ON STABILITY OF CALL/PUT OPTION PRICES IN INCOMPLET MODELS UNDER STATISTICAL ESTIMATIONS

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Plan

General model

We consider the model with two assets. The model of risky asset is a semimartingale model of the type $S = (S_t)_{t\geq 0}$:

$$S_t = S_0 \exp(X_t)$$

where $X = (X_t)_{t\geq 0}$ is a semi-martingale. The model for non-risky asset is given by:

$$B_t = B_0 \exp(rt)$$

where r is a positive constant.
To simplify the presentation we assume without restrictions that

$$S_0 = B_0 = 1,\ r = 0$$

Some exemples

- Usually the law of this semi-martingale depends on unknown parameter, say $\theta \in \Theta$, where Θ is some space.

- In Black-Scholes model we have:

$$X_t = (\mu - \sigma^2/2)t + \sigma W_t$$

where $W = (W_t)_{t \geq 0}$ is Wiener process. In this case $\theta = (\mu, \sigma)$ and $\Theta = \mathbb{R} \times \mathbb{R}^{+,\star}$.

- In Geometric Variance Gamma model , as well known,

$$X_t = \mu\tau_t + \sigma W_{\tau_t}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, W is Wiener process $(\tau_t)_{t \geq 0}$ is, independent from W, Gamma process with parameters $(1, \nu)$, $\nu > 0$. In this case $\theta = (\mu, \sigma, \nu)$ and $\Theta = \mathbb{R} \times \mathbb{R}^{+,*} \times R^{+,*}$.

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Some exemples

- In GMY model, as well known the process X has the same structure as in VG model but with $(\tau_t)_{t\geq 0}$ being Levy process with Levy measure

$$\nu(dx) = \begin{cases} \frac{C\exp(-Mx)}{x^{1+\alpha}}dx & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x < 0. \end{cases}$$

where $\alpha < 2$, $C > 0$ and $M \geq 0$. Then, obviously, $\theta = (C, M, \alpha)$ and $\Theta = \mathbb{R}^{+,*} \times \mathbb{R}^{+} \times] - \infty, 2[$.

Call/put options

The classical procedure of calculus of call/put option price $\mathbb{C}_T$ of maturity time T consists to choose the type of the option given by a continuous in the space $D([0, T])$ functional $g(\cdot)$, then to choose in the set of equivalent martingale measures $\mathcal{M}(P)$, supposed non-empty, a "good" one, say Q, and to put:

$$\mathbb{C}_T = \mathbb{E}_Q(g(S)).$$

Equivalent martingale measures

As we know, there exist many approaches to chose a "good" equivalent martingale measure:

1. minimisation of the risk in L^2-sense (Follmer, Schweizer (1995)),
2. minimisation of Hellinger integrals (Chouli, Stricker (2007)),
3. minimisation of entropy (Miyahara, Fudjara (2003))
4. minimisation of f-divergence (Jeanblanc, Kloppel, Miyahara (2007))

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Estimation and calibration

- We remark that since the law of X^θ depends on θ, the price $\mathbb{C}_T$ does it as well. To ajust the "good" value of θ one perform then so called **calibration** which is equivalent, from statistical point of view, to find a minimal distance estimator or contrast estimator with very special contrast (Millar, Kutoyants)
- One can use also another approach and consider maximum likelihood estimators or Bayesion estimators for the unknown parameters (Ibragimov, Hasminskij, Kutoyants). The conditions for weak convergence of these processes in terms of Hellinger processes can be found in Vostrikova (1986).
- When the density of the law of X with respect to some majorating measure can not be expressed explicitely or when it is too complicated, one can use moment estimators.

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Our goal

- The estimation procedure change the value of the price $\mathbb{C}_T(\theta)$ which become $\mathbb{C}_T(\hat{\theta})$ where $\hat{\theta}$ is an estimator of θ. So, it is important from point of view of stability of the procesure to measure the distance between estimated $\mathbb{C}_T(\hat{\theta})$ and "true" price $\mathbb{C}_T(\theta)$.
- Our goal is to evaluate L^1 distance between these quantities, namely $\mathbb{E}^{\theta}|\mathbb{C}_T(\hat{\theta}) - \mathbb{C}_T(\theta)|$ where the expectation is taken with respect to "physical" measure P_{θ}.

