Parameter Estimation for Stochastic Navier-Stokes Equations

Igor Cialenco

Department of Applied Mathematics
Illinois Institute of Technology, USA
igor@math.iit.edu

joint with N. Glatt-Holtz (IU)

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SPDE

$$du(t,x) = \theta Au(t,x)dt + Fu(t,x)dt + \sigma M(u(t,x))dW(t,x)$$

u in some "suitable" Hilbert spaces and stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$

A - linear operator

$F, \ M$ - some (could be nonlinear) operators

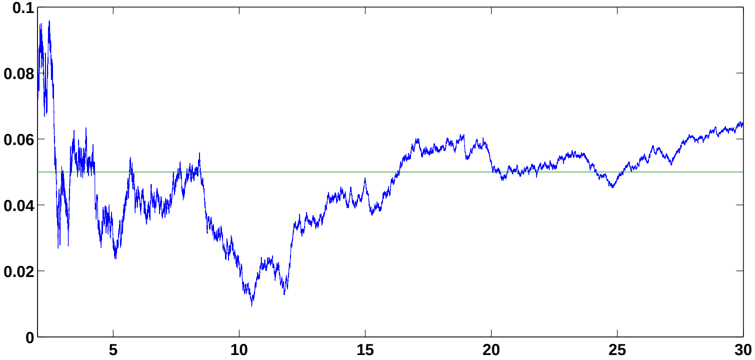
$W(t,x)$ - cylindrical Brownian motion

u is known for all $x \in G, \ t \in [0,T]$ - **continuous observations**

$\theta, \ \sigma$ - parameters (scalars), unknown

Goal:

Find estimators $\hat{\theta}(\omega), \ \hat{\sigma}(\omega)$, $\omega \in \Omega$, for parameters $\theta, \ \sigma$ by **observing a single outcome** $u = u(\omega)$ over a finite time horizon $[0,T]$.



Girsanov Th, maximize Log-Likelihood Ratio w.r.t. θ

$$du = \theta u(t)dt + \sigma u(t)dW(t), \qquad \widehat{\theta}_t = \frac{1}{t}\int_0^t \frac{du(s)}{u(s)} = \frac{1}{t}\log\frac{u(t)}{u(0)} - \frac{\sigma^2}{2}$$

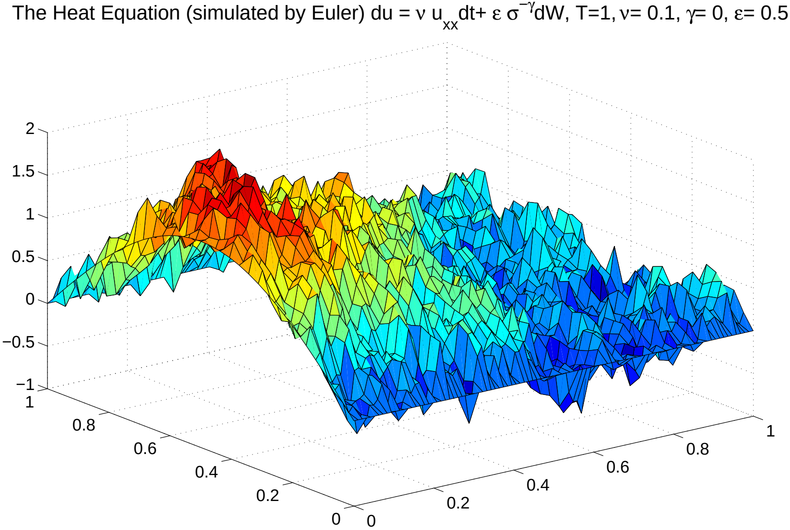
$$dy = y(t)dt + \sigma dW(t), \quad 0 \le t \le T; \ \sigma > 0$$

$\langle y\rangle_t = \sigma^2 t \ \Rightarrow \ \sigma = \sqrt{\frac{\langle y\rangle_T}{T}}$; In practice: $\sigma^2 \approx \dfrac{\sum_{k=1}^{N}\left(y(\frac{kT}{N})-y(\frac{(k-1)T}{N})\right)^2}{T}$

the drift θ - approximated, the volatility σ - exactly WHY?

