Statistical problems on the delays measure density in small diffusion processes

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Consider observations $(X^\varepsilon(t), t \in [0,T])$ of SDE with delays

$$dX^\varepsilon(t) = \left(\int_0^\delta X^\varepsilon(t-s)\mu(ds)\right)dt + \varepsilon dW(t),$$

Conditions
$X^\varepsilon(s) = x_0(s), -\delta \le s \le 0,$
x_0 positive function on $[-\delta, 0]$,
$(W(t), t \in [0,T])$ a Wiener process
μ signed bounded measure on $[0,\delta]$ and $\delta > 0$.
ε small in (0,1)
Deterministic equation

$$\frac{dX^0(t)}{dt} = \int_0^\delta X^0(t-s)\mu(ds), \quad 0 \le t \le T$$

$X^0(s) = x_0(s),\ -\delta \le s \le 0.$

Plan.

the delays measure μ has a density with a truncated series Fourier

Estimation of the Fourier coefficients

1. Minimum distance estimates MDE
(W. Millar (1984) Y. Kutoyants (1994))

2. Maximum likelihood and Bayes estimates
(Ibragimov-Hasminskii (1981) Y. Kutoyants (1994))

Estimation of the density of the delays measure μ .

In stationary case :

μ discrete measure

(Kuchler - Kutoyants (2000)- Kuchler-Gushchin (2003))

μ has a density g (Reiss (2003))

In small diffusion :

μ discrete measure

Kutoyants-Bosq-Mourid (1992)

Estimation of the delays measure density

the density $g \in L^2$ admits a truncated Fourier series

$$g = \sum_{k=1}^{p} a_j(g) e_j$$

$a_j(g)$ coefficients and (e_j) orthonormal basis in L^2

Consider the family of densities with a truncated Fourier series : for $p \in N^*$

$$\mathcal{F}_0(p) = \{g : g \in L^2, a_p(g) = 0, a_j(g) = 0, j > p\}$$

and the space

$$\mathcal{F}_0 = \bigcup_{p \geq 1} \mathcal{F}_0(p)$$

we choose g in $\mathcal{F}_0$

(typically whenever the coefficients become small after a truncating index)

Consider g on [0,1] ($\delta = 1$) with a truncated trigonometric Fourier series

$$g(s) = c_0 1_{[0,1]}(s) + \sum_{k=1}^{p} c_k \sqrt{2} \cos(2\pi ks)$$

$$+ \sum_{k=1}^{p} d_k \sqrt{2} \sin(2\pi ks)$$

The coefficients

$$(c_0, c_1, ...c_p, d_1, ...d_p)$$

a (2p+1)-dimensional vector of unknown parameters to estimate

the integer p is supposed known

Set $\theta = (c_0, c_1, ...c_p, d_1, ...d_p)$.

Minimum distance estimates for

$$\theta = (c_0, c_1, ...c_p, d_1, ...d_p).$$

(Millar(1984) Kutoyants (1994))
Recall results of Kutoyants-Mourid (1994)
estimation of the deterministic drift

$$f(t) := \int_0^\delta X^0_\theta(t-s)\mu(ds)$$

by the kernel estimate

$$\widehat{f}_\varepsilon(t) = \frac{1}{\psi_\varepsilon}\int_0^T K\left(\frac{\tau - t}{\psi_\varepsilon}\right) dX^\varepsilon(\tau) \qquad (1)$$

where the windows ψ_ε tends to 0 with ε

Kernel K bounded and

$$\int_{-\infty}^{+\infty} K(u)du = 1, \quad K(u) = 0, \quad u \notin [A, B], \quad A.B < 0$$

(Ibragimov-Hasminski 1981, Kutoyants (1984))

Asymptotic behavior of estimator $\widehat{f}_\varepsilon$ as $\varepsilon \to 0$
For $L > 0$ the space

$$\Theta_1(L) = \{\mu \ / \ \exists \delta > 0, Supp(\mu) = [0, \delta], \quad \|\mu\|_{TV} \leq L \quad \}$$

Convergence and rate

Proposition *let* $[a, b] \subset [0, T]$. *If* $\psi_\varepsilon \to 0$ *and* $\varepsilon^2 \psi_\varepsilon^{-1} \to 0$ *whenever* $\varepsilon \to 0$, *then*

$$\lim_{\varepsilon \to 0} \sup_{\mu \in \Theta_1(L)} \sup_{a \leq t \leq b} \mathbf{E}_\theta^\varepsilon |\widehat{f}_\varepsilon(t) - f(t)|^2 = 0$$

