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On the estimation of analytic  
intensity densities of Poisson  
processes.

## 1. The Problem

We are observing a Poisson  
random measure  $X_\varepsilon(G)$  in  $\mathbb{R}^d$   
(or a Poisson random  
set  $\Pi_\varepsilon$ )

$$X_\varepsilon(G) = \#(\Pi_\varepsilon \cap G)$$

$$P\{X_\varepsilon(G) = k\} = \frac{(\Lambda_\varepsilon(G))^k}{k!} e^{-\Lambda_\varepsilon(G)}$$

The intensity measure  $\Lambda_\varepsilon$  is a. c. w. r. t. Lebesgue

$$\frac{d\Lambda_\varepsilon}{dm}(x) = \frac{1}{\varepsilon} \theta(x)$$

The small parameter  $\varepsilon$  is supposed to be known.

The unknown  $\theta \in \Theta$

where  $\Theta$  is a known set of functions

The problem: to estimate  $\theta$  and to study the behaviour of risk function

$$\underline{E_\theta \| \hat{\theta}_\varepsilon - \theta \|_p}$$

Dependence on  $\Theta$

Example:  $\Theta$  is 1-dim.,  $\theta$ -  
- constants

MLE:

$$\hat{\theta}_\varepsilon = \frac{\varepsilon}{\text{mes } G} X_\varepsilon(G)$$

$$E_\theta |\hat{\theta}_\varepsilon - \theta|^2 = \frac{\varepsilon \theta}{\text{mes } G}$$

Example:  $\Pi_\varepsilon$  in  $\mathbb{R}^1$

$$\Theta = \{\theta: \sup |\theta'(x)| \leq 1\}$$

$$\sup_\theta E_\theta |\hat{\theta}_\varepsilon - \theta(x_0)|^2 \geq c \varepsilon^{\frac{2}{3}}$$



Kutoyants Yu., Statistical Inference for Spatial Poisson Processes, Lecture Notes in Statistics, v.134, 1998.

## 2. Classes of analytic functions

a)  $\Pi_\varepsilon$  is observed in  $G \subseteq \mathbb{R}^d$ .  
The set  $D \subseteq A(D, M)$ ,  $D \subseteq \mathbb{C}^d$  is a bounded region.

$A(D, M)$  consists of functions analytic in  $D \supseteq G$  and  
$$\sup_{z \in D} |f(z)| \leq M, \quad z = (z_1, \dots, z_d)$$

Example:  $d=1, G = [a, b]$

b)  $\Pi_z \subseteq G \subseteq \mathbb{R}^d$ .  $\Theta \subseteq \mathcal{E}(M, \sigma, \rho)$ ,

$$\sigma = (\sigma_1, \dots, \sigma_d), \quad \rho = (\rho_1, \dots, \rho_d).$$

$\mathcal{E}(M, \sigma, \rho)$  - the class of entire functions  $f$ :

$$\sup_{|z_j| \leq R_j} |f(z)| \leq M \exp \left\{ \sum_1^d \sigma_j R_j^{\rho_j} \right\}$$

c)  $\Pi_z \subseteq G \subseteq \mathbb{R}^d$ ;  $\Theta \subseteq \mathcal{E}_K$

$\mathcal{E}_K$  consists of  $f$ :

$$f(x) = \frac{1}{(2\pi)^d} \int_K e^{-i(t, x)} \varphi(t) dt,$$

$K \subset \mathbb{R}^d$  is bounded,  $\varphi \in L_2(K)$

evidently

$$\mathcal{E}_K \subseteq \mathcal{E}(M, \sigma, \rho), \quad \rho = (1, \dots, 1)$$

Theorem 2. Let  $\Theta \supseteq A(\mathcal{D}, M)$ .

Then for any  $\theta_\varepsilon$

$$\sup_{\theta} E_{\theta} \|\theta_\varepsilon - \theta\|_p \geq c_p \sqrt{\varepsilon} \left( \sqrt{\ln \frac{1}{\varepsilon}} \right)^d, \quad p < 4$$

$$\sup_{\theta} E_{\theta} \|\theta_\varepsilon - \theta\|_4 \geq c_4 \varepsilon \left( \sqrt{\ln \frac{1}{\varepsilon}} \sqrt[4]{\ln \ln \frac{1}{\varepsilon}} \right)$$

$$\sup_{\theta} E_{\theta} \|\theta_\varepsilon - \theta\|_p \geq c_p \sqrt{\varepsilon} \left( \ln \frac{1}{\varepsilon} \right)^{\left(1 - \frac{2}{p}\right)}$$

The constants depend on  $M, \mathcal{D}$ .

Remark. Observe iid

$X_1, \dots, X_n$

Poisson with int. density

$\theta$ . The statistics  $\sum_{j=1}^n X_j$

### 3. Results.

Th. 1. Let  $\Theta \subseteq A(M, \mathbb{D})$ .

Then  $\exists \hat{\Theta}_\varepsilon$

$$E_\Theta \|\hat{\Theta}_\varepsilon - \Theta\|_p \leq c_p \sqrt{\varepsilon} \left( \sqrt{\ln \frac{1}{\varepsilon}} \right)^d, \quad 1 \leq p < 4$$

$$E_\Theta \|\hat{\Theta}_\varepsilon - \Theta\|_4 \leq c_4 \sqrt{\varepsilon} \left( \sqrt{\ln \frac{1}{\varepsilon}} \sqrt[4]{\ln \ln \frac{1}{\varepsilon}} \right)^d$$

$$E_\Theta \|\hat{\Theta}_\varepsilon - \Theta\|_p \leq c_p \sqrt{\varepsilon} \left( \ln \frac{1}{\varepsilon} \right)^{\left(1 - \frac{2}{p}\right)d},$$

$\forall 4 < p \leq \infty$ .

