

Exact asymptotic bias for estimators of the Ornstein-Uhlenbeck process

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Outline

- 1 The Ornstein-Uhlenbeck process
- 2 Asymptotically efficient estimators
- 3 Asymptotic Bias
- 4 Reducing the bias
- 5 The discrete case
- 6 Some Applications

Definition

Consider a real stationary markov zero mean gaussian process $X = (X_t, t \in \mathbb{R})$ with a continuous nondegenerated autocorrelation $(\rho(h), h \geq 0)$, then, there exists $\theta > 0$ such that

$$\rho(h) = \exp(-\theta h), h \geq 0,$$

this is the so-called **Ornstein-Uhlenbeck process (OU)**.

One may also define OU as the unique stationary solution of the stochastic differential equation

$$dX_t = -\theta X_t dt + \sigma dW_t$$

where W is a bilateral standard Wiener process.

Interpretation: X is the speed of a particle submitted to brownian motion.

Finally, another simple form of OU is

$$X_t = \frac{e^{-\theta t}}{\sqrt{2\theta}} W_1(e^{2\theta t}), t \geq 0,$$

where W_1 is a standard Wiener process.

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The information inequality

In the following we suppose that $\sigma = 1$ and we intend to estimate θ and $g(\theta)$ from the observation of $X_{(\mathcal{T})} = (X_t, 0 \leq t \leq T)$. The **information inequality** (or Fréchet-Darmois-Cramer-Rao inequality) is

$$E_{\theta} (g(\theta_{\mathcal{T}}) - g(\theta))^2 \geq \frac{(b'_{\mathcal{T}}(\theta) + g'(\theta))^2}{I_{\mathcal{T}}(\theta)} + b_{\mathcal{T}}^2(\theta),$$

where $I_{\mathcal{T}}(\theta)$ is the Fisher information and $b_{\mathcal{T}}(\theta)$ the bias of the estimator $g(\theta_{\mathcal{T}})$. Thus, in order to compute the above lower bound, it is necessary to study the bias and the bias derivative of $g(\theta_{\mathcal{T}})$. This study allows to evaluate precisely the difference between the mean square error and the bound.

A family of asymptotically efficient estimators

Consider the family F of estimators of the form

$$\theta_T = \theta_T(\alpha, \beta, \Delta_T) = \frac{T - \alpha X_0^2 - \beta X_T^2}{2B_T} + \Delta_T,$$

where $B_T = \int_0^T X_t^2 dt$, $\alpha, \beta \in$

The **empirical estimator (EE)** is given by

$$\bar{\theta}_T = \frac{T}{2B_T}$$

The conditional likelihood of $X_{(T)}$ is

$$L = \exp\left(A_T\theta - B_T\frac{\theta^2}{2}\right)$$

where $A_T = \frac{T+X_0^2-X_T^2}{2}$, hence, the **conditional maximum likelihood estimator (CMLE)**:

$$\hat{\theta}_T = \frac{A_T}{B_T}$$

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Now, the likelihood is

$$\tilde{L} = \sqrt{\frac{\theta}{\pi}} \exp(-\theta X_0^2) \cdot L,$$

and the **maximum likelihood estimator (MLE)**:

$$\tilde{\theta}_T = \frac{(A_T - X_0^2) + \sqrt{(A_T - X_0^2)^2 + 2B_T}}{2B_T},$$

Finally, the **reverse conditional maximum likelihood estimator (RCMLE)** has the form

$$\check{\theta}_T = \frac{A'_T}{B_T},$$

where $A'_T = \frac{T + X_T^2 - X_0^2}{2}$.

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These estimators belong to F :

$$\bar{\theta}_T = \theta_T(0,0,0)$$

$$\hat{\theta}_T = \theta_T(-1,1,0)$$

$$\check{\theta}_T = \theta_T(1,-1,0)$$

$$\tilde{\theta}_T = (1,1,\Delta_T),$$

where $\Delta_T = \frac{T}{4B_T} \left[\left(\Gamma_T^2 + \frac{8B_T}{T^2} \right)^{\frac{1}{2}} - \Gamma_T \right]$, with $\Gamma_T = 1 - \frac{X_0^2 + X_T^2}{T}$.

Note that F is a convex set.

Asymptotic efficiency

Proposition

For each $\theta_T \in F$ one has

$$T^{\frac{p}{2}} E_{\theta} |\theta_T - \theta|^p \rightarrow (2\theta)^{\frac{p}{2}} E_{\theta} |N|^p, \quad p \geq 1,$$

and

$$T^{\frac{1}{2}} (\theta_T - \theta) \Longrightarrow (2\theta)^{\frac{1}{2}} N,$$

where $N \sim \mathcal{N}(0, 1)$.

