Exact asymptotic bias for estimators of the Ornstein-Uhlenbeck process

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Outline

Definition

Consider a real stationary markov zero mean gaussian process $X = (X_t, t \in \mathbb{R})$ with a continuous nondegenerated autocorrelation $(\rho(h), h \geq 0)$, then, there exists $\theta > 0$ such that

$$\rho(h) = \exp(-\theta h),\ h \geq 0,$$

this is the so-called **Ornstein-Uhlenbeck process (OU)**.

One may also define OU as the unique stationary solution of the stochastic differential equation

$$dX_t = -\theta X_t\, dt + \sigma dW_t$$

where W is a bilateral standard Wiener process.
Interpretation: X is the speed of a particle submitted to brownian motion.

Finally, another simple form of OU is

$$X_t = \frac{e^{-\theta t}}{\sqrt{2\theta}} W_1(e^{2\theta t}),\ t \geq 0,$$

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The information inequality

In the following we suppose that $\sigma = 1$ and we intend to estimate θ and $g(\theta)$ from the observation of $X_{(T)} = (X_t, 0 \leq t \leq T)$. The information inequality (or Fréchet-Darmois-Cramer-Rao inequality) is

$$E_\theta \left(g(\theta_T) - g(\theta)\right)^2 \geq \frac{\left(b_T'(\theta) + g'(\theta)\right)^2}{I_T(\theta)} + b_T^2(\theta),$$

where $\mathrm{I}_T(\theta)$ is the Fisher information and $\mathrm{b}_T(\theta)$ the bias of the estimator $g(\theta_T)$. Thus, in order to compute the above lower bound, it is necessary to study the bias and the bias derivative of $g(\theta_T)$. This study allows to evaluate precisely the difference between the mean square error and the bound.

A family of asymptotically efficient estimators

Consider the family $\mathcal{F}$ of estimators of the form

$$\theta_T = \theta_T(\alpha, \beta, \triangle_T) = \frac{T - \alpha X_0^2 - \beta X_T^2}{2B_T} + \triangle_T,$$

where $B_T = \int_0^T X_t^2\, dt$, $\alpha, \beta \in \mathbb{R}$ and $\triangle_T$ is a statistic satisfying

$$\textbf{(C)} \quad \Delta_T \xrightarrow{a.s.} 0,\ T^{\frac{p}{2}} E_\theta \left|\triangle_T\right|^p \to 0, p \geq 1,\ TE_\theta(\triangle_T) \to \delta_\theta, \quad T \to \infty$$

where δ_θ depends on $(\triangle_T)$ and θ

The empirical estimator (EE) is given by

$$\bar{\theta}_T = \frac{T}{2B_T}$$

The conditional likelihood of $X_{(T)}$ is

$$L = \exp\left(A_T\theta - B_T\frac{\theta^2}{2}\right)$$

where $A_T = \frac{T+X_0^2-X_T^2}{2}$, hence, the conditional maximum likelihood estimator (CMLE):

$$\hat{\theta}_T = \frac{A_T}{B_T}$$

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Now, the likelihood is

$$\tilde{L} = \sqrt{\frac{\theta}{\pi}} \exp(-\theta X_0^2) \,.L,$$

and the maximum likelihood estimator (MLE):

$$\tilde{\theta}_T = \frac{(A_T - X_0^2) + \sqrt{(A_T - X_0^2)^2 + 2B_T}}{2B_T},$$

Finally, the reverse conditional maximum likelihood estimator(RCMLE) has the form

$$\check{\theta}_T = \frac{A_T'}{B_T},$$

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$$\check{\theta}_T = \frac{A_T^{'}}{B_T},$$

where $A_T^{'} = \frac{T + X_T^2 - X_0^2}{2}$.

These estimators belong to $\mathcal{F}$:

$$\bar{\theta}_T = \theta_T(0,0,0)$$

$$\hat{\theta}_T = \theta_T(-1,1,0)$$

$$\check{\theta}_T = \theta_T(1,-1,0)$$

$$\widetilde{\theta}_T = (1,1,\triangle_T),$$

where $\triangle_T = \frac{T}{4B_T}\left[\left(\Gamma_T^2 + \frac{8B_T}{T^2}\right)^{\frac{1}{2}} - \Gamma_T\right]$, with $\Gamma_T = 1 - \frac{X_0^2+X_T^2}{T}$.
Note that $\mathcal{F}$ is a convex set.

Asymptotic efficiency

Proposition

For each $\theta_T \in \mathcal{F}$ *one has*

$$T^{\frac{p}{2}} E_\theta \, |\theta_T - \theta|^p \to (2\theta)^{\frac{p}{2}} E_\theta |N|^p, \; p \geq 1,$$

and

$$T^{\frac{1}{2}} (\theta_T - \theta) \Longrightarrow (2\theta)^{\frac{1}{2}} N,$$

where $N \sim \mathcal{N}(0,1)$.

