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On the estimation of analytic
intensity densities of Poisson
processes.

1. The Problem

We are observing a Poisson
random measure $X_\varepsilon(G)$ in \mathbb{R}^d
(or a Poisson random
set Π_ε)

$$X_\varepsilon(G) = \#(\Pi_\varepsilon \cap G)$$

$$P\{X_\varepsilon(G) = k\} = \frac{(\Lambda_\varepsilon(G))^k}{k!} e^{-\Lambda_\varepsilon(G)}$$

The intensity measure Λ_ε is a. c. w. r. t. Lebesgue

$$\frac{d\Lambda_\varepsilon}{dm}(x) = \frac{1}{\varepsilon} \theta(x)$$

The small parameter ε is supposed to be known.

The unknown $\theta \in \Theta$

where Θ is a known set of functions

The problem: to estimate θ and to study the behaviour of risk function

$$\underline{E_\theta \|\hat{\theta}_\varepsilon - \theta\|_p}$$

Dependence on Θ

Example: Θ is 1-dim., θ -
- constants

MLE:

$$\hat{\theta}_\varepsilon = \frac{\varepsilon}{\text{mes } G} X_\varepsilon(G)$$

$$E_\theta |\hat{\theta}_\varepsilon - \theta|^2 = \frac{\varepsilon \theta}{\text{mes } G}$$

Example: Π_ε in \mathbb{R}^1

$$\Theta = \{\theta: \sup |\theta'(x)| \leq 1\}$$

$$\sup_\theta E_\theta |\hat{\theta}_\varepsilon - \theta(x_0)|^2 \geq c \varepsilon^{\frac{2}{3}}$$

In this talk it is
supposed that \odot
consists of analytic
functions.

Why analytic?

two quotations:

D. Hilbert, MATHEMATISCH

Problemen.

S. Bernstein.

Kutoyants Yu., Statistical Inference for Spatial Poisson Processes, Lecture Notes in Statistics, v.134, 1998.

2. Classes of analytic functions

a) Π_ε is observed in $G \subseteq \mathbb{R}^d$.
The set $D \subseteq A(D, M)$, $D \subseteq \mathbb{C}^d$ is a bounded region.

$A(D, M)$ consists of functions analytic in $D \supseteq G$ and
$$\sup_{z \in D} |f(z)| \leq M, \quad z = (z_1, \dots, z_d)$$

Example: $d=1$, $G = [a, b]$

b) $\Pi_z \subseteq G \subseteq \mathbb{R}^d$. $\Theta \subseteq \mathcal{E}(M, \sigma, \rho)$,

$$\sigma = (\sigma_1, \dots, \sigma_d), \quad \rho = (\rho_1, \dots, \rho_d).$$

$\mathcal{E}(M, \sigma, \rho)$ - the class of entire functions f :

$$\sup_{|z_j| \leq R_j} |f(z)| \leq M \exp \left\{ \sum_1^d \sigma_j R_j^{\rho_j} \right\}$$

c) $\Pi_z \subseteq G \subseteq \mathbb{R}^d$; $\Theta \subseteq \mathcal{E}_K$

\mathcal{E}_K consists of f :

$$f(x) = \frac{1}{(2\pi)^d} \int_K e^{-i(t, x)} \varphi(t) dt,$$

$K \subset \mathbb{R}^d$ is bounded, $\varphi \in L_2(K)$

evidently

$$\mathcal{E}_K \subseteq \mathcal{E}(M, \sigma, \rho), \quad \rho = (1, \dots, 1)$$

Theorem 2. Let $\Theta \supseteq A(\mathcal{D}, M)$.

Then for any θ_ε

$$\sup_{\theta} E_{\theta} \|\theta_\varepsilon - \theta\|_p \geq c_p \sqrt{\varepsilon} \left(\sqrt{\ln \frac{1}{\varepsilon}} \right)^d, \quad p < 4$$

$$\sup_{\theta} E_{\theta} \|\theta_\varepsilon - \theta\|_4 \geq c_4 \varepsilon \left(\sqrt{\ln \frac{1}{\varepsilon}} \sqrt[4]{\ln \ln \frac{1}{\varepsilon}} \right)$$

$$\sup_{\theta} E_{\theta} \|\theta_\varepsilon - \theta\|_p \geq c_p \sqrt{\varepsilon} \left(\ln \frac{1}{\varepsilon} \right)^{\left(1 - \frac{2}{p}\right)}$$

The constants depend on M, \mathcal{D} .

Remark. Observe iid

X_1, \dots, X_n

Poisson with int. density

θ . The statistics $\sum_{j=1}^n X_j$

3. Results.

Th. 1. Let $\Theta \subseteq A(M, \mathbb{D})$.

Then $\exists \hat{\Theta}_\varepsilon$

$$E_\Theta \|\hat{\Theta}_\varepsilon - \Theta\|_p \leq c_p \sqrt{\varepsilon} \left(\sqrt{\ln \frac{1}{\varepsilon}} \right)^d, \quad 1 \leq p < 4$$

$$E_\Theta \|\hat{\Theta}_\varepsilon - \Theta\|_4 \leq c_4 \sqrt{\varepsilon} \left(\sqrt{\ln \frac{1}{\varepsilon}} \sqrt[4]{\ln \ln \frac{1}{\varepsilon}} \right)^d$$

$$E_\Theta \|\hat{\Theta}_\varepsilon - \Theta\|_p \leq c_p \sqrt{\varepsilon} \left(\ln \frac{1}{\varepsilon} \right)^{\left(1 - \frac{2}{p}\right)d},$$

$\forall 4 < p \leq \infty$.

