On the convergence in law of some empirical estimators

Dominique Dehay

Institut de Recherche MAthématique de Rennes,
Rennes, France

dominique.dehay@univ-rennes2.fr

1 – Problem

2 – Local time

3 – Karhunen-Loève series expansion

4 – Occupation time formula

5 – Convergence in law of the random mean

6 – An application

1 - Problem

$$\frac{1}{T}\int_0^T \Phi_T(X_t)\,\mathrm{d}t, \qquad \text{as} \quad T \to \infty$$

$$\frac{1}{T}\int_0^T \Phi_T(X_t)g(X_t)\,\mathrm{d}t.$$

2 – Local time

$X \triangleq \{X_t,\ t > 0\}$ real-valued measurable process on $(\Omega, \mathcal{F}, \mathsf{P})$

- Occupation measure

$$\nu_T(B,\omega) \triangleq \int_0^T \mathbb{I}_{\{X_t(\omega)\in B\}}\,\mathrm{d}t$$

- Local time (occupation density)

$$\nu_T(B,\omega) = \int_B l_T(x,\omega)\,\mathrm{d}x$$

- Classical occupation time formula

$$\int_0^T \varphi(X_t)\,\mathrm{d}t = \int_{\mathbb{R}} \varphi(x) l_T(x)\,\mathrm{d}x, \qquad \text{a.e.}$$

Case of a continuous semi-martingale

- ”Local time” (Tanaka Meyer formula)

$$\Lambda_T(x) \stackrel{\Delta}{=} |X_T - x| - |X_0 - x| - \int_0^T \text{sgn}(X_t - x)\,\mathrm{d}X_t.$$

$$\int_0^T \varphi(X_t)\,\mathrm{d}[X_t] = \int_{\mathbb{R}} \varphi(x)\Lambda_T(x)\,\mathrm{d}x, \qquad \text{a.s.}$$

local time of the quadratic variation process [X]

- X diffusion : $\mathrm{d}X_t = S(X_t)\mathrm{d}t + \sigma(X_t)dW_t$

$$l_T(x) = \frac{\Lambda_T(x)}{\sigma(x)^2}$$

3 – Karhunen-Loève series expansion

$\Phi \triangleq \{\Phi(x),\ x \in \mathbb{R}\}$ second order, measurable, continuous in quadratic mean process.

- Then for any $M > 0$

$$\Phi(x) = \sum_{k=1}^{\infty} \xi_k \phi_k(x)$$

convergence in quadratic mean uniform with respect to $x \in [-M, M]$

- If in addition

$$\mathsf{E}\left[\left|\Phi(x) - \Phi(y)\right|^2\right] \leq c|x-y| \left|\ln|x-y|\right|^{-3-r}, \quad |x-y| \leq \delta_M$$

then

$$\lim_{n\to\infty} \mathsf{P}\left[\sup_{|x|\leq M}\left|\Phi(x) - \sum_{k=1}^{n} \xi_k \phi_k(x)\right| > \epsilon\right] = 0. \qquad (unif.K.L.)$$

Conditions

$$\mathsf{E}[|\Phi(x)-\Phi(y)|^2] \leq c|x-y|^{1+\alpha} \quad \text{if} \quad |x|,|y| \leq M \quad \text{and} \quad |x-y| \leq \delta_M.$$

On the other side, the condition can be weakened to

$$\mathsf{E}\left[\left|\Phi(x)-\Phi(y)\right|^2\right] \leq c|x-y|\left|\ln|x-y|\right|^{-3}\left(\ln|\ln|x-y||\right)^{-1-r}.$$

- The previous condition is not necessary.

Indeed, for any zero-mean Gaussian process $\{\Phi(x),\, x \in \mathbb{R}\}$ such that

i) P-almost its paths are continuous, and

ii) $(x,y) \mapsto \mathsf{E}[\Phi(x)\Phi(y)]$ is continuous,

K.L.-expansion is uniformly almost surely convergent on $[-M, M]$, and so satisfies relation (unif.K.L.)

4 – Occupation time formula

Theorem .

$\{X_t,\, t \in [0,T]\}$ real-valued measurable process with a local time l_T,
$\{\Phi(x),\, x \in \mathbb{R}\}$ real-valued separable measurable process, continuous in quadratic mean, and such that

i) P-almost all the paths of the process Φ are locally bounded,

ii) Karhunen-Loève series expansion converges uniformly on $[-M, M]$ in probability, for any $M > 0$.

Then

$$\int_0^T \Phi(X_t) g(X_t)\,\mathrm{d}t = \int_{\mathbb{R}} \Phi(x) g(x) l_T(x)\,\mathrm{d}x \quad \text{a.e.}$$

for any bounded Borelian function $g : \mathbb{R} \to \mathbb{R}$ with bounded support.

Corollary .

$\{X_t,\, t \in [0,T]\}$ and $\{\Phi(x), x \in \mathbb{R}\}$ satisfy the hypotheses of the previous theorem.