- *Regular model*
  1) $\frac{d\mathbb{P}_\theta}{dQ}$ exists; 2) has a special form (LAN)
  Same procedure for all
  Find MLE by maximizing likelihood ratio

- *Singular model* otherwise
  Individual approach
  In particular, if $\mathbb{P}_{\theta_1} \perp \mathbb{P}_{\theta_2}$ for $\theta_1 \neq \theta_2$, then one may find θ exactly



What do we have for SPDEs?   Mostly Singular Models

We will try to understand the singularity and explore it to find the exact θ.

- *additive noise*: Huebner-Khasminskii-Rozovskii '92, '95
- *Bayesian*: Bishwal ('02)
- *Several parameters*: Huebner ('97)
- *Discrete-time observations*: Piterbarg-Rozovskii ('97)
  $q = \frac{2(m_1 - 2m)}{d} \geq 1$
- $\theta(t)$*-random*: Lototsky ('04)
- *Small noise*: Huebner ('97), Ibragimov-Khasminskii ('98,'99)
- *"almost" commutative case* or *"almost" diagonalizable model*- Rozovskii-Lototsky ('97), Lototsky ('01)
- additive fractional noise: IgC, Lototsky, Pospisil ('09)
- multiplicative noise: IgC and Lototsky ('08), IgC ('10)
- **nonlinear SPDE: IgC and Glatt-Holtz** ('11)

$$dU(t) = \theta AU(t)dt + F(U)dt + \sigma dW(t), \quad U(0) = U_0$$

$(-A)$ a linear, selfadjoint, positive-defined (think Laplace) in $\mathcal{H}$

with eigenvalues $\{\Phi_k\}_{k\geq 1}$ CONS in $\mathcal{H}$

F maybe nonlinear

$\sigma dW(t) = \sum_{k\geq 1} \sigma_k \Phi_k dW_k(t)$, $W_k, k \in \mathbb{N}$ ind. Brownian Motions

σ known

θ - parameter/scalar of interest

$U(t) \in \mathcal{H}$ is observed/measured/known for all $t \in [0, T]$ but one $\omega \in \Omega$;
Observe one path during $t \in [0, T]$

Formal Procedure to Derive an Estimator

- Project the full system down to N dimensions $P_N(\mathcal{H}) = \mathcal{H}_N \simeq \mathbb{R}^N$

$$dU^N = (\theta A U^N + \Psi_N)dt + P_N \sigma dW, \quad U(0) = U_0$$

  - Formally treat $\Psi_N = P_N F(U)$ as an external and known quantity (independent of θ)
  - Assume that $P_N \sigma$ is invertible on H_N and take $G = (P_N \sigma)^{-1}$
- Consider $\mathbb{P}_\nu^{N,T}(\cdot) = \mathbb{P}(U^N \in \cdot)$, the measure on $C([0,T];\mathbb{R}^N)$ generated by U^N.
- For a reference values θ_0, apply (formally) Girsanov Theorem and get the 'likelihood ratio' (Radon-Nikodym derivative) $d\mathbb{P}_\theta^{N,T}/d\mathbb{P}_{\theta_0}^{N,T}$
- Maximize the Log-Likelihood Ratio $\hat{\theta}(\omega) := \arg\max_\theta d\mathbb{P}_\theta^{N,T}/d\mathbb{P}_{\theta_0}^{N,T}(\omega)$

$$\frac{d\mathbb{P}_{\theta}^{N,T}}{d\mathbb{P}_{\theta_0}^{N,T}} = \exp\Big[-\int_0^T (\theta-\theta_0)\langle AU^N, G^2 dU^N(t)\rangle \\ -\frac{1}{2}\int_0^T (\theta^2-\theta_0^2)\langle AU^N, G^2 AU^N dt\rangle \\ -\int_0^T (\theta-\theta_0)\langle AU^N, G^2\psi^N dt\rangle\Big],$$

$$\widetilde{\theta}_N = \frac{\int_0^T \langle AU^N, G^2 dU^N\rangle + \int_0^T \langle AU^N, G^2 P_N \mathbf{F}(\mathbf{U})\rangle dt}{\int_0^T |GAU^N|^2 dt}$$

