Penalized nonparametric drift estimation in a continuous time one-dimensional diffusion.

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1. Penalized nonparametric drift estimation
2. Polynomial deviation inequality in the ergodic theorem for one-dimensional diffusion and integrability of hitting times

Let $(X_t)_{t\geq 0}$ be one-dimensional positive recurrent diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x,$$

observed continuously during $[0, t]$.

We want to estimate non-parametrically the drift function b on $[0, 1]$. The estimator we propose is obtained by a penalized least square approach:

1. Consider families of finite-dimensional linear subspaces of $\mathbb{L}^2([0, 1])$. Compute for each sub-space an associated least-squares estimator (non-adaptive estimation).
2. Adaptive procedure chooses among the resulting collection of estimators the "best one" via some penalization device (adaptive estimation).

Assumptions on X.

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x,$$

- b and σ are locally Lipschitz and there is some $C > 0$ such that $|b(x)| \leq C(1+|x|)$.
- There exist $0 < \sigma_0^2 \leq \sigma_1^2 < \infty$ such that for all x,

$$\sigma_0^2 \leq \sigma^2(x) \leq \sigma_1^2.$$

- There exist γ, $2\gamma > 31\sigma_1^2$ and $M_0 > 0$, such that $[0,1] \subset [-M_0, M_0]$ and

$$xb(x) \leq -\gamma \quad \text{for} \quad |x| \geq M_0.$$

The third hypothesis means “polynomial β-mixing case”. The analogous problem was considered by Genon-Catalot, Compte, Rozenholc for X starting from stationary distribution and under the assumption of exponential β-mixing.

Assumptions on the collection of models

We aim to estimate b on $[0,1]$ using a data-driven procedure. For this purpose we consider a collection $(S_m)_{m\in\mathcal{M}_t}$ of finite dimensional sub-spaces of $\mathbb{L}^2([0,1],dx)$. Denote D_m dimension of S_m.

- $\forall m \in \mathcal{M}_t\ S_m \subseteq S_t$. (All models belong to a "biggest" model.)
- The dimension of "biggest space" verifies $D_t \leq t$ and $Card\mathcal{M}_t \leq D_t$.
- Let $\{\phi_1, \ldots \phi_{D_t}\}$ be an orthonormal basis of $S_t \subset \mathbb{L}^2([0,1])$. Then there exists a positive constant Φ_1 such that for all i,

$$\mathrm{Card}\{j:\ \|\phi_i\phi_j\|_\infty \neq 0\} \leq \Phi_1.$$

Non-adaptive estimation.

Fix a subspace $S_m \subset \mathbb{L}^2([0,1], dx)$. The estimator $\hat{b}_m$ of b on $[0,1]$ is a minimizer on S_m of the following contrast function:

$$\gamma_t(h) = \frac{1}{t}\int_0^t h^2(X_s)ds - \frac{2}{t}\int_0^t h(X_s)dX_s.$$

Denote

$$\|h\|_t^2 = \frac{1}{t}\int_0^t h^2(X_s)ds, \quad \text{and} \quad T_m(h,f) = \frac{1}{t}\int_0^t h(X_s)f(X_s)ds$$

the “empirical $\mathbb{L}^2$ norm” and the corresponding quadratic form.
Remark that by Ito

$$\gamma_t(h) = \|h-b\|_t^2 - \frac{2}{t}M_t - \|b\|_t^2,$$

hence $M_t/t \to 0$, minimization of $\gamma_t(h) \sim$ minimization of $\|h-b\|_t^2$.

To ensure the existence of $\hat{b}_m$:

Let

$$A_t = \{\forall m \in \mathcal{M}_t;\ \rho_m \geq t^{-1/2}\}$$

where ρ_m is a smallest eigenvalue of T_m.