Proposition . *Let* $[a_\varepsilon, b_\varepsilon] \subset]0, T[$, *such that*

$$a_\varepsilon \to 0, \quad b_\varepsilon \to T, \quad a_\varepsilon \psi_\varepsilon^{-1} \to +\infty, \quad (T - b_\varepsilon)\psi_\varepsilon^{-1} \to +\infty,$$

$$\psi_\varepsilon \longrightarrow 0, \quad \varepsilon^2 \psi_\varepsilon^{-1} \longrightarrow 0$$

then $\lim_{\varepsilon \to 0} \sup_{\mu \in \Theta_1(L)} \sup_{a_\varepsilon \leq t \leq b_\varepsilon} \mathbf{E}_\theta^\varepsilon |\widehat{f}_\varepsilon(t) - f(t)|^2 = 0.$

Proposition . *under the above conditions, we have :*

$$\limsup_{\varepsilon\to 0} \sup_{\mu\in\Theta_1(L)} \sup_{a_\varepsilon\le t\le b_\varepsilon} \mathbf{E}_\theta^\varepsilon \left\{\varepsilon^{-\frac{4}{3}} |\widehat{f}_\varepsilon(t) - f(t)|^2\right\} < \infty.$$

Limit law

Proposition . *Let* (a_ε), (b_ε) *be satisfy the above conditions .*

For $t \in [a_\varepsilon, b_\varepsilon]$ *we define the process*

$$Y^\varepsilon(t) = \left(\int_{-\infty}^{+\infty} uK(u)du\right)\left(\int_0^\delta \dot{X}^0(t-s)\mu(ds)\right) + \xi^\varepsilon(t)$$

where

$$\xi^\varepsilon(t) = \frac{1}{\psi_\varepsilon^{1/2}} \int_0^T K\left(\frac{\tau - t}{\psi_\varepsilon}\right) dW(\tau), \quad \psi_\varepsilon = \varepsilon^{2/3}$$

then

$$\lim_{\varepsilon\to 0}\ \sup_{\mu\in\Theta_1(L)}\ \sup_{a_\varepsilon\le t\le b_\varepsilon} |\varepsilon^{-2/3}(\hat{f}_\varepsilon(t)-f(t))-Y^\varepsilon(t)| = 0, \quad a.s.$$

Corollary . *Let us define the process :*

$$(V^{\varepsilon}(t) := \varepsilon^{-\frac{2}{3}}\left(\widehat{f}_{\varepsilon}(t) - f(t)\right),\ t \in [a_{\varepsilon}, b_{\varepsilon}])$$

Then for all $t_1, .., t_l \in]0, T[$*,*$l \in N^*$*, we have*

$$(V^{\varepsilon}(t_1), .., V^{\varepsilon}(t_l)) \underset{\varepsilon \longrightarrow 0}{\Longrightarrow} \mathcal{N}_l$$

where $\mathcal{N}_l$ *is a gaussian vector in* $I\!R^l$ *with mean vector* $(m(t_1), \ldots, m(t_l))$ *:*

$$m(t_i) = \left(\int uK(u)du\right)\left(\int_0^{\delta} \dot{X}^0(t_i - s)\mu(ds)\right)$$

i=1,.., l, and covariance matrix $\Sigma = \sigma^2 I_l$*, where* $\sigma^2 = \int K^2(u)du$*, and* I_l *is the identity matrix of* $I\!R^l$*.*

MDE of the Fourier coefficients of g

The delays measure density g of μ has a truncated Fourier expansion

$$g(s) = c_0 1_{[0,1]}(s) + \sum_{k=1}^{p} c_k \sqrt{2} \cos(2\pi ks)$$

$$+ \sum_{k=1}^{p} d_k \sqrt{2} \sin(2\pi ks)$$

The parameters to estimate is

$$\theta := (c_0, c_1, ...c_p, d_1, ...d_p)) \in \Theta \subset R^{2p+1}$$

the integer p is supposed known

Minimum distance estimators MDE θ_ε^* defined by

$$\theta_\varepsilon^* := \arg \inf_{\theta \in \bar{\Theta}} \int_{a_\varepsilon}^{b_\varepsilon} \left(\widehat{f}_\varepsilon(t) - \int_0^\delta X^0(t-s)g(s)ds \right)^2 \nu(dt)$$

