Estimation of the volatility for stochastic differential equations

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Plan of the talk

1. Introduction: Ibragimov-Has'minskii's program

2. Polynomial type large deviation inequality for a random field derived from a quasi-likelihood of a discretely observed SDE on a fixed interval

3. Asymptotic properties of both maximum contrast estimator and Bayes type estimator

4. Example and simulation result

1. Introduction

In order to study asymptotic properties of statistics in likelihood analysis, Ibragimov and Has'minskii (1972, 1973, 1981) established a new paradigm of likelihood analysis.

$\mathcal{E}^\epsilon = \{\mathcal{X}^\epsilon, \mathcal{A}^\epsilon, (P^\epsilon_\theta)_{\theta\in\Theta}\}$: a sequence of statistical experiments, where $\epsilon \in (0,1]$ and a parameter space $\Theta \subset \mathbf{R}^p$.

For a $\theta^* \in \Theta$, a statistical (likelihood ratio) random field Z_ϵ

$$Z_\epsilon(u) = \frac{dP^\epsilon_{\theta^*+\phi(\epsilon)u}}{dP^\epsilon_{\theta^*}}(X^\epsilon)$$

for $u \in \mathbf{R}^p$, where $\phi(\epsilon)$ is a positive normalizing factor tending to zero as $\epsilon \to 0$.

$\widehat{C}(\mathbf{R}^p)$: the space of continuous functions on $\mathbf{R}^p$ that tends to 0 at the infinity.

$C_\uparrow(\mathbf{R}^p)$: the space of continuous functions on $\mathbf{R}^p$ of at most polynomial growth.

A simplified version of their result is as follows.

Theorem 1 (Ibragimov and Has'minskii (1972, 1973, 1981)) *Suppose that Z_ϵ meets the following conditions.*

(i) There exist $\alpha > p$ and $k \geq \alpha$ such that for some constant $C > 0$,

$$E^\epsilon_{\theta^*}\left[\left|Z_\epsilon(u_2)^{1/k} - Z_\epsilon(u_1)^{1/k}\right|^k\right] \leq C|u_2 - u_1|^\alpha \quad \text{for all } u_1, u_2, \epsilon.$$

(ii) For some $\gamma > 0$ and $c > 0$,

$$P^\epsilon_{\theta^*}\left[Z_\epsilon(u) \geq e^{-c|u|^\gamma}\right] \leq e^{-c|u|^\gamma}.$$

(iii) Finite-dimensional convergence: $Z_\epsilon \to^{d_f} Z$, where Z is a $\widehat{C}(\mathbf{R}^p)$-valued random variable.

Then, $(P^\epsilon_{\theta^*})^{Z_\epsilon} \to \mathcal{L}\{Z\}$. Moreover,

$$P^\epsilon_{\theta^*}\left[\sup_{u:|u|\geq r} Z_\epsilon(u) \geq e^{-c_1 r^\gamma}\right] \leq e^{-c_1 r^\gamma}.$$

If $\widehat{u}$ uniquely attains the maximum of $Z(u)$, then for any sequence of the maximum likelihood estimator $\widehat{\theta}_\epsilon$ for θ, $\widehat{u}_\epsilon := \phi(\epsilon)^{-1}(\widehat{\theta}_\epsilon - \theta^*) \to^d \widehat{u}$ and

$$E^\epsilon_{\theta^*}\left[f(\widehat{u}_\epsilon)\right] \to E[f(\widehat{u})]$$

for all $f \in C_\uparrow(\mathbf{R}^p)$.

The most essential part of their method is the use of the exponential type large deviation (ELP) inequality

$$P_{\theta^*}^{\epsilon}\left[Z_\epsilon(u) \geq e^{-c|u|^\gamma}\right] \leq e^{-c|u|^\gamma} \tag{1}$$

for some $\gamma > 0$ and $c > 0$, and strong properties such as the convergence of moments of the estimator are derived from it.

Kutoyants (1984, 1994, 1998, 2004) applied the I-H approach to stochastic processes including diffusion type processes and point processes.