Proof.

It is an easy consequence of Kutoyants (2004,2009). □

Bias of $\bar{\theta}_T$

First note that

$$\bar{\theta}_T = \frac{1}{2}\left(\hat{\theta}_T + \check{\theta}_T\right)$$

Then, since X is gaussian stationary, the three estimators have the same bias. Moreover this bias is positive:

$$E_\theta(\bar{\theta}_T) > \frac{1}{E_\theta(2T^{-1}B_T)} = \frac{1}{2(2\theta^{-1})} = \theta$$

In order to study this bias one may use the representation of X as the transform of a Wiener process for obtaining

$$b_T(\theta, X) = \theta b_{\theta T}(1, Y)$$

where $Y_t = \sqrt{\theta}\, X_{t/\theta}$.

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It follows that

$$b'_T(\theta) = \mathcal{O}\left(\frac{\ln T}{T}\right)$$

and

$$T\, b_T(\theta) \to 2$$

The general case

For the general θ_T one obtains

$$\frac{\partial}{\partial\theta} E_\theta(\theta_T - \theta) \to 0,$$

and

$$T\,.\,E_\theta\left(\theta_T(\alpha,\beta,\Delta_T) - \theta\right) \to 2 - \frac{\alpha+\beta}{2} + \delta_\theta .$$

For the MLE one has again

$$T\,.\,E_\theta(\widetilde{\theta}_T - \theta) \to 2.$$

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Asymptotic efficiency of $g(\theta_T^*)$

Let $g : \mathbb{R}_+^* \mapsto \mathbb{R}$, in order to estimate $g(\theta)$, one sets

$$\theta_T^* = \max\left(\theta_T, e^{-T}\right), \theta_T \in \mathscr{F}, \ T > 0.$$

Clearly θ_T^* and θ_T have the same asymptotic behaviour and, under mild conditions, $g(\theta_T^*)$ is asymptotically efficient.

For example, if g is derivable, one has

$$T^{1/2}\left(g(\theta_T^*) - g(\theta)\right) \Rightarrow (2\theta)^{1/2}\left|g^{'}(\theta)\right| N,$$

and if, in addition $\left|g^{'}(\theta)\right| \leq c_\theta\, \theta^m$, $m \geq 0$, then

$$E_\theta\left(T^{p/2}\left|g(\theta_T^*) - g(\theta)\right|^p\right) \to (2\theta)^{p/2}\left|g^{'}(\theta)\right|^p E_\theta\left[|N|^p\right] \quad p \geq 1.$$

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For example, if g is derivable, one has

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Asymptotic bias for $g(\theta)$

Proposition

If g has three continuous derivatives with

$$|g^{'''}(\theta)| \leq c\,\theta^m, \theta > 0 \ \ (c > 0, m \geq 0)$$

and $\theta_T = \theta_T(\alpha, \beta, \Delta_T)$ *then*

$$T.E_\theta\left(g(\theta_T^*) - g(\theta)\right) \to \left(2 - \frac{\alpha+\beta}{2} + \delta_\theta\right) g'(\theta) + \theta g''(\theta).$$

Again, the four "classical" estimators have the same asymptotic bias: $2\,g^{'}(\theta) + \theta\, g^{''}(\theta)$.

Bias derivative

Proposition

If g has one continuous derivative such that

$$\left|g'(u)\right| \leq c\,|u|^m \quad u \in \mathbb{R},$$

for some $c > 0$ *and* $m \geq 0$, *and if* $E\left(g(\theta.\bar{\theta}_{\theta T}(Y)\right)$ *is differentiable under expectation, then*

$$\frac{\partial}{\partial \theta} E_\theta\left(g(\bar{\theta}_T) - g(\theta)\right) \xrightarrow[T \to \infty]{} 0,$$

and the same property holds for each θ_T *in* F.

Examples

1. If $g(\theta)$ is a polynomial the result applies. In particular $T.E_\theta(\bar{\theta}_T^2 - \theta^2) \to 6\theta$.
2. If $g(\theta) = \exp(\text{-}\theta h) = \rho(h)$, $(h > 0)$, one obtains
$$TE_\theta\left(\exp(-\theta_T^* h) - \exp(-\theta h)\right) \to (\theta h - 2 + \frac{\alpha+\beta}{2} - \delta_\theta) h \exp(-\theta h).$$
3. If $g(\theta) = \sqrt{\frac{\theta}{\pi}} \exp(-\theta x^2)$, then
$$TE_\theta\left(\sqrt{\frac{\bar{\theta}_T^*}{\pi}} \exp(-\bar{\theta}_T^* x^2) - \sqrt{\frac{\theta}{\pi}} \exp(-\theta x^2)\right) \to I(x,\theta)$$
where $I(x,\theta) = \frac{\exp(-\theta x^2)}{\sqrt{\pi}} \left[x^4 \theta^{\frac{3}{2}} - 3x^2 \theta^{\frac{1}{2}} - \frac{3}{4} \theta^{-\frac{1}{2}} \right]$

