Statistical properties of affine stochastic delay differential equations

Uwe Küchler, Humboldt University, Berlin

Abstract

Affine stochastic delay differential equations (SDDE's) are time continuous analogues of autoregressive sequences. They are defined as

$$dX(t) = \int_{-r}^{0} X(t+s)a(ds)dt + dW(t), \quad t \geq 0,$$

with initial conditions

$$X(s) = \xi(s), \quad s \in [-r, 0].$$

Here, $W = (W(t), t \geq 0)$ denotes a standard Wiener process, $a(\cdot)$ is a finite signed measure on $[-r, 0]$ and $\xi(.)$ a given initial condition. The number $r \geq 0$ is the length of the delay, or the "memory."

For affine SDDE's there are several powerful tools to treat them analytically, in opposite to more general SDDE's. In the case where the measure $a(\cdot)$ or the parameter $\vartheta \in \Theta$ in the parametrised case $(a_\vartheta, \vartheta \in \Theta)$ are unknown, one can derive estimators for them from continuous- or discrete-time sampling of $X = (X(t), t \geq 0)$.
The talk will present a review on construction and asymptotic properties of such estimators.

First, conditions on the function $\vartheta \to a_\vartheta$ are formulated which ensure the local asymptotic normality of the normalized likelihood function. They consist mainly in a weak differentiability of the function $\vartheta \to a_\vartheta$, and its Hölder continuity with respect to the total variation norm [1].

If they are not satisfied, a great variety of limits is possible. For example, for every $H \in (\frac{1}{2}, 1]$ the limit

$$\exp\left[B^H(u) - \frac{1}{2}E(B^H(u))^2\right], \quad u \in R_1,$$

can occur, where B^H is a fractional Brownian motion with Hurst index H [2], [3]. The local asymptotic normality corresponds to $H = \frac{1}{2}$.

The talk also treats sequential procedures $(\tau, \hat{\vartheta}_\tau)$ for estimating ϑ and their asymptotic properties. In the following simple case:

$$dX(t) = (\vartheta_0 X(t) + \vartheta_1 X(t-r))dt + dW(t),$$

slightly extending the well-known Ornstein-Uhlenbeck case ($\vartheta_1 = 0$), the constructions of the relevant stopping time τ and the sequential plan $\hat{\vartheta}_\tau$ are already much more complex [5].

In the case of discrete-time observations of X at

$$\{k\triangle : k = 1, 2, \ldots, N\}$$

for $\triangle > 0$, the likelihood function can be calculated from the covariance function of the (Gaussian) solution X. By neglecting terms being far away in the past, we obtain pseudo-likelihood estimators. They turn out to be consistent and asymptotically normal [4].

The methods we employ here can be extended to affine differential equations with fractional derivatives.

References

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