The results are asymptotically exact.



is sufficient. It is Poisson  
with  $n, \theta$

Thus in this case

$$\varepsilon = \frac{1}{n}$$

4. On proofs.

4.1. Construction of estimates.

$$G = \Gamma = [-1, 1]^d$$

$P_n(x)$  - Legendre polynomials.

Any  $f \in L_2(\Gamma)$

$$f(x) = \sum_j f_j P_j(x)$$

$$f_j = \int f(x) P_j(x) dx$$

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$$\theta(x) = \sum \theta_j P_j,$$

estimate  $\theta_j$  by

$$\hat{\theta}_j = \varepsilon \int P_j(t) dX_\varepsilon(t) =$$

$$= \varepsilon \sum_{x \in \Pi_\varepsilon} P_j(x)$$

$$x \in \Pi_\varepsilon$$

Set

$$\hat{\theta}_{\varepsilon N} = \sum_{1 \leq j \leq N} \hat{\theta}_j P_j, \quad \vec{j} = (j_1, \dots, j_d)$$

Then

$$E \|\hat{\theta}_{\varepsilon N} - \theta\|_2^2 = \sum_{1 \leq j \leq N} E |\hat{\theta}_j - \theta_j|^2 + \sum_{j \notin \{1, \dots, N\}} \theta_j^2.$$

The analysis  $\rightarrow$

$$\rightarrow |a_j| \leq c_1 e^{-c_2 |j|}, \quad c_2 > 0.$$

Hence

$$\sup_{\theta} E_{\theta} \|\hat{\theta}_{\varepsilon} - \theta\|^2 \leq$$

$$\leq c_3 \left( \varepsilon N^d + e^{-c_2 N} \right).$$

Take

$$N \sim \frac{1}{c_2} \ln \frac{1}{\varepsilon}$$

4.2. Lower bounds

$\Pi_2 = \{X\}$  - Poisson r. set

$(\mathcal{X}, \mathcal{Q}, \mu)$  with  $\theta_{\varepsilon} = \frac{\theta}{\varepsilon}$  (w.r.p)

$\Theta$  is equipped with a metric  $\rho$ .

Theorem. Suppose that for any  $\delta > 0$  there exist a set  $\{\theta_{i\delta}, i=1, 2, \dots, N(\delta)\} \subset \Theta$ ,  $\rho(\theta_i, \theta_j) > \delta$

Set

$$\delta(\varepsilon, \Theta) = \sup_{\theta_0, \{\theta_{i\delta}\}} \{ \delta :$$

$$\frac{1}{\ln N(\delta)} \max_i \left\| \frac{\theta_{i\delta} - \theta_0}{\sqrt{\theta_0}} \right\|_{L_2(d_{P_1})} \leq$$

$$\left. \leq \frac{1}{2} \varepsilon \right\}$$

Then

$$\sup_{\theta} E_{\theta} \left[ \ln \left( \frac{p(\theta_{\varepsilon}, \theta)}{\delta(\varepsilon, \Theta)} \right) \right] \geq \frac{1}{2} \ln \left( \frac{1}{2} \right)$$

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$$\left. \leq \frac{1}{2} \varepsilon \right\}$$

Then

$$\sup_{\theta} E_{\theta} \ln \left( \frac{p(\theta_{\varepsilon}, \theta)}{\delta(\varepsilon, \Theta)} \right) \geq \frac{1}{2} \ln \left( \frac{1}{2} \right)$$

Application to the proof of Th. 2.

$$\theta_a = e^{-\gamma n} \sum_0^{n-1} a_j P_j$$

$$a = (a_0, \dots, a_{n-1}), \quad a_j = 0, \pm 1$$

$$\#\{a\} = 2^n$$

The distance

$$\|\theta_a - \theta_{a'}\|_2 = e^{-\gamma n} \left( \sum |a_j - a'_j| \right)^{1/2}$$

$$\geq c\sqrt{n} e^{-\gamma n}$$

and

$$N \geq 2^{cn}, \quad c > 0$$

$$c \frac{1}{\ln N} e^{-2\gamma n} \cdot n \leq \frac{\epsilon}{2\epsilon}$$

$$\text{if } e^{-\gamma n} \leq c_1 \epsilon \Rightarrow$$

$$c\sqrt{n} e^{-\gamma n} \sim c_2 \sqrt{\epsilon} \sqrt{\ln \frac{1}{\epsilon}}$$

6.  $\Theta \subseteq B(M, \sigma, \rho), \rho_j \geq 1.$

$[ |\theta(x+iy)| \leq M e^{\sum \sigma_j |y_j|^\rho} ]$

Th. 5.

$$\sup_{\theta} E_{\theta} \|\theta - \theta_{\varepsilon}\|_p \leq c_p \left( \frac{(\ln \frac{1}{\varepsilon})^{\frac{p-1}{p}}}{\varepsilon} \right)^{\frac{p-1}{p}}$$

$$1 \leq p \leq 2$$

$$\sup_{\theta} E_{\theta} \|\theta - \theta_{\varepsilon}\|_p \leq c_p \left( \varepsilon \left( \ln \frac{1}{\varepsilon} \right)^{\frac{p-1}{p}} \right)^{\frac{p-1}{p}}$$

$$2 \leq p < \infty$$

$$\begin{aligned} \sup_{\theta} E_{\theta} \|\theta - \theta_{\varepsilon}\|_{\infty} &\leq \\ &\leq c_{\infty} \left( \varepsilon \left( \ln \frac{1}{\varepsilon} \right)^{\frac{p-1}{p}} \right)^{\frac{1}{2}} \left( \ln \ln \frac{1}{\varepsilon} \right) \end{aligned}$$

