Estimation of Spectral Functionals for Continuous-time Stationary Models[2]

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The Problem

- Suppose we observe a realization $\mathbf{X}_T = \{X(t), 0 \leq t \leq T\}$ of a zero mean real-valued continuous-time stationary Gaussian process $X(t)$ with an **unknown** spectral density (SD) $\theta(\lambda)$, $\lambda \in \mathbb{R}$.
- We assume that $\theta(\lambda)$ belongs to a given class $\mathbf{\Theta} \subset L^p = L^p(\mathbb{R})$ $(p > 1)$ of spectral densities possessing some smoothness conditions.
- Let $\Phi(\cdot)$ be some *known* functional, the domain of definition of which contains $\mathbf{\Theta}$. The distribution of $X(t)$ is completely determined by the SD, and we consider $\theta(\lambda)$ as an infinite-dimensional "parameter" on which the distribution of $X(t)$ depends.
- **The problem** is to estimate the value $\Phi(\theta)$ of the functional $\Phi(\cdot)$ at an unknown point $\theta \in \mathbf{\Theta}$ on the basis of an observation $\mathbf{X}_T$, and to investigate the asymptotic (as $T \to \infty$) properties of the suggested estimators (unbiasedness, consistency, asymp. normality).
- The main objective is construction of **asymptotically efficient estimators (AEE)** for $\Phi(\theta)$.

Some References

- The problem of AE nonparametric estimation of different kind of spectral functionals for discrete-time processes has been considered by
  Millar (1985),
  Ibragimov and Khas'minskii (1986, 1991),
  Dahlhaus and Wefelmeyer (1996),
  Ginovian (1988, 1994, 2003), and others.
- The objective of the present talk is to state the corresponding results for continuous-time processes.
- The problem is less investigated for continuous-time processes
  Ginovian (1988), estimation of linear spectral functionals,
  Haberzettl (1997), estimation of linear covariance functionals.
- For construction of AEE's we use a general method suggested by
  Ibragimov and Khas'minskii (1986, 1991),
  (applied for disc.-time case in Ginovian (2003)).
- Our plan will be as follows:

An Approach

1. We define the concept of local asymptotic normality (LAN) in the spirit of Ibragimov & Khas'minskii (1991), and give conditions under which the underlying family of distributions is LAN at $\theta_0 \in \boldsymbol{\Theta}$.
2. Using LAN we state variants of Hájek–Le Cam local asymptotic minimax theorem and Hájek–Ibragimov–Khas'minskii convolution theorem.
3. We define the concepts of H_0– and IK– AEE's, and show that
4. the statistic $\Phi(I_T)$, where $I_T = I_T(\lambda)$ is the periodogram of the underlying process $X(t)$ with an unknown SD $\theta(\lambda)$, $\lambda \in \mathbb{R}$, is H_0– and IK– AEE for a linear functional $\Phi(\theta)$, while
5. for a nonlinear functional $\Phi(\theta)$ a H_0– and IK– AEE is $\Phi(\widehat{\theta}_T)$, where $\widehat{\theta}_T$ is a suitable sequence of $T^{1/2}$–consistent estimators of $\theta(\lambda)$.

- Problems involving statistical analysis of Gaussian stationary processes usually require **two** type of conditions imposed on the SD $\theta(\lambda)$.
- The first type of these conditions controls the singularities of SD $\theta(\lambda)$, and describe the dependence structure of the underlying process $X(t)$.
- The second type conditions require smoothness of SD $\theta(\lambda)$.
- To specify the model we need the following definitions.
- **Definition** (Muckenhoupt condition (A_2)). We say that a nonnegative locally integrable function $f(\lambda)$ $(\lambda \in \mathbb{R})$ satisfies the *Muckenhoupt condition* $(\mathcal{A}_2)$, if

$$\sup \frac{1}{|J|^2} \int_J f(\lambda)\, d\lambda \int_J \frac{1}{f(\lambda)}\, d\lambda < \infty, \qquad (A_2)$$

where the supremum is over all intervals J, and $|J|$ stands for the length of J. ($J := J(\mu, \varepsilon) = \{\lambda \in \mathbb{R} : |\lambda - \mu| < \varepsilon\}$.)
- The class of functions $f(\lambda)$ satisfying (A_2) we denote by $\mathcal{A}_2$.