Motivated by MLE type estimator

$$\widetilde{\theta}_N = \frac{\int_0^T \langle AU^N, G^2 dU^N\rangle + \int_0^T \langle AU^N, G^2 P_N F(U)\rangle dt}{\int_0^T |GAU^N|^2 dt}$$

$$\check{\theta}_N = \frac{\int_0^T \langle AU^N, G^2 dU^N\rangle + \int_0^T \langle AU^N, G^2 F^N(U^N)\rangle dt}{\int_0^T |GAU^N|^2 dt}$$

$$\widehat{\theta}_N = \frac{\int_0^T \langle AU^N, G^2 dU^N\rangle}{\int_0^T |GAU^N|^2 dt}$$

$$
\check{\theta}_N = \theta + \frac{\int_0^T \langle AU^N, G^2 \sum_{j=1}^N \sigma_j(U)\Phi_j dW_j(t)\rangle}{\int_0^T |GAU^N|^2 dt} + \frac{\int_0^T \langle AU^N, G^2(F^N(U) - F^N(U^N))\rangle dt}{\int_0^T |GAU^N|^2 dt}
$$

$$
\widehat{\theta}_N = \theta + \frac{\int_0^T \langle AU^N, G^2 \sum_{j=1}^N \sigma_j(U)\Phi_j dW_j(t)\rangle}{\int_0^T |GAU^N|^2 dt} + \frac{\int_0^T \langle AU^N, G^2 F^N(U)\rangle dt}{\int_0^T |GAU^N|^2 dt}
$$

- Successfully applied to:
  - Stochastic Navier Stokes Equations, 2D, additive noise
  - Stochastic Reaction-Diffusion Equation, additive noise
  - Stochastic Fractional Burgers Equation, additive noise
- Work in progress to derive general conditions on A, F, G that guarantee consistency and asymptotic normality of $\check{\theta}_N$ and $\widehat{\theta}_N$ as number of modes $N \to \infty$

$$dU + ((U\cdot\nabla)U - \nu\Delta U)dt = \sum_k \lambda_k^{-\gamma}\Phi_k dW_k$$

$$\nabla\cdot U = 0, \quad U(0) = U_0$$

- Models a viscous $2D$ incompressible fluid. U – velocity field, ν – kinematic viscosity
- Periodic or Dirichlet boundary conditions
- Φ_k, $\lambda_k \sim \lambda_1 k$ eigenfunctions, eigenvalues of the Stokes operator A
- $(P_N\sigma)^{-1} \cong A^{\gamma}$ on $H_N = \mathrm{Span}\{\Phi_1,\dots,\Phi_N\}$
- When $\gamma > 1$, $\exists!$ Strong, Pathwise solutions i.e. $U \in C([0,\infty), H^1)$ $\cap L^2_{loc}([0,\infty)H^2)$. Higher regularity for larger γ.

Put $G = A^{\alpha-\gamma}$, for some $\alpha \in \mathbb{R}$ (determined later). Then,

$$\check{\nu}_N = -\frac{\int_0^T \langle A^{1+2\alpha} U^N, dU^N + P_N B(U^N) dt \rangle}{\int_0^T |A^{1+\alpha} U^N|^2 dt}$$

$$= \frac{\sum_{k=1}^{N} \lambda_k^{1+2\alpha} \left[\int_0^T u_k(t) du_k(t) + \int_0^T u_k(t) b_k(t) dt \right]}{\sum_{k=1}^{N} \lambda_k^{2(1+\alpha)} \int_0^T |u_k(t)|^2 dt}$$

$$\hat{\nu}_N = -\frac{\int_0^T \langle A^{1+2\alpha} U^N, dU^N \rangle}{\int_0^T |A^{1+\alpha} U^N|^2 dt} = \frac{\sum_{k=1}^N \lambda_k^{1+2\alpha} \int_0^T u_k(t) du_k(t)}{\sum_{k=1}^N \lambda_k^{2(1+\alpha)} \int_0^T |u_k(t)|^2 dt}$$

$$= \frac{\sum_{k=1}^N \lambda_k^{1+2\alpha} [u_k^2(T) - u_k^2(0) - T\lambda_k^{-2\gamma}]}{2\sum_{k=1}^N \lambda_k^{2(1+\alpha)} \int_0^T |u_k(t)|^2 dt}$$

Theorem (Ig.C. & N. Glatt-Holtz – 2011)

Let $U = U(\omega, t, x)$ is a solution of the $2D$ SNSE with Periodic or Dirichlet BCs. If $\gamma > 1$ (also $\gamma - 1 < 1/4$ in the Dirichlet case) then:

(i) *Consistency: If $\alpha > \gamma - 1$, then*

$$\lim_{N\to\infty} \check{\nu}_N = \nu, \qquad \lim_{N\to\infty} \hat{\nu}_N \to \nu$$

in Probability.

