Estimating discontinuous periodic signals in a time inhomogeneous diffusion

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introduction: a statistical problem in a time inhomogeneous diffusion

observation: one-dimensional diffusion $(X_t)_{t\geq 0}$

$$dX_t = [S(\vartheta, t) + b(X_t)]\, dt + \sigma(X_t) dW_t \quad , \quad t \geq 0$$

observed continuously in time, over a long time interval

drift involves a time dependent deterministic input

$$S(\vartheta, t) = \left(\lambda + 1_{(\vartheta, \vartheta+a)} \lambda^*\right)(i_T(t)) \quad , \quad i_T(t) := t \text{ modulo } T$$

where $t \to \lambda(t)$, $t \to \lambda^*(t)$ are known continuous T-periodic functions

view function $\lambda^* > 0$ as additional signal occuring periodically on intervals

$$(kT+\vartheta, kT+\vartheta+a) \quad , \quad k \in \mathbb{N}_0$$

fixed duration a : $\hookrightarrow$ one-dimensional unknown parameter $\vartheta \in \Theta$ where

$$\Theta \;:=\; (0, T-a)$$

assume suitable 'ergodicity conditions' (??, ex.: OU with additional $S(\vartheta, t) dt$)

Q^ζ law of the time inhomogeneous diffusion

$$dX_t = [S(\zeta,t) + b(X_t)]\,dt + \sigma(X_t)dW_t \quad , \quad X_0 \equiv x_0$$

on canonical path space $(C, \mathcal{C}, \mathbb{G})$, $C = C([0,\infty), \mathbb{R})$, can. proc. $\eta = (\eta_t)_{t\geq 0}$

$L^{\zeta'/\zeta}$ likelihood ratio process of $Q^{\zeta'}$ to Q^ζ relative $\mathbb{G}$, assuming $\sigma(\cdot) > 0$:

for ζ' sufficiently close to ζ (here in case $\zeta' < \zeta < \zeta' + a$, similiar else)

$$\begin{aligned} L_t^{\zeta'/\zeta} \quad = \quad \exp\Bigg\{ & \left[\int_0^t \frac{\lambda^*(s)}{\sigma(\eta_s)} 1_{(\zeta',\zeta)}(i_T(s))\,dB_s \;-\; \frac{1}{2}\langle\ldots\rangle_t\right] \\ & + \left[\int_0^t \left(-\frac{\lambda^*(s)}{\sigma(\eta_s)}\right) 1_{(\zeta'+a,\zeta+a)}(i_T(s))\,dB_s \;-\; \frac{1}{2}\langle\ldots\rangle_t\right] \Bigg\} \end{aligned}$$

where B is a $\mathbb{G}$–Brownian motion under all Q^ζ, $\zeta \in \Theta$

localization: fix ϑ, consider as $n \to \infty$ $\;L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta}$ in 'local parameter' u

localization heuristics: $\log L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta}$ under Q^ϑ as $n \to \infty$

angle bracketts of martingale terms (case $u<0$, similiar for $u>0$):

$$\int_0^{nT} \left(\frac{\lambda^*(s))}{\sigma(\eta_s)} \right)^2 \underbrace{1_{(\vartheta+\frac{u}{n},\vartheta)}(i_T(s))}_{\text{case } u<0} \, ds \quad \approx \quad \sum_{k=0}^{n-1} \left(\frac{\lambda^*(kT+\vartheta))}{\sigma(\eta_{kT+\vartheta})} \right)^2 \cdot \frac{|u|}{n}$$

$$= \ |u| \cdot \left[\frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\sigma^2(\eta_{kT+\vartheta})} \right] (\lambda^*(\vartheta))^2 \quad \stackrel{(??)}{\longrightarrow} \quad |u| \cdot c(\vartheta, \sigma^2)$$

need suitable SLLN (??) adapted to the T-periodic structure of the drift
need suitable control on the fluctuations in $t \longrightarrow \eta_t$
martingale terms:

$$\int_0^{nT} \frac{\lambda^*(s)}{\sigma(\eta_s)} \underbrace{1_{(\vartheta+\frac{u}{n},\vartheta)}(i_T(s))}_{\text{case } u<0} \, dB_s \quad \approx \quad \sum_{k=0}^{n-1} \frac{\lambda^*(kT+\vartheta))}{\sigma(\eta_{kT+\vartheta})} \left(B_{kT+\vartheta} - B_{kT+\vartheta+\frac{u}{n}} \right)$$