1. The SD's of **short memory** processes belong to $\mathcal{A}_2$.
   $X(t)$ is called a short memory process if its SD $\theta(\lambda)$ is separated from zero and infinity:
   $$0 < C_1 \le \theta(\lambda) \le C_2 < \infty.$$
2. The class $\mathcal{A}_2$ contains SD's possessing singularities In particular, it can be shown (Böttcher & Karlovich (1997)) that if $\lambda_k, \alpha_k \in \mathbb{R}$, $k = \overline{1, n}$, $\lambda_k \neq \lambda_j$, $k \neq j$, then functions of the form
   $$f(\lambda) = \prod_{k=1}^{n} |\lambda - \lambda_k|^{\alpha_k}$$
   belong to $\mathcal{A}_2$ if and only if $-1 < \alpha_k < 1$ for all $k = \overline{1, n}$.
3. (A_2) is a **regularity condition** for the process $X(t)$, and means that the maximal coefficient of correlation between the "past" and the "future" of $X(t)$ is less than 1. (Ibragimov & Rozanov (1970)).
4. Condition $\mathcal{A}_2$ was introduced by Muckenhoupt in 1972 (Hunt, Muckenhoupt and Wheeden (1973)). Into statistics: Solev (1980).

The Model Assumption. Hölder class.

- **Definition** (Hölder class). Given numbers $0 < \alpha < 1$, $r \in \mathbb{N}_0$, where $\mathbb{N}_0$ stands for the set of nonnegative integers. We put $\beta = r + \alpha$, and denote by $H_p(\beta)$ the Hölder class of functions, that is, the class of functions $\psi(\lambda) \in L^p$, which have r-th derivatives in L^p and satisfy

$$||\psi^{(r)}(\cdot + u) - \psi^{(r)}(\cdot)||_p \leq C|u|^\alpha,$$

  where C is a positive constant.

- Also by $\Sigma_p(\beta)$ we denote the set of all spectral densities which belong to the class $H_p(\beta)$.

  The assumption on the observed process $X(t)$ is the following.

- **The Model Assumption (M).** $X(t)$ $(t \in \mathbb{R})$ is a zero mean real-valued stationary Gaussian process with a SD $\theta(\lambda)$ satisfying Muckenhoupt condition $(\mathcal{A}_2)$ and belonging to a Hölder class $\Sigma_p(\beta)$. Thus, $\theta(\lambda) \in \mathcal{A}_2 \cap \Sigma_p(\beta)$.

Local asymptotic normality (LAN)

- LAN of families of distributions plays an important role in asymptotic estimation theory. Hájek, Le Cam, Ibragimov and Khas'minskii and others have shown (see, e.g., Ibragimov and Khas'minskii (1979)) that many important properties of statistical estimators (characterization of limiting distributions, lower bounds on the accuracy, asymptotic efficiency, etc) follow in fact from LAN.
- The importance of LAN for nonparametric estimation problems has been emphasized by Levit (1974, 1978), Millar (1985), Ibragimov and Khas'minskii (1991) and others.
- LAN for families of distributions generated by a stationary Gaussian process with SD depending on a finite-dimensional parameter has been studied by Davies (1973), Dzhaparidze (1986), Ginovian (1988).
- Ibragimov and Khas'minskii (1991) suggested a new definition of LAN for families of distributions in the case where the parametric set is a subset of an infinite-dimensional normed space.