(ii) *Asymptotic normality (rate N): If $\alpha > \gamma - \frac{1}{2}$, then*

$$N(\tilde{\nu}_N - \nu) \xrightarrow{d} \eta \sim \mathcal{N}\left(0, \frac{2\nu(\alpha - \gamma + 1)^2}{\lambda_1 T(\alpha - \gamma + 1/2)}\right)$$

Note: In particular we can take $\alpha = \gamma$. This corresponds to formal MLE.

$$\hat{\nu}_N - \nu = -\frac{\int_0^T \langle A^{1+2\alpha-\gamma}(\bar{U}^N + R^N), \sum_k \Phi_k dW_k + A^\gamma P_N B(U) dt\rangle}{\int_0^T |A^{1+\alpha}(\bar{U}^N + R^N)|^2 dt}$$

$$= -\frac{\mathbb{E}\int_0^T |A^{1+\alpha}\bar{U}^N|^2 dt}{\int_0^T |A^{1+\alpha}(\bar{U}^N + R^N)|^2 dt} \cdot$$

$$\cdot \frac{\int_0^T \langle A^{1+2\alpha-\gamma}(\bar{U}^N + R^N), \sum_k \Phi_k dW_k + A^\gamma P_N B(U) dt\rangle}{\mathbb{E}\int_0^T |A^{1+\alpha}\bar{U}^N|^2 dt}$$

Decompose $U = \bar{U} + R =$ linear + nonlinear

$$d\bar{U} + \nu A\bar{U} dt = \sigma dW, \quad \bar{U}(0) = 0.$$

$$\partial_t R + \nu A R = -B(U), \quad R(0) = U_0.$$

- Find explicit and exact rates for moments of the linear part
- R is 'more regular' in comparison to $\bar{U}$

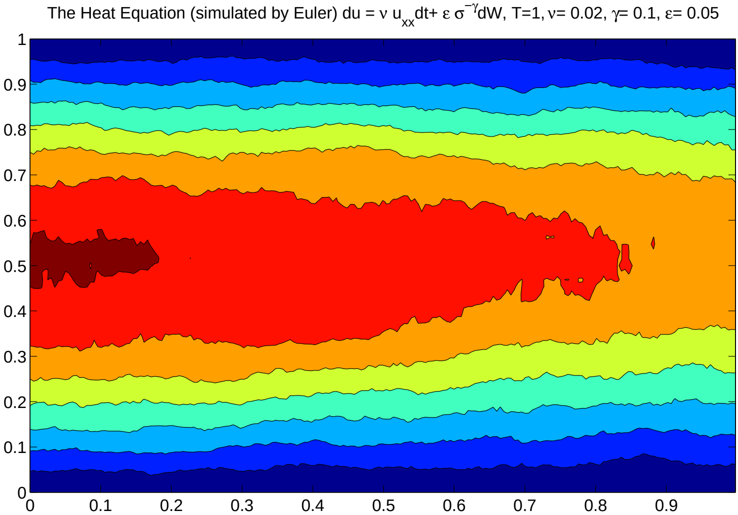
- $$\frac{\int_0^T |A^{1+\alpha}(\bar{U}^N + R^N)|^2 dt}{\mathbb{E}\int_0^T |A^{1+\alpha}\bar{U}^N|^2 dt} \to 1$$

- $$N^{\delta_1} \frac{\int_0^T \langle A^{1+2\alpha-\gamma}\overline{U}^N, \sum_{k=1}^N \Phi_k dW_k\rangle}{\int_0^T |A^{1+\alpha}\bar{U}^N|^2 dt} \to 0$$