$\widehat{f}_\varepsilon(t)$ the kernel estimate
ν bounded positive measure on $]0, T[$
denote

$$S(t, \theta, X_\theta) := \int_0^\delta X_\theta(t-s)g(s)ds$$

For all $\beta > 0$ and compact K of Θ define

$$g_\varepsilon(\beta) = \inf_{\theta_0 \in K} \inf_{||\theta - \theta_0|| > \beta > 0} \left(\int_{a_\varepsilon}^{b_\varepsilon} \left(S(t, \theta, X_\theta^0) - S(t, \theta_0, X_{\theta_0}^0) \right)^2 \nu(dt) \right)^{\frac{1}{2}}$$

$||.||$ a norm of $I\!R^{2k+1}$
$I(\theta)$ the matrix

$$I(\theta) := < \dot{S}(t, \theta, X_\theta^0), \dot{S}(t, \theta, X_\theta^0)^T >_{L^2(\nu)}$$

$\dot{S}(t, \theta, X_\theta^0)$ vector of derivatives w.r.t. θ

$< .,. >_{L^2(\nu)}$ the inner product of $L^2(\nu)$

Convergence

Proposition *If for all* $\beta > 0$, $g_\varepsilon(\beta) > 0$, *and the matrix* $I(\theta)$ *is defined positive then there exists* $\varepsilon_2 > 0$, *and* $C > 0$ *such that* $0 < \varepsilon < \varepsilon_2$

$$\sup_{\theta_0 \in K} P^{\varepsilon}_{\theta_0} \left(||\theta^*_\varepsilon - \theta_0|| > \beta \right) \leq 3 \exp \left(-C \frac{g_\varepsilon(\beta)^2}{\varepsilon^{\frac{4}{3}}} \right)$$

Remarks

1. We have result uniformly on $\Theta_1(L)$ *: if there exists* $\kappa > 0$, *such that for all* $\beta > 0$ *and compact K of* Θ, *we have*

$$\inf_{\mu \in \Theta_1(L)} g_\varepsilon(\beta) > \kappa > 0$$

and the matrix $I(\theta)$ *is defined positive then there exists*

$\varepsilon_2 > 0$, $C > 0$ *such that* $\forall \varepsilon < \varepsilon_2$

$$\sup_{\mu \in \Theta_1(L)} \sup_{\theta_0 \in K} P_\theta^\varepsilon \left(||\theta_\varepsilon^* - \theta_0|| > \beta \right) \leq 3 \exp \left(-C \frac{\kappa^2}{\varepsilon^{4/3}} \right)$$

2. Find sufficient conditions to have $\forall \beta > 0$,

$$\forall \beta > 0 \quad g_\varepsilon(\beta) > 0$$

and the matrix $I(\theta)$ *defined positive ?*

3. Choice another base ?

Limit Law

Proposition . *If* ν *is a discrete measure on* $]0, T[$ *and for all* $\beta > 0$, $g_\varepsilon(\beta) > 0$, *then*

$$\varepsilon^{-\frac{2}{3}} (\theta^*_\varepsilon - \theta_0) \underset{\varepsilon \longrightarrow 0}{\Longrightarrow} \zeta$$

where

$$\zeta = I^{-1}(\theta_0) \int_0^T \left[\left(\int uK(u)du \right) \left(\int_0^\delta \dot{X}^0(t-s)\mu(ds) \right) \right.$$

$$\left. + Y^0(t) \dot{S}(t, \theta_0, X^0_{\theta_0}) \nu(dt) \right.$$

$Y^0(t)$ *is a zero mean gaussian rv with variance* $\int K^2(u)du$ *et* $E(Y^0(s)Y^0(t)) = 0$, *if* $s \neq t$.