- Let $\mathbb{P}_{T,\theta}$ be the distribution of $\mathbf{X}_T = \{X(t), 0 \le t \le T\}$ with SD $\theta(\lambda)$. Following Ibragimov and Khas'minskii (1991)
- **Definition.** A family $\{\mathbb{P}_{T,\theta}, \theta \in \mathbf{\Theta}\}$ is called LAN at a point $\theta_0 \in \mathbf{\Theta}$ in the direction L^2 with norming factors $A_T = A_T(\theta_0)$ if there exist (a) a linear manifold $H_0 = H_0(\theta_0) \subset L^2$ with closure $\overline{H_0} = L^2$, and (b) a family $\{A_T\}$ of linear operators $A_T : L^2 \to L^2$ that satisfy:
  1) for any $h \in H_0$, $||A_T h||_2 \to 0$ as $T \to \infty$, where $||\cdot||_2$ is the L_2-norm;
  2) for any $h \in H_0$ there is a natural $T(h)$ such that $\theta_0 + A_T h \in \mathbf{\Theta}$ for all $T > T(h)$;
  3) for any $h \in H_0$ and $T > T(h)$ the representation

$$\ln \frac{d\mathbb{P}_{T,\theta_0+A_T h}}{d\mathbb{P}_{T,\theta_0}}(\mathbf{X}_T) = \Delta_T(h,\theta_0) - \frac{1}{2}||h||_2^2 + \phi(T,h,\theta_0) \qquad (4.1)$$

  is valid, where
  (1) $\Delta_T(h) = \Delta_T(h,\theta_0)$ is a random linear function on H_0 which is asymptotically $N(0, \|h\|_2^2)$-normal for any $h \in H_0$ and
  (2) $\phi(T,h,\theta_0) \to 0$ as $T \to \infty$ in $\mathbb{P}_{T,\theta_0}$-probability.

- **Definition.** We say that a pair of functions (f, g) satisfies condition $(\mathcal{H}_1)$, if $f \in \Sigma_p(\beta)$, $1 < p \le 2$, $\beta > 1/p$, and $g \in L^q$, $1/p + 1/q = 1$.
- The parametric set $\mathbf{\Theta}$ is assumed to be a subset of L^p $(p \ge 1)$ consisting of spectral densities belonging to $\mathcal{A}_2 \cap \Sigma_p(\beta)$.
- Define H_0 to be the linear manifold consisting of bounded functions $h(\lambda)$ such that the pair $(\theta, h\theta^{-1})$ satisfies condition $(\mathcal{H}_1)$.
- Define $A_T : L^2 \to L^2$ by $A_T h = [T^{-1/2}\theta] \cdot h$.
- **Theorem 1.** Let $\mathbf{\Theta}$, H_0 and A_T be defined as above. Then the family $\{\mathbb{P}_{T,\theta}, \theta \in \mathbf{\Theta}\}$ satisfies LAN at any point $\theta \in \mathbf{\Theta}$ in the direction L^2 with norming factors A_T and

$$\Delta_T(h) = \frac{T^{1/2}}{4\pi} \int_{-\infty}^{\infty} \left[\frac{I_T(\lambda)}{\theta(\lambda)} - 1\right] h(\lambda)\, d\lambda, \tag{4.2}$$

where

$$I_T(\lambda) = \frac{1}{2\pi T} \left| \int_0^T X(u) e^{-iu\lambda}\, du \right|^2 \tag{4.3}$$

is the periodogram of the process $X(t)$.
- **Remark.** Solev & Zerbet (2003), Ginovian (1999, 2003).

- Consider the problem of estimating a functional $\Phi(\cdot)$ at an unknown point $\theta \in \mathbf{\Theta}$ on the basis of $\mathbf{X}_T$, which has distribution $\mathbb{P}_{T,\theta}$. We assume that the family $\{\mathbb{P}_{T,\theta},\ \theta \in \mathbf{\Theta}\}$ satisfies LAN at $\theta_0 = f \in \mathbf{\Theta}$, and $\Phi(\theta)$ is Fréchet differentiable with derivative $\Phi'(\theta)$ satisfying