As an important observation, the ELD inequality like (1) is much stronger than our use. (It was written in Ibragimov-Has'minskii's papers.)

In order to develop a theory, it is sufficient to obtain the following polynomial type large deviation (PLD) inequality.

$$P_{\theta^*}^{\epsilon}\left[\sup_{u:|u|\geq r} Z_\epsilon(u) \geq r^{-N}\right] \leq \frac{C_N}{r^N}.$$

In particular, Kutoyants (2004) presented a PLD inequality for one-dimensional diffusion process by means of the local time.

In case that the I-H-K program is applied to discretely observed diffusion type processes, there is a serious problem that we do not generally have an explicit form of likelihood function.

The PLD inequality for an abstract statistical random field based on the (partially) local asymptotically quadratic (LAQ) sequence of experiments was obtained by Yoshida (2005).

This approach enables us to connect
the I-H-K program for stochastic processes
to the quasi-likelihood analysis for discretely sampled SDEs.

As an example, for multi-dimensional mixing diffusion processes, the convergence of moments of quasi-MLE, asymptotic normality of Bayes type estimator and the convergence of moments of it were shown in Yoshida (2005).

Consider a d-dimensional Itô process Y defined by the SDE

$$dY_t = b_t dt + \sigma(X_t, \theta) dw_t, \quad t \in [0, T], \tag{2}$$

where T is fixed,

w is an r-dimensional standard Wiener process on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$,

b and X are progressively measurable processes with values in $\mathbf{R}^d$ and $\mathbf{R}^{d_1}$, respectively,

σ is an $\mathbf{R}^d \otimes \mathbf{R}^r$-valued function defined on $\mathbf{R}^{d_1} \times \Theta$,

Θ is a bounded convex domain in $\mathbf{R}^p$.

θ^* denotes the true value of θ.

Data : $\mathbf{Z}_n = (X_{t_k}, Y_{t_k})_{0 \le k \le n}$ with $t_k = kh$ for $h = h_n = T/n$.

Asymptotics : $n \to \infty$, which means that $\mathbf{Z}_n$ are high frequency data.

Note that if an argument of X_t is t, then the volatility in the model (2) is time dependent. Furthermore, if we set that $b_t = b(Y_t, t)$ and $X_t = (Y_t, t)$, then Y is the time-inhomogeneous diffusion process.

History:

Asymptotic theory of estimation for an unknown parameter θ in the volatility based on high frequency data observed on a fixed interval has been developed.

- Dohnal (1987) has shown the local asymptotic mixed normality (LAMN) property for the likelihood in the case of one-dimensional diffusions.
- For the LAMN property in the case of multi-dimensional diffusions, see Genon-Catalot and Jacod (1994) and Gobet (2001).
- Genon-Catalot and Jacod (1993, 1994) proposed contrast functions for diffusion type processes and they proved the asymptotic mixed normality of the minimum contrast estimator.

In this talk, asymptotic mixed normality and convergence of moments of both the maximum contrast estimator and the Bayes type estimator for an unknown parameter θ in the volatility of the SDE (2) will be shown.

The key point is to obtain the PLD inequality for the statistical random field.