MDE of the number of Fourier coefficients of g.

estimate p the number of Fourier coefficients in

$$g(s) = c_0 1_{[0,1]}(s) + \sum_{k=1}^{p} c_k \sqrt{2} \cos(2\pi k s)$$

$$+ \sum_{k=1}^{p} d_k \sqrt{2} \sin(2\pi k s)$$

Fourier coefficients are known : $c_i = h_l(i)$, $d_i = k_l(i)$

the parameter is $p \in \{1, 2, ..., K\}$ where $K \in N^*$

Define

$$p_\varepsilon^* = \arg \min_{p \in \{1,2,..,K\}} \int_{a_\varepsilon}^{b_\varepsilon} \left(\widehat{f}_\varepsilon(t) - \int_0^\delta X^0(t-s) g(s)(ds) \right)^2 \nu(dt)$$

$\widehat{f}_\varepsilon(.)$ the kernel estimate

ν is a bounded measure on $]0, T[$

Define for all $u \in N^*$

$$l_\varepsilon(u) = \min_{p_0 \in \{1,2,..,L\}} \min_{|p-p_0| \geq u} \left(\int_{a_\varepsilon}^{b_\varepsilon} \left(S(t,p,X_p^0) - S(t,p_0,X_{p_0}^0) \right)^2 \nu(dt) \right)^{1/2}$$

Proposition . *If for all $u \in N^*$ we have $l_\varepsilon(u) > 0$, then there exists $C > 0$ such that*

$$\max_{p_0 \in \{1,2,..,L\}} P_{p_0}^\varepsilon \left(p_\varepsilon^* \neq p_0 \right) \leq 3 \exp \left(-C \frac{l_\varepsilon(u)}{\varepsilon^{\frac{4}{3}}} \right)$$

Maximum likelihood estimates MLE

Consider observations $(X^\varepsilon(t), t \in [0,T])$ of SDE with delays

$$dX^\varepsilon(t) = \left(\int_0^\delta X^\varepsilon(t-s)\mu(ds)\right) dt + \varepsilon dW(t),$$

conditions

$X^\varepsilon(s) = x_0(s), -\delta \leq s \leq 0,$

x_0 positive function on $[-\delta, 0]$,

$(W(t), t \in [0,T])$ Wiener process

μ signed bounded measure on $[0,\delta]$, $\delta > 0$.

deterministic equation

$$\frac{dX^0(t)}{dt} = \int_0^\delta X^0(t-s)\mu(ds), \quad 0 \leq t \leq T$$

$X^0(s) = x_0(s)$, if $-\delta \leq s \leq 0$

estimation of the delays measure density g of μ with a truncated Fourier expansion

$$g(s) = \sum_{k=1}^{p} c_k e_k(s)$$

e_k some known positive functions

$(c_1, ...c_p)$ unknown parameters

integer p is supposed known

$\theta = (c_1, ...c_p) \in \Theta =]0, D[^p$

The solution of SDE induces a measure P_θ^ε on $(C_{[0,T]}, \mathcal{C})$ the maximum likelihood estimates $\widehat{\theta}_\varepsilon$ solution

$$\frac{dP^\varepsilon_{\widehat{\theta}_\varepsilon}}{dP^\varepsilon_{\theta_0}}(X^\varepsilon) = \sup_{\theta \in \bar{\Theta}} \frac{dP^\varepsilon_\theta}{dP^\varepsilon_{\theta_0}}(X^\varepsilon)$$

where θ_0 is a value in $\bar{\Theta}$.

Local asymptotic normality (LAN condition)

$q(t,\theta,X^0)$ the vector of $I\!R^p$ with composantes

$$q_i(t,\theta,X^0) = \int_0^\delta X^0(t-s)e_i(s)ds +$$

$$\sum_{j=1, j\neq i}^{p} a_j \int_0^\delta \frac{\partial X^0}{\partial a_i}(t-s)e_j(s)ds$$

$i = 1, .., p$

$I(\theta)$ the matrix with entries $(I_{i,j}(\theta),\ i,j = 1, .., p)$

$$I_{i,j}(\theta) = \int_0^T q_i(t,\theta,X^0)q_j(t,\theta,X^0)dt$$

Proposition *for all compact K of* Θ*, the family of measures* $(P_\theta^\varepsilon, \theta \in \Theta)$ *satisfy the LAN condition uniformly in* $\theta \in K$*.*
The matrix of normalization $\phi_\varepsilon(\theta) = \varepsilon I^{-1/2}(\theta)$*, and the rv*

$$\Delta_\varepsilon(\theta, X^\varepsilon) = \varepsilon^{-1} I^{-\frac{1}{2}}(\theta) \int_0^T q(\theta, t, X^0)(dX^\varepsilon(t)$$

$$- \left(\int_0^\delta X^\varepsilon(t-s)\mu(ds) \right) dt)$$

and $\Delta_\varepsilon(\theta, X^\varepsilon)$ *is gaussian rv in* $\mathbb{R}^p$ *.*

- **Minimax bound - Hajek inequality** .