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Inconsistent estimators

- We notice the importance of use of consistent estimators of θ in this procesure. In fact, usually $\mathbb{C}_T(\theta) \neq \mathbb{C}_T(\theta')$ for $\theta \neq \theta'$. If the sequence of estimators is not consintent, then under some mild conditions one can extract a subsequence $(\hat{\theta}^n)$ converging $P - a.s.$ to $\theta + \delta$ with $\delta \neq 0$.
- Then $\mathbb{E}^\theta |\mathbb{C}_T(\hat{\theta}^n) - \mathbb{C}_T(\theta)|$ is converging to $|\mathbb{C}_T(\theta + \delta) - \mathbb{C}_T(\theta)|$ which is different from zero. It means that without arbitrage condition we can have asumptotic arbitrage consequences if $\mathbb{C}_T(\theta + \delta) \neq \mathbb{C}_T(\theta)$.

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Binary models

- We suppose that we are given with a filtered canonical space of cadlag functions $(\Omega, \mathcal{F}, \mathbb{F})$ where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is the right-continuous filtration such that $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let P and $\tilde{P}$ be two equivalent probability measures on $(\Omega, \mathcal{F})$ and we denote by P_t and $\tilde{P}_t$ the restrictions of these measures on the σ-algebra $\mathcal{F}_t$, $t \geq 0$.
- The measures P and $\tilde{P}$ corresponds to the laws of our semimartingale $X = (X_t)_{t \geq 0}$ under two fixed values of parameter.
- We suppose that X has predictable represetation property with respect to P and the caracteristics of X are (B, C, ν) and $(\tilde{B}, \tilde{C}, \tilde{\nu})$ respectively.

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Notations

- As usual we denote by $||P - \tilde{P}||$ the variation distance between the measures P and $\tilde{P}$, i.e.

$$||P - \tilde{P}|| = 2 \sup_{A \in \mathcal{F}} |P(A) - \tilde{P}(A)|$$

- Let $\mathcal{M}(P)$ and $\mathcal{M}(\tilde{P})$ be the sets of equivalent martingale measures which are supposed to be non-empty. Let g be measurable functional in $D([0, T]$.

- We choose, then, using some procedure, two martingale measures: Q and $\tilde{Q}$ to calculate call/put option prices: $\mathbb{C}_T$ and $\tilde{\mathbb{C}}_T$ of maturity time T:

$$\mathbb{C}_T = E_Q[g(S)], \qquad \tilde{\mathbb{C}}_T = E_{\tilde{Q}}[g(S)].$$

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- We introduce also dual measures Q' and $\tilde{Q}'$ (Eberlein, Papapantoleon, Shiryaev) by:

$$\frac{dQ'_T}{dQ_T} = S_T, \qquad \frac{d\tilde{Q}'_T}{d\tilde{Q}_T} = S_T.$$

- So, the measures involved in calculation can be represented by the following diagrammes containing initial measure, martingale measure and dual measure:

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Lemma 1

Lemma

Let g be measurable functional in $D([0,T],\mathbb{R}^+)$ *verifying:*

$$|g(x)| \leq c|x_T| + d$$

where c, d *are positif constants. Then for call/ put options price corresponding to* g *we have:*

$$|\mathbb{C}_T - \tilde{\mathbb{C}}_T| \leq c||Q'_T - \tilde{Q}'_T|| + d||Q_T - \tilde{Q}_T||$$

where $||\cdot||$ *is a variation distance between the restriction of the corresponding measures on* σ*-algebra* $\mathcal{F}_T$*.*

Lemma2

Lemma

We have the following estimation for the variation distance via Hellinger processes: for $\epsilon > 0$*:*

$$||Q_T - \tilde{Q}_T|| \leq 4\sqrt{E_Q\, h_T(\frac{1}{2}, Q, \tilde{Q})}$$

$$||Q_T - \tilde{Q}_T|| \leq 3\sqrt{2\epsilon} + 2Q(\, h_T(\frac{1}{2}, Q, \tilde{Q}) \geq \epsilon\,)$$

Lemma 3

Lemma

a) The predictable caracteristics of X with respect to the measure Q' via P are given by:

$$\begin{cases} B' = B + \beta^Q \bullet C + l \cdot (e^x \cdot Y^Q - 1) \star \nu \\ C' = C \\ \nu' = e^x \cdot Y^Q \star \nu \end{cases}$$

where $l(\cdot)$ is a truncation function and (β^Q, Y^Q) are Girsanov parameters to pass from P to Q.