2. Polynomial type large deviation inequality

Let $C_{\uparrow}^{k,l}(\mathbf{R}^d \times \Theta; \mathbf{R}^d)$ denote the space of all functions f satisfying the following conditions: (i) $f(x,\theta)$ is an $\mathbf{R}^d$-valued function on $\mathbf{R}^d \times \Theta$, (ii) $f(x,\theta)$ is continuously differentiable with respect to x up to order k for all θ, and their derivatives up to order k are of polynomial growth in x uniformly in θ. (iii) for $|\mathbf{n}| = 0, 1, \ldots, k$, $\partial_x^{\mathbf{n}} f(x,\theta)$ is continuously differentiable with respect to θ up to order l for all x. Moreover, for $|\nu| = 0, 1, \ldots, l$ and $|\mathbf{n}| = 0, 1, \ldots, k$, $\partial_\theta^\nu \partial_x^{\mathbf{n}} f(x,\theta)$ is of polynomial growth in x uniformly in θ. Here $\mathbf{n} = (n_1, \ldots, n_d)$ and $\nu = (\nu_1, \ldots, \nu_p)$ are multi-indices, $p = \dim(\Theta)$, $|\mathbf{n}| = n_1 + \ldots + n_d$, $|\nu| = \nu_1 + \ldots + \nu_p$, $\partial_x^{\mathbf{n}} = \partial_{x_1}^{n_1} \cdots \partial_{x_d}^{n_d}$, $\partial_{x_i} = \partial/\partial x_i$, and $\partial_\theta^\nu = \partial_{\theta_1}^{\nu_1} \cdots \partial_{\theta_m}^{\nu_m}$, $\partial_{\theta_i} = \partial/\partial\theta_i$. Let P_θ denote the law of the process Y defined by (2). We denote by $\to^p$ and $\to^{d_s(\mathcal{F}_T)}$ the convergence in probability and the $\mathcal{F}_T$-stable convergence in distribution, respectively. For matrices A and B of the same size, we write $A^{\otimes 2} = AA^\star$ and $A[B] = tr(AB^\star)$, where $\star$ means the transpose.

2. Polynomial type large deviation inequality

$$\text{(stat. model)} \quad dY_t \;=\; b_t dt + \sigma(X_t,\theta)dw_t, \quad t \in [0,T].$$

Set $S(x,\theta) = \sigma(x,\theta)^{\otimes 2} := \sigma\sigma^{\star}(x,\theta)$ and $\Delta_k Y = Y_{t_k} - Y_{t_{k-1}}$.

Let $\mathbb{U}_n = \{u \in \mathbf{R}^p \;;\; \theta^* + (1/\sqrt{n})u \in \Theta\}$.

We make the following assumption.

$[H1]$ (i) $E_{\theta^*}[|X_0|^q] < \infty$ for all $q > 0$. For every $q > 0$, there exists $C > 0$, $E_{\theta^*}[|X_t - X_s|^q] \le C|t-s|^{q/2}$ for all $t, s \in [0,T]$.
(ii) $\sup_{0\le t\le T} E_{\theta^*}[|b_t|^q] < \infty$ for all $q > 0$.
(iii) $\sigma \in C^{2,4}_{\uparrow}(\mathbf{R}^d \times \Theta; \mathbf{R}^d \otimes \mathbf{R}^r)$ and $\inf_{x,\theta} \det S(x,\theta) > 0$.

We define the random field $\mathbb{Z}_n(u)$ for $u \in \mathbb{U}_n$ by

$$\mathbb{Z}_n(u) = \exp\left\{\mathbb{H}_n\left(\theta^* + \frac{1}{\sqrt{n}}u\right) - \mathbb{H}_n(\theta^*)\right\},$$

where

$$\mathbb{H}_n(\theta) = -\frac{nd}{2}\log(2\pi h) - \frac{1}{2}\sum_{k=1}^{n}\left\{\log\det S(X_{t_{k-1}},\theta) + h^{-1}S^{-1}(X_{t_{k-1}},\theta)[(\Delta_k Y)^{\otimes 2}]\right\}.$$

Note that for $u \in \mathbb{U}_n$,

$$\mathbb{Z}_n(u) = \exp\left(\Delta_n[u] - \frac{1}{2}\Gamma(\theta^*)[u,u] + r_n(u)\right),$$