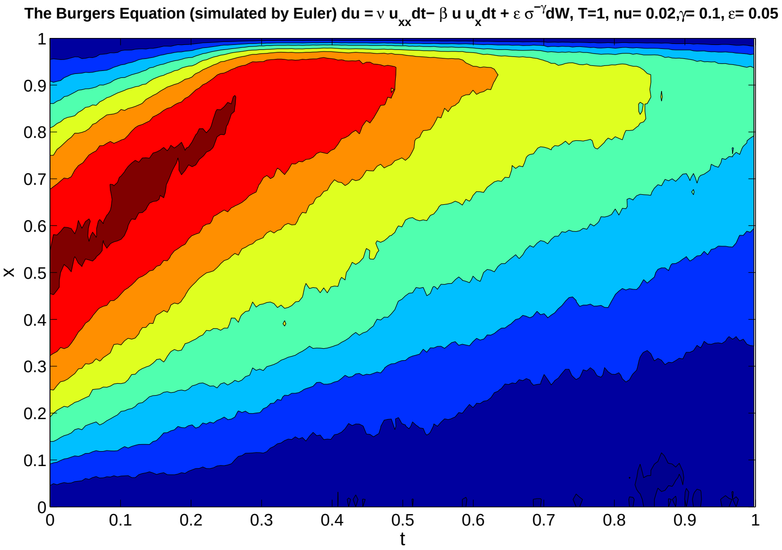
- $$N^{\delta_2} \frac{\int_0^T \langle A^{1+2\alpha-\gamma} R^N, \sum_{k=1}^N \Phi_k dW_k\rangle}{\int_0^T |A^{1+\alpha}\bar{U}^N|^2 dt} \to 0$$

- $$N^{\delta_3} \frac{\langle A^{1+2\alpha} U^N, P_N(B(U))\rangle}{\int_0^T |A^{1+\alpha}\bar{U}^N|^2 dt} \to 0$$

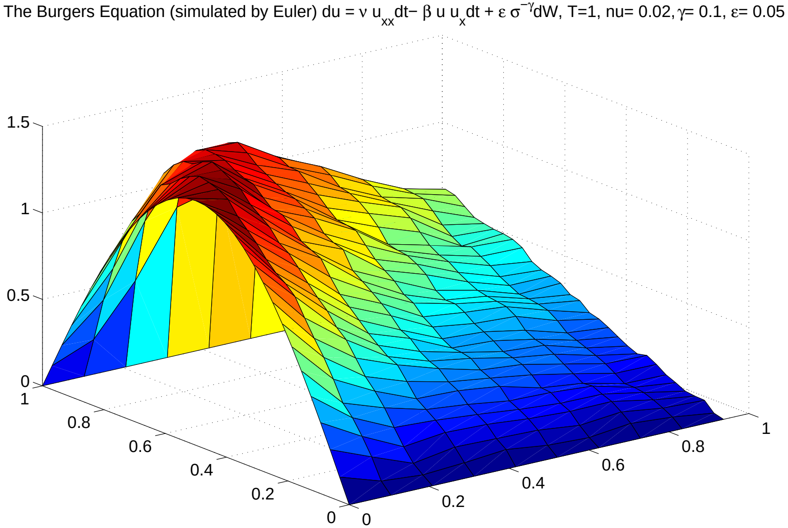
- $$N \frac{\int_0^T \langle A^{1+2\alpha-\gamma}\overline{U}^N, \sum_{k=1}^N \Phi_k dW_k\rangle}{\mathbb{E}\int_0^T |A^{1+\alpha}\overline{U}^N|^2 dt} \sim \mathcal{N}\Big(0, \frac{2\gamma(\alpha-\gamma+1)^2}{\lambda_1 T(\alpha-\gamma+1/2)}\Big)$$