$$\stackrel{d}{=} \ \mathcal{N}\left(0\,,\ |u| \cdot \left[\frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\sigma^2(\eta_{kT+\vartheta})} \right] (\lambda^*(\vartheta))^2 \right) \quad \longrightarrow \quad \mathcal{N}\left(0\,, |u| \cdot c(\vartheta, \sigma^2) \right)$$

for local models $\left\{ Q^{\vartheta + \frac{u}{n}} \mid \mathcal{G}_{nT} \;:\; u \in \mathbb{R} \right\}$ at points $\vartheta \in \Theta$

expect convergence as $n \to \infty$ to a limit model $\widetilde{\mathcal{E}} := \left\{ \widetilde{P}_u \;:\; u \in \mathbb{R} \right\}$ s.t.

$$\widetilde{L}^{u/0} := \frac{d\widetilde{P}_u}{d\widetilde{P}_0} = \exp\left\{ \widetilde{W}_u - \frac{1}{2}|u| \right\} \;,\; u \in \mathbb{R}$$

with double sided standard BM $\widetilde{W} = (\widetilde{W}_u)_{u \in \mathbb{R}}$, up to scaling cst. $c(\vartheta, \sigma^2)$

$\widetilde{\mathcal{E}}$ has been studied by Ibragimov and Khasminskii (1981) and others
some unusual properties:

- likelihood is Hölder-$\frac{1+\varepsilon}{2}$ in the parameter, $\varepsilon > 0$
- Hellinger dist. $H(\widetilde{P}_{u'}, \widetilde{P}_u) = \left(1 - e^{-\frac{1}{8}|u'-u|}\right)^{\frac{1}{2}} \sim \sqrt{\frac{1}{8}|u'-u|}$ as $u' \to u$
- BE (Bayes w.r.t. squared loss) and MLE are equivariant estimators for u
- MLE variance is $= 26$ (Ibragimov and Khasminskii 1981)
- BE variance is ≈ 19.3 (Golubev 1979, Rubin and Song 1995)
- no sufficient statistic except the full observation $(\widetilde{W}_u)_{u \in \mathbb{R}}$ ...

striking contrast to world of quadratic experiments
or L^2-differentiable experiments ($\hookrightarrow$ LAN)
well studied since LeCam (1968), Hájek (1970), LeCam and Yang (1999)

$\widetilde{\mathcal{E}}$ occurred as a limit model e.g. the following contexts:
signal in white noise problem (special case $\sigma(\cdot) \equiv 1$, $b(\cdot) \equiv 0$ of SDE $(*)$, Ibragimov and Khasminskii 1981)
iid changepoint problem (Deshayes and Picard 1984)
delay equations (Küchler and Kutoyants 2000)
spatial discontinuity in the drift of an ergodic diffusion (Kutoyants 2004)
changepoint problem in discretely observed diffusions (Yoshida, ongoing work)

main problem in our context: have to prove convergence of experiments / convergence of estimators; need new and different probabilistic tools

- notion of 'ergodicity' for diffusions X with T-periodic semigroup
- limit theorems for martingales and certain functionals of $X = (X_t)_{t \geq 0}$ (which are not additive functionals of X)
- control of 'fluctuations' via exponential inequalities ($\hookleftarrow$ Brandt 2005, Dzhaparidze and vanZanten 2003)

on a probability space carrying independent standard BM's $\widetilde{W}^+$ and $\widetilde{W}^-$, define double sided BM $\widetilde{W}$ by $\widetilde{W}^+_u$ if $u \geq 0$, and by $\widetilde{W}^-_{|u|}$ if $u \leq 0$.

on 'bivariate' path space $C([0,\infty),\mathbb{R}^2)$, define laws

$$\widetilde{P}_u \;:=\; \begin{cases} \mathcal{L}\left(\left(\widetilde{W}^+_v + v \wedge u \,,\; \widetilde{W}^-_v \right)_{v \geq 0} \mid \widetilde{P}_0 \right) & \text{in case } \; u \geq 0 \\ \mathcal{L}\left(\left(\widetilde{W}^+_v \,,\; \widetilde{W}^-_v + v \wedge |u| \right)_{v \geq 0} \mid \widetilde{P}_0 \right) & \text{in case } \; u \leq 0 \end{cases}$$

observation over the *infinite* time interval $[0,\infty)$ $\hookrightarrow$ likelihood ratios (use Liptser and Shiryaev 1981, Jacod and Shiryaev 1987)

$$\widetilde{L}^{u/0} \;=\; \frac{d\widetilde{P}_u}{d\widetilde{P}_0} \;=\; \exp\left\{ \widetilde{W}_u - \frac{1}{2}|u| \right\} \quad , \quad u \in \mathbb{R}$$

interpretation 'switch known drift off at unknown time' of experiment $\widetilde{\mathcal{E}}$:

