}\sqrt{\mathcal{Z}_T(\vartheta_1, \vartheta_2)}\right) \\
&\leq 2\left(1 - \exp\left(-\chi\left(u_2 - u_1\right)^2\right)\right) \\
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SATISFYING CONDITION (L)

The computation of the Laplace transform is based on the Cameron-Martin formula.

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Let us recall that, for $\vartheta = \vartheta_1$, the optimal filter $\pi_t^{\vartheta_1}(\zeta)$ and the difference $\delta_{\vartheta_1,\vartheta_2}(t) = \pi_t^{\vartheta_2}(\zeta) - \pi_t^{\vartheta_1}(\zeta)$ are governed by:

$$d\tilde{\pi}_t = \mathcal{A}(t)\tilde{\pi}_t d\langle N\rangle_t + \mathcal{B}(t) d\nu_t^{\vartheta_1} \,,$$

where $\tilde{\pi}_t = \begin{pmatrix} \pi_t^{\vartheta_1}(\zeta) \\ \delta_{\vartheta_1,\vartheta_2} \end{pmatrix}$, $\underline{\mathbf{A}}^{\vartheta_2}(t) = -\vartheta_2 \lambda \mathbf{A}(t) - \mu^2 \lambda^2 \gamma_{\zeta,\zeta}^{\vartheta_2} I I^*$,

$$\mathbf{D}_\gamma^{\vartheta_1,\vartheta_2} = \gamma_{\zeta\zeta}^{\vartheta_2}(t) - \gamma_{\zeta\zeta}^{\vartheta_1}(t)\,,$$

$$\mathcal{A}(t) = \begin{pmatrix} -\vartheta_1 \lambda \mathbf{A} & 0 \\ -(\vartheta_2 - \vartheta_1)\lambda \mathbf{A} & \underline{\mathbf{A}}^{\vartheta_2} \end{pmatrix} \quad \text{and} \quad \mathcal{B}(t) = \mu\lambda \begin{pmatrix} \gamma_{\zeta\zeta}^{\vartheta_1}(t) \\ \mathbf{D}_\gamma^{\vartheta_1,\vartheta_2} \end{pmatrix} I(t)\,.$$

SATISFYING CONDITION (L)

Then, $L_T(a, \vartheta_1, \vartheta_2) =$

$$
\begin{aligned}
&= \mathbf{E}_{\vartheta_1} \exp\left\{-a\frac{\mu^2\lambda^2}{2}\int_0^T \delta^*_{\vartheta_1,\vartheta_2} ll^* \delta_{\vartheta_1,\vartheta_2} d\langle N\rangle_t\right\} \\
&= \mathbf{E}_{\vartheta_1} \exp\left\{-a\frac{\mu^2\lambda^2}{2}\int_0^T \begin{pmatrix} \pi_t^{\vartheta_1}(\zeta) \\ \delta_{\vartheta_1,\vartheta_2} \end{pmatrix}^* \mathcal{M}(t) \begin{pmatrix} \pi_t^{\vartheta_1}(\zeta) \\ \delta_{\vartheta_1,\vartheta_2} \end{pmatrix} d\langle N\rangle_t\right\} \\
&= \exp\left\{-a\frac{\mu^2\lambda^2}{2}\int_0^T \operatorname{trace}(\mathcal{H}(t)\mathcal{M}(t)) d\langle N\rangle_t\right\},
\end{aligned}
$$

where $\mathcal{M}(t) = \begin{pmatrix} 0 & 0 \\ 0 & ll^* \end{pmatrix}$ and $\mathcal{H}(t)$ is the solution of Ricatti differential equation: $\mathcal{H}(0) = 0$,

$$
\frac{d\mathcal{H}(t)}{d\langle N\rangle_t} = \mathcal{A}(t)\mathcal{H}(t) + \mathcal{H}(t)\mathcal{A}(t)^* + \mathcal{B}(t)\mathcal{B}(t)^* - a\lambda^2\mu^2\mathcal{H}(t)\mathcal{M}(t)\mathcal{H}(t)\,.
$$

SATISFYING CONDITION (L)

It is known that solution $\mathcal{H}(t)$ can be written as $\mathcal{H}(t) = \Psi_1^{-1}(t)\Psi_2(t)$, where the pair of 4×4 matrices (Ψ_1, Ψ_2) satisfies the system of linear differential equations:

$$\begin{aligned} \frac{d\Psi_1(t)}{d\langle N\rangle_t} &= -\Psi_1(t)\mathcal{A}(t) + a\lambda^2\mu^2\Psi_2(t)\mathcal{M}(t) \quad \Psi_1(0) = \mathcal{I}d \\ \frac{d\Psi_2(t)}{d\langle N\rangle_t} &= \Psi_1(t)\mathcal{B}(t)\mathcal{B}(t)^* + \Psi_2(t)\mathcal{A}^*(t) \quad \Psi_2(t) = 0\,, \end{aligned}$$

and $\mathcal{I}d$ is the 4×4 identity matrix.