$$0 < ||\Phi'(f)f||_2 < \infty. \tag{5.1}$$

- Let $\widehat{\Phi}_T$ be an estimator of $\Phi(\theta)$. To state a variant of Hájek–Ibragimov–Khas'minskii convolution theorem we need the following
- **Definition.** An estimator $\widehat{\Phi}_T$ of $\Phi(\theta)$ is called H_0–regular at $\theta_0 \in \mathbf{\Theta}$, if for any $h \in H_0$ there exists a proper limit distribution function F of $T^{1/2}\left(\widehat{\Phi}_T - \Phi(\theta_h)\right)$, where $\theta_h = \theta_0 + A_T h$, and this limit distribution does not depend on h, that is, we have the following weak convergence

$$\mathcal{L}\left\{T^{1/2}\left(\widehat{\Phi}_T - \Phi(\theta_h)\right)\Big|\,\mathbb{P}_{T,\theta_h}\right\} \Longrightarrow F \quad \text{as} \quad T \to \infty.$$

The next result follows from Th. 1 and Th. 3.1 in Ibragimov-Khas'minskii (1991).

- **Theorem 2.** Let $\widehat{\Phi}_T$ be a H_0–regular estimator of $\Phi(\theta)$ at $f \in \mathbf{\Theta}$. Let the SD $f(\lambda)$ and $\Phi(\cdot)$ be such that the pair $(f, \Phi'(f))$ satisfies (5.1). Then under the assumptions of Theorem 1 the limit distribution F of $T^{1/2}\left(\widehat{\Phi}_T - \Phi(f)\right)$ is a convolution of some probability distribution G and a centered normal distribution with variance $||\Phi'(f)f||_2^2$:

$$F = N(0, ||\Phi'(f)f||_2^2) * G. \quad (5.2)$$

- By a well–known lemma of Anderson the distribution F in (5.2) is less concentrated in symmetric intervals than $N(0, ||\Phi'(f)f||_2^2)$. This justifies the following definition of H_0–efficiency.
- **Definition.** Let $\{\mathbb{P}_{T,\theta},\ \theta \in \mathbf{\Theta}\}$ be LAN at $f \in \mathbf{\Theta}$. An estimator $\widehat{\Phi}_T$ of $\Phi(\theta)$ is called H_0–*asymptotically efficient* at f (in the class of H_0–regular estimators) with asymptotic variance $\sigma^2 = ||\Phi'(f)f||_2^2$, if

$$\mathcal{L}\left\{T^{1/2}\left(\widehat{\Phi}_T - \Phi(\theta_h)\right)\Big|\,\mathbb{P}_{T,\theta_h}\right\} \Longrightarrow N(0,\sigma^2) \quad \text{as} \quad T \to \infty,$$

that is, the distribution G in (5.2) is degenerate.

- Denote by $\mathbf{\Phi}_T$ the set of all estimators of $\Phi(\theta)$ constructed on the basis of $\mathbf{X}_T$, and by $\mathbf{W}$ the set of loss functions $w : \mathbb{R}^1 \to \mathbb{R}^1$, which are symmetric and non–decreasing on $(0, \infty)$, and satisfy $w(x) \geq 0$, $w(0) = 0$.
- The next result, which follows from of Th. 1 and Th. 4.1 in Ibragimov-Khas'minskii (1991), contains a minimax lower bound for the risk of all possible estimators $\widehat{\Phi}_T$ of $\Phi(\cdot)$ in a vicinity of $f \in \mathbf{\Theta}$.
- **Theorem 3.** Let $\mathbf{\Phi}_T$ and $\mathbf{W}$ be defined as above. Let the SD $f(\lambda)$ and the functional $\Phi(\cdot)$ be such that the pair $(f, \Phi'(f))$ satisfies $(\mathcal{H}_1)$ and (5.1). Then under the assumptions of Th. 1, we have for all $w \in \mathbf{W}$,

$$\liminf_{\delta\to 0} \lim_{T\to\infty} \inf_{\widehat{\Phi}_T \in \mathbf{\Phi}_T} \sup_{||\theta-f||_2<\delta} \mathbb{E}_\theta\{w(T^{1/2}(\widehat{\Phi}_T - \Phi(f)))\} \geq \mathbb{E}w(\xi) \quad (6.1)$$

where $\xi \sim N\left(0, ||\Phi'(f)f||_2^2\right)$, and $\mathbb{E}_\theta\{\cdot\}$ stands for the expectation w.r.t. measure corresponding to SD $\theta(\lambda)$.