- add constant drift 1 to $\widetilde{W}^+$ if $u > 0$ and to $\widetilde{W}^-$ if $u < 0$
- switch this drift off at time $|u|$, $u \in \mathbb{R}$

diffusions with T-periodic semigroup

consider time inhomogeneous diffusion (ϑ fixed + suppressed from notation)

$$(*) \qquad dX_t \;=\; [S(t) + b(X_t)]\,dt + \sigma(X_t)dW_t \quad , \quad t \geq 0$$

$t \longrightarrow S(t)$ deterministic, piecewise continuous, T-periodic

semigroup $(P_{s,t})_{s<t}$ of transition probabilities, with Lebesgue densities $(p_{s,t})_{s<t}$ time inhomogeneous and T-periodic in the sense that

$$p_{s,t}(x,y) \;=\; p_{kT+s,kT+t}(x,y) \;, \quad k \in \mathbb{N}_0 \;.$$

question: asymptotics as $t \to \infty$ of class of functionals of the process $(X_t)_{t\geq 0}$ more general than the class of additive functionals: need

$$A = (A_t)_{t\geq 0} \quad , \quad A_t \;=\; \int_0^t f(X_s)\,\Lambda_T(ds) \;,\; t \geq 0$$

for measures $\Lambda_T(ds)$ which are σ-finite and T-periodic:

$$\Lambda_T(B) \;=\; \Lambda_T(B + kT) \quad \text{for all } k \in \mathbb{Z},\, B \in \mathcal{B}(\mathbb{R}) \;.$$

example 1 : for $0 \leq r < r' < T$ fixed consider

$$A_t = \sum_{k \in \mathbb{N}_0, kT+r \leq t} f(X_{kT+r}) \quad , \quad \Lambda_T(ds) = \sum_{k \in \mathbb{Z}} \epsilon_{kT+r}(ds)$$

$$A_t = \int_0^t f(X_s) 1_{(r,r')}(i_T(s))\, ds \quad , \quad \Lambda_T(ds) = \sum_{k \in \mathbb{Z}} 1_{(kT+r, kT+r')}(s)\, ds$$

main idea: obtain limit theorems for functionals $A = (A_t)_{t \geq 0}$ of this type through the chain $\mathbb{X}$ of T-segments in the path of $(X_t)_t$:

$$\mathbb{X} := (\mathbb{X}_k)_k \quad : \quad \mathbb{X}_k := (X_{(k-1)T+s})_{0 \leq s \leq T} \;,\; k = 1, 2, \ldots$$

discrete-time Markov chain, time-homogeneous, $C([0,T], \mathbb{R})$-valued theorem 1 : (ergodicity property for diffusions with T-periodic semigroup)

$$(X_{kT})_k \text{ on } \mathbb{R} \text{ positive Harris } \Longrightarrow (\mathbb{X}_k)_k \text{ on } C([0,T],\mathbb{R}) \text{ positive Harris}$$

invariant probabilities :

$$\mu \text{ on } \mathbb{R} \text{ for } (X_{kT})_k \ , \ \ m \text{ on } C([0,T],\mathbb{R}) \text{ for } (\mathbb{X}_k)_k$$

where m is determined from μ and $(p_{s,t})_{0\le s<t\le T}$; we assume from now on

$$(X_{kT})_k \text{ is positive Harris with invariant probability } \mu \text{ on } \mathbb{R} \ .$$

theorem 2 : for every $\mathbb{F}^X$-increasing process $A=(A_t)_{t\ge 0}$ with the property

$$\left\{ \begin{array}{l} \exists \text{ some } F : C([0,T],\mathbb{R}) \to \mathbb{R}, \text{ measurable, in } L^1(m), \text{ nonnegative, s.t.} \\ A_{kT} \ = \ \sum_{j=1}^k F(\mathbb{X}_k) \ = \ \sum_{j=1}^k F\left((X_{(k-1)T+s})_{0\le s\le T} \right) \ , \ \ k=1,2,\ldots \end{array} \right.$$

