Estimating a Wiener process first-passage time from noisy or delayed observations

M. V. Burnashev

Institute for Information Transmission Problems,

Russian Academy of Sciences, Moscow, Russia

A. Tchamkerten

Telecom ParisTech, France

1. Problem statement

Consider the process

$$X: \quad X_0 = 0, \qquad X_t = st + W_t, \qquad t \geq 0,$$

where $s > 0$ – given constant, W_t – standard Wiener process.
For a given threshold level $A > 0$ consider the first-passage time

$$\tau_A = \min\{t > 0 : X_t = A\}.$$

We can not observe the process X directly.
We observe another rand. process $Y = \{Y_t, t \geq 0\}$, correlated with X.
Based on observations Y we want to estimate the moment τ_A using a stopping time η over Y.
We want to find the value $\inf\limits_{\eta} \mathbf{E}|\eta - \tau_A|^p$, $p > 0$ over estimates η.

Probably, that kind of problem statement (i.e. to estimate a stopping time τ of X from correlated observations Y) appeared first in [Niesen, Tchamkerten, 2009].

Also in [Niesen, Tchamkerten, 2009] a number of possible applications are described (communications, finance, economy, etc.).

Example. $X = \{X_t\}$ represents an objective price of some shares. For optimal investment we need to know the moment τ_A for some $A > 0$.
On a stock market we can observe only Y = corrupted version of X (due to human factors, delays, etc.).

Based on observations Y we should estimate the moment τ_A.

We consider two cases of the observation process $Y = \{Y_t\}$: *Noisy observations* and *Delayed observations*.

Noisy observations. The observation process Y has the form

$$Y: \quad Y_0 = 0, \qquad Y_t = X_t + \varepsilon V_t = st + W_t + \varepsilon V_t, \qquad t \geq 0,$$

where V_t – Wiener process (independent of $\{W_t\}$), $\varepsilon > 0$ is known.

For given A and an estimate η for τ_A, introduce the function

$$q(A, p, \eta) = \mathbf{E}|\eta - \tau_A|^p.$$

We are interested in the minimal possible function $q(A, p)$

$$q(A, p) = \inf_{\eta} \mathbf{E}|\eta - \tau_A|^p,$$

where infimum is taken over all stopping times η with respect to Y. We investigate only asymptotics, when s, ε are fixed and $A \to \infty$.

Delayed observations. We are given a fixed delay $d = d(A) > 0$ and Y has the form

$$Y: \quad Y_t = 0, \quad 0 \le t \le d; \qquad Y_t = X_{t-d}, \qquad t \ge d.$$

Introduce the functions

$$q(A, p, d, \eta) = \mathbf{E}|\eta - \tau_A|^p,$$

and

$$q(A, p, d) = \inf_{\eta} q(A, p, d, \eta),$$

where infimum is taken over all stopping times η with respect to Y.

That problem is an example of traditional estimation problems.
But it has also some unpleasant feature.

Usually, in estimation theory we consider the likelyhood ratio process

$$Z(t, u, \theta_0) = \ln \frac{d\mathbf{P}_{\theta_0+u}}{d\mathbf{P}_{\theta_0}} \left(Y_0^t\right),$$

where θ_0 is the true parameter value, and $Y_0^t = \{Y_v, 0 \leq v \leq t\}$.
Most of estimation problems reduce to investigation of certain properties of the random process $Z(t, u, \theta_0)$.
In many cases the process $Z(t, u, \theta_0)$ has a convenient for investigation form (e.g. sum of independent random variables, stochastic integral, etc.).
Here the parameter $\theta =$ Markov stopping time.
$\Rightarrow$ Process Z has a non-convenient form.

2. Main results

Introduce the value

$$m_p = \mathbf{E}|\xi|^p = \frac{2^{p/2}}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right), \qquad \xi \sim \mathcal{N}(0,1).$$

Noisy observations

Theorem 1. *For any $p > 0$ the formula holds*

$$q(A,p) = \left[\frac{\varepsilon^2 A}{s^3(1+\varepsilon^2)}\right]^{p/2} (m_p + o(1)), \qquad A \to \infty.$$

In particular, we have

Statement 1. *For any $p > 0$ the lower bound holds (as $A \to \infty$)*

$$q(A,p) \geq \inf_{\eta(Y_0^\infty)} \mathbf{E}|\eta - \tau_A|^p \geq \left[\frac{\varepsilon^2 A}{s^3(1+\varepsilon^2)}\right]^{p/2} (m_p + o(1)).$$

Note that in this lower bound infimum is taken over all observation process Y_0^∞ up to $t = \infty$.