where

$$\begin{aligned}
\Delta_n[u] &= \frac{1}{\sqrt{n}}\partial_\theta \mathbb{H}_n(\theta^*)[u] \\
&= -\frac{1}{2\sqrt{n}}\sum_{k=1}^{n}\left\{(\partial_\theta \log\det S(X_{t_{k-1}},\theta^*))[u] + h^{-1}(\partial_\theta S^{-1})(X_{t_{k-1}},\theta^*)[u,(\Delta_k Y)^{\otimes 2}]\right\}, \\
\Gamma_n(\theta)[u,u] &= -\frac{1}{n}\partial_\theta^2 \mathbb{H}_n(\theta)[u,u] \\
&= \frac{1}{2n}\sum_{k=1}^{n}\left\{(\partial_\theta^2 \log\det S(X_{t_{k-1}},\theta))[u^{\otimes 2}] + h^{-1}(\partial_\theta^2 S^{-1})(X_{t_{k-1}},\theta))[u^{\otimes 2},(\Delta_k Y)^{\otimes 2}]\right\}, \\
\Gamma(\theta^*)[u,u] &= \frac{1}{2T}\int_0^T tr\left((\partial_\theta S)S^{-1}(\partial_\theta S)S^{-1}(X_t,\theta^*)[u^{\otimes 2}]\right)dt, \\
r_n(u) &= \int_0^1 (1-s)\left\{\Gamma(\theta^*)[u,u] - \Gamma_n(\theta^* + s(1/\sqrt{n})u)[u,u]\right\}ds.
\end{aligned}$$

Let

$$\mathbb{Y}_n(\theta) = \frac{1}{n}\left\{\mathbb{H}_n(\theta) - \mathbb{H}_n(\theta^*)\right\},$$

$$\mathbb{Y}(\theta) = -\frac{1}{2T}\int_0^T \left\{\log\left(\frac{\det S(X_t,\theta)}{\det S(X_t,\theta^*)}\right) + tr\left(S^{-1}(X_t,\theta)S(X_t,\theta^*) - I_d\right)\right\} dt.$$

Let $\chi_0 = \inf_{\theta\neq\theta^*} \frac{-\mathbb{Y}(\theta)}{|\theta-\theta^*|^2}$.

$[H2]$ For every $L > 0$, there exists $c_L > 0$ such that for all $r > 0$,

$$P_{\theta^*}\left[\chi_0 \leq r^{-1}\right] \leq \frac{c_L}{r^L}.$$

A sufficient condition for $[H2]$.

For simplicity, we set $d = d_1 = r = 1$.

Let $\mathcal{X}_0$ be a subset of $\mathbf{R}$ satisfying that $\mathrm{supp}\mathcal{L}(X_0) \subset \mathcal{X}_0$.

We assume that there exist a set $\hat{\mathcal{X}}$ including $\mathcal{X}_0$ and a function $f(x,\theta) : \hat{\mathcal{X}} \times \Theta \to \mathbf{R}$ such that for every $(x,\theta) \in \hat{\mathcal{X}} \times \Theta$,

$$\left(\frac{S(x,\theta^*)}{S(x,\theta)} - 1 - \log \frac{S(x,\theta^*)}{S(x,\theta)}\right) |\theta - \theta^*|^{-2} \geq c|f(x,\theta)|^2$$

for some $c > 0$ independent of (x,θ). Moreover, we assume that f has the following form:

$$f(x,\theta) = \sum_{j=0}^{J-1} c_j(x_0,\theta)[(x-x_0)^j] + c_J(x_0,x,\theta)[(x-x_0)^J],$$

where $c_j : \mathcal{X}_0 \times \Theta \to \mathbf{R}$ for $j = 0, ..., J-1$, and $c_J : \mathcal{X}_0 \times \hat{\mathcal{X}} \times \Theta \to \mathbf{R}$.

Lemma 1 *Suppose that the following conditions are satisfied:*

(i) $\hat{\mathcal{X}}$, $\mathcal{X}_0$ *and* Θ *are compact.*

(ii) c_j $(j = 0, ..., J-1)$ *are continuous on* $\mathcal{X}_0 \times \Theta$*, and* c_J *is bounded on* $\mathcal{X}_0 \times \hat{\mathcal{X}} \times \Theta$*.*

(iii) For each $(x_0, \theta) \in \mathcal{X}_0 \times \Theta$, $\max_{j=0,...,J-1} \left| c_j(x_0, \theta) \right| > 0$.