T_m positively defined on A_t, hence minimizer exists and is unique.

$$\hat{b}_m = argmin_{h \in S_m} \gamma_t(h) \quad \text{on} \quad A_t \quad \text{and} \quad \hat{b}_m = 0 \quad \text{on} \quad A_t^c.$$

We define the risk as the expectation of the empirical norm:

$$\mathbb{E}_x \|\hat{b}_m - b\|_t^2 = \mathbb{E}_x \left(\frac{1}{t} \int_0^t (\hat{b}_m - b)^2 (X_s) ds \right).$$

Let b_m be the $\mathbb{L}^2([0,1],dx)$ projection of b onto S_m and $\|.\|$ be the $\mathbb{L}^2([0,1],dx)$ norm.

Theorem

for $t \geq t_0 = \frac{4}{p_0^2}$,

$$E_x\|\hat{b}_m - b\|_t^2 \leq 3\kappa\|b_m - b\|^2 + 16\sigma_1^2 \frac{\kappa}{p_0}\frac{D_m}{t} + C\left(b_0^2 + \sigma_1^2\right) t^{-1}$$

Here, $\kappa = \kappa(t) = \frac{2}{\sigma_0^2}(\frac{2}{t} + \frac{2\sigma_1}{\sqrt{t}} + 2b_0 + \frac{\sigma_1^2}{2})$ *and* $p_0 = const(\sigma_1, b_0)$.

in some cases we can bound $\|b_m - b\|^2$. For Besov space $B_{\alpha,2,\infty}([0,1])$, $\|b_m - b\|^2 \leq CteD_m^{-2\alpha}$. Then the best choice of D_m : $D_m = t^{\frac{1}{2\alpha+1}}$; then we obtain

$$E_x\|\hat{b}_m - b\|_t^2 \leq Ct^{-\frac{2\alpha}{2\alpha+1}} + C\left(b_0^2 + \sigma_1^2\Phi_0^2\right) t^{-1},$$

and this yields exactly the classical non-parametric rate $t^{-\frac{2\alpha}{2\alpha+1}}$.

Adaptive estimation.

The previous choice supposes the knowledge of the regularity α of the unknown drift. Adaptive estimation scheme choose automatically the best dimension D_m when α is unknown. Recall for $m \in \mathcal{M}_t$

$$\hat{b}_m = argmin_{h \in S_m} \gamma_t(h) \quad \text{on} \quad A_t \quad \text{and} \quad \hat{b}_m = 0 \quad \text{on} \quad A_t^c.$$

Put

$$\hat{m} := \arg \min_{m \in \mathcal{M}_t} \left[\gamma_t(\hat{b}_m) + pen(m) \right]. \tag{1}$$

Then the estimator that we propose is the following adaptive estimator

$$\hat{b}_{\hat{m}} := \begin{cases} \sum_n 1_{\{\hat{m}=n\}} \hat{b}_n & \text{on } A_t \\ 0 & \text{on } A_t^c \end{cases}. \tag{2}$$

Theorem

Let

$$pen(m) \geq \chi \sigma_1^2 \frac{D_m}{t},$$

where χ *is a universal constant. Then we have for all* $b \in \mathcal{M}(M_0, b_0, \gamma)$,

$$E_x ||\hat{b}_{\hat{m}} - b||_t^2 \leq 3\kappa \inf_{m \in \mathcal{M}_t} \left(||b_m - b||^2 + pen(m) \right) + \frac{K'}{t}, \qquad (3)$$

where $\kappa = \kappa(t) = \frac{2}{\sigma_0^2}(\frac{2}{t} + \frac{2\sigma_1}{\sqrt{t}} + 2b_0 + \frac{\sigma_1^2}{2})$

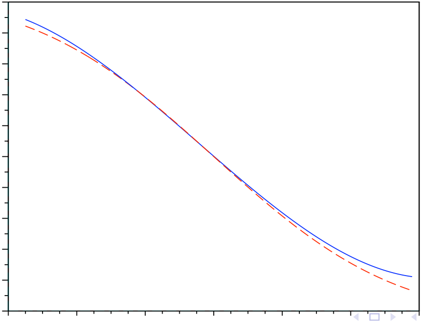
The adaptive estimator achieves automatically the bias-varance equilibrium.