Lemma 3

b) The predictable caracteristics of X with respect to the measure $\tilde{Q}$ via P are given by:

$$\begin{cases} B^{\tilde{Q}} = B + (\beta + \beta^{\tilde{Q}}) \bullet C + I \cdot (Y^{\tilde{Q}} \cdot Y - 1) \star \nu \\ C^{\tilde{Q}} = C \\ \nu^{\tilde{Q}} = Y^{\tilde{Q}} \cdot Y \star \nu \end{cases}$$

where $I(\cdot)$ is a truncation function, $(\beta^{\tilde{Q}}, Y^{\tilde{Q}})$ and (β, Y) are Girsanov parameters which permit us to pass from $\tilde{P}$ to $\tilde{Q}$ and from P to $\tilde{P}$ respectively.

Lemma 3

c) The predistable caracteristics of X with respect to the measure $\tilde{Q}'$ via P are given by:

$$\begin{cases} B^{\tilde{Q}'} = B + (1 + \beta + \beta^{\tilde{Q}}) \bullet C + l \cdot (e^x \cdot Y^{\tilde{Q}} \cdot Y - 1) \star \nu \\ C^{\tilde{Q}'} = C \\ \nu^{\tilde{Q}'} = e^x \cdot Y^{\tilde{Q}} \cdot Y \star \nu \end{cases}$$

Lemma 4

Lemma

let X be a process without fixed points of discontinuity with respect to P. We assume that there exists a kernel $K(dx,t)$ such that we have a desintegration formula:

$$d\nu = K(dx,t)dC_t$$

where C is predictable variation of continuous martingale part of X if it is not zero, and some increasing predictable process if not.

Lemma 4

Then the Hellinger Processes of order 1/2 of the measures P and $\tilde{P}$, Q and $\tilde{Q}$, Q' and $\tilde{Q}'$ are given respectively by:

$$h(\frac{1}{2}, P, \tilde{P}) = \frac{1}{8}(\beta)^2 \bullet C + \frac{1}{2}\left(1 - \sqrt{Y}\right)^2 \star \nu$$

$$h(\frac{1}{2}, Q, \tilde{Q}) = \frac{1}{8}(\beta^{Q} - \beta^{\tilde{Q}} - \beta)^2 \bullet C + \frac{1}{2}\left(\sqrt{Y^{Q}} - \sqrt{Y^{\tilde{Q}} \cdot Y}\right)^2 \star \nu$$

Lemma 4

$$h(\frac{1}{2}, Q', \tilde{Q}') = \frac{1}{8}(\beta^{Q} - \beta^{\tilde{Q}} - \beta)^2 \bullet C + \frac{\exp(x)}{2}\left(\sqrt{Y^{Q}} - \sqrt{Y^{\tilde{Q}} \cdot Y}\right)^2 \star \nu$$

In addition we have ($P \times \lambda_C$ -a.s.)

$$\beta^{Q} - \beta^{\tilde{Q}} - \beta = (\exp(x) - 1)\left(Y^{\tilde{Q}} \cdot Y - Y^{Q}\right) \star K(dx, \cdot)$$

Important processes

Let us introduce the processes $\rho(Q,\tilde{Q})$ and $\rho(P,\tilde{P})$ which are closely related with the Hellinger processes, namely with their integral part with respect to the compensator of the jump measure of X: for all $t \geq 0$

$$\rho_t(Q,\tilde{Q}) = \int_0^t \int_{\mathbb{R}^*} \left(\sqrt{Y^{\tilde{Q}}} - \sqrt{Y^{Q}}\right)^2 d\nu,$$

$$\rho_t(P,\tilde{P}) = \int_0^t \int_{\mathbb{R}^*} \left(1 - \sqrt{Y}\right)^2 d\nu.$$