(iv) The process $X = (X_t)$ *is a diffusion type process that is uniformly elliptic on* $\mathcal{X}_0$*.*

Then, $[H2]$ *holds.*

Example 1. $d = d_1 = r = 1$, $X_0 = x_0 = 0$, $\hat{\mathcal{X}} = [-\epsilon, \epsilon]$, $\Theta = [-\pi, \pi]$, $\theta^* \in \mathrm{Int}[\Theta]$.

$$S(x, \theta) = \sigma^2(x, \theta) = \exp(\theta \sin^2 x).$$

For small $\hat{\mathcal{X}}$, for every $(x, \theta) \in \hat{\mathcal{X}} \times \Theta$,

$$|S(x,\theta)^{-1}S(x,\theta^*) - 1 - \log\{S(x,\theta)^{-1}S(x,\theta^*)\}||\theta - \theta^*|^{-2} \geq c|f(x,\theta)|^2$$

for some $c > 0$ independent of (x, θ), and

$$\begin{aligned} f(x,\theta) &= (\theta - \theta^*)^{-1} \log \frac{S(x,\theta^*)}{S(x,\theta)} = -\sin^2 x \\ &= \sum_{j=0}^{2} c_j(x_0,\theta)[(x-x_0)^j] + c_3(x_0,x,\theta)[(x-x_0)^3], \end{aligned}$$

where

$$\begin{aligned} c_0(x_0,\theta) &= -\sin^2 x_0 = 0, \\ c_1(x_0,\theta) &= -2\sin x_0 \cos x_0 = 0, \\ c_2(x_0,\theta) &= -\cos^2 x_0 + \sin^2 x_0 = -1, \end{aligned}$$

and $c_3(x_0, x, \theta)$ is bounded on $\hat{\mathcal{X}} \times \Theta$.

Thus, by Lemma 1, $[H2]$ holds if X is a non-degenerate diffusion.

Example 2. $d = d_1 = r = 1$, $X_0 = x_0 = 0$, $\hat{\mathcal{X}} = [-\epsilon, \epsilon]$, $\Theta = [0, \pi]$, $\theta^* \in \mathrm{Int}[\Theta]$.

$$S(x,\theta) = \sigma^2(x,\theta) = \exp\left(\sin\theta \; \sin x - \theta^2 \sin^2 x \right).$$

For small $\hat{\mathcal{X}}$, for every $(x,\theta) \in \hat{\mathcal{X}} \times \Theta$,

$$|S(x,\theta)^{-1}S(x,\theta^*) - 1 - \log\{S(x,\theta)^{-1}S(x,\theta^*)\}||\theta - \theta^*|^{-2} \geq c|f(x,\theta)|^2$$

for some $c > 0$ independent of (x,θ), and

$$\begin{aligned} f(x,\theta) &= \frac{\sin\theta - \sin\theta^*}{\theta - \theta^*} \sin x - (\theta + \theta^*) \sin^2 x \\ &= \sum_{j=0}^{2} c_j(x_0,\theta)[(x - x_0)^j] + c_3(x_0, x, \theta)[(x - x_0)^3], \end{aligned}$$

where

$$\begin{aligned} c_0(x_0,\theta) &= \frac{\sin\theta - \sin\theta^*}{\theta - \theta^*} \sin x_0 - (\theta + \theta^*) \sin^2 x_0 = 0, \\ c_1(x_0,\theta) &= \frac{\sin\theta - \sin\theta^*}{\theta - \theta^*} \cos x_0 - (\theta + \theta^*) \sin x_0 \cos x_0 = \frac{\sin\theta - \sin\theta^*}{\theta - \theta^*}, \\ c_2(x_0,\theta) &= -\frac{1}{2}\frac{\sin\theta - \sin\theta^*}{\theta - \theta^*} \sin x_0 - (\theta + \theta^*)(\cos^2 x_0 - \sin^2 x_0) = -(\theta + \theta^*) \neq 0, \end{aligned}$$

and $c_3(x_0, x, \theta)$ is bounded on $\hat{\mathcal{X}} \times \Theta$. Note that $\inf_\theta |c_1(x_0,\theta)| = 0$ if $\theta^* = \pi/2$.