Proof.

It is an easy consequence of Kutoyants (2004, 2009). □

Bias of $\bar{\theta}_T$

First note that

$$\bar{\theta}_T = \frac{1}{2} (\hat{\theta}_T + \check{\theta}_T)$$

Then, since X is gaussian stationary, the three estimators have the same bias. Moreover this bias is **positive**:

$$E_{\theta}(\bar{\theta}_T) > \frac{1}{E_{\theta}(2T^{-1}B_T)} = \frac{1}{2(2\theta^{-1})} = \theta$$

In order to study this bias one may use the representation of X as the transform of a Wiener process for obtaining

$$b_T(\theta, X) = \theta b_{\theta T}(1, Y)$$

where $Y_t = \sqrt{\theta} X_{t/\theta}$.

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It follows that

$$b'_T(\theta) = \mathcal{O}\left(\frac{\ln T}{T}\right)$$

and

$$T b_T(\theta) \rightarrow 2$$

The general case

For the general θ_T one obtains

$$\frac{\partial}{\partial \theta} E_{\theta}(\theta_T - \theta) \rightarrow 0,$$

and

$$T \cdot E_{\theta}(\theta_T(\alpha, \beta, \Delta_T) - \theta) \rightarrow 2 - \frac{\alpha + \beta}{2} + \delta_{\theta}.$$

For the MLE one has again

$$T \cdot E_{\theta}(\tilde{\theta}_T - \theta) \rightarrow 2.$$

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Asymptotic efficiency of $g(\theta_T^*)$

Let $g : \mathbb{R}_+^* \mapsto \mathbb{R}$, in order to estimate $g(\theta)$, one sets

$$\theta_T^* = \max(\theta_T, e^{-T}), \theta_T \in \mathcal{F}, T > 0.$$

Clearly θ_T^* and θ_T have the same asymptotic behaviour and, under mild conditions, $g(\theta_T^*)$ is asymptotically efficient.

For example, if g is derivable, one has

$$T^{1/2} (g(\theta_T^*) - g(\theta)) \Rightarrow (2\theta)^{1/2} |g'(\theta)| N,$$

and if, in addition $|g'(\theta)| \leq c_\theta \theta^m$, $m \geq 0$, then

$$E_\theta \left(T^{p/2} |g(\theta_T^*) - g(\theta)|^p \right) \rightarrow (2\theta)^{p/2} |g'(\theta)|^p E_\theta [|N|^p] \quad p \geq 1.$$

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Asymptotic bias for $g(\theta)$

Proposition

If g has three continuous derivatives with

$$|g'''(\theta)| \leq c\theta^m, \theta > 0 \quad (c > 0, m \geq 0)$$

and $\theta_T = \theta_T(\alpha, \beta, \Delta_T)$ then

$$T.E_{\theta}(g(\theta_T^*) - g(\theta)) \rightarrow \left(2 - \frac{\alpha + \beta}{2} + \delta_{\theta}\right) g'(\theta) + \theta g''(\theta).$$

Again, the four “classical” estimators have the same asymptotic bias: $2g'(\theta) + \theta g''(\theta)$.

Bias derivative

Proposition

If g has one continuous derivative such that

$$|g'(u)| \leq c |u|^m \quad u \in \mathbb{R},$$

for some $c > 0$ and $m \geq 0$, and if $E(g(\theta \cdot \bar{\theta}_T(Y)))$ is differentiable under expectation, then

$$\frac{\partial}{\partial \theta} E_{\theta} (g(\bar{\theta}_T) - g(\theta)) \xrightarrow{T \rightarrow \infty} 0,$$

and the same property holds for each θ_T in F .

Examples

① If $g(\theta)$ is a polynomial the result applies. In particular
 $T.E_{\theta}(\bar{\theta}_T^2 - \theta^2) \rightarrow 6\theta$.

