Parameter estimation for an Ornstein Uhlenbeck process with a periodic in time drift

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Introduction

Discrete time : PAR(1) model (Gladyshev 1961, Paggano 1978, Vecchia 1985, Hurd, Makagon & Miamee 2002)

$$\xi_{k+1} = \phi_k \xi_k + \epsilon_k \quad \text{where} \quad \phi_{k+T} = \phi_k$$

$$\xi_{k+1} - \xi_k = \theta_k \xi_k + \epsilon_k \qquad \theta_k = \phi_k - 1$$

Continuous time model :

$$d\xi_t = \theta f(t) \xi_t dt + dW_t \qquad f(t+T) = f(t)$$

Examples :

$$d\xi_t = \theta \sin(t) \xi_t dt + dW_t \qquad f(t) = \sin(t)$$

$$d\xi_t = \theta(\sin(t) - 0.5) \xi_t dt + dW_t \qquad f(t) = \sin(t) - 0.5$$

$$d\xi_t = \theta(\sin(t) + 0.5) \xi_t dt + dW_t \qquad f(t) = \sin(t) + 0.5$$

Höpfner & Kutoyants (2009): Estimating discontinuous periodic signal

$$d\xi_t = [S(\theta, t) + b(\xi_t)]dt + \sigma(\xi_t)dW_t$$

periodic drift $t \mapsto S(\theta, t) = \lambda(t) + \lambda^*(t)\mathbb{I}_{(\theta,\theta+a)}(i_T(t))$,

with $i_T(t) = t\ [T]$

$b(\cdot)$ and $\sigma(\cdot)$ are Lipschitz

T-periodic semi-group

Let $(\xi_t)_{t\geq 0}$ Markov process with continuous paths, inhomogeneous in time with T-periodic semigroup density

$(P_{s,t}(x,dy) = p_{s,t}(x,y)dy)$

$$p_{s,t}(x,y) = p_{s+kT,t+kT}(x,y) \qquad \text{for all } k \in \mathbb{N}^*, 0 \leq s,t < \infty.$$

$\xi = (\xi_t)_{t\in\mathbb{R}+}$ continuous Markov process

$\rightarrow$ induced Markov chain $\mathbf{X} = (X_k)_{k\in\mathbb{N}}$, $\qquad X_k := (\xi_{kT+s})_{0\leq s\leq T}$.

chain of T-segments in the paths of ξ

The state space of the Markov chain X is the space $C[0,T]$ of continuous functions defined on $[0,T]$.

The chain X is time homogeneous with one-step-transition kernel

$$Q(\alpha, F) = P\Big[(\xi_s)_{0\leq s\leq T} \in F \,\Big|\, \xi_0 = \alpha(T)\Big], \quad \alpha \in C[0,T], \quad F \in \mathcal{C}_T.$$

Example . With $\sigma > 0, \gamma > 0$ and
some function $S(\cdot)$ which is T-periodic and piecewise continuous, consider Ornstein-Ulhenbeck type diffusion with T-periodic drift

$$d\xi_t = (S(t) - \gamma\xi_t)dt + \sigma dW_t.$$

Solution with initial value ξ_0

$$\xi_t = \xi_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)}(S(s)ds + \sigma dW_s).$$

Transition semigroup density of ξ

$$p_{s,t}(x, \cdot) = \mathcal{N}\left(xe^{-\gamma(t-s)} + \int_0^{t-s} e^{-\gamma v} S(t-v)\, dv\,,\ \frac{1 - e^{-2\gamma(t-s)}}{2\gamma}\sigma^2\right)$$

Orstein Uhlenbeck model with periodic in time drift

$$d\xi_t = f(t)\xi_t dt + dW_t$$

where $f(\cdot)$ is some known T-periodic function

Deterministic equation

$$x'(t) = f(t)x(t), \qquad x(t) = x_0 \exp\left\{\int_0^t f(s)ds\right\} = x_0 \exp\{F(t)\}$$

$$F(t) = \int_0^t f(s)ds = n_t F(T) + F(t - n_t T), \qquad n_t = \text{integer part of } \frac{t}{T}.$$

If $F(T) = \int_0^T f(t)dt = 0$ then $x_t = x_{t+kT}$ for any $k \in \mathbb{N}$

Otherwise

$$\lim_{t\to\infty} |x_t| = \begin{cases} 0 & \text{if} \quad F(T) < 0 \\ \infty & \qquad F(T) > 0 \end{cases} \qquad \text{(when } x_0 \neq 0\text{).}$$

$$d\xi_t = f(t)\xi_t dt + dW_t$$

Solution

$$\xi_t = x_{0,t}\left(\xi_0 + \int_0^t x_{0,s}^{-1}\, dW_s\right) = x_{0,t}\xi_0 + \int_0^t x_{s,t}\, dW_s$$

where

$$x_{0,t} = \exp\left\{\int_0^t f(u)du\right\} = \exp\{F(t)\} \qquad 0 \le s < t < \infty$$