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Examples

4. If $g(\theta) = P_\theta(X_0 \leq a)$, the constant in the asymptotic bias is

$$\lambda(x,\theta) = \frac{a\exp(-a^2\theta)}{\sqrt{\pi}}\left[(1-\frac{a^2}{2})\frac{1}{\sqrt{\theta}} + \frac{1}{2\sqrt{\pi}}\right]$$

5. If $g(\theta) = \frac{c}{\theta} + d$ (c and d constants), assumption in the previous Proposition is not satisfied and we have

$$2g'(\theta) + \theta g''(\theta) = 0.$$

Actually, a slight modification of the proof gives

$$T.E_\theta(g(\bar{\theta}_T) - g(\theta)) \to 0,$$

which is natural since $g(\bar{\theta}_T)$ is an unbiased estimator of $g(\theta)$!

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Reducing the bias for θ

If $\delta_\theta = \delta$ does not depends on θ, one may reduce the bias of θ_T by putting

$$\theta_T^{(1)} = \theta_T - T^{-1}(2 - \frac{\alpha+\beta}{2} + \delta),$$

then, clearly, $\theta_T^{(1)}$ remains asymptotically efficient and

$$T.E_\theta(\theta_T^{(1)} - \theta) \to 0.$$

Note that $\theta_T^{(1)} \in \mathscr{F}$, actually $\theta_T^{(1)} = \theta_T\left(\alpha, \beta, \Delta_T - \frac{2-\frac{\alpha+\beta}{2}+\delta}{T}\right)$. In particular, for $\hat{\theta}_T, \check{\theta}_T, \tilde{\theta}_T$ and$\bar{\theta}_T$, $\theta_T^{(1)}$ is obtained by substracting $\frac{2}{T}$. Moreover one has

$$T\left|E_\theta(\bar{\theta}_T - \frac{2}{T}) - \theta\right| = \mathscr{O}(\frac{1}{\sqrt{T}})$$

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Reducing the bias for $g(\theta)$

The situation is somewhat different for $g(\theta)$; putting

$$\tilde{\bar{\theta}}_T = \max\left(\bar{\theta}_T - \frac{2}{T}, \exp(-T)\right)$$

one obtains

$$TE_\theta\left(g(\tilde{\bar{\theta}}_T) - g(\theta)\right) \to \theta g''(\theta).$$

If $g'.g''$ is positive, the absolute value of the asymptotic bias is reduced, but it is not the case in a general situation (cf $g(\theta)$=exp(-θ) at $\theta = 3$).

The discrete case

Suppose that one only observes $X_\delta, \ldots, X_{n\delta}$ ($\delta > 0$) and uses the estimator

$$\bar{\theta}_n = \left(\frac{2}{n} \sum_{i=1}^{n} X_{i\delta}^2 \right)^{-1}$$

then

$$n\delta \, E_\theta(\bar{\theta}_n - \theta) \xrightarrow[n\to\infty]{} 2\delta\theta \, \frac{1+\exp(-2\delta\theta)}{1-\exp(-2\delta\theta)}$$

Now, if $\delta = \delta_n \to 0$ and $n\delta_n \to \infty$ one has again

$$n\delta_n \, E_\theta(\bar{\theta}_n - \theta) \xrightarrow[n\to\infty]{} 2.$$

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Some Applications

Collecting the above results one obtains

$$0 < E_\theta(\bar{\theta}_T - \theta)^2 - m_T(\theta) < c_\theta T^{-\frac{3}{2}}$$

where $m_T(\theta)$ is the Frechet- Darmois- Cramer- Rao bound.

Another application involves Statistical prediction. Consider the predictor of X_{T+h} $(h > 0)$ defined by

$$\hat{X}_{T+h} = \exp(-\bar{\theta}_{(T-a\ln T)}h)\, X_T$$

then, using the fact that the O.U. process is geometrically strongly mixing and chosing a suitably, we get

$$TE_\theta(\hat{X}_{T+h}) \underset{T\to\infty}{\longrightarrow} 0.$$

As a consequence, one obtains asymptotic efficiency of the predictor : consider the genuine inequality

$$E_\theta(p-g)^2 \geq E_\theta(E_\theta^X(g)-g)^2,$$

where p is the statistical predictor of g. Here, $p = \hat{X}_{T+h}$, $g = X_{T+h}$ and the bound $E_\theta(E_\theta^X(g)-g)^2 = \frac{1-e^{-2\theta h}}{2\theta}$.

Then, we have

$$T\left[E_\theta(\hat{X}_{T+h}-X_{T+h})^2 - \frac{1}{2\theta}(1-e^{-2\theta h})\right] \xrightarrow[T\to\infty]{} h^2 e^{-2\theta h}.$$

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