Nonlinear functionals estimation for the Poisson observed processes.

1. Overview of known results on nonlinear functionals estimation.
2. Two models of the Poisson observed process.
3. Linear functionals estimation.
4. Nonlinear functionals estimation for model 1
5. Nonlinear functionals estimation for model 2
6. Generalization: nonlinear functional estimation of intensity Poisson random measure at R^n

1. $X_1, \dots, X_n$ are i.i.d. observations with unknown distribution density $p(x)$, $x \in R^1$.

$\Phi(p)$ is the differentiable functional with the derivative $\Phi'(p)(\cdot) \in L_2(F)$.

Then (Levit (1973))

$$\liminf_{n\to\infty} \sup_{p\in \mathcal{O}_\delta(p_0)} E_p\, \ell\left(\sqrt{n}\,(\Phi_n - \Phi(p))\right) \geq E\,\ell\left(\sigma(p_0)\zeta\right),$$

here ζ is $\mathcal{N}(0,1)$, $\ell(\cdot) \in \mathcal{L}$: $\ell(-x) = \ell(x)$, $\ell(x) \uparrow$ for $x > 0$.

$$\sigma^2(p) = \int_R \left[\Phi'(p)(x) - E_p \Phi'(p)(X_1)\right]^2 p(x)\, dx.$$

Creation of asympt. efficient estimator (I K (1978))

<u>Theorem 1</u>. Let $p(x) \in W(\beta)$ - class of densities having smoothness β in $L_2(F)$-norm. Let $\Phi'(p)(\cdot) \in L_2(F)$ and

$$\|\Phi'(p_2)(\cdot) - \Phi'(p_1)(\cdot)\|_{L_2(R)} \leq C\,\|p_2 - p_1\|^{\gamma}_{L_2(R)}.$$

Assume that $\gamma > \frac{1}{2\beta}$. Then $\exists$ estimate Φ_n s.t.

$$\limsup_{n\to\infty} \sup_{p\in W(\beta)} \left\{ n E_p |\Phi_n - \Phi(p)|^2 - \int_R \left[\Phi'(p)(x) - E_p \Phi'(p)(X_1)\right]^2 p(x)\,dx \right\}$$

Two observation models.

Model 1. $X_1(t), \dots X_n(t)$, $t \in [0,T]$ are i.i.d. Poisson processes with unknown intensity function $S(t)$, $n \to \infty$.

Model 2. $X_\varepsilon(t)$ is the Poisson process with intensity

$S_\varepsilon(t) = \varepsilon^{-1} S(t)$, $\varepsilon \to 0$.

For both models we have to estimate $\Phi(S)$.

Remark. Model 1 can be reduced to the Model 2.

Estimation of linear functional.

$$L(S) = \int_0^T f(t)\, S(t)\, dt.$$

Model 1 : $L_n = \frac{1}{n} \sum_{i=1}^{n} \int_0^T f(t)\, dX_i(t)$

Model 2 : $L_\varepsilon = \varepsilon \int_0^T f(t)\, dX_\varepsilon(t)$

$$E \left| L_\varepsilon - L(S) \right|^2 = \varepsilon \int_0^T f^2(t)\, S(t)\, dt$$

$$\cdot\,\cdot\; L_n \;\cdot\,\cdot\;\; \cdot\,\cdot\;\; \frac{1}{n} \;\; \cdot\,\cdot\,\cdot\,\cdot$$

Lower bound.

Assume that $\Phi(S)$ is weakly differentiable:

$$\Phi(S+\lambda h)=\Phi(S)+\lambda\int_0^T\Phi'(S)(t)\,h(t)\,dt+o(\lambda)\quad(\lambda\to 0$$

Theorem 2.

Let $\Phi(S)$ be weakly differentiable in $\mathcal{O}_\delta(S_0)$, and denote $\mathcal{F}_\varepsilon$ the set of estimates based on $X_\varepsilon(t)$, $0\le t\le T$. Then for $\forall\, \ell\in\mathcal{L}$

$$\liminf_{\varepsilon\to 0}\ \inf_{\Phi_\varepsilon\in\mathcal{F}_\varepsilon}\ \sup_{S\in\mathcal{O}_\delta(S_0)} E\,\ell\left(\varepsilon^{-\frac{1}{2}}\left(\Phi_\varepsilon-\Phi(S)\right)\right)\ge E\,\ell\left(\sigma(S_0)\,\zeta\right),$$

here ζ is $\mathcal{N}(0,1)$ random variable, and

$$\sigma^2(S_0)=\int_0^T\left(\Phi'(S_0)(t)\right)^2 S_0'(t)\,dt.$$

Idea of the proof: similar Levit (1973) we choose parametric family $S_\varepsilon(t,\theta)$, s.t.