- The next definition, which is based on Theorem 3, defines the notion of asymptotically efficient estimators in the spirit of Ibragimov and Khas'minskii (IK–efficiency) (1986, 1991).
- **Definition.** Let the family $\{\mathbb{P}_{T,\theta},\, \theta \in \mathbf{\Theta}\}$ satisfy LAN at $f \in \mathbf{\Theta}$. An estimator $\widehat{\Phi}_T$ of $\Phi(\theta)$ is called *IK–asymptotically efficient* at f for a loss $w(x) \in \mathbf{W}$, with asymptotic variance $\sigma^2 = ||\Phi'(f)f||_2^2$, if

$$\liminf_{\delta\to 0} \lim_{T\to\infty} \sup_{||\theta-f||_2<\delta} \mathbb{E}_\theta\{w(T^{1/2}(\widehat{\Phi}_T - \Phi(f)))\} = \mathbb{E}w(\xi), \quad (6.2)$$

  where $\xi \sim N\left(0, ||\Phi'(f)f||_2^2\right)$.
- **Remark.** Both definitions of efficiency, roughly speaking, require from the asymp. efficient estimator $\widehat{\Phi}_T$ the uniformity (local) of the convergence of $T^{1/2}\left(\widehat{\Phi}_T - \Phi(f)\right)$ to a RV $\xi \sim N\left(0, ||\Phi'(f)f||_2^2\right)$, and for bounded loss functions $w(\cdot)$ they are rather close.
  The difference: the Def. 4 compares only regular estimators, while the Def. 5 compares all estimators constructed on the basis of $\mathbf{X}_T$.

- Assume that the functional $\Phi(f)$ to be estimated is linear and continuous in $L^p(\mathbb{R})$, $p > 1$. Then $\Phi(f)$ admits the representation

$$\Phi(f) = \int_{\mathbb{R}} f(\lambda) g(\lambda) d\lambda, \tag{7.1}$$

where $g(\lambda) \in L^q(\mathbb{R})$; $1/p + 1/q = 1$. As an estimator for $\Phi(f)$ we consider the averaged periodogram statistics:

$$\widehat{\Phi}_T = \Phi(I_T) = \int_{\mathbb{R}} I_T(\lambda) g(\lambda) d\lambda. \tag{7.2}$$

Denote $\mathbf{W}_e = \{w \in \mathbf{W} : \ w(x) \leq C_1 \exp\{C_2|x|\}, \ C_1 > 0, \ C_2 > 0\}$.

- **Theorem 4.** Let $\Phi(f)$ and $\widehat{\Phi}_T$ be defined by (7.1) and (7.2). Assume that the pair (f, g) satisfies the conditions $(\mathcal{H}_1)$ and $0 < ||fg||_2 < \infty$. Then under the assump's of Th. 1 the statistics $\widehat{\Phi}_T$ is:
a) H_0–regular and H_0–AEE of $\Phi(f)$ with asymptotic variance $||fg||_2^2$;
b) IK–AEE of $\Phi(f)$ for $w(x) \in \mathbf{W}_e$ with asymptotic variance $||fg||_2^2$.

