On the estimation of analytic intensity density functions of Poisson random fields

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Abstract

1. The aim of this talk is to present some results about non-parametric estimation of analytic functions. We consider the following problem. We are observing a Poisson random field $X_\varepsilon(t)$ (a Poisson random measure $X_\varepsilon(A)$) on a region $G \subset R^d$). The non-homogeneous random field $X_\varepsilon(t)$ has the intensity measure $\epsilon^{-1}\Lambda$ where $\varepsilon > 0$ is a known small parameter and Λ is an unknown measure. It is supposed that the measure Λ is absolutely continuous with respect to the Lebesgue measure and has the density (the intensity density) function $\lambda(t)$ and that the unknown density λ belongs to a known class F of analytic functions. The basic problem is to estimate λ on the base of the observations X_ε. Denote $||.||_p$ the norm in $L_p(G)$. Set

$$\Delta_p(\varepsilon, F) = \Delta_p(\varepsilon) = \inf \sup \mathbf{E}_\lambda ||\hat{\lambda} - \lambda||_p$$

where sup is taken over all $\lambda \in F$ and inf is taken over all possible estimates $\hat{\lambda}$ of λ. We are interesting in asymptotic behavior of estimates when $\varepsilon \to 0$ and in particular in the rate of convergence of Δ to zero.

The rate depends on F. We suppose that F consists of analytic functions and consider the following well known classes of analytic functions.

(i) The classes $A(M, \Gamma)$ of functions analytic in a bounded region $\Gamma, G \subset \Gamma$ and bounded there by a constant M.

(ii) The classes $E(M, \sigma, \rho)$ of entire functions satisfying the following restrictions on their growth

$$\sup_{z=(z_1,\dots,z_d),|z_i|\le R_i} |\lambda(z)| \le M \exp\{\sum_1^d \sigma_i |R_i|^{\rho_i}\}$$

.

2. Let F= A(M, G) where M, G are supposed to be known.

Theorem 0.1 *If $F = A(M, G)$, then*

$$\Delta_p(\varepsilon, F) \asymp (\varepsilon \ln(\varepsilon)^{-1})^{1d/2},\ p < 4,$$
$$\Delta_4(\varepsilon, F) \asymp (\varepsilon \ln(\varepsilon)^{-1})^{d/2} (\ln\ln(\varepsilon)^{-1})^{1d/4},$$
$$\Delta_p(\varepsilon, F) \asymp (\varepsilon)^{1d/2} (\ln(\varepsilon)^{-1})^{d(1-2/p)},\ p > 4.$$

3. An analytic function $f(z)$ posses a remarkable property: being observed on an interval it becomes immediately known throughout its domain of analyticity. Of course the problem of recovering $f(z)$ is an ill posed problem and it would be interesting to know what will be happen if the observation are disturbed by random errors. Below is an example of such problem. We suppose that the Poisson random field X_ε is observed on a bounded set $G \subset R^d$. The intensity function λ is supposed to belong to a class $E(M,\sigma,\rho)$ with known M,σ,ρ. The question is how far from the set G the consistent estimation of λ is still possible. The answer is that when $\varepsilon \to 0$ this distance behaves itself roughly speaking as $\left(\ln \frac{1}{\varepsilon}\right)^{\frac{1}{\varrho}}$ where $\varrho = \max \rho_i$. We give two following examples.

Theorem 0.2 *Let $F = E(M,\sigma,\rho)$ where M,σ,ρ are supposed to be known. Let $G_\epsilon(\alpha)$ denote the T_ε - neighborhood of the set G where $T_\varepsilon = T_\varepsilon(\alpha) = \ln(\frac{1}{\varepsilon}^{\alpha/\varrho}$. Here $\varrho = \max \rho_i, \alpha > 0$. Then there exists an estimator $\hat{\lambda}$ of λ such that for any $\beta < \alpha < 1$*

$$\sup_{\lambda \in F} \mathbf{E}_\lambda \{ \sup_{G_\epsilon(\beta)} |\lambda(x) - \hat{\lambda}(x)|^2 \} \leq C_{\alpha,\beta} \varepsilon^{(1-\alpha)/d}.$$

Theorem 0.3 *Let under the conditions of the previous theorem x is an exterior point of the set $G_\varepsilon(\alpha), \alpha > 1$. Then for all sufficiently small $\varepsilon, \varepsilon < c_\alpha$,*

$$\inf_{\hat{\lambda}} \sup_{\lambda \in F} \mathbf{E}_\lambda |\lambda(x) - \hat{\lambda}(x)|^2 \geq 1/2.$$