First property

$$(x_{0,t})^{-1}\xi_t - \xi_0 = \int_0^t (x_{0,s})^{-1}\, dW_s$$

so $\{(x_{0,t})^{-1}\xi_t - \xi_0 : t \ge 0\}$ is a zero-mean martingale

Markov properties

Transition semigroup density of ξ

$$p_{s,t}(x,\cdot) = \mathcal{N}\left(x_{s,t}\,,\,\int_s^t x_{u,t}^2\,du\right)$$

T-periodic semigroup :

$$p_{s+T,t+T}(x,\cdot) = p_{s,t}(x,\cdot)$$

Chain of T-segments on $C[0,T]$: $\mathbf{X}_k := (\xi_{kT+t})_{0\leq t\leq T}$

Let $\mathbf{W}_k(t) = \int_0^t x_{s,T}\, dW_s^{(kT)}$ where $W_s^{(kT)} = W_{s+kT}$.

$$\begin{aligned} \xi_{t+kT} &= x_{0,T}\xi_{t+(k-1)T} + \int_0^T x_{s,t}\, dW_s^{(t+(k-1)T)} \\ &= x_{0,T}\xi_{t+(k-1)T} + x_{0,t}\int_t^T x_{s,T}\, dW_s^{((k-1)T)} + \int_0^t x_{s,t}\, dW_s^{(kT)} \end{aligned}$$

$t \in [0, T]$.

Then

$$\mathbf{X}_{k+1}(t) = x_{0,T}\mathbf{X}_k(t) + x_{0,t}(\mathbf{W}_k(T) - \mathbf{W}_k(t)) + x_{t,T}^{-1}\mathbf{W}_{k+1}(t)$$

If $F(T) < 0$, the chain $\mathbf{X}$ is Harris positive recurrent (ergodic)

If $F(T) > 0$, the chain $\mathbf{X}$ is not recurrent.

Conjecture : if $F(T) = 0$, the chain $\mathbf{X}$ is recurrent null .

Autoregressive representation

The $C[0,T]$-valued variable $\mathbf{W}_{k+1}$ is independent from $\mathbf{X}_k$ and $\mathbf{W}_k(T)-\mathbf{W}_k$. Futhermore, we can say that

$$\mathbf{Y}_{k+1} = \Phi(\mathbf{Y}_k, \mathbf{W}_{k+1}) = \mathbf{A}(\mathbf{Y}_k) + (x_{\cdot,T})^{-1}\mathbf{W}_{k+1}(\cdot),$$

where $\mathbf{Y}_k = (\mathbf{X}_k, \mathbf{W}_k)$ and

$$\mathbf{A}(\mathbf{Y}_k) = \mathbf{A}(\mathbf{X}_k, \mathbf{W}_k) = x_{0,T}\mathbf{X}_k(\cdot) + x_{0,\cdot}(\mathbf{W}_k(T) - \mathbf{W}_k(\cdot))$$

The operator $\mathbf{A}$ is linear on $C[0,T] \times C[0,T]$ with values in $C[0,T]$, bounded for sup norm.

$(\mathbf{Y}_k)_{k\in\mathbb{Z}}$ is a Markov chain.

Asymptotics of ξ_t as $t \to \infty$ Let $G(t) = \int_0^t e^{-2F(s)}\, ds$

If $F(T) < 0$ then

$$\xi_{t+nT} \Longrightarrow e^{F(t)} \mathcal{N}\Big(0\,,\, \frac{G(T)}{e^{-2F(T)} - 1} + G(t)\Big) \quad \text{as} \quad n \to \infty, t \in [0, T]$$

If $F(T) = 0$ then

$$\frac{1}{n} \xi_{t+nT} \xrightarrow{a.e.} 0$$

$$\frac{1}{\sqrt{n}} \xi_{t+nT} \Longrightarrow e^{F(t)} \mathcal{N}\Big(0, G(T)\Big) \quad \text{as} \quad n \to \infty, t \in [0, T]$$

If $F(T) > 0$ then

$$e^{-nF(T)} \xi_{t+nT} \xrightarrow{a.e.} e^{F(t)} (\xi_0 + Y) \quad \text{as} \quad n \to \infty, t \in [0, T]$$

and $Y \sim \mathcal{N}\Big(0\,,\, \frac{G(T)}{1 - e^{-2F(T)}}\Big)$

Parameter estimation for Ornstein-Uhlenbeck process

$$d\xi_t = \theta\xi_t dt + dw_t, \quad \xi_0 = 0, \qquad t \in [0, A], \qquad A \to \infty$$