Delayed observations

Theorem 2. *For any $p > 0$ the formula holds*

$$q(A, d, p) = \left(\frac{d(A)}{s^2}\right)^{p/2} (m_p + o(1)), \qquad A \to \infty.$$

3. Sketch of proofs

3.1. Theorem 1 - upper bound for $q(A,p)$. It is sufficient to choose a good estimate η_A for τ_A and to evaluate its performance $\mathbf{E}|\eta_A - \tau_A|^p$. The most straightforward way is as follows.

We construct some reasonable estimates $\hat{X}_t = \hat{X}_t(Y_0^t), t > 0$ for the process $\{X_t\}$ and use the following estimate η_A

$$\eta_A = \min\{t : \hat{X}_t = A\}.$$

As a result, we set

$$\hat{X}_t = s(1-\alpha)t + \alpha Y_t,$$

where

$$\alpha = \frac{1}{1+\varepsilon^2}.$$

Then we get

Statement 2. *The difference $\eta_A - \tau_A$ can be represented as*

$$\eta_A - \tau_A = \sqrt{\frac{\varepsilon^2 A}{s^3(1+\varepsilon^2)}}\left[1 + O\left(\sqrt{\frac{\ln A}{As}}\right)\right]\zeta + \xi_1,$$

where:

1) $\zeta \sim \mathcal{N}(0,1)$;

2) *remaining term ξ_1 is "small"; in particular,*

$$\mathbf{E}|\xi_1|^p \le C(p)s^{-2p}\left[1 + (sA\ln A)^{p/4}\right], \quad p > 0.$$

From Statement 2 we get the upper bound (as $A \to \infty$)

$$q(A,p) \le \mathbf{E}|\eta_A - \tau_A|^p = \left[\frac{\varepsilon^2 A}{s^3(1+\varepsilon^2)}\right]^{p/2}(m_p + o(1)).$$

3.2. Theorem 1 - lower bound for $\mathbf{E}|\eta - \tau_A|^p$.

To get the lower bound for $\mathbf{E}|\eta - \tau_A|^p$ it is convenient to replace $\mathbf{E}|\eta - \tau_A|^p$ by some related inaccuracy $\mathbf{E}|\hat{X}_T - X_T|^p$ when estimating the value X_T at a fixed moment T.

Note that $\mathbf{E}\tau_A \approx A/s$. We introduce the fixed time moment T

$$T = \frac{A}{s} - ct_0, \qquad t_0 = \sqrt{\frac{2A\ln A}{s^3}}$$

with sufficiently large $c > 0$.
$\Longrightarrow$ with high probability $\tau_A \in (T, T + 2ct_0)$, since

$$\mathbf{P}\left\{\left|\tau_A - \frac{A}{s}\right| \geq ct_0\right\} \lesssim A^{-c^2}.$$

We replace the observation process Y_t by a better process Y_t'

$$Y_t' = \begin{cases} Y_t = X_t + \varepsilon V(t), & t \leq T, \\ Y_T + X_t - X_T, & t \geq T. \end{cases}$$

In other words, for the process Y_t' the additional observation noise $\varepsilon V(t)$, $t \geq T$ disappears after the moment T. It is easier to evaluate τ_A based on new observation process $\{Y_t'\}$ than on $\{Y_t\}$.

If $\tau_A > T$ then difficulty in estimating the moment τ_A is equivalent to the difficulty in estimating the value X_T. In particular, we have

$$\inf_{\eta(Y_0^\infty)} \mathbf{E}|\eta - \tau_A|^p \geq \frac{1}{s^p} \inf_{\mu(Y_T)} \left[\mathbf{E}|\mu - X_T|^p; \tau_A > T\right](1 + o(1))\,.$$

After some calculations we get Statement 1. $\square$

4. Why do we need $s \neq 0$?

Proposition 1. *Assume that* $s = 0$. *Then for any* $A > 0$, $\varepsilon > 0$ *and any* $p \geq 1/2$

$$\inf_{\eta(Y_0^\infty)} \mathbf{E}|\eta - \tau_A|^p = \infty.$$