Then the occupation time formula is valid, for any Borelian function $g : \mathbb{R} \to \mathbb{R}$ such that

> i) P-a.e. path of the process $\{\Phi(x)g(x)l_T(x), x \in \mathbb{R}\}$ is integrable on $\mathbb{R}$

or

> ii) P-a.e. path of the process $\{\Phi(X_t)g(X_t), t \in [0,T]\}$ is integrable on $[0,T]$.

5 – Convergence in law of the random mean

Let F_b be some complete separable space of bounded real-valued Borelian functions on $\mathbb{R}$ endowed with the uniform metric.

Proposition .
$\{X_t,\, t \geq 0\}$ measurable separable process with local times l_T, $T > 0$
$\{\Phi_T(x),\, x \in \mathbb{R}\}$, $T > 0$, measurable separable processes on $\mathbb{R}$ such that

i) for any $T > 0$, P-a.e path of the process l_T belongs to F_b,

ii) $\lim_{T\to\infty} l_T/T = f$ in probability in F_b, for $f \in F_b$ non-random

iii) the process Φ_T satisfies (unif.K.L.) and P-a.e path is in F_b,

iv) Φ_T converges weakly in F_b to some $\{\Phi(x),\, x \in \mathbb{R}\}$.

Then for any $g \in L^1(\mathbb{R})$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi_T(X_t)g(X_t)\,\mathrm{d}t = \int_{\mathbb{R}} \Phi(x)g(x)f(x)\,\mathrm{d}x \qquad \text{in law in } \mathbb{R}\,. \tag{1}$$

VI – Application

$X \triangleq \{X_t,\, t \in \mathbb{R}\}$ strong solution of SDE

$$\mathrm{d}X_t = S(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t,$$

where

$$\limsup_{|x|\to\infty} \mathsf{sgn}(x) S(x) < 0 \qquad \text{and} \qquad 0 < \sigma_1 \le \sigma(x) \le \sigma_2 < \infty.$$

– marginal stationary law μ

– distribution function $F(\,\cdot\,)$

– stationary density function $f(\,\cdot\,)$

- Empirical distribution function

$$\hat{F}_T(x) \triangleq \frac{1}{T}\int_0^T \mathbb{I}_{\{X_t \le x\}}\,\mathrm{d}t$$

– consistent estimator of the distribution function $F(x)$.

– weak convergence in $\mathcal{C}_0(\mathbb{R})$ of the process

$$\left\{\zeta_T(x) \triangleq \sqrt{T}\big(\hat{F}_T(x) - F(x)\big),\ x \in \mathbb{R}\right\}$$

to a zero mean Gaussian process ζ as $T \to \infty$

– $\mathrm{E}\left[\left|\zeta_T(x) - \zeta_T(y)\right|^2\right] \le c\,|x-y|^2 \qquad |x|, |y| \le M.$

Thus

$$\lim_{T\to\infty} \frac{1}{\sqrt{T}}\int_0^T \big(\hat{F}_T(X_t) - F(X_t)\big)g(X_t)\,\mathrm{d}t = \int_{\mathbb{R}} \zeta(x)g(x)f(x)\,\mathrm{d}x$$

in law in $\mathbb{R}$, for any $g \in L^1(\mathbb{R})$.

- Local time density estimator $\dfrac{l_T(\cdot)}{T}$

– consistent estimator of the density function $f(\cdot)$

– weak convergence in $\mathcal{C}_0(\mathbb{R})$ of the process

$$\left\{\eta_T(x) \triangleq \sqrt{T}\big(l_T(x)/T - f(x)\big),\ x \in \mathbb{R}\right\}$$

to a zero mean Gaussian process η as $T \to \infty$

– weak convergence in $\mathcal{C}_0(\mathbb{R}) \times \mathcal{C}_0(\mathbb{R})$ of the process (ζ_T, η_T) to (ζ, η) as $T \to \infty$

Thus

$$\lim_{T\to\infty} \sqrt{T}\left(\frac{1}{T}\int_0^T \hat{F}_T(X_t)g(X_t)\,\mathrm{d}t - \int_{\mathbb{R}} F(x)f(x)g(x)\,\mathrm{d}x\right) = \int_{\mathbb{R}} \big(\zeta(x)f(x) - F(x)\eta(x)\big)g(x)\mathrm{d}x$$

in law in $\mathbb{R}$, for any $g \in L^1(\mathbb{R})$.

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Let $\Phi_T = \left\{\Phi_T(x),\, x \in \mathbb{R}\right\}$, $T > 0$, be a family of measurable real-valued processes on $\mathbb{R}$ with paths in $\mathcal{B}_b$, the class of real-valued Borelian bounded functions, which converges in law in $\mathcal{B}_b$ to a process Φ, as T goes to ∞. The aim of this work is to establish conditions on the processes Φ_T, $T > 0$, and on the real-valued process $(X_t)_{t\geq 0}$ for the weak convergence of the average

$$\frac{1}{T}\int_0^T \Phi_T(X_t)\,dt,$$

as T goes to ∞. Our method is based on local time and on a stochastic version of the occupation time formula proved with Karhunen Loève series expansion. Then we present an application of this result to the estimation of the distribution of an ergodic diffusion process.