Simulations

$$dX_t = -\frac{\gamma X_t}{1+X_t^2}\,dt + dW_t, \quad X_0 = x, \quad \gamma > \frac{1}{2}.$$

This diffusion is positive recurrent with stationary distribution

$$\mu(dx) \sim \frac{dx}{(1+x^2)^{\gamma}}.$$



Polynomial deviation inequality.

Denote by μ the invariant measure of X. The main tool in the proof of the statistical result is the following deviation inequality:

$$\mathbb{P}_\nu \left(\left| \frac{1}{t} \int_0^t f(X_s) ds - \mu(f) \right| \geq \epsilon \right) \leq K(p) \frac{1}{t^{p/2}} \frac{1}{\epsilon^p} A(f)^p$$

where ν is an initial distribution, f bounded or bounded and compactly supported and $A(f) = \|f\|_\infty$ when f is bounded and $A(f) = \mu(|f|)$ when f is bounded and compactly supported.

What is the speed p ? To formulate conditions on p we start by introducing the notion of polynomial degree of recurrence.

Polynomial degree of recurrence.

Assume: $\sigma^2(x) > 0$ for all x, β, σ continuous, non exploding solution exist.

$$dX_t = \beta(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x,$$

Denote $T_y = \inf\{t \geq 0,\ X_t = y\}$.

X is recurrent if and only if $\mathbb{P}_x(T_y < \infty) = 1$ for all x, y.
X is positive recurrent if and only if $E_x T_y < \infty$ for all x, y.

Definition

The degree of recurrence of X is polynomial of order $p \geq 1$, if for all x, y we have $\mathbb{E}_x T_y^p < \infty$ and $\mathbb{E}_x T_y^{p+1} = \infty$.

It is known (Maryama, Tanaka) $\mathbb{E}_x T_y^p < \infty$ or $= \infty$ simultaneously for all couples x, y : We show the following theorem:

Theorem

1. *Let $x < b$ and $n \in \mathbb{N}^*$.*
   - (1i) $\mathbb{E}_x T_b^n < \infty$ *if and only if* $\int_{-\infty}^{x} \mathbb{E}_\xi T_b^{n-1} m(\xi) d\xi < \infty$.
   - (1ii) *If for one couple $x < b$, $\mathbb{E}_x T_b^n < \infty$, then for all couples $x' < b'$, $\mathbb{E}_{x'} T_{b'}^n < \infty$.*
     Moreover, $x' \to \mathbb{E}_{x'} T_{b'}^n$ is continuous.
2. *Let $a < x$ and $n \in \mathbb{N}^*$.*
   - (2i) $\mathbb{E}_x T_a^n < \infty$ *if and only if* $\int_{x}^{+\infty} \mathbb{E}_\xi T_a^{n-1} m(\xi) d\xi < \infty$.
   - (2ii) *If for one couple $a < x$, $\mathbb{E}_x T_a^n < \infty$, then for all couples $a' < x'$, $\mathbb{E}_{x'} T_{a'}^n < \infty$.*
     Moreover, $x' \to \mathbb{E}_{x'} T_{a'}^n$ is continuous.

here $m(x)$ is the density of the invariant distribution.

If X has the degree of recurrence at least p, the deviation inequality is true with this p.

$$\mathbb{P}_\nu\left(|\frac{1}{t}\int_0^t f(X_s)ds - \mu(f)| \geq \epsilon\right) \leq K(p)\frac{1}{t^{p/2}}\frac{1}{\epsilon^p}A(f)^p$$

Fix two points $a < b$. Define: $S_0 = 0,\ R_0 = 0,$

$$S_1 = \ \{t \geq 0 : X_t = b\}, \quad R_1 := \inf\{t \geq S_1 : X_t = a\},$$

and for $n \geq 1$,

$$S_{n+1} := R_n + S_1 \circ \partial_{R_n}, \quad R_{n+1} := R_n + R_1 \circ \partial_{R_n}.$$

For any measurable and bounded function $f : \mathbb{R} \to \mathbb{R}$, we put

$$\xi_n = \int_{R_n}^{R_{n+1}} f(X_s)ds, \quad n \geq 1. \tag{4}$$

Lemma

For any initial distribution ν, the sequence $(\xi_n)_{n\geq 1}$ is an i.i.d. sequence under $\mathbb{P}_\nu$. For all $n \geq 1$, the law of ξ_n under $\mathbb{P}_\nu$ is equal to the law of $\int_0^{R_1} f(X_s)ds$ under $\mathbb{P}_a$.