- The problem of AE estimation becomes somewhat more complicated for non-linear functionals. In this case the statistics $\Phi(I_T)$ is not necessary a consistent estimator for $\Phi(f)$, and hence instead of the periodogram $I_T(\lambda)$ we need to use a suitable sequence of consistent estimators $\widehat{f}_T$ of f.
  On the other hand, if $\widehat{f}_T$ is a sequence of consistent estimators for SD f, the estimators $\Phi(\widehat{f}_T)$, in general, will converge to $\Phi(f)$ too slowly to be AE (Ibragimov - Khasminskii (1986)).
- We consider a sequence $\{\widehat{f}_T\}$ of estimators for f which are consistent of order of $T^{2\alpha}$ $(0 < \alpha \leq 1/2)$, and derive conditions under which the statistics $\widehat{\Phi}_T = \Phi(\widehat{f}_T)$ is AEE for $\Phi(f)$.
- Recall that an estimator $\widehat{f}_T$ of f is called $T^{2\alpha}$–consistent with asymptotic variance σ^2, if (Parzen (1957))

$$\lim_{T\to\infty} T^{2\alpha}\mathbb{E}(\widehat{f}_T - f)^2 = \sigma^2.$$

- We assume $f \in \Sigma_p(\beta)$, and as an estimator for unknown f consider the statistics $\widehat{f}_T$ given by

$$\widehat{f}_T(\lambda) = \int_{-\infty}^{\infty} W_T(\lambda - \mu) I_T(\mu)\, d\mu. \tag{8.1}$$

For the kernel $W_T(\lambda)$ we set down the following assumptions.

- **K1.** $W_T(\lambda) = M_T W(M_T \lambda)$, where $M_T = O\left(T^{\alpha}\right)$. The choice of α $(0 < \alpha < 1)$ depends on the appriori knowledge about f and Φ.
- **K2.** $W(\lambda) \geq 0$ is bounded, even with $W(\lambda) \equiv 0$ for $|\lambda| > 1$ and

$$\int_{-1}^{1} W(\lambda)\, d\lambda = 1, \quad \int_{-1}^{1} \lambda^k W(\lambda)\, d\lambda = 0, \quad k = 1, 2, \ldots, r,$$

where $r = [\beta]$ is the integer part of β.

- We assume the functional $\Phi(\cdot)$ to be Fréchet differentiable with derivative $\Phi'(\cdot)$ satisfying (5.1) and a Hölder condition: there exist positive constants C and δ such that for any $f_1, f_2 \in L^2$,

$$||\Phi'(f_1) - \Phi'(f_2)|| \leq C||f_1 - f_2||_2^{\delta}. \tag{8.2}$$

- **Theorem 5.** Let the SD $f(\lambda)$ and the functional $\Phi(\cdot)$ be such that:
(i) the pair $(f, \Phi'(f))$ satisfies conditions $(\mathcal{H}_1)$ and (5.1);
(ii) $\Phi(\cdot)$ satisfies condition (8.2) with $\delta \geq (2\beta - 1)^{-1}$.
Let the estimator $\widehat{f}_T$ for f be defined by (8.1) with the kernel $W_T(\lambda)$ satisfying assumptions K1 and K2 with $\frac{1}{2\beta} < \alpha < \frac{\delta}{\delta+1}$.
Then under the conditions given in Th. 1 the statistics $\Phi(\widehat{f}_T)$ is:
a) H_0–regular and H_0–AEE for $\Phi(f)$ with asymp. variance $||\Phi'(f)f||_2^2$;
b) IK-AEE of $\Phi(f)$ for $w(x) \in \mathbf{W}_e$ with asymp. variance $||\Phi'(f)f||_2^2$.

- We return to the problem of estimation of linear, continuous in L^p functional $\Phi(f)$. If $p \leq 2$ the functional $\Phi(f)$ is continuous in L^2, and so we can apply Theorem 4 to construct an AEE for $\Phi(f)$. If $p > 2$ the right hand side of (6.1) goes to infinity, and it becomes of interest to estimate the rate of decrease of the minimax risk:

$$\Delta_T(\mathbf{\Phi}_T, \Sigma, w) = \inf_{\widehat{\Phi}_T \in \mathbf{\Phi}_T} \sup_{f \in \Sigma} \mathbb{E}_f\{w(\widehat{\Phi}_T - \Phi(f))\},$$

where Σ is a given class of spectral densities and $\mathbf{\Phi}_T$ is the set of all estimators of $\Phi(f)$ constructed on the basis of an observation $\mathbf{X}_T$. The next theorem gives exact asymptotic bounds for the risk $\Delta_T^2 = \Delta_T(\mathbf{\Phi}_T, \Sigma, w)$ with $w(x) = |x|^2$.

