

Discretization error of stochastic integration

Masaaki Fukasawa

Forschungsinstitut für Mathematik, ETH Zürich
Japan Science and Technology Agency, CREST

22 March 2011

Problem

We consider the limit distribution of Z^n defined as

$$Z_t^n = \int_0^t X_s dY_s - \sum_{j=0}^{\infty} X_{\tau_j^n} (Y_{\tau_{j+1}^n \wedge t} - Y_{\tau_j^n \wedge t})$$

as $\sup_{j \geq 0} |\tau_{j+1}^n \wedge t - \tau_j^n \wedge t| \rightarrow 0$ in probability for each $t \in [0, T)$, where X, Y are continuous semimartingale and $\tau^n = \{\tau_j^n\}$ is a sequence of **stopping times** with

$$0 = \tau_0^n < \tau_1^n < \dots$$

and $\tau_j^n \rightarrow T$ as $j \rightarrow \infty$. Here $T \in (0, \infty]$ is fixed.

Rootzén (1980) for the case $\tau_j^n = j/n$.

Example: Realized Volatility

For a continuous semimartingale X ,

$$\begin{aligned} & \sum_{j=0}^{\infty} (X_{\tau_{j+1}^n \wedge t} - X_{\tau_j^n \wedge t})^2 - \langle X \rangle_t \\ &= 2 \left\{ \int_0^t X_s dX_s - \sum_{j=0}^{\infty} X_{\tau_j^n} (X_{\tau_{j+1}^n \wedge t} - X_{\tau_j^n \wedge t}) \right\}. \end{aligned}$$

This is the special case $X = Y$. The stopping times τ_j^n are **sampling times** in this context.

In financial tick data, time stamps for prices correspond to transaction times or quote revision times, which are not deterministic.

Example: Estimating Function

Consider a parametric model

$$dX_t = \mu_t dt + \sigma(X_t, \theta) dW_t$$

and Z -estimation by the estimating function

$$F_n^i(\theta) = \sum_{j=0}^{N_T^n - 1} w^i(X_{\tau_j^n}, \theta) \left\{ (X_{\tau_{j+1}^n} - X_{\tau_j^n})^2 - \sigma(X_{\tau_j^n}, \theta)(\tau_{j+1}^n - \tau_j^n) \right\},$$

where $w = (w^1, w^2, \dots, w^d)$ are weight functions and

$$N_T^n = \max\{j \geq 0; \tau_j^n \leq T\}.$$

(Recall that T is fixed.)

Example: Estimating Function

Then the asymptotic distribution of the estimator is determined by that of

$$Z_t^n = \int_0^t X_s dY_s - \sum_{j=0}^{\infty} X_{\tau_j^n} (Y_{\tau_{j+1}^n \wedge t} - Y_{\tau_j^n \wedge t})$$

with

$$Y_t = \int_0^t w(X_s, \theta) dX_s.$$

Here Y is d -dimensional but X is one-dimensional.

Example: Hedging derivatives

It is often the case that a hedging strategy for an option payoff H_T is given as a continuous trading strategy:

$$H_T = \int_0^T \Pi_u dS_u$$

In practice, rebalancing portfolio is done discretely: the hedging error is

$$\int_0^T \Pi_u dS_u - \sum_{j=0}^{\infty} \Pi_{\tau_j^n} (S_{\tau_{j+1}^n \wedge t} - S_{\tau_j^n \wedge t}).$$

In this context τ_j^n are the transaction times.

Structure condition

We suppose that $\exists \psi, \varphi$ such that

$$X = X_0 + \int_0^\cdot \psi_s d\langle M \rangle_s + M, \quad Y = Y_0 + \int_0^\cdot \varphi_s d\langle M \rangle_s + M^Y,$$

with continuous local martingales M and M^Y and that $\exists \kappa$ such that

$$\langle M^Y \rangle = \int_0^\cdot \kappa_s d\langle M \rangle_s.$$

Suppose in addition that $\mathbb{E}[\langle X \rangle_T^6] < \infty$ for simplicity.

Theorem A

If $\exists \epsilon_n > 0$ with $\epsilon_n \rightarrow 0$ and $\exists a, b$ loc. bounded cag processes such that

$$G_{j,n}^4/G_{j,n}^2 = \epsilon_n^2 a_{\tau_j^n} + o_p(\epsilon_n^2), \quad G_{j,n}^3/G_{j,n}^2 = \epsilon_n b_{\tau_j^n} + o_p(\epsilon_n)$$

and

$$G_{j,n}^6/G_{j,n}^2 = O_p(\epsilon_n^2), \quad G_{j,n}^{12}/G_{j,n}^2 = o_p(\epsilon_n^8)$$

uniformly in j with $\tau_j^n \leq t$ for each $t \in [0, T)$, where

$G_{j,n}^k = \mathbb{E}[(M_{\tau_{j+1}^n} - M_{\tau_j^n})^k | \mathcal{F}_{\tau_j^n}]$, then

$$\epsilon_n^{-1} Z^n \rightarrow \frac{1}{3} \int_0^\cdot b_s dY_s + \frac{1}{\sqrt{6}} \int_0^\cdot \left\{ a_s^2 - \frac{2}{3} b_s^2 \right\}^{1/2} dY'_s$$

stably, where $Y' = W_{\langle Y \rangle}$ and W is a BM independent of \mathcal{F} .