Let us introduce two following processes U and V:

$$U_T = \int_0^T \int_{\mathbb{R}^*} p(x) d\rho_s(Q, \tilde{Q}) + \int_0^T \int_{\mathbb{R}^*} a e^{kx} p(x) d\rho_s(P, \tilde{P})$$

$$V_T = \int_0^T \int_{\mathbb{R}^*} q(x) d\rho_s(Q, \tilde{Q}) + \int_0^T \int_{\mathbb{R}^*} a e^{kx} q(x) d\rho_s(P, \tilde{P})$$

where $p(x) = A\frac{|e^x - 1|}{4} + 1, \qquad q(x) = A\frac{|e^x - 1|}{4} + e^x$

Lemma 5

Lemma

We suppose that Y^Q and $Y^{\tilde{Q}}$ are bounded by ae^{kx} where a, k are constants. Then we have:

$$h_T(\frac{1}{2}, Q, \tilde{Q}) \leq U_T, \quad h_T(\frac{1}{2}, Q', \tilde{Q}') \leq V_T$$

Theorem 1

Theorem

Suppose that X is a process without fixed points of discontinuity under P. Assume some integrability conditions and boundeness of second Girsanov parameters by ae^{kx}. Then we have:

$$|\mathbb{C}_T - \tilde{\mathbb{C}_T}| \leq 4c\,[E_Q U_T]^{1/2} + 4d\left[E_{\tilde{Q}} V_T)\right]^{1/2},$$

and, for $\epsilon > 0$,

$$|\mathbb{C}_T - \tilde{\mathbb{C}_T}| \leq 3\sqrt{2\epsilon}(c+d) + 2cQ\,(\,U_T \geq \epsilon\,) + 2d\,\tilde{Q}\,(\,V_T \geq \epsilon\,)$$

Corollary 1

Corollary

Let X be the process with independent increments under P and $\tilde{P}$ and let the conditions of Theorem 1 are satisfied. Let $f(x) = \frac{A}{2}|e^x - 1| + max(1, e^x)$ *and*

$$R_T = \int_0^T \int_{\mathbb{R}^*} f(x) d\rho_s(Q, \tilde{Q}) + \int_0^T \int_{\mathbb{R}^*} a e^{kx} f(x) d\rho_s(P, \tilde{P})$$

If under the measures $Q, \tilde{Q}$ the process X remains the process with independent increments then

$$|\mathbb{C}_T - \tilde{\mathbb{C}_T}| \leq 3(c+d)\sqrt{2R_T}$$

Levy processes

- In this part we consider X being Levy process with parameters (b, c, ν) under the measure P. We emphasize that here ν is no more the compensator of the measure of jumps of X but a Levy measure, i.e. positive σ-finite measure on $\mathbb{R}$ such that

$$\int_{\mathbb{R}^*} (x^2 \wedge 1) d\nu < \infty.$$

- We recall that the caracteristic function of X_t for $t \geq 0$ and $\lambda \in \mathbb{R}$ is given by:

$$\phi_t(\lambda) = \exp(t\psi(\lambda))$$

where $\psi(\lambda)$ is a caracteristic exponent of Levy process,

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$$\psi(\lambda) = ib\lambda - \frac{1}{2}\lambda^2 c + \int_{\mathbb{R}^*} (\exp(i\lambda x) - 1 - i\lambda l(x)) d\nu,$$

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- $$\psi(\lambda) = ib\lambda - \frac{1}{2}\lambda^2 c + \int_{\mathbb{R}^*} (\exp(i\lambda x) - 1 - i\lambda l(x)) d\nu,$$

General setting

Let now $\tilde{P}$ be the measure coresponding to the parameter $(\tilde{b}, \tilde{c}, \tilde{\nu})$. According to Colorary 1 we have to find, for chosen equivalent martingale measures Q and $\tilde{Q}$, the Girsanov parameters (β^Q, Y^Q) and $(\beta^{\tilde{Q}}, Y^{\tilde{Q}})$ and write the expressions for the processes $\rho(Q, \tilde{Q})$ and $\rho(P, \tilde{P})$.

