On limiting likelihood ratio processes of some change-point type statistical models

Sergueï Dachian, Université Blaise Pascal, Clermont-Ferrand

Abstract

We consider the random process Z_ρ on $\mathbb{R}$ defined by

$$\ln Z_\rho(x) = \begin{cases} \rho\,\Pi_+(x) - x, & \text{if } x \geq 0, \\ -\,\rho\,\Pi_-(-x) - x, & \text{if } x \leq 0, \end{cases}$$

where $\rho > 0$, and Π_+ and Π_- are two independent Poisson processes on $\mathbb{R}_+$ with the intensities $1/(e^\rho - 1)$ and $1/(1 - e^{-\rho})$.

The process Z_ρ (up to a time change) occurs in some change-point type statistical models (i.i.d. observations, inhomogeneous Poisson processes) as the limiting likelihood ratio process. So, in these models, the random variables

$$\zeta_\rho = \frac{\int_{\mathbb{R}} x\, Z_\rho(x)\; dx}{\int_{\mathbb{R}} Z_\rho(x)\; dx} \quad \text{and} \quad \xi_\rho = \operatorname*{argsup}_{x\in\mathbb{R}} Z_\rho(x)$$

are (up to a multiplicative constant) the limiting distributions of the Bayesian estimators and of the maximum likelihood estimator respectively, and in particular, $B_\rho = \mathbf{E}\zeta_\rho^2$ and $M_\rho = \mathbf{E}\xi_\rho^2$ (up to the square of the above multiplicative constant) are the limiting variances of these estimators. Moreover, the Bayesian estimators being asymptotically efficient, the ratio $E_\rho = B_\rho/M_\rho$ is the asymptotic efficiency of the maximum likelihood estimator.

We show, that as $\rho \to 0$, the process $Z_\rho(y/\rho)$, $y \in \mathbb{R}$, converges weakly in the space $\mathcal{D}_0(-\infty, +\infty)$ to the process

$$Z_0(x) = \exp\left\{W(x) - \frac{1}{2}\,|x|\right\}$$

which is the limiting likelihood ratio process of many other change-point type statistical models (signal in a white Gaussian noise, dynamical systems with small noise, ergodic diffusion processes, etc.) So, the random variables $\rho\,\zeta_\rho$ and $\rho\,\xi_\rho$ converge weakly to the random variables

$$\zeta_0 = \frac{\int_{\mathbb{R}} x\, Z_0(x)\; dx}{\int_{\mathbb{R}} Z_0(x)\; dx} \quad \text{and} \quad \xi_0 = \operatorname*{argsup}_{x\in\mathbb{R}} Z_0(x)$$

respectively, and in particular, $\rho^2 B_\rho \to \mathbf{E}\zeta_0^2 = 16\,\zeta(3)$, $\rho^2 M_\rho \to \mathbf{E}\xi_0^2 = 26$ and $E_\rho \to 8\,\zeta(3)/13$.

We equally discuss the second possible asymptotics $\rho \to +\infty$ and illustrate the results by numerical simulation.