Discretization error of stochastic integration

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Problem

We consider the limit distribution of Z^n defined as

$$Z^n_t = \int_0^t X_s \mathrm{d}Y_s - \sum_{j=0}^{\infty} X_{\tau^n_j}(Y_{\tau^n_{j+1}\wedge t} - Y_{\tau^n_j \wedge t})$$

as $\sup_{j\geq 0} |\tau^n_{j+1} \wedge t - \tau^n_j \wedge t| \to 0$ in probability for each $t \in [0, T)$, where X, Y are continuous semimartingale and $\tau^n = \{\tau^n_j\}$ is a sequence of stopping times with

$$0 = \tau^n_0 < \tau^n_1 < \dots$$

and $\tau^n_j \to T$ as $j \to \infty$. Here $T \in (0, \infty]$ is fixed.

Rootzén (1980) for the case $\tau^n_j = j/n$.

Example: Realized Volatility

For a continuous semimartingale X,

$$\sum_{j=0}^{\infty}(X_{\tau^n_{j+1}\wedge t} - X_{\tau^n_j\wedge t})^2 - \langle X\rangle_t$$
$$= 2\left\{\int_0^t X_s \mathrm{d}X_s - \sum_{j=0}^{\infty} X_{\tau^n_j}(X_{\tau^n_{j+1}\wedge t} - X_{\tau^n_j\wedge t})\right\}.$$

This is the special case $X = Y$. The stopping times τ^n_j are sampling times in this context.
In financial tick data, time stamps for prices correspond to transaction times or quote revision times, which are not deterministic.

Example: Estimating Function

Consider a parametric model

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma(X_t, \theta)\mathrm{d}W_t$$

and Z-estimation by the estimating function

$$F_n^i(\theta) = \sum_{j=0}^{N_T^n - 1} w^i(X_{\tau_j^n}, \theta) \left\{ (X_{\tau_{j+1}^n} - X_{\tau_j^n})^2 - \sigma(X_{\tau_j^n}, \theta)(\tau_{j+1}^n - \tau_j^n) \right\},$$

where $w = (w^1, w^2, \ldots, w^d)$ are weight functions and

$$N_T^n = \max\{j \geq 0; \tau_j^n \leq T\}.$$

(Recall that T is fixed.)

Example: Estimating Function

Then the asymptotic distribution of the estimator is determined by that of

$$Z_t^n = \int_0^t X_s \mathrm{d}Y_s - \sum_{j=0}^{\infty} X_{\tau_j^n}(Y_{\tau_{j+1}^n \wedge t} - Y_{\tau_j^n \wedge t})$$

with

$$Y_t = \int_0^t w(X_t, \theta)\mathrm{d}X_s.$$

Here Y is d-dimensional but X is one-dimensional.

Example: Hedging derivatives

It is often the case that a hedging strategy for an option payoff H_T is given as a continuous trading strategy:

$$H_T = \int_0^T \Pi_u \mathrm{d}S_u$$

In practice, rebalancing portfolio is done discretely: the hedging error is

$$\int_0^T \Pi_u \mathrm{d}S_u - \sum_{j=0}^{\infty} \Pi_{\tau_j^n}(S_{\tau_{j+1}^n \wedge t} - S_{\tau_j^n \wedge t}).$$

In this context τ_j^n are the transaction times.

Structure condition

We suppose that $\exists \psi, \varphi$ such that

$$X = X_0 + \int_0^{\cdot} \psi_s \mathrm{d}\langle M \rangle_s + M, \quad Y = Y_0 + \int_0^{\cdot} \varphi_s \mathrm{d}\langle M \rangle_s + M^Y,$$

with continuous local martingales M and M^Y and that $\exists \kappa$ such that

$$\langle M^Y \rangle = \int_0^{\cdot} \kappa_s \mathrm{d}\langle M \rangle_s.$$

Suppose in addition that $\mathbb{E}[\langle X \rangle_T^6] < \infty$ for simplicity.

Theorem A

If $\exists \epsilon_n > 0$ with $\epsilon_n \to 0$ and $\exists a, b$ loc. bounded cag processes such that

$$G^4_{j,n}/G^2_{j,n} = \epsilon_n^2 a_{\tau^n_j} + o_p(\epsilon_n^2), \quad G^3_{j,n}/G^2_{j,n} = \epsilon_n b_{\tau^n_j} + o_p(\epsilon_n)$$

and

$$G^6_{j,n}/G^2_{j,n} = O_p(\epsilon_n^2), \quad G^{12}_{j,n}/G^2_{j,n} = o_p(\epsilon_n^8)$$

uniformly in j with $\tau^n_j \leq t$ for each $t \in [0, T)$, where $G^k_{j,n} = \mathbb{E}[(M_{\tau^n_{j+1}} - M_{\tau^n_j})^k | \mathcal{F}_{\tau^n_j}]$, then

$$\epsilon_n^{-1} Z^n \to \frac{1}{3}\int_0^{\cdot} b_s \mathrm{d}Y_s + \frac{1}{\sqrt{6}}\int_0^{\cdot} \left\{a_s^2 - \frac{2}{3}b_s^2\right\}^{1/2} \mathrm{d}Y'_s$$

stably, where $Y' = W_{\langle Y \rangle}$ and W is a BM independent of $\mathcal{F}$.

Pearson's inequality

In general, for a random variable X with $E[X] = 0$, we have

$$\frac{E[X^4]}{|E[X^2]|^2} - \frac{|E[X^3]|^2}{|E[X^2]|^3} \geq 1.$$

Proof:

$|E[X^3]|^2 = |E[X(X^2 - E[X^2])]|^2 \leq E[X^2](E[X^4] - |E[X^2]|^2)$.