we have

$$\lim_{t\to\infty} \frac{A_t}{t} \ = \ \frac{m(F)}{T} \ \text{ almost surely .}$$

example 1 continued : consider $f:\mathbb{R}\to\mathbb{R}$ measurable, then

$$\lim_{t\to\infty}\frac{1}{t}\sum_{k\in\mathbb{N}_0\,,\,kT+r\le t} f\left(X_{kT+r}\right) \;=\; \frac{(\mu P_{0,r}f)}{T}$$

almost surely if $f\in L^1\left(\mu P_{0,r}\right)$,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X_s)\,1_{(r,r')}(i_T(s))\,ds \;=\; \frac{1}{T}\int_0^T\int_r^{r'}(\mu P_{0,s}f)\,ds$$

almost surely if $f\in L^1\left(\int_r^{r'}(\mu P_{0,s})ds\right)$, and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f\left(X_{kT+s}\right)\Lambda_T(ds) \;=\; \frac{1}{T}\int_0^T(\mu P_{0,s}f)\Lambda_T(ds)$$

almost surely if $f\in L^1\left(\int_0^T \Lambda_T(ds)(\mu P_{0,s})\right)$, for T-periodic measure $\Lambda_T(ds)$.

example 2 : for arbitrary input $S : \mathbb{R} \to \mathbb{R}$ piecewise cont. and T-periodic, for $\sigma > 0$, $\gamma > 0$, the OU type diffusion

$$dX_t \;=\; (S(t) - \gamma X_t)\, dt + \sigma dW_t \;, \quad t \geq 0$$

has all 'ergodicity properties' described above: $(X_{kT})_k$ is positive Harris;

$$\mu \;=\; \mathcal{N}\left(\int_0^\infty e^{-\gamma v} S(-v) dv \;,\; \frac{\sigma^2}{2\gamma} \right)$$

is invariant for $(X_{kT})_k$; for every $0 < s < T$ we have

$$\mu P_{0,s} \;=\; \mathcal{N}\left(\int_0^\infty e^{-\gamma v} S(s-v) dv \;,\; \frac{\sigma^2}{2\gamma} \right)$$

(s-marginal of the invariant measure m on $C([0,T],\mathbb{R})$), and thus know

$$\int_0^T (\mu P_{0,s}) \Lambda_T(ds) \;=\; \mathcal{N}\left(\int_0^\infty e^{-\gamma v}\, \widehat{S}_\Lambda(-v)\, dv \;,\; \frac{\sigma^2}{2\gamma} \right)$$

with notation $\widehat{S}_\Lambda(\cdot) := \int_0^T S(s+\cdot)\Lambda_T(ds)$, for T-periodic measures $\Lambda_T(ds)$

convergence of local models: main limit theorem

assume from now on

- $\sigma(\cdot)$ bounded away from 0 and ∞, Lipschitz conditions on $b(\cdot)$ and $\sigma(\cdot)$
- $(X_{kT})_k$ is positive Harris recurrent with invariant measure μ on $\mathbb{R}$

to study asymptotics of log-likelihood ratios $L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta}$ in local models at ϑ

fluctuations of $(X_t)_{t\geq 0}$ solution to $(*)$ over small time intervals of length Δ controlled via exponential inequality due to Brandt (PhD thesis 2005):

lemma 1 : fix $0 < \lambda < \frac{1}{2}$ and $\frac{1}{2} < \eta < 1-\lambda$; then there is some Δ_0 such that

$$P\left(\sup_{t_1 \leq t \leq t_1+\Delta} |X_t - X_{t_1}| > \Delta^\lambda \, , \, |X_{t_1}| \leq (\frac{1}{\Delta})^\eta \right) \;\leq\; c_1 \, \exp\{-c_2 \, (\frac{1}{\Delta})^{1-2\lambda}\}$$

for all $0 < \Delta \leq \Delta_0$ and all $0 \leq t_1 < \infty$.