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and $\mathcal{I}d$ is the 4×4 identity matrix.

Now,

$$L_T(a, \vartheta_1, \vartheta_2) = \exp\left\{-\frac{1}{2}\int_0^T \operatorname{trace} \mathcal{A}(t) d\langle N\rangle_t\right\} (\det \Psi_1(T))^{-\frac{1}{2}}\ .$$

The explicit representation of $\gamma^{\vartheta}_{\zeta,\zeta}$ imply that we can replace $\gamma^{\vartheta}_{\zeta,\zeta}$ by $\Delta^{-1}\gamma_{\infty}\Delta^{-1}$ in the coefficients of Equation and therefore

$$\lim_{T\to\infty} L_T(a,\vartheta_1,\vartheta_2) = \lim_{T\to\infty} \exp\left\{-\frac{1}{2}\int_0^T \text{trace}\,\mathcal{A}_{\infty}(t)d\langle N\rangle_t\right\}(\det\Psi_{1,\infty}(T))^{-\frac{1}{2}}$$

where

$$\begin{aligned}
\frac{d\Psi_{1,\infty}(t)}{d\langle N\rangle_t} &= -\Psi_{1,\infty}(t)\mathcal{A}_{\infty}(t) + a\lambda^2\mu^2\Psi_{2,\infty}(t)\mathcal{M}(t) \quad \Psi_{1,\infty}(0) = \mathcal{I}d \\
\frac{d\Psi_{2,\infty}(t)}{d\langle N\rangle_t} &= \Psi_1(t)\mathcal{B}_{\infty}(t)\mathcal{B}_{\infty}(t)^* + \Psi_{2,\infty}(t)\mathcal{A}^*_{\infty}(t) \quad \Psi_{2,\infty}(t) = 0\,,
\end{aligned}$$

with

$$\mathcal{A}_{\infty}(t) = \begin{pmatrix} -\vartheta_1 & 0 \\ -(\vartheta_2-\vartheta_1) & -\alpha_2 \end{pmatrix} \otimes \lambda\mathbf{A}\,, \quad \mathcal{B}_{\infty}\mathcal{B}^*_{\infty} = \begin{pmatrix} g_1^2 & g_1 g_2 \\ g_1 g_2 & g_2^2 \end{pmatrix} \otimes \lambda\mathbf{AJ}\,,$$

$\alpha_2 = \sqrt{\vartheta_2^2+\mu^2}$, $g_1 = \frac{\mu}{\sqrt{\lambda}(\alpha_1+\vartheta_1)}$, $g_2 = \frac{\mu}{\sqrt{\lambda}(\alpha_2+\vartheta_2)} - \frac{\mu}{\sqrt{\lambda}(\alpha_1+\vartheta_1)}$ and $\otimes$ is the Kronecker product.

Linear system (1) can be rewritten as

$$\frac{d\left(\Psi_1(t), \Psi_2(t) \otimes \mathbf{J}\right)}{d\langle N\rangle_t} = \left(\Psi_1(t), \Psi_2(t) \otimes \mathbf{J}\right) \cdot \left(\beth \otimes \lambda \mathbf{A}(t)\right)$$

where

$$\beth = \begin{pmatrix} \vartheta_1 & 0 & g_1^2 & g_1 g_2 \\ (\vartheta_2 - \vartheta_1) & \alpha_2 & g_1 g_2 & g_2^2 \\ 0 & 0 & -\vartheta_1 & -(\vartheta_2 - \vartheta_1) \\ 0 & a\lambda\mu^2 & 0 & -\alpha_2 \end{pmatrix}.$$

Clearly, system (5) has an explicit solution:

$$(\Psi_1(t), \Psi_2(t) \otimes \mathbf{J}) = (\mathcal{I}d, 0) \cdot (\mathcal{P} \otimes \mathbf{Id})\, \mathcal{G} \left(\mathcal{P}^{-1} \otimes \mathbf{Id}\right)$$

where $\mathcal{G} = \operatorname{diag}(\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3, \mathbf{G}_4)$ and

$$\frac{d\mathbf{G}_i(t)}{d\langle N\rangle_t} = \lambda x_i \mathbf{G}_i \mathbf{A} \quad \mathbf{G}_i(0) = \mathbf{Id}\,, \quad i = 1 \ldots 4\,,$$

with $(x_i)_{i=1\ldots4}$ the eigenvalues of matrix $\beth$ and $\mathcal{P}$ the matrix of its eigenvectors.