The successive visits (R_n) cut trajectory on i.i.d. cycles:

$$\int_0^t f(X_s)ds = \int_0^{R_1} f(X_s)ds + \sum_{n=1}^{N_t} \xi_n + \int_{N_t}^t f(X_s)ds$$

where $N_t = sup\{n : \quad R_n \leq t\}$ is a number of visits R_n before t.

$$\text{Main term:} \quad \sum_{n=1}^{N_t} \xi_n$$

To control the main term $\sum_{n=1}^{N_t} \xi_n$ we need to control N_t and ξ_n

- The processes $(N_t)_{t\geq 0}$ and $(R_n)_{n\in\mathbb{N}}$ are mutually inverse: $\{N_t \geq n\} = \{R_n \leq t\}$ and $\{N_t \leq n\} = \{R_n \geq t\}$.
- $R_n = R_0 + (R_1 - R_0) + \ldots + (R_n - R_{n-1})$, sum i.i.d.
- $\xi_n = \int_{R_n}^{R_{n+1}} f(X_s)ds$

Conditions in ergodic theorem involve integrability of cycles $(R_k - R_{k-1})$, but

$$E_\nu(R_k - R_{k-1})^p \leq 2^{p-1}\left(E_a T_b^p + E_b T_a^p\right).$$

Theorem

Let f be a measurable bounded function, or measurable bounded with compact support, such that $\mu(f) \neq 0$. Suppose that (X) is recurrent of degree at least $p \geq 1$. For all x

(a) *For a bounded function f such that $\|f\|_\infty \leq M$:*

$$\mathbb{P}_x\left(\left|\frac{1}{t}\int_0^t f(X_s)ds - \mu(f)\right| > \epsilon\right) \leq K(p)\frac{1}{t^{p/2}}\frac{1}{\epsilon^p}M^p, \quad (5)$$

where $K(p)$ does not depend on f, t, ϵ.

(b) *For a bounded function f with compact support:*

$$\mathbb{P}_x\left(\left|\frac{1}{t}\int_0^t f(X_s)ds - \mu(f)\right| > \epsilon\right) \leq K(p)\frac{1}{t^{p/2}}\frac{1}{\epsilon^p}(\mu(|f|))^p, \quad (6)$$

where $K(p)$ does not depend on f, t, ϵ.

To have this theorem under $\mathbb{P}_\nu$ we need to ensure $\mathbb{E}_\nu T_a^p < \infty$.

Existence of hitting times moments

Assume:

- There exist positive constants σ_0 and σ_1 such that

$$\sigma_0 \leq \sigma(x) \leq \sigma_1, \text{ for all } x \in \mathbb{R}. \quad (7)$$

- There exist two constant $R \geq r > 0$ such that

$$r \leq -x\beta(x) \leq R \quad \text{for } |x| > M_0. \quad (8)$$

It is known (Veretennikov, Balaji and Ramasubramanian) that under these assumptions for $x > a > M_0$ $\mathbb{E}_x T_a^p$ is finite for $p < r/\sigma_1^2 + 1/2$ and infinite for $p > R/\sigma_0^2 + 1/2$,

but we need a finer control on $\mathbb{E}_x T_a^p$ to be able to estimate $\mathbb{E}_\nu T_a^p$.

Theorem

If $x > a > M_0$ (or $x < a < -M_0$) and $n < r/\sigma_1^2 + 1/2$ then

$$\frac{(x-a)^{2n}}{R_n} \le \mathbb{E}_x T_a^n \le \frac{x^{2n}}{r_n} + \ldots = P_{2n}(x)$$

where R_n and r_n depend explicitly on σ_0, σ_1 and n.

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