asymptotics: angle brackett terms in the log-likelihood ratios $L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta}$:

theorem 3A : for arbitrary $0 < r < T$ and $h > 0$ fixed

$$\int_0^{nT} \frac{1}{\sigma^2(X_s)} 1_{(r-\frac{h}{n},r)}(i_T(s))\, ds \;=\; h \underbrace{\frac{1}{n}\sum_{j=0}^{n-1} \frac{1}{\sigma^2(X_{jT+r})}}_{\longrightarrow (\mu P_{0,r})(\frac{1}{\sigma^2})} + o_P(1)$$

$$\int_0^{nT} \frac{1}{\sigma^2(X_s)} 1_{(r,r+\frac{h}{n})}(i_T(s))\, ds \;=\; h \overbrace{\frac{1}{n}\sum_{j=0}^{n-1} \frac{1}{\sigma^2(X_{jT+r})}} + o_P(1)$$

as $n \to \infty$; recall $i_T(s) = s$ modulo T.

martingale terms in the likelihoods $L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta}$:

theorem 3B : for different points $r_j, r_{j'}, \ldots$ in $(0,T)$, there are indep. BM's

$$\widetilde{B}^{(r_j,+)}, \widetilde{B}^{(r_{j'},+)}, \ldots, \widetilde{B}^{(r_j,-)}, \widetilde{B}^{(r_{j'},-)}, \ldots$$

such that weak convergence in $\mathbb{R}$ holds as $n \to \infty$ for fixed $h > 0$:

$$\int_0^{nT} \frac{1}{\sigma(X_s)} 1_{(r_j - \frac{h}{n}, r_j)}(i_T(s))\, dB_s \longrightarrow \widetilde{B}^{(r_j,+)}\left(h\, (\mu P_{0,r_j})(\frac{1}{\sigma^2}) \right)$$

$$\int_0^{nT} \frac{1}{\sigma(X_s)} 1_{(r_{j'}, r_{j'} + \frac{h}{n})}(i_T(s))\, dB_s \longrightarrow \widetilde{B}^{(r_{j'},-)}\left(h\, (\mu P_{0,r_{j'}})(\frac{1}{\sigma^2}) \right)$$

important: on the right hand side, $h \geq 0$ begins to play the role of time !!

thm extends to convergence of finite dimensional distributions in time $h \geq 0$

proof of theorems 3A+B: from lemma 1 and theorem 2, note the following:

in order to apply the martingale CLT with $0 \le h \le H$ playing the role of time, cf. Jacod and Shiryaev 1987, replace – thanks to lemma 1 – contributions

$$\frac{1}{\sigma(X_s)}\, 1_{(kT+r_j-\frac{h}{n},kT+r_j)}(s)\, dB_s \quad , \quad \frac{1}{\sigma(X_s)}\, 1_{(kT+r_{j'},kT+r_{j'}+\frac{h'}{n})}(s)\, dB_s$$

to the stochastic integrals of theorem 3B by

$$\frac{1}{\sigma(X_{kT+r_j-\frac{H}{n}})}\, 1_{(kT+r_j-\frac{h}{n},kT+r_j)}(s)\, dB_s \quad , \quad \frac{1}{\sigma(X_{kT+r_{j'}-\frac{H}{n}})}\, 1_{(kT+r_{j'},kT+r_{j'}+\frac{h'}{n})}(s)\, dB_s$$

and introduce a filtration $\mathbb{H}^n = (\mathcal{H}^n_h)_{0\le h\le H}$ where $\mathcal{H}^n_h$ is generated from

$$X_{s_0}\, ,\, (B_{s_2}-B_{s_1}) \;:\; \left\{ \begin{array}{l} s_0 = kT+r_j-\frac{H}{n} \;,\; kT+r_j-\frac{h}{n} \le s_2 < s_1 \le kT+r_j+\frac{h}{n} \\ k=0,1,\ldots,n-1 \;,\;\; \text{different points } r_j, r_{j'}, \ldots \end{array} \right\}$$

then the martingale property in time $0 \le h \le H$, linked to increasing intervals, is obtained from independent increments in the BM B

with these tools: back to the statistical problem

recall from above: $\Theta = (0, T-a)$, for $\zeta \in \Theta$: Q^ζ is the law of

$$dX_t = [S(\zeta, t) + b(X_t)]\, dt + \sigma(X_t) dW_t \quad , \quad X_0 \equiv x_0$$

under Lipschitz conditions on $b(\cdot)$ and $\sigma(\cdot)$, with

$$S(\zeta, t) = \left(\lambda + 1_{(\zeta, \zeta+a)} \lambda^*\right)(i_T(t))$$

for $\lambda(\cdot)$, $\lambda^*(\cdot)$ continuous and T-periodic; assume always