Esscher measures

- Esscher measures play very important role in actuary theory as well as in the option pricing theory
- Let

$$\mathbf{D} = \{\lambda \in \mathbb{R} \,|\, E_P e^{\lambda X_1} < \infty\}$$

where E_P is the expectation with respect to the physical measure P.

- Then for $\lambda \in \mathbf{D}$ we define Esscher measure P^{ES} of the parameter λ and risque process $(X_t)_{t \geq 0}$ by : for $t \geq 0$

$$\frac{dP_t^{ES}}{dP_t} = \frac{e^{\lambda X_t}}{E_P[e^{\lambda X_t}]}$$

- But this is equivalent to

$$\psi(-i(1+\lambda)) - \psi(-i\lambda) = r$$

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- Then for $\lambda \in \mathbf{D}$ we define Esscher measure P^{ES} of the parameter λ and risque process $(X_t)_{t\geq 0}$ by : for $t \geq 0$

$$\frac{dP_t^{ES}}{dP_t} = \frac{e^{\lambda X_t}}{E_P[e^{\lambda X_t}]}$$

- But this is equivalent to

$$\psi(-i(1+\lambda)) - \psi(-i\lambda) = r$$

Esscher measures

- Esscher measures play very important role in actuary theory as well as in the option pricing theory
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- Then consider the equation:

$$b + (\frac{1}{2} + \lambda)c + \int_{\mathbb{R}^*} ((e^x - 1)\, e^{\lambda x} - I(x))d\nu = r$$

- Suppose again that X is Levy process with parameters (b, c, ν) under P, and that it has the parameters $(\tilde{b}, c, \tilde{\nu})$ under $\tilde{P}$.
- Suppose that the solution of mentionned equation exists as well as the solution of the same equation with the replacement (b, c, ν) by $(\tilde{b}, c, \tilde{\nu})$ denoted λ^* and $\tilde{\lambda}^*$ respectively.
- Then $Q = P^{ES}(\lambda^*)$ and $\tilde{Q} = P^{ES}(\tilde{\lambda}^*)$.

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Now we show that the Girsanov parameters for Q and $\tilde{Q}$ are: $\beta^Q = \lambda^*,\ Y^Q = e^{\lambda^* x}$ and $\beta^{\tilde{Q}} = \tilde{\lambda}^*,\ Y^Q = e^{\tilde{\lambda}^* x}$ respectively.
Now, we have to write the expression of $\rho_T(Q,\tilde{Q})$ and $\rho_T(P,\tilde{P})$:

$$\rho_T(Q,\tilde{Q}) = T\int_{\mathbb{R}^*}(\sqrt{e^{\lambda^* x}} - \sqrt{e^{\tilde{\lambda}^* x}})^2 d\nu$$

$$\rho_T(P,\tilde{P}) = T\int_{\mathbb{R}^*}(1-\sqrt{Y})^2 d\nu$$

where $Y = \dfrac{d\tilde{\nu}}{d\nu}$.

Esscher measures

In the case when $\lambda^* < 0$ and $\tilde{\lambda}^* < 0$ we can find easily that the conditions of Lemmas 4, 5 are verified with $k = 0$ and $a = 1$. We remark that mean value theorem gives:

$$(\sqrt{e^{\lambda^* x}} - \sqrt{e^{\tilde{\lambda}^* x}})^2 \leq |x|^2(\lambda^* - \tilde{\lambda}^*)^2$$

So, we obtain the estimation:

$$|\mathbb{C}_T - \tilde{\mathbb{C}}_T| \leq T(\lambda^* - \tilde{\lambda}^*)^2 \int_{\mathbb{R}^*} f(x)x^2 d\nu + T \int_{\mathbb{R}^*} f(x)(\sqrt{d\nu} - \sqrt{d\tilde{\nu}})^2$$

In the case when λ^* and/or $\tilde{\lambda}^*$ are not negatif we can obtain similar estimations but with different constants.