② If $g(\theta) = \exp(-\theta h) = \rho(h)$, ($h > 0$), one obtains

$$TE_{\theta}(\exp(-\theta_T^* h) - \exp(-\theta h)) \rightarrow (\theta h - 2 + \frac{\alpha + \beta}{2} - \delta_{\theta}) h \exp(-\theta h).$$

③ If $g(\theta) = \sqrt{\frac{\theta}{\pi}} \exp(-\theta x^2)$, then

$$TE_{\theta}(\sqrt{\frac{\bar{\theta}_T^*}{\pi}} \exp(-\bar{\theta}_T^* x^2) - \sqrt{\frac{\theta}{\pi}} \exp(-\theta x^2)) \rightarrow l(x, \theta)$$

$$\text{where } l(x, \theta) = \frac{\exp(-\theta x^2)}{\sqrt{\pi}} \left[x^4 \theta^{\frac{3}{2}} - 3x^2 \theta^{\frac{1}{2}} - \frac{3}{4} \theta^{-\frac{1}{2}} \right]$$

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Examples

- 4 If $g(\theta) = P_\theta(X_0 \leq a)$, the constant in the asymptotic bias is

$$\lambda(x, \theta) = \frac{a \exp(-a^2 \theta)}{\sqrt{\pi}} \left[\left(1 - \frac{a^2}{2}\right) \frac{1}{\sqrt{\theta}} + \frac{1}{2\sqrt{\pi}} \right]$$

- 5 If $g(\theta) = \frac{c}{\theta} + d$ (c and d constants), assumption in the previous Proposition is not satisfied and we have

$$2g'(\theta) + \theta g''(\theta) = 0.$$

Actually, a slight modification of the proof gives

$$T \cdot E_\theta(g(\bar{\theta}_T) - g(\theta)) \rightarrow 0,$$

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Reducing the bias for θ

If $\delta_\theta = \delta$ does not depend on θ , one may reduce the bias of θ_T by putting

$$\theta_T^{(1)} = \theta_T - T^{-1} \left(2 - \frac{\alpha + \beta}{2} + \delta \right),$$

then, clearly, $\theta_T^{(1)}$ remains asymptotically efficient and

$$T \cdot E_\theta(\theta_T^{(1)} - \theta) \rightarrow 0.$$

Note that $\theta_T^{(1)} \in \mathcal{F}$, actually $\theta_T^{(1)} = \theta_T \left(\alpha, \beta, \Delta_T - \frac{2 - \frac{\alpha + \beta}{2} + \delta}{T} \right)$. In particular, for $\hat{\theta}_T, \check{\theta}_T, \tilde{\theta}_T$ and $\bar{\theta}_T$, $\theta_T^{(1)}$ is obtained by subtracting $\frac{2}{T}$. Moreover one has

$$T \left| E_\theta(\bar{\theta}_T - \frac{2}{T}) - \theta \right| = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$$

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Reducing the bias for $g(\theta)$

The discrete case

Suppose that one only observes $X_\delta, \dots, X_{n\delta}$ ($\delta > 0$) and uses the estimator

$$\bar{\theta}_n = \left(\frac{2}{n} \sum_{i=1}^n X_{i\delta}^2 \right)^{-1}$$

then

$$n\delta E_\theta(\bar{\theta}_n - \theta) \xrightarrow{n \rightarrow \infty} 2\delta\theta \frac{1 + \exp(-2\delta\theta)}{1 - \exp(-2\delta\theta)}$$

Now, if $\delta = \delta_n \rightarrow 0$ and $n\delta_n \rightarrow \infty$ one has again

$$n\delta_n E_\theta(\bar{\theta}_n - \theta) \xrightarrow{n \rightarrow \infty} 2.$$

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Some Applications

Collecting the above results one obtains

$$0 < E_{\theta}(\bar{\theta}_T - \theta)^2 - m_T(\theta) < c_{\theta} T^{-\frac{3}{2}}$$

where $m_T(\theta)$ is the Frechet- Darmois- Cramer- Rao bound.

Another application involves **Statistical prediction**. Consider the predictor of X_{T+h} ($h > 0$) defined by

$$\hat{X}_{T+h} = \exp(-\bar{\theta}_{(T-a \ln T)} h) X_T$$

then, using the fact that the O.U. process is geometrically strongly mixing and choosing a suitably, we get

$$TE_{\theta}(\hat{X}_{T+h}) \xrightarrow{T \rightarrow \infty} 0.$$

As a consequence, one obtains **asymptotic efficiency of the predictor** : consider the genuine inequality

$$E_{\theta}(p - g)^2 \geq E_{\theta}(E_{\theta}^X(g) - g)^2,$$

where p is the statistical predictor of g . Here, $p = \hat{X}_{T+h}$, $g = X_{T+h}$ and the bound $E_{\theta}(E_{\theta}^X(g) - g)^2 = \frac{1 - e^{-2\theta h}}{2\theta}$.

Then, we have

$$T \left[E_{\theta}(\hat{X}_{T+h} - X_{T+h})^2 - \frac{1}{2\theta}(1 - e^{-2\theta h}) \right] \xrightarrow{T \rightarrow \infty} h^2 e^{-2\theta h}.$$

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