- for all $\zeta \in \Theta$, $(X_{kT})_k$ is pos. Harris rec. with invariant $\mu^{(\zeta)}$ on $\mathbb{R}$
- $\sigma(\cdot)$ is bounded away from 0 and ∞

work on canonical path space $(C, \mathcal{C}, \mathbb{G})$, $C = C([0,\infty), \mathbb{R})$, canonical process $\eta = (\eta_t)_{t \geq 0}$, likelihood ratio process of $Q^{\zeta'}$ to Q^ζ relative $\mathbb{G}$:

$$L^{\zeta'/\zeta} = \mathcal{E}_\zeta \left(\int_0^\cdot \delta_s^{\zeta'/\zeta} dB_s \right) \quad , \quad \delta_s^{\zeta'/\zeta} = \frac{S(\zeta', s) - S(\zeta, s)}{\sigma(X_s)}$$

in exponential representation of $L^{\zeta'/\zeta}$, in case $\zeta' < \zeta < \zeta' + a$ (similiar else)

$$\delta_s^{\zeta'/\zeta} = \frac{S(\zeta', s) - S(\zeta, s)}{\sigma(X_s)} = \underbrace{\frac{\lambda^*(s)}{\sigma(\eta_s)}}_{\text{bounds}} \underbrace{\left[1_{(\zeta',\zeta)} - 1_{(\zeta'+a,\zeta+a)}\right](i_T(s))}_{\text{deterministic}}$$

very simple structure !

from Ito formula for $\left(L^{\zeta'/\zeta}\right)^{1/p}$, with $p = 2, 4$, exploiting the above bounds:

$$(1) \quad E_\vartheta\left(\left[1 - \left(L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta)}\right)^{1/p}\right]^p\right) \leq C_{p,K}\,|u|^{p/2}\ ,\quad p = 2, 4\ ,\ u \in K$$

$$(2) \quad E_\vartheta\left(\left[L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta)}\right]^{1/2}\right) \leq \exp\{-k\,|u|\,\}$$

for all ϑ, any compact K in $\mathbb{R}$, with constants not depending on u, n, ϑ

(1)+(2) provide the basis to follow Ibragimov and Khasminskii's (1981) approach to convergence of likelihoods and convergence of estimators

eqn. (1) with $p = 2$: taking sqrt $\hookrightarrow$ Hellinger distances are of order $|u|^{1/2}$!!!

for $\vartheta \in \Theta$, consider local models at ϑ with local scale $\frac{1}{n}$ as $n \to \infty$

theorem 4 : a) on compacts $[-K, K]$, K arbitrarily large: likelihood ratios

$$(L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta})_{u\in[-K,K]} \quad \text{under } Q^\vartheta$$

converge as $n \to \infty$ to the likelihoods in the limit model $\widetilde{\mathcal{E}}$

$$(\widetilde{L}^{u/0})_{u\in[-K,K]} \text{ under } \widetilde{P}_0 \ , \quad \widetilde{L}^{u/0} = \exp\{ \widetilde{W}(uJ_\vartheta) - \frac{1}{2}|uJ_\vartheta| \}$$

(weak conv. in $C([-K, K])$, for K arbitrarily large) with scaling factor J_ϑ

$$J_\vartheta := [\lambda^*(\vartheta)]^2 \, (\mu^{(\vartheta)} P_{0,\vartheta}^{(\vartheta)} \frac{1}{\sigma^2}) + [\lambda^*(\vartheta+a)]^2 \, (\mu^{(\vartheta)} P_{0,\vartheta+a}^{(\vartheta)} \frac{1}{\sigma^2})$$

b) outside large compacts $[-K, K]$, $K \geq K_0$: have exponential bounds

$$Q^\vartheta(\sup_{|u|>K} |u|^p \, L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta} \geq 1) \leq b_1(p, K_0) \, e^{-\, b_2 K}$$

$$E_\vartheta(\int_{\{|u|>K\}} |u|^p \, \frac{L_{nT}^{(\vartheta+\frac{u}{n})/\vartheta}}{\int L_{nT}^{(\vartheta+\frac{u'}{n})/\vartheta} du'} \, du) \leq b_1(p, K_0) \, e^{-\, b_2 K}$$

for arbitrary order $p \in \mathbb{N}_0$, with constants not depending on n, ϑ, K.