Pearson's inequality

In general, for a random variable X with $E[X] = 0$, we have

$$\frac{E[X^4]}{|E[X^2]|^2} - \frac{|E[X^3]|^2}{|E[X^2]|^3} \geq 1.$$

Proof:

$$|E[X^3]|^2 = |E[X(X^2 - E[X^2])]|^2 \leq E[X^2](E[X^4] - |E[X^2]|^2).$$

Therefore, $G_{j,n}^4 / G_{j,n}^2 = O_p(\epsilon_n^2)$ implies

$$G_{j,n}^3 / G_{j,n}^2 = O_p(\epsilon_n) \text{ and } G_{j,n}^2 = O_p(\epsilon_n^2).$$

Theorem B

If in addition, $\exists q$ loc. bounded cag process which is loc. bounded also away from 0 such that $G_{j,n}^2 = \epsilon_n^2 q_{\tau_j^n}^2 + o_p(\epsilon_n^2)$, then,

$$\epsilon_n^2 N_t^n \rightarrow \int_0^t q_s^{-2} d\langle X \rangle_s, \quad N_t^n = \max\{j \geq 0; \tau_j^n \leq t\}.$$

Moreover, $\sqrt{N_t^n} Z_t^n$ converges stably and the asymptotic conditional variance has a lower bound

$$\frac{1}{6} \left| \int_0^t \sqrt{\kappa_s} d\langle X \rangle_s \right|^2$$

attained by $\tau_{j+1}^n = \inf\{t > \tau_j^n : |M_t - M_{\tau_j^n}| \geq \epsilon_n \kappa_{\tau_j^n}^{-1/4}\}$.

Kurtosis-Skewness inequalities

In general for a random variable X with $E[X] = 0$, we have

$$\frac{E[X^4]}{|E[X^2]|^2} - \frac{|E[X^3]|^2}{|E[X^2]|^3} \geq 1$$

and

$$\frac{E[X^4]}{|E[X^2]|^2} - \frac{3|E[X^3]|^2}{4|E[X^2]|^3} \geq \frac{E[X^2]}{|E[X]|^2}.$$

The equalities are attained if and only if the support of X consists of two points.

Theorem C

If in addition, $\exists \zeta$ loc. bounded cag process which is loc. bounded also away from 0 such that

$$\epsilon_n \mathbb{E}[|M_{\tau_{j+1}^n} - M_{\tau_j^n}| | \mathcal{F}_{\tau_j^n}] / G_{j,n}^2 = \epsilon_n^2 q_{\tau_j^n}^2 + o_p(\epsilon_n^2),$$

then for any loc. bounded cag process u ,

$$\epsilon_n U_t^n \rightarrow \int_0^t |u_s| \zeta_s d\langle X \rangle_s, \quad U_t^n = \sum_{j=0}^{\infty} |u_{\tau_j^n}| |M_{\tau_{j+1}^n \wedge t} - M_{\tau_j^n \wedge t}|.$$

Moreover, $U_t^n Z_t^n$ converges stably and the asymptotic conditional variance has a lower bound

$$\frac{1}{6} \left| \int_0^t |u_s|^{2/3} \kappa_s^{1/3} d\langle X \rangle_s \right|^3$$

attained by $\tau_{j+1}^n = \inf\{t > \tau_j^n : |M_t - M_{\tau_j^n}| \geq \epsilon_n |u_{\tau_j^n}|^{1/3} \kappa_{\tau_j^n}^{-1/3}\}.$

Theorem D

Suppose κ is continuous and denote by \mathcal{T} the class of the sequences of stopping times satisfying

- for any stopping time $\tau < T$,
 $\sup_{j \geq 0} |\tau_{j+1}^n \wedge \tau - \tau_j^n \wedge \tau| \rightarrow 0$ in \mathbb{P} ,
- there exists a sequence of stopping times σ^m such that $\sigma^m < T$, $\sigma^m \rightarrow T$, $m \rightarrow \infty$ and $E[N_{\sigma^m}^n] \langle Z^n \rangle_{\sigma^m}$ is uniformly integrable for each m .

Then for any $\tau^n \in \mathcal{T}$ and any stopping time $\tau \leq T$,

$$\liminf_{n \rightarrow \infty} E[N_{\tau^n}^n] E[\langle Z^n \rangle_{\tau}] \geq \frac{1}{6} E\left[\int_0^{\tau} \sqrt{\kappa_s} d\langle X \rangle_s \right]^2.$$

The lower bound is attained by

$$\tau_{j+1}^n = \inf\{t > \tau_j^n : |X_t - X_{\tau_j^n}| \geq \epsilon_n \kappa_{\tau_j^n}^{-1/4}\}.$$

Theorem E

Suppose κ is continuous. Let

$$C^n(\alpha, \beta) = \sum_{j=0}^{\infty} \alpha_{\tau_j^n} |X_{\tau_{j+1}^n \wedge t} - X_{\tau_j^n \wedge t}|^\beta$$

and denote by $\mathcal{T}^{\alpha, \beta}$ the class of the sequences satisfying

- $\forall \tau < T, \sup_{j \geq 0} |\tau_{j+1}^n \wedge \tau - \tau_j^n \wedge \tau| \rightarrow 0$ in \mathbb{P} ,
- $\exists \sigma^m$ such that $\sigma^m < T, \sigma^m \rightarrow T, m \rightarrow \infty$ and $E[C^n(\alpha, \beta)_{\sigma^m}]^{2/(2-\beta)} \langle Z^n \rangle_{\sigma^m}$ is u.i. for each m .

Then for any $\tau^n \in \mathcal{T}^{\alpha, \beta}$ and any stopping time $\tau \leq T$,

$$\begin{aligned} & \liminf_{n \rightarrow \infty} E[C^n(\alpha, \beta)_\tau]^{2/(2-\beta)} E[\langle Z^n \rangle_\tau] \\ & \geq \frac{1}{6} E\left[\int_0^\tau \alpha^{2/(4-\beta)} \kappa_s^{2(2-\beta)/(4-\beta)} d\langle X \rangle_s \right]^{(4-\beta)/(2-\beta)}. \end{aligned}$$