Minimal entropy measures

Let Q and P be two equivalent probability measures then the relative entropy of Q with respect to P (or Kulback-Leibler information in Q with respect to P) is:

$$H(Q|P) = E_Q\left(\ln(\frac{dQ}{dP})\right) = E_P\left(\frac{dQ}{dP}\ln(\frac{dQ}{dP})\right)$$

We are interested in minimal entropy martingale measure, i.e. the measure P^{ME} such that $(e^{-rt}S_t)_{t\geq 0}$ is a P^{ME}-martingale, and that for all Q martingale measures

$$H(P^{ME}|P) \leq H(Q|P)$$

Minimal entropy measures

It turns out that in the case of Levy processes P^{ME} is nothing else as Esscher measure but for another risque process $(\hat{X}_t)_{t\geq 0}$, namely for the process appearing in the representation:

$$S_t = S_0\,\mathcal{E}(\hat{X})_t$$

where $\mathcal{E}(\cdot)$ is Dolean's-Dade exponential,

$$\mathcal{E}(\hat{X})_t = \exp(\hat{X}_t - \frac{1}{2}\langle \hat{X}\rangle_t) \prod_{0\leq s\leq t} (1+\Delta\hat{X}_s)e^{-\Delta\hat{X}_s}$$

Minimal entropy measures

The parameter λ of minimal entropy martingale measure satisfies:

$$\hat{\psi}(-i(1+\lambda)) - \hat{\psi}(-i\lambda) = r$$

and the last equation is equivalent to:

$$b + (\frac{1}{2} + \lambda)c + \int_{\mathbb{R}^*} ((e^x - 1)\, e^{\lambda(e^x - 1)} - l(x))\, d\nu = r$$

We denote the left hand side of the last equality by $\hat{f}$. In Kallsen, Shiryaev(2002) it was shown that if the solution of the equation exists, it is unique. Let us suppose that the solution λ^* of the equation exists as well the solution $\tilde{\lambda}^*$ of the similar equation with replacing (b, c, ν) by $(\tilde{b}, c, \tilde{\nu})$. If these solutions are negatifs we obtain the estimation:

$$|\mathbb{C}_T - \tilde{\mathbb{C}}_T| \leq T(\lambda^* - \tilde{\lambda}^*)^2 \int_{\mathbb{R}^*} f(x)(e^x - 1)^2 d\nu + T \int_{\mathbb{R}^*} f(x)(\sqrt{d\nu} - \sqrt{d\tilde{\nu}})^2$$

Example 1

Example 1 *Geometric Variance Gamma model*

In Geometric Variance Gamma model the parameters (b, c, ν) are equal to $(0, 0, \nu)$. The Levy measure of this model has the following form:

$$\nu(dx) = \frac{C(\mathbb{I}_{\{x<0\}}e^{-M|x|} + \mathbb{I}_{\{x>0\}}e^{-Nx})}{|x|}dx$$

where $C > 0$ and $M, N \geq 0$. It is known from Novikov, Miyahara (2002) that if $0 \leq N \leq 1$, or $N > 1$ and $\hat{f}(0) \geq r$, then λ^* is negatif. If $N > 1$ and $\hat{f}(0) < r$, then λ^* does not exist. So, in the case of existence of minimal entropy measures we have the estimation given before.

Example 2

Example 2 *Geometric CGMY model*

In Geometric CGMY model the parameters (b, c, ν) are equal to $(0, 0, \nu)$. The Levy measure of this model has the following form:

$$\nu(dx) = \frac{C(\mathbb{I}_{\{x<0\}}e^{-M|x|} + \mathbb{I}_{\{x>0\}}e^{-Nx})}{|x|^{1+\alpha}}dx$$

where $\alpha < 1$, $C > 0$ and $M, N \geq 0$.
We recall that the case of $\alpha = 0$ corresponds to Geometric Variance Gamma model and it was already considered. It is known from Novikov, Miyahara (2002) that if $M = N = 0$ and $-1 < \alpha < 1$ then X is symetric stable process and if $C > 0$ then λ^* is negatif. If $0 \leq N \leq 1$ or if $N > 1$ and $\hat{f}(0) \geq r$ then λ^* is again negatif. If $N > 1$ and $\hat{f} < r$ the equation has no solution.