Maximum likelihood estimator (MLE) $\qquad \widehat{\theta}_A = \int_0^A \xi_s d\xi_s \Big/ \int_0^A \xi_s^2 ds.$

(i) $\theta < 0$: the process is positive recurrent, ergodic with invariant measure $\mathcal{N}\left(0, \frac{-1}{2\theta}\right)$, and $\sqrt{A}\left(\widehat{\theta}_A - \theta\right) \Longrightarrow \mathcal{N}(0, 2\theta)$.

(ii) $\theta = 0$: the process is null recurrent

$$\sqrt{A}\left(\widehat{\theta}_A - \theta\right) = \sqrt{A}\widehat{\theta}_A \Longrightarrow \frac{\displaystyle\int_0^1 w_t dw_t}{\displaystyle\int_0^1 w_t^2 dt} = \frac{w_t^2 - 1/2}{\displaystyle\int_0^1 w_t^2 dt}$$

where $(w_t)_{0 \le t \le 1}$ is some standard Wiener process.

(iii) $\theta > 0$: the process is not recurrent : $|\xi_t| \to \infty$ with probability 1

$$\frac{1}{\sqrt{2\theta}} e^{\theta A}\left(\widehat{\theta}_A - \theta\right) \Longrightarrow \frac{\nu}{\xi^0}$$

where $\nu \sim \mathcal{N}(0, 1)$, and $\xi^0 \sim \mathcal{N}\left(0, \frac{1}{2\theta}\right)$

Parameter estimation

$$d\xi_t = \theta f(t)dt + dW_t \quad \xi_0 = 0, \qquad 0 \le t \le A$$

Solution

$$\xi_t(\theta) = x^\theta_{0,t}\left(\xi_0 + \int_0^t (x^\theta_{0,s})^{-1}\, dW_s\right) = x^\theta_{0,t}\xi_0 + \int_0^t x^\theta_{s,t}\, dW_s$$

where

$$x^\theta_{s,t} = (x_{s,t})^\theta = \exp\left\{\theta \int_s^t f(u)du\right\} = \exp\{\theta(F(t) - F(s))\} = e^{\theta(F(t)-F(s))}$$

$$0 \le s < t < \infty$$

Here the condition is on $\theta F(T)$

Maximum likelihood estimator of θ

$$\widehat{\theta}_A = \frac{\int_0^A f(t)\xi_t\, d\xi_t}{\int_0^A f(t)^2 \xi_t^2\, dt} = \theta + \frac{\int_0^A f(t)\xi_t\, dW_t}{\int_0^A f(t)^2 \xi_t^2\, dt}$$

1) If $\theta F(T) < 0$ then

$$\lim_{A\to\infty} \frac{1}{A}\int_0^A f(t)^2\,\xi_t^2\,dt \quad \overset{\text{a.e.}}{\longrightarrow} \quad \frac{-G_\theta(T)}{T(1-e^{-2\theta F(T)})}\int_0^T f(t)^2\,dt$$
$$+\frac{1}{T}\int_0^T f(t)^2\,G_\theta(t)\,dt = \sigma_{\theta,1}^{-2}$$

$$\sqrt{A}\Big(\widehat{\theta}_A - \theta\Big) \Longrightarrow \mathcal{N}\Big(0, \sigma_{\theta,1}^2\Big)$$

2) If $\theta F(T) > 0$ then

$$(x_{0,t})^{-2}\xi_t^2 \overset{\text{a.e.}}{\longrightarrow} Y_\theta^2$$

$$e^{n\theta F(T)}\Big(\widehat{\theta}_{nT} - \theta\Big) \Longrightarrow \frac{\mathcal{N}(0,\sigma_{\theta,2}^2)}{|Y_\theta|}$$

where $\sigma_{\theta,2}^{-2} = \frac{e^{2\theta F(T)}-1}{\int_0^T e^{2\theta F(t)} f(t)^2\,dt}$

3) If $\theta = 0$: $\xi_t = W_t$

$$\sqrt{A}\Big(\widehat{\theta}_A - \theta\Big) = \sqrt{A}\widehat{\theta}_A \Longrightarrow \frac{\displaystyle\int_0^1 w_t dw_t}{\displaystyle\int_0^1 w_t^2 dt} = \frac{w_t^2 - 1/2}{\int_0^1 w_t^2 dt}$$

where $(w_t)_{0 \le t \le 1}$ is some standard Wiener process.

4) If $F(T) = 0$ open problem

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