$\Phi(S_\varepsilon(t,\theta))=\theta+o(1)$, and use LeCam-Hajek lower bound for parametric families.

Estimation of the intensity function.

Model 2: intensity $\frac{S(t)}{\varepsilon}$, and $S(t) \in \Sigma(\beta, L)$:

$\beta = k + \alpha$; k is integer, $0 < \alpha \le 1$.

$\exists\, S^{(k)}(t)$ and

$$|S^{(k)}(t+h) - S^{(k)}(t)| \le L\, |h|^{\alpha}.$$

Consider kernels $g_1(t), g_2(t)$:

$\int_R g_i(t)dt = 1$; $\int t^j g_i(t)dt = 0$, for $j = 1, 2, \ldots, k$,

g_i have compact support, smooth enough, and

$g_1(t) = 0$ for $t < 0$, $g_2(t) = 0$ for $t > 0$.

Choose $h_\varepsilon = \varepsilon^{\frac{1}{2\beta+1}}$

Then the estimator

$$\hat{S}_\varepsilon(t) = \frac{\varepsilon}{h_\varepsilon} \int_0^T \left[g_1\left(\frac{t-s}{h_\varepsilon}\right) \mathbb{1}_{\{t \le \frac{T}{2}\}} + g_2\left(\frac{t-s}{h_\varepsilon}\right) \mathbb{1}_{\{t > \frac{T}{2}\}} \right] X_\varepsilon(ds)$$

has the property: for $\forall$ loss function $\ell(x) < c_1 e^{c_2|x|}$

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} E\,\ell\left(\varepsilon^{-\frac{\beta}{2\beta+1}}\left(\hat{S}_\varepsilon(t) - S(t)\right)\right) < \infty$$

Model 1 – analogous: $\varepsilon = \frac{1}{n}$;

$$X_\varepsilon(\cdot) = \sum_{i=1}^{n} X_i(\cdot).$$

Functional estimation for the Model 1. [6

Let $X_1(t), X_2(t), \dots, X_n(t)$ be i.i.d. Poisson random processes with the intensity function $S(t) \in \Sigma(\beta, L)$. Let $\Phi(S)$ be Frechet differentiable functional with the Frechet derivative $\Phi'(S, \cdot)$ satisfying the conditions $\|\Phi'(S, \cdot)\| < C$; $\|\Phi'(S_2, \cdot) - \Phi'(S_1, \cdot)\| < C \|S_2 - S_1\|^{\gamma}$ (

1. Use $[n^{\delta}]$, $0 < \delta < 1$, observations for the estimation of $S(t)$. Then we have for $\forall r > 0$

$$\sup_{S \in \Sigma(\beta, L)} E \|S_n(\cdot) - S\|^r \leq C_r n^{-r\beta\delta/(2\beta+1)}.$$

2. Assuming that $\gamma > \frac{1}{2\beta}$ choose $\delta \in \left(\frac{1+\frac{1}{2\beta}}{1+\gamma}, 1\right)$ and consider the estimate

$$\Phi_n = \Phi(S_n) + \frac{1}{n - [n^{\delta}]} \sum_{i=[n]^{\delta}+1}^{n} \int_0^T \Phi'(S_n, t)\left(X_i(dt) - S_n(t)\,dt\right) \quad (*$$

Theorem 3.

Let $S \in \Sigma(\beta, L)$, $\Phi(S)$ satisfies the conditi[on]

(•) and $\gamma > \frac{1}{2\beta}$. Then

$\sqrt{n}\,(\Phi_n - \Phi(S))$ is asymptotically

$\mathcal{N}(0, \sigma^2(S'))$ and asymptotically

efficient for any loss function $l(\cdot)$

with polynomial majorant.

Functional estimation for the model 2.