f divergence

These measures take part of the measures minimising so called f-divergence between two probability measures. Let Q and P be two probability measures, $Q << P$, and f be a convex function with the values in $\mathbb{R}^{+,*}$. Then f-divergence of Cizsar of Q given P, denoted $f(Q\,|\,P)$ is given by:

$$f(Q\,|\,P) = E_P[\,f\left(\frac{dQ}{dP}\right)]$$

- If $f(x) = x\ln x$ we obtain as $f(Q\,|\,P)$ the entropy or Kulback-Leibler information,
- If $f(x) = |1-x|$ we obtain the variation distance,
- If $f(x) = (1-x)^2$ we obtain variance squared distance,
- If $f(x) = (1-\sqrt{x})^2$ we obtain Hellinger distance.

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f^q martingale measures

In the papers mentionned before the authors consider the case of

$$f(x) = \begin{cases} -x^q, & \text{if} \quad 0 < q < 1, \\ x^q, & \text{if} \quad q < 0 \text{ or } q > 1. \end{cases}$$

It is not difficult to see that such f is a convex function. It was shown that in the case of Levy processes the Girsanov parameters (β_q, Y_q) of the measure $P^{(q)}$ minimising f-divergence with f given by the above expression, are deterministic. So, X is also Levy process under $P^{(q)}$.

f^q martingale measures

The Girsanov parameters of this measure can be obtained as a solution of the following minimisation problem. Let

$$k(\beta, Y) = \frac{q(q-1)}{2}\beta^2 c + \int_{\mathbb{R}^*} (Y^q - 1 - q(Y-1))d\nu(x)$$

and

$$\mathcal{A} = \{(\beta, Y) \,|\, \beta \in \mathbb{R}, Y > 0, \; b + c\beta \int_{\mathbb{R}^*} (xY(x) - l(x))d\nu = 0\}$$

We suppose that $\mathcal{A} \neq \emptyset$ and the following minimum is atteint. Then (β_q, Y_q) satisfies:

$$k(\beta_q, Y_q) = \min_{(\beta, Y) \in \mathcal{A}} k(\beta, Y)$$

f^q martingale measures

It was shown that some times there exists the solution of this minimisation problem in a special form, namely

$$Y_q = ((q-1)\lambda x + 1)^{1/(q-1)}$$

where λ is some constant, and $\beta_q = \lambda$.
Let λ^* be the value of λ which solves the minimisation problem with $Y_q = ((q-1)\lambda^* x + 1)^{1/(q-1)}$ and $\beta_q = \lambda^*$, and let $\tilde{\lambda}^*$ the value of the constant which solves the same problem with replacement of (b, c, ν) by $(\tilde{b}, c, \tilde{\nu})$.

f^q martingale measures

To evaluate $\rho(P^{(q)}, \tilde{P}^{(q)})$ we remark that if $q > 1$ or $q \leq 1/2$ then

$$(\sqrt{Y_q(x)} - \sqrt{\tilde{Y}_q(x)})^2 \leq (q-1)^2\, x^2(\lambda^* - \tilde{\lambda}^*)^2$$

and if $1/2 < q < 1$ then

$$(\sqrt{Y_q(x)} - \sqrt{\tilde{Y}_q(x)})^2 \leq C(q, \lambda^*, \tilde{\lambda}^*)(q-1)^2\, x^2(\lambda^* - \tilde{\lambda}^*)^2$$

with $C(q, \lambda^*, \tilde{\lambda}^*) = (|q-1| \max(\lambda^*, \tilde{\lambda}^*) + 1)^{(1-2q)/(2q-2)}$. So, we have the estimations similar to given before estimation but with the different constants.