- **Theorem 6.** Assume that $\Sigma = \Sigma_p(\beta)$ and let the linear functional $\Phi(f)$ be continuous in L^p $(p > 2)$. Then as $T \to \infty$ we have

$$\Delta_T^2 \asymp \begin{cases} T^{(-2p\beta)/(p+2p\beta-2)} & \text{for } \beta > 1/p, \\ T^{-2\beta} & \text{for } \beta \leq 1/p \end{cases}$$

- **Lemma 1.** Assume that the family $\{\mathbb{P}_{T,\theta},\, \theta \in \mathbf{\Theta}\}$ is LAN at $f \in \mathbf{\Theta}$. Then an estimator $\widehat{\Phi}_T$ of $\Phi(f)$ is H_0–regular and H_0–AEE at f with asymptotic variance $||f\Phi'(f)||_2^2$ if and only if

$$T^{1/2}[\widehat{\Phi}_T - \Phi(f)] - \Delta_T(f\Phi'(f)) = o_P(1) \quad \text{as} \quad T \to \infty,$$

where $\Delta_T(f\Phi'(f))$ is defined by (4.2) with $h = f\Phi'(f)$.

- **Lemma 2.** Let $\Phi(f)$ and $\widehat{\Phi}(I_T)$ be defined by (7.1) and (7.2), and let the pair (f, g) satisfies the conditions $(\mathcal{H}_1)$ and $0 < ||fg||_2 < \infty$. Then for $w(x) \in \mathbf{W}_e$

$$\lim_{T\to\infty} \mathbb{E}_f\{w(T^{1/2}(\widehat{\Phi}(I_T) - \Phi(f)))\} = \mathbb{E}w(\xi),$$

where ξ is a centered normal RV with variance $||fg||_2^2$.

- **Lemma 3.** Assume that $f \in \Sigma_p(\beta)$, and let $\widehat{f}_T$ for f be defined by (8.1) with the kernel $W_T(\lambda)$ satisfying assumptions K1 and K2 with $\frac{1}{2\beta} < \alpha < \frac{\delta}{\delta+1}$, then

$$T^{1/2}||\widehat{f}_T - f||_2^{1+\delta} = o_P(1) \qquad \text{as} \quad T \to \infty.$$

- **Lemma 4.** Assume that $f \in \Sigma_p(\beta)$, and the assumptions K1 and K2 are satisfied. Let $\psi(\lambda)$ be a continuous even function such that the pair (f, ψ) satisfies the conditions $(\mathcal{H}_1)$ and $0 < ||f\psi||_2 < \infty$. Then the distribution of the random variable

$$\eta_T = T^{1/2} \int_{-\infty}^{\infty} \psi(\lambda) \left[\widehat{f}_T(\lambda) - f(\lambda)\right] d\lambda.$$

as $T \to \infty$ tends to the normal distribution $N(0, \sigma^2)$, where

$$\sigma^2 = 4\pi \int_{-\infty}^{\infty} \psi^2(\lambda) f^2(\lambda)\, d\lambda.$$

An Example.
Consider the problem of estimation of the integrated squared spectral density functional $\Phi(f)$:

$$\Phi(f) = \int_{-\infty}^{\infty} f^2(\lambda)\, d\lambda. \tag{11.1}$$

It follows from Theorems 4 and 5 that the statistics

$$\widehat{\Phi}_T = \Phi(\widehat{f}_T) = \int_{-\infty}^{\infty} \widehat{f}_T^2(\lambda)\, d\lambda$$

is H_0– and IK–asymptotically efficient estimator for functional (11.1) with asymptotic variance $||f^2||_2^2$.

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