Therefore, by Lemma 1, $[H2]$ holds if X is a non-degenerate diffusion.

Let $V_n(r) = \{u \in \mathbb{U}_n \ ; \ r \leq |u|\}$.

Theorem 2 *Assume* $[H1]$-$[H2]$. *Then, for every* $L > 0$, *there exists a positive constant* C_L *such that*

$$P_{\theta^*}\left[\sup_{u \in V_n(r)} \mathbb{Z}_n(u) \geq e^{-r}\right] \leq \frac{C_L}{r^L}$$

for all $r > 0$ *and* $n \in \mathbf{N}$.

Sketch of Proof. By Theorem 1 in Yoshida (2005), it is enough to show the following lemmas.

Lemma 2 *Assume* $[H1]$*. Then, for every* $q > 0$*,*
(i)

$$\sup_{n \in \mathbf{N}} E_{\theta^*} \left[|\Delta_n|^q \right] < \infty.$$

(ii)

$$\sup_{n \in \mathbf{N}} E_{\theta^*} \left[\left(\sup_{\theta \in \Theta} \sqrt{n} \left| \mathbb{Y}_n(\theta) - \mathbb{Y}(\theta) \right| \right)^q \right] < \infty.$$

Lemma 3 *Assume* $[H1]$*. Then, for every* $q > 0$*,*
(i)

$$\sup_{n \in \mathbf{N}} E_{\theta^*} \left[\left(\sqrt{n} \left| \Gamma_n(\theta^*) - \Gamma(\theta^*) \right| \right)^q \right] < \infty.$$

(ii)

$$\sup_{n \in \mathbf{N}} E_{\theta^*} \left[\left(\frac{1}{n} \sup_{\theta \in \Theta} \left| \partial_\theta^3 \mathbb{H}_n(\theta) \right| \right)^q \right] < \infty.$$

3. Maximum contrast estimator and Bayes type estimator

Let $\widehat{\theta}_n^{(M)}$ be the maximum contrast estimator defined as

$$\mathbb{H}_n(\widehat{\theta}_n^{(M)}) = \sup_{\theta \in \Theta} \mathbb{H}_n(\theta). \tag{3}$$

Let $\widetilde{\theta}_n^{(B)}$ be the Bayes type estimator for a prior density $\pi : \Theta \to \mathbf{R}_+$ defined as

$$\widetilde{\theta}_n^{(B)} = \left(\int_\Theta \exp(\mathbb{H}_n(\theta))\pi(\theta)d\theta \right)^{-1} \int_\Theta \theta \exp(\mathbb{H}_n(\theta))\pi(\theta)d\theta. \tag{4}$$

We assume that π is continuous and
$0 < \inf_{\theta \in \Theta} \pi(\theta) \leq \sup_{\theta \in \Theta} \pi(\theta) < \infty$.

By setting $\tilde{u}_n^{(B)} = \sqrt{n}(\tilde{\theta}_n^{(B)} - \theta^*)$,

$$\tilde{u}_n^{(B)} = \left(\int_{\mathbb{U}_n} \mathbb{Z}_n(u)\pi(\theta^* + (1/\sqrt{n})u)du\right)^{-1} \int_{\mathbb{U}_n} u\mathbb{Z}_n(u)\pi(\theta^* + (1/\sqrt{n})u)du. \quad (5)$$

Let

$$\mathbb{Z}(u) = \exp\left(\Gamma(\theta^*)^{1/2}\zeta[u] - \frac{1}{2}\Gamma(\theta^*)[u,u]\right),$$

where ζ is a p-dimensional standard normal random variable independent of $\Gamma(\theta^*)$.

Let

$$\tilde{u}^{(B)} = \left(\int_{\mathbf{R}^p} \mathbb{Z}(u)du\right)^{-1} \int_{\mathbf{R}^p} u\mathbb{Z}(u)du \ \left(= \Gamma(\theta^*)^{-1/2}\zeta\right). \quad (6)$$

In order to obtain the weak convergence of the statistical random field on compact sets, we make the following assumption.

