Example of LDP approach to smoothing problem

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(joint work with F. Klebaner)

Abstract

We consider an example of absorption at zero diffusion process, relative to Brownian motion B_t and small positive parameter ε, having with non-Lipschitz diffusion coefficient "x^γ"

$$dX^\varepsilon_t = \mu X^\varepsilon_t dt + (X^\varepsilon_t)^\gamma \varepsilon dB_t,$$

with $X^\varepsilon_0 = x(> 0)$ and $\gamma \in \big[\frac{1}{2}, 1\big)$, is known as "Constant Elasticity of Variance Model", introduced by Cox (1996); $\gamma = \frac{1}{2}$ correspond to branching diffusion.

Denote $\tau^\varepsilon_0 = \inf\{t : X^\varepsilon_t = 0\}$. The main problem is how to find a smoothing estimate of $(X^\varepsilon_{t\wedge\tau^\varepsilon_0})_{t\in[0,T]}$ To this end, we decide the Large Deviation Principle (LDP) is the most convenient tool for such analysis and found that the estimate

$$\widehat{X}_t = \begin{cases} e^{\mu t} x\Big[1 - \frac{1-e^{-2\mu(1-\gamma)t}}{1-e^{-2\mu(1-\gamma)T}}\Big]^{1/(1-\gamma)}, & \text{if } \mu \neq 0 \\ x\Big[1 - \frac{t}{T}\Big]^{1/(1-\gamma)}, & \text{if } \mu = 0 \end{cases}$$

is optimal in LDP scale in the following sense: for essentially small (ε, δ) and any deterministic function $\widetilde{X}_t \neq \widehat{X}_t$ with $\widetilde{X}_0 = x$ and absorbing on $[0, T]$,

$$\mathsf{P}_x\Big(\sup_{t\in[0,T]} |X^\varepsilon_t - \widetilde{X}_t| > \delta\Big) \gtrsim \mathsf{P}_x\Big(\sup_{t\in[0,T]} |X^\varepsilon_t - \widehat{X}_t| > \delta\Big)$$

$$\mathsf{P}_x\Big(\sup_{t\in[0,T]} |X^\varepsilon_t - \widehat{X}_t| \le \delta\Big) \asymp \exp\Big(-\frac{1}{\varepsilon^2}\mathsf{P}_x\big(\tau^\varepsilon_0 \le T\big)\Big).$$