Therefore, $G^4_{j,n}/G^2_{j,n} = O_p(\epsilon_n^2)$ implies $G^3_{j,n}/G^2_{j,n} = O_p(\epsilon_n)$ and $G^2_{j,n} = O_p(\epsilon_n^2)$.

Theorem B

If in addition, $\exists q$ loc. bounded cag process which is loc. bounded also away from 0 such that $G^2_{j,n} = \epsilon_n^2 q^2_{\tau_j^n} + o_p(\epsilon_n^2)$, then,

$$\epsilon_n^2 N_t^n \to \int_0^t q_s^{-2} \mathrm{d}\langle X\rangle_s, \quad N_t^n = \max\{j \geq 0; \tau_j^n \leq t\}.$$

Moreover, $\sqrt{N_t^n} Z_t^n$ converges stably and the asymptotic conditional variance has a lower bound

$$\frac{1}{6}\left|\int_0^t \sqrt{\kappa_s} \mathrm{d}\langle X\rangle_s\right|^2$$

Kurtosis-Skewness inequalities

In general for a random variable X with $E[X] = 0$, we have

$$\frac{E[X^4]}{|E[X^2]|^2} - \frac{|E[X^3]|^2}{|E[X^2]|^3} \geq 1$$

and

$$\frac{E[X^4]}{|E[X^2]|^2} - \frac{3}{4}\frac{|E[X^3]|^2}{|E[X^2]|^3} \geq \frac{E[X^2]}{|E[|X|]|^2}.$$

The equalities are attained if and only if the support of X consists of two points.

Theorem C

If in addition, $\exists\zeta$ loc. bounded cag process which is loc. bounded also away from 0 such that

$$\epsilon_n \mathbb{E}[|M_{\tau^n_{j+1}} - M_{\tau^n_j}||\mathcal{F}_{\tau^n_j}]/G^2_{j,n} = \epsilon_n^2 q^2_{\tau^n_j} + o_p(\epsilon_n^2),$$

then for any loc. bounded cag process u,

$$\epsilon_n U^n_t \to \int_0^t |u_s|\zeta_s \mathrm{d}\langle X\rangle_s, \quad U^n_t = \sum_{j=0}^{\infty} |u_{\tau^n_j}||M_{\tau^n_{j+1}\wedge t} - M_{\tau^n_j \wedge t}|.$$

Moreover, $U^n_t Z^n_t$ converges stably and the asymptotic conditional variance has a lower bound

$$\frac{1}{6}\left|\int_0^t |u_s|^{2/3}\kappa_s^{1/3}\mathrm{d}\langle X\rangle_s\right|^3$$

attained by $\tau^n_{j+1} = \inf\{t > \tau^n_j : |M_t - M_{\tau^n_j}| \geq \epsilon_n |u_{\tau^n_j}|^{1/3}\kappa^{-1/3}_{\tau^n_j}\}$.

Theorem D

Suppose κ is continuous and denote by $\mathcal{T}$ the class of the sequences of stopping times satisfying

- for any stopping time $\tau < T$, $\sup_{j\geq 0} |\tau^n_{j+1} \wedge \tau - \tau^n_j \wedge \tau| \to 0$ in p,
- there exists a sequence of stopping times σ^m such that $\sigma^m < T$, $\sigma^m \to T$, $m \to \infty$ and $E[N^n_{\sigma^m}]\langle Z^n\rangle_{\sigma^m}$ is uniformly integrable for each m.

Then for any $\tau^n \in \mathcal{T}$ and any stopping time $\tau \leq T$,

$$\liminf_{n\to\infty} E[N^n_\tau]E[\langle Z^n\rangle_\tau] \geq \frac{1}{6}E[\int_0^\tau \sqrt{\kappa_s}\mathrm{d}\langle X\rangle_s]^2.$$

The lower bound is attained by
$\tau^n_{j+1} = \inf\{t > \tau^n_j : |X_t - X_{\tau^n_j}| \geq \epsilon_n \kappa^{-1/4}_{\tau^n_j}\}$.

Theorem E

Suppose κ is continuous. Let

$$C^n(\alpha,\beta) = \sum_{j=0}^{\infty} \alpha_{\tau_j^n} |X_{\tau_{j+1}^n \wedge t} - X_{\tau_j^n \wedge t}|^\beta$$

and denote by $\mathcal{T}^{\alpha,\beta}$ the class of the sequences satisfying

- $\forall \tau < T$, $\sup_{j\geq 0} |\tau_{j+1}^n \wedge \tau - \tau_j^n \wedge \tau| \to 0$ in p,
- $\exists \sigma^m$ such that $\sigma^m < T$, $\sigma^m \to T$, $m \to \infty$ and $E[C^n(\alpha,\beta)_{\sigma^m}]^{2/(2-\beta)} \langle Z^n \rangle_{\sigma^m}$ is u.i. for each m.

Then for any $\tau^n \in \mathcal{T}^{\alpha,\beta}$ and any stopping time $\tau \leq T$,

$$\liminf_{n\to\infty} E[C^n(\alpha,\beta)_\tau]^{2/(2-\beta)} E[\langle Z^n \rangle_\tau] \geq \frac{1}{6} E[\int_0^\tau \alpha^{2/(4-\beta)} \kappa_s^{2(2-\beta)/(4-\beta)} \mathrm{d}\langle X \rangle_s]^{(4-\beta)/(2-\beta)}.$$