$[H1']$ Assumption $[H1]$ holds and

$$X_t = X_0 + \int_0^t \tilde{b}_s ds + \int_0^t a_s dw_s + \int_0^t \tilde{a}_s d\tilde{w}_s,$$

where $\tilde{b}$, a and $\tilde{a}$ are locally bounded predictable with values in $\mathbf{R}^{d_1}$, $\mathbf{R}^{d_1} \otimes \mathbf{R}^r$ and $\mathbf{R}^{d_1} \otimes \mathbf{R}^{r_1}$, respectively, b is locally bounded and $\tilde{w}$ is an r_1-dimensional Wiener process independent of w.

We denote by $B(R) = \{u \in \mathbf{R}^p \ ; \ |u| \leq R\}$.

Lemma 4 *Assume* $[H1']$. *Then, for every* $R > 0$,

$$\mathbb{Z}_n(u) \to^{d_s(\mathcal{F}_T)} \mathbb{Z}(u)$$

in $C(B(R))$ *as* $n \to \infty$.

Theorem 3 *Assume* $[H1']$ *and* $[H2]$. *Then,*

$$\sqrt{n}(\widehat{\theta}_n^{(M)} - \theta^*) \to^{d_s(\mathcal{F}_T)} \Gamma(\theta^*)^{-1/2}\zeta$$

and

$$E_{\theta^*}\left[f(\sqrt{n}(\widehat{\theta}_n^{(M)} - \theta^*))\right] \to \mathbb{E}\left[f(\Gamma(\theta^*)^{-1/2}\zeta)\right]$$

as $n \to \infty$ *for all* $f \in C_{\uparrow}(\mathbf{R}^p)$.

Theorem 4 *Assume* $[H1']$ *and* $[H2]$. *Then,*

$$\sqrt{n}(\widetilde{\theta}_n^{(B)} - \theta^*) \to^{d_s(\mathcal{F}_T)} \Gamma(\theta^*)^{-1/2}\zeta$$

and

$$E_{\theta^*}\left[f(\sqrt{n}(\widetilde{\theta}_n^{(B)} - \theta^*))\right] \to \mathbb{E}\left[f(\Gamma(\theta^*)^{-1/2}\zeta)\right]$$

as $n \to \infty$ *for all* $f \in C_{\uparrow}(\mathbf{R}^p)$.

4. Example and simulation result

As an example, we consider the one-dimensional diffusion process

$$dX_t \;=\; X_t dt + \exp\{\theta \sin^2 X_t\} dw_t, \quad t \in [0,1], \quad X_0 = 0, \quad (7)$$

where $\theta \in [-\pi, \pi]$.

Here we examine the asymptotic behaviour of both

the MCE $\hat{\theta}_n^{(M)}$ and the BE $\tilde{\theta}_n^{(B)}$ w.r.t. the uniform prior $\pi(\theta)$

through the simulations, which were done

for each $h_n = 1/50, 1/250, 1/500$.

For the true model (7) with $\theta^* = 1$, 10000 independent sample paths are generated by the Milstein scheme, and the means and the standard deviations of the estimators are computed and shown in Table 1 below.

$$dX_t = X_t dt + \exp\{\theta \sin^2 X_t\} dw_t, \quad t \in [0,1], \quad X_0 = 0.$$

Table 1. The mean and standard deviation (s.d.) of the estimators for 10000 independent simulated sample paths with $\theta^* = 1$.

h_n	$\widehat{\theta}_n^{(M)}$ mean	$\widehat{\theta}_n^{(M)}$ s.d.	$\widetilde{\theta}_n^{(B)}$ mean	$\widetilde{\theta}_n^{(B)}$ s.d.
1/50	0.90938	0.55704	0.97465	0.47647
1/250	0.98181	0.23022	0.99714	0.22370
1/500	0.99354	0.16436	1.00164	0.16236

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