General setting

- We suppose that $(\Omega, \mathcal{F}, \mathbb{F})$ is filtered space endowed by the measures $P_\theta, \theta \in \Theta$, where θ is unknown parameter.
- We suppose that for each $\theta \in \Theta$, there exists an equivalent martingale measure Q_θ.
- We denote as before by $\mathbb{C}_T(\theta)$ the price of risky asset obtained using as physical measure P_θ.
- Let $\hat{\theta}$ be an estimator of θ.
- Let $\mathbb{C}_T(\hat{\theta})$ be the result of the replacement in $\mathbb{C}_T(\theta)$ of the unknown parameter by its estimator.

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From general model to binary model

- How to pass from general model to binary model ?

$$E_\theta|\mathbb{C}_T(\hat{\theta}) - \mathbb{C}_T(\theta)| \leq$$

$$2(c+d)P_\theta(|\hat{\theta} - \theta| > \epsilon) + \sup_{|\theta' - \theta| \leq \epsilon} |\mathbb{C}_T(\theta') - \mathbb{C}_T(\theta)|$$

- Let now replace P_θ by P and $P_{\theta'}$ by $\tilde{P}$ and use previous results.

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Theorem 2

Theorem

Suppose that the conditions of Theorem 1 are satisfied for each pair of measures P_θ and $P_{\theta'}$ with $\theta, \theta' \in \Theta$. Then we have:

$$\mathbb{E}^{\theta}|\mathbb{C}_T(\hat{\theta}) - \mathbb{C}_T(\theta)| \;\leq\; 2(c+d)\, P_\theta\left(|\hat{\theta} - \theta| > \epsilon\right) +$$

$$\sup_{|\theta-\theta'|\leq\epsilon}\left(4c\left[E_{Q_\theta}\left(U_T(Q_\theta, Q_{\theta'}\right)\right]^{1/2} + 4d\left[E_{\tilde{Q}_\theta}\left(\,V_T(Q'_\theta, Q'_{\theta'})\right)\right]^{1/2}\right)$$

where Q_θ is a martingale measure corresponding to P_θ and Q'_θ is the corresponding dual measure.

Theorem 2

For any $\epsilon > 0$ we have:

$$\mathbb{E}^{\theta}|\mathbb{C}_T(\hat{\theta}) - \mathbb{C}_T(\theta)| \leq 2(c+d)\, P_{\theta}\left(|\hat{\theta} - \theta| > \epsilon\right) + 3\sqrt{2\epsilon}\,(c+d)$$

$$\sup_{|\theta-\theta'|\leq\epsilon} \left(cQ_{\theta}\left(U_T(Q_{\theta}, Q_{\theta'}) \geq \epsilon \right) + dQ'_{\theta}\left(V_T(Q'_{\theta}, Q'_{\theta'}) \geq \epsilon \right)\right)$$

Corollary 2

Corollary

Suppose that X is a processes with independent increments under (P_θ), $\theta \in \Theta$, as well as under the corresponding martingale measures (Q_θ), $\theta \in \Theta$,. We suppose that the conditions of Theorem 2 are satisfied. Then we have:

$$\mathbb{E}^\theta|\mathbb{C}_T(\hat{\theta}) - \mathbb{C}_T(\theta)| \leq$$

$$2(c+d)\, P_\theta\left(|\hat{\theta}-\theta| > \epsilon\right) + 3\sqrt{2\epsilon}\,(c+d)[\sup_{|\theta-\theta'|\leq\epsilon} R_T(\theta,\theta')]^{1/2}$$

where $R(\theta,\theta')$ is the analog of R process for binary model.

Corollary 3

Corollary

Suppose that we have a sequence of processes with independent increments involving the physical measures $(P^n_\theta)_{n\geq 1}$, $\theta \in \Theta$, the corresponding martingale measures $(Q^n_\theta)_{n\geq 1}$, $\theta \in \Theta$, and the respective sequence of the consistent estimators $(\hat{\theta}^n)_{n\geq 1}$. If uniformly in the neighbourhood of θ as $n \to \infty$

$$R^n_T(\theta, \theta') \to 0$$

then

$$\mathbb{E}^n_\theta |\mathbb{C}_T(\hat{\theta}^n) - \mathbb{C}_T(\theta)| \to 0.$$