We consider the estimation of $\Phi(S)$ satisfying conditions formulated above, on the base of the observation $X_\varepsilon(t)$ - Poisson process with intensity function $\frac{S(t)}{\varepsilon}$, $0 \le t \le T$.

"Thining" of the Poisson process will be used.

Let $Y(t)$ be the Poisson process with intensity function $\lambda(t)$. Let $\tau_1 < \tau_2 < \ldots$ be times of "jumps" of this process, s.t. $Y(t) = \sum_i \mathbb{1}\{\tau_i < t\}$.

Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables s.t. $P\{\xi_i = 1\} = p = 1 - P\{\xi_i = 0\}$. Then the process $Z(t) = \sum_i \xi_i \mathbb{1}\{\tau_i < t\}$ is also Poissonian, with intensity f-n $p\lambda(t)$.

Processes $Z(t)$ and $Y(t)-Z(t)$ are independent.

Creation of the estimator.

Step 1. Creation of the process $Z_\varepsilon(t)$ by thining of $X_\varepsilon(t)$.

$$p=\varepsilon^{\delta}, \quad 0<\delta<\frac{\gamma-(2\beta)^{-1}}{1+\gamma}. \qquad (\cdot\cdot$$

$Z_\varepsilon(t)$ is the Poisson r.p. with the intensity function $\frac{1}{\varepsilon^{1-\delta}} S(t)$.

Step 2. Create a kernel estimate of $S(t)$ on the base of $Z_\varepsilon(t)$. Then due to previous results we have

$$E\|S_\varepsilon(t)-S(t)\|^r \le C_r\,\varepsilon^{\frac{(1-\delta)r\beta}{2\beta+1}}$$

Step 3. Create the estimate Φ_ε of $\Phi(S)$ by formula

$$\Phi_\varepsilon=\Phi(S_\varepsilon)+\int_0^T \Phi'(S_\varepsilon,t)\left[\frac{\varepsilon}{1-\varepsilon^{\delta}}\left(X_\varepsilon(dt)-Z_\varepsilon(dt)\right)-S_\varepsilon(t)\,dt\right] \qquad (**)$$

Theorem 4. Let $S\in\Sigma(\beta,L)$, the functional $\Phi(S)$ satisfies conditions $(\cdot)$ with $\gamma>(2\beta)^{-1}$. Let $Z_\varepsilon(t)$, δ and $S_\varepsilon(t)$ are chosen as before. Then

The estimator $(**)$ is asymptotically efficient for any loss function with polynomial majorant, and

$\varepsilon^{-\frac{1}{2}}\left(\Phi_\varepsilon-\Phi(S)\right)$ is asymptotically $\mathcal{N}\left(0,\sigma^2(S)\right)$

Generalisation.

Let $X_\varepsilon(A)$ be the Poisson random measure in $D \subset R^n$ with the intensity function $\varepsilon^{-1} S(x)$. Let $S(x) \in \Sigma(\beta_1, \beta_2, \dots, \beta_n, L)$; here $\beta_j = k_j + \alpha_j$, $0 < \alpha_j \le 1$, the function S has derivative $\frac{\partial^{k_j} S(x)}{\partial x_j}$ and this derivative satisfies Hölder condition in D:

$$\left| \frac{\partial^{k_j} S}{\partial x_j}(x_1, \dots, x_j + h, x_{j+1}, \dots, x_n) - \frac{\partial^{k_j} S}{\partial x_j}(x_1, \dots, x_j, \dots x_n) \right| \le L |h|^{\alpha_j}.$$

Denote $\beta = \left(\sum_{j=1}^{n} \frac{1}{\beta_j} \right)^{-1}$.

Then analolously the similar results for the density estimation for i.i.d. case:

There exists the kernel estimate $S_\varepsilon(x)$, based on observation of $X_\varepsilon(A)$, $A \in D$, s.t.

$$\limsup_{\varepsilon \to 0} \sup_{x \in D} E\left(| S_\varepsilon(x) - S(x) | \varepsilon^{-\frac{\beta}{2\beta+1}} \right)^r < \infty$$

for any $r > 0$.

The procedure of thining also can be applied for Poisson random measure. From these two facts follows that Theorem 4 is also valid for the P.r.m.