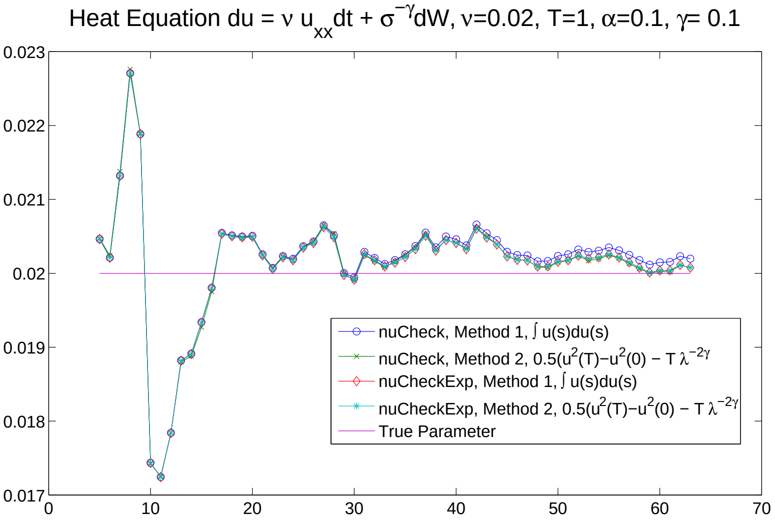
$$du = \nu u_{xx} dt + dW(t,x), \; u(0,x) = u_0, \; u(t,0) = u(t,1) = 0$$



$$du = \nu u_{xx} dt - u u_x dt + dW(t,x), \;\; u(0,x) = u_0, \;\; u(t,0) = u(t,1) = 0$$



$$du = \nu u_{xx}dt - uu_x dt + dW(t,x),\ u(0,x) = u_0,\ u(t,0) = u(t,1) = 0$$



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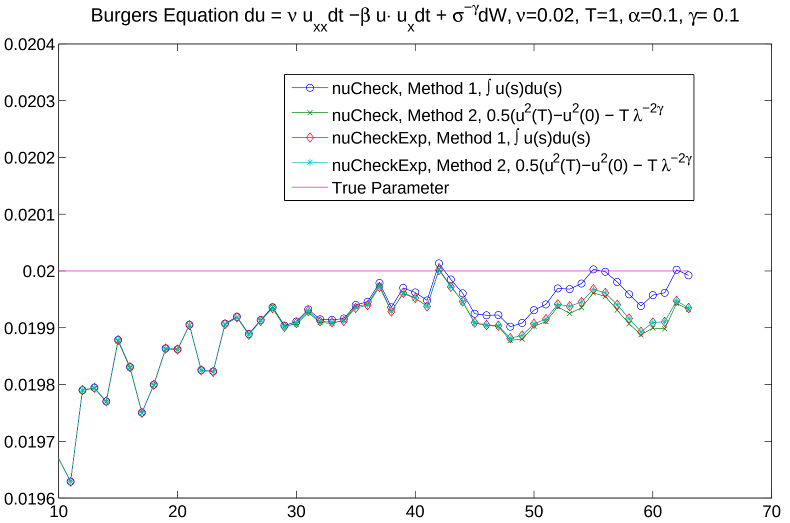

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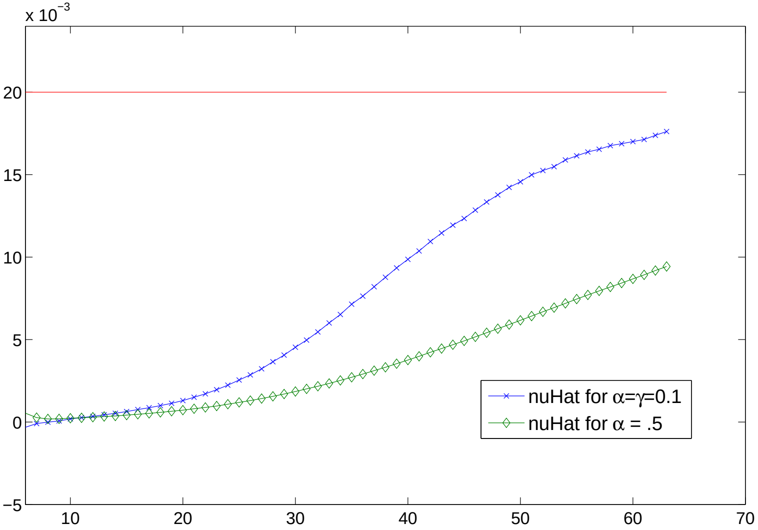

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Future Work

Future Work

... a lot

Thank You!

The end of the talk

But not of the story ...