consider MLE and BE based on observation of X up to time nT

$$\widehat{\vartheta}_{nT} = \underset{\zeta\in\overline{\Theta}}{\text{argmax}}\ L_{nT}^{\zeta/\zeta_0} \quad , \quad \vartheta^*_{nT} = \frac{\int_\Theta \zeta\, L_{nT}^{\zeta/\zeta_0} d\zeta}{\int_\Theta L_{nT}^{\zeta/\zeta_0} d\zeta}$$

and MLE and BE in the limit experiment $\widetilde{\mathcal{E}} = \{\widetilde{P}_u : u \in \mathbb{R}\}$

$$\widehat{u} = \underset{u\in\mathbb{R}}{\text{argmax}}\ \widetilde{L}^{u/0} \quad , \quad u^* = \frac{\int_{-\infty}^{\infty} u\, \widetilde{L}^{u/0} du}{\int_{-\infty}^{\infty} \widetilde{L}^{u/0} du}$$

(BE under quadratic loss, in the limit experiment with 'uniform prior over $\mathbb{R}$')
using theorem 4, we can show convergence of MLE and BE as $n \to \infty$

$$\mathcal{L}\left(n(\widehat{\vartheta}_{nT} - \vartheta) \mid Q^\vartheta \right) \longrightarrow \mathcal{L}\left(\widehat{u} \mid \widetilde{P}_0 \right)$$
$$\mathcal{L}\left(n(\vartheta^*_{nT} - \vartheta) \mid Q^\vartheta \right) \longrightarrow \mathcal{L}\left(u^* \mid \widetilde{P}_0 \right)$$

together with convergence of their moments of arbitrary order p

Ibragimov and Khasminskii (1981) and Rubin and Song (1995) have shown:
BE limit variance is better than MLE limit variance (ratio $\approx$ 19.3 to 26) !!

focus on the better sequence: behaviour under contiguous alternatives ?
here: use tools due to Le Cam, mainly process versions of his 'third lemma':

theorem 5 : a) in the limit experiment, the BE u^* is equivariant:

$$\text{for every } u \in \mathbb{R}: \quad \mathcal{L}\left(u^* - u \mid \widetilde{P}_u\right) = \mathcal{L}\left(u^* \mid \widetilde{P}_0\right)$$

b) the BE sequence $(\vartheta^*_{nT})_n$ is locally asymptotically equivariant at $\vartheta \in \Theta$:
for loss functions ℓ cont. + subconvex + polynomial maj., for $C < \infty$ fixed:

$$\lim_{n\to\infty} \sup_{|u|\leq C} \left| E_{\vartheta+\frac{u}{n}}\left(\ell\left(n(\vartheta^*_{nT} - (\vartheta+\frac{u}{n}))\right)\right) - E_{\widetilde{P}_0}\left(\ell\left(u^*\right)\right) \right| = 0 \,.$$

contiguous alternatives + squared loss: wish to compare this BE sequence
to arbitrary sequences of $\mathcal{G}^{\xi}_{nT}$–measurable estimators $\widetilde{\vartheta}_{nT}$ for the unknown ϑ

we obtain at every point $\vartheta \in \Theta$:

theorem 6 : a) local asymptotic efficiency bound with respect to squared loss:

$$\lim_{C\uparrow\infty} \liminf_{n\to\infty} \inf_{\widetilde{\vartheta}_{nT}} \sup_{|u|\leq C} E_{\vartheta+\frac{u}{n}} \left(\left[n \left(\widetilde{\vartheta}_{nT} - (\vartheta + \frac{u}{n}) \right) \right]^2 \right) \quad \geq \quad E_{\widetilde{P}_0} \left([u^*]^2 \right)$$

b) the BE sequence $(\vartheta^*_{nT})_n$ attains the bound in a).

warning: this assertion is NOT of the same power as the well known local asymptotic minimax bound in LAN/LAMN situations

in our framework here, there is nothing like a 'central sequence'

our guess: other choices [loss function $\ell(\cdot)$ combined with $\ell(\cdot)$-BE sequence]
$\hookrightarrow$ other local asymptotic efficiency bounds of parallel structure (always with rhs of type [loss $\ell(\cdot)$ combined with $\ell(\cdot)$-Bayesian], in the limit experiment $\mathcal{E}$)
$\hookrightarrow$ a 'zoo' of different rhs's depending on choice of $\ell(\cdot)$, no tool for comparison