The results are asymptotically exact.

is sufficient. It is Poisson
with n, θ

Thus in this case

$$\varepsilon = \frac{1}{n}$$

4. On proofs.

4.1. Construction of estimates.

$$G = \Gamma = [-1, 1]^d$$

$P_n(x)$ - Legendre polynomials.

Any $f \in L_2(\Gamma)$

$$f(x) = \sum_j f_j P_j(x)$$

$$f_j = \int f(x) P_j(x) dx$$

18

$$\theta(x) = \sum \theta_j P_j,$$

estimate θ_j by

$$\hat{\theta}_j = \varepsilon \int P_j(t) dX_\varepsilon(t) =$$

$$= \varepsilon \sum_{x \in \Pi_\varepsilon} P_j(x)$$

Set

$$\hat{\theta}_{\varepsilon N} = \sum_{1 \leq j \leq N} \hat{\theta}_j P_j, \quad \vec{j} = (j_1, \dots, j_d)$$

Then

$$E \|\hat{\theta}_{\varepsilon N} - \theta\|_2^2 = \sum_{1 \leq j \leq N} E |\hat{\theta}_j - \theta_j|^2 + \sum_{j \notin \{1, \dots, N\}} \theta_j^2$$

The analysis \rightarrow

$$\rightarrow |a_j| \leq c_1 e^{-c_2 |j|}, \quad c_2 > 0.$$

Hence

$$\sup_{\theta} E_{\theta} \|\hat{\theta}_{\varepsilon} - \theta\|^2 \leq$$

$$\leq c_3 \left(\varepsilon N^d + e^{-c_2 N} \right).$$

Take

$$N \sim \frac{1}{c_2} \ln \frac{1}{\varepsilon}$$

4.2. Lower bounds

$\Pi_2 = \{X\}$ - Poisson r. set

$(\mathcal{X}, \mathcal{Q}, \mu)$ with $\theta_{\varepsilon} = \frac{\theta}{\varepsilon}$ (w.r.p)

Θ is equipped with a metric ρ .

Theorem. Suppose that for any $\delta > 0$ there exist a set

$$\{\theta_{i\delta}, i=1, 2, \dots, N(\delta)\} \subset \Theta, \underline{\rho(\theta_i, \theta_j)} > \delta$$

Set

$$\delta(\varepsilon, \Theta) = \sup_{\theta_0, \{\theta_{i\delta}\}} \{ \delta :$$

$$\frac{1}{\ln N(\delta)} \max_i \left\| \frac{\theta_{i\delta} - \theta_0}{\sqrt{\theta_0}} \right\|_{L_2(d_{\rho})} \leq$$

$$\leq \frac{1}{2} \varepsilon \}$$

Then

$$\sup_{\theta} E_{\theta} \left(\frac{p(\theta_{\varepsilon}, \theta)}{\delta(\varepsilon, \Theta)} \right) \geq \frac{1}{2} \ln \left(\frac{1}{2} \right)$$

Application to the proof
of Th. 2.

$$\theta_a = e^{-rn} \sum_0^{n-1} a_j P_j$$

$$a = (a_0, \dots, a_{n-1}), \quad a_j = 0, 1$$

$$\#\{a\} = 2^n$$

The distance

$$\|\theta_a - \theta_{a'}\|_2 = e^{-rn} \left(\sum |a_j - a'_j| \right)^{1/2}$$

$$\geq c\sqrt{n} e^{-rn}$$

$$\text{and } N \geq 2^{cn}, \quad c > 0$$

$$c \frac{1}{\ln N} e^{-2rn} \cdot n \leq \frac{\epsilon}{2\epsilon}$$

$$\text{if } e^{-rn} \leq c_1 \epsilon \Rightarrow$$

$$c\sqrt{n} e^{-rn} \sim c_2 \sqrt{\epsilon} \sqrt{\ln \frac{1}{\epsilon}}$$

$$6. \quad \Theta \subseteq \mathcal{B}(M, \sigma, p), \quad p \geq 1.$$

$$\left[| \theta(x+iy) | \leq M e^{\sum \sigma_j |y_j|^p} \right]$$

Th. 5.

$$\sup_{\theta} E_{\theta} \| \theta - \theta_{\epsilon} \|_p \leq c_p \left(\frac{(\ln \frac{1}{\epsilon})^{\frac{p-1}{p}}}{\epsilon} \right)^{\frac{p-1}{p}}$$

$$1 \leq p \leq 2$$

$$\sup_{\theta} E_{\theta} \| \theta - \theta_{\epsilon} \|_p \leq c_p \left(\epsilon \left(\ln \frac{1}{\epsilon} \right)^{\frac{p-1}{p}} \right)^{\frac{p-1}{p}}$$

$$2 \leq p < \infty$$

$$\begin{aligned} \sup_{\theta} E_{\theta} \| \theta - \theta_{\epsilon} \|_{\infty} &\leq \\ &\leq c_{\infty} \left(\epsilon \left(\ln \frac{1}{\epsilon} \right)^{\frac{p-1}{p}} \right)^{\frac{1}{2}} \left(\ln \ln \frac{1}{\epsilon} \right) \end{aligned}$$