For $\vartheta_1 = \vartheta + \frac{u_1}{\sqrt{T}}$ and $\vartheta_2 = \vartheta + \frac{u_2}{\sqrt{T}}$, we can compute the eigenvalues and we obtain:

$$\begin{aligned}
x_1 &= \vartheta_1 + C_1 \frac{(u_2 - u_1)^2}{T} + o\left(\frac{1}{T}\right) , \\
x_2 &= -\vartheta_1 + C_2 \frac{(u_2 - u_1)^2}{T} + o\left(\frac{1}{T}\right) , \\
x_3 &= \alpha_2 + C_3 \frac{(u_2 - u_1)^2}{T} + o\left(\frac{1}{T}\right) , \\
x_4 &= -\alpha_2 + C_4 \frac{(u_2 - u_1)^2}{T} + o\left(\frac{1}{T}\right) ,
\end{aligned}$$

where $C_1 = \frac{a\mu^2}{2\vartheta\alpha^2(\alpha+\vartheta)^2}\left(\alpha^2 + 2\alpha\vartheta\right)$ and $C_3 = \frac{a\mu^2}{2\alpha^3(\alpha+\vartheta)^2}\left(-\vartheta^2 - 2\alpha\vartheta\right)$.

It can be easily checked that

$$\begin{aligned} \det\Psi_{1,\infty}(T) &= \det\left(G_1.G_3\right)\left(1+o\left(\frac{1}{T}\right)\right) \\ &= \exp\left(\left(x_1+x_3\right)T\right)\left(1+o\left(\frac{1}{T}\right)\right). \end{aligned}$$

Finally,

$$\begin{aligned} \lim_{T\to\infty} L_T(a,\vartheta+\frac{u_1}{\sqrt{T}},\vartheta+\frac{u_2}{\sqrt{T}}) &= \lim_{T\to\infty}\exp\left\{-\frac{1}{2}(x_1+x_3-\alpha_2-\vartheta_1)T\right\} \\ &= \lim_{T\to\infty}\exp\left\{-\frac{1}{2}\left(C_1+C_3\right)\left(u_2-u_1\right)^2\right\} \\ &= \exp\left(-a\frac{\left(u_2-u_1\right)^2}{2}\mathcal{I}\left(\vartheta\right)\right). \end{aligned}$$

Case $H < \frac{1}{2}$

Thanks to Jost, for $H < 1/2$, we have the link between fBm processes of indexes H and $1 - H$:

$$W_t^H = \left(\frac{2H}{\Gamma(2H)} \Gamma(3 - 2H) \right)^{\frac{1}{2}} \int_0^t (t - s)^{2H-1} \, dW_s^{1-H} .$$

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The observation model becomes:

$$\begin{cases} d\tilde{X}_t & = & -\vartheta \tilde{X}_t dt + dw_t^{1-H} \, , \quad \tilde{X}_0 = 0 \, , \\ d\tilde{Y}_t & = & \mu \tilde{X}_t dt + dv_t^{1-H} \, , \quad \tilde{Y}_0 = 0 \, , \end{cases}$$

with, for instance,

$$\tilde{X}_t = \left(\frac{2H}{\Gamma(2H)} \Gamma(3 - 2H) \right)^{\frac{1}{2}} \int_0^t (t - s)^{1-2H} \, dX_s \, .$$

Dependent noises case

$$\begin{cases} dX_t &= -\vartheta X_t dt + dW_t\,, \quad X_0 = 0\,, \\ Y_t &= \mu X_t + V_t\,, \end{cases}$$

DEPENDENT NOISES CASE

$$\begin{cases} dX_t &= -\vartheta X_t dt + dW_t\,, \qquad\qquad X_0 = 0\,, \\ dY_t &= -\mu\vartheta X_t dt + \mu dW_t + dV_t\,, \end{cases}$$

DEPENDENT NOISES CASE

$$\begin{cases} dX_t &= -\vartheta X_t dt + dW_t\,, & X_0 = 0\,, \\ dY_t &= -\mu\vartheta X_t dt + \mu dW_t + dV_t\,, \end{cases}$$

In this case, we can show that,

$$(1+\mu^2)\frac{d\gamma_{\zeta,\zeta}(t)}{d\langle N\rangle_t} = -\vartheta\lambda\left(\mathbf{A}\gamma_{\zeta,\zeta} + \gamma_{\zeta,\zeta}\mathbf{A}^*\right) + bb^* - \left(\mu\vartheta\right)^2\lambda^2\gamma_{\zeta,\zeta}ll^*\gamma_{\zeta,\zeta}\,.$$

REFERENCES

- M.L. Kleptsyna and A. Le Breton (2002) *Extension of the Kalman-Bucy filter to Elementary Linear Systems with Fractional Brownian Noises*, Stat. Inf. Stoch. Proc.,**5**:249–271.
- A. Brouste and M.L. Kleptsyna, *Asymptotic properties of Maximum Likelihood Estimator for partially observed fractional diffusion system*, submitted, `http://www.univ-lemans.fr/sciences/statist/download/Kleptsyna/BrKl.pdf`.
- A. Brouste, *Asymptotic properties of MLE for partially observed fractional diffusion system with dependent noises*, preprint, `http://www.univ-lemans.fr/sciences/statist/download/Brouste/B09a.pdf`

Thank you for your attention