Proposition . *For all* $\theta_0 \in \Theta$ *and* $w \in W_{e,2}$ *we have*

$$\lim_{\delta \to 0} \liminf_{\varepsilon \to 0} \inf_{\theta^*_\varepsilon} \sup_{\|\theta - \theta_0\| < \delta} E^\varepsilon_\theta(w(\phi_\varepsilon^{-1}(\theta_0)(\theta^*_\varepsilon - \theta))) \geq Ew(\xi)$$

where ξ *is gaussian rv* $N(0, I)$ *in* $\mathbb{R}^p$, $\phi_\varepsilon(\theta_0)$ *is the matrix of normalization and* $w \in W_{e,2}$ *class of loss functions*

Proposition *Let K be a compact of Θ. Then the MLE $\widehat{\theta}_\varepsilon$ satisfy uniformly for all $\theta_0 \in K$, the following properties :*

1. Under $P^\varepsilon_{\theta_0}$

$$\lim_{\varepsilon \to 0} \widehat{\theta}_\varepsilon = \theta_0 \qquad \textit{in probability.}$$

$$\forall \beta > 0, \quad \sup_{\theta_0 \in K} P^\varepsilon_{\theta_0}\left(\|\widehat{\theta}_\varepsilon - \theta_0\| > \beta\right) \le C_1 \exp\left(-\frac{C_2(\beta)}{\varepsilon^2}\right)$$

where $C_1 > 0$ $C_2 > 0$

2)

$$\phi_\varepsilon^{-1}(\theta_0)(\widehat{\theta}_\varepsilon - \theta_0) \underset{\varepsilon \longrightarrow 0}{\Longrightarrow} \xi$$

where ξ is a zero mean gaussian rv $\mathcal{N}(0, I_d)$ in $I\!R^p$ and $\phi_\varepsilon(\theta_0) = \varepsilon I^{-1/2}(\theta_0)$

Bayes estimates .

The parameter θ is a rv with values in $I\!R^p$ with a prior density $\pi(\theta)$ on Θ
Bayes estimator $\tilde{\theta}_\varepsilon$ with respect to loss functions $w(\theta - y) = ||\theta - y||^d$, $d > 1$ satisfy

$$\int_\Theta w(\theta - \tilde{\theta}_\varepsilon) p(\theta|X^\varepsilon) d\theta = \inf_{y\in\Theta} \int_\Theta w(\theta - y) p(\theta|X^\varepsilon) d\theta$$

$p(\theta|X^\varepsilon)$ is the posterior density of θ given X^ε
In quadratic loss Bayes estimator

$$\tilde{\theta}_\varepsilon(X^\varepsilon) = E(\theta|X^\varepsilon) = \int_\Theta y p(y|X^\varepsilon) dy$$

We have similar asymptotic properties than of MLE

Proposition *If the density* $\pi(\theta)$ *is positive bounded and continuous then the estimator* $\tilde{\theta}_\varepsilon$ *for quadratic loss have the same properties than of the MLE*

- MLE of the number of Fourier coefficients

The density g admits a truncated Fourier expansion

$$g(s) = \sum_{k=1}^{p} c_k e_k(s)$$

e_k are known functions

$(c_1, ...c_p)$ are known ($c_i = f_l(i)$)

the parameter is $p \in \Theta = \{1, 2, .., L\}$, $L \in N^*$.

The solution of SDE induces a measure P_p^ε on $(C_{[0,T]}, \mathcal{C})$

the maximum likelihood estimates $\widehat{p}_\varepsilon$ satisfy

$$\frac{dP^\varepsilon_{\widehat{p}_\varepsilon}}{dP^\varepsilon_{p_0}}(X^\varepsilon) = \max_{p\in\{1,2,..,L\}} \frac{dP^\varepsilon_p}{dP^\varepsilon_{p_0}}(X^\varepsilon) \qquad (2)$$

where $p_0 \in \Theta$

Proposition . *We have*

$$\max_{p_0 \in \Theta} P^{\varepsilon}_{p_0} (\widehat{p}_{\varepsilon} \neq p_0) \leq C_1 \exp\left(-\frac{C_2}{\varepsilon^2}\right)$$

where $C_1 > 0$ *and* $C_2 > 0$

Remark. The same result holds for the Bayes estimates