On some estimation problems in periodic diffusions I

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Le Mans 2011

introduction : some statistical problems in periodic diffusions

parametric model : unknown $\vartheta \in \Theta$, $\Theta \subset \mathbb{R}^d$, observe solution ξ of

$$d\xi_t = [S(\vartheta, t) + b(\xi_t)]\, dt + \sigma(\xi_t) dW_t \quad , \quad t \geq 0$$

continuously in time, over a long time interval (asymptotics $n \to \infty$)

T-periodicity : in the drift, deterministic periodic input gouverned by $\vartheta \in \Theta$

$$t \longrightarrow S(\vartheta, t) = S(\vartheta, i_T(t)) \ , \quad i_T(t) := t \text{ modulo } T$$

aim : asymptotic inference about ϑ

- local models at ϑ, local scale, convergence of local models (this talk)
- convergence of estimators (talk by Yury Kutoyants)

under assumptions which provide periodic ergodicity (talk by Eva Löcherbach)

- positive Harris recurrence of the grid chain $X := (\xi_{kT})_{k \in \mathbb{N}_0}$
- pos. Harris recurrence of the segment chain $\mathbb{X} := ((\xi_{kT+s})_{0 \leq s \leq T})_{k \in \mathbb{N}_0}$

examples A+B : with some 1-periodic function $S_0(\cdot)$, fixed and known,

$$S(\vartheta, t) = S_0\left(\frac{1}{\vartheta}\, t\right) \quad : \quad \text{period } T = \vartheta,\ \Theta = (0, \infty)$$

A : if S_0 is continuous [H-Ku. MMS 11] : local scale $\delta_n(\vartheta) = n^{-3/2}$
obtain local asymptotic normality (LAN) in the sense of Le Cam (1969), Hájek (1970), Le Cam and Yang (1990), ...

B : if S_0 has discontinuities [H-Ku. MMS 11] : local scale $\delta_n(\vartheta) = n^{-2}$
obtain convergence of local models at ϑ to a limit experiment

$$\widetilde{\mathcal{E}} := \left\{ \widetilde{P}_u \ : \ u \in \mathbb{R} \right\}$$

studied by Ibragimov and Khasminskii (1981, from 'signal in white noise'), Golubev (1979), Rubin and Song (1995), Dachian (2010), ..., with likelihoods

$$\widetilde{L}^{u/0} := \frac{d\widetilde{P}_u}{d\widetilde{P}_0} = \exp\left\{ \widetilde{W}_u - \frac{1}{2}|u| \right\} \ , \ u \in \mathbb{R}$$

where $\widetilde{W} = (\widetilde{W}_u)_{u \in \mathbb{R}}$ is double sided standard BM (neglecting scaling cst.)

Ibragimov and Khasminskii limit experiment $\widetilde{\mathcal{E}}$ arises when switching off known drift after an unknown amount of time (w.r. to 'both past and future') :

on a probability space carrying independent standard BM's $\widetilde{W}^+$ and $\widetilde{W}^-$, create double sided BM from $\widetilde{W}_v := \widetilde{W}^+_v$ if $v \geq 0$, $\widetilde{W}_v := \widetilde{W}^-_{|v|}$ if $v \leq 0$, put

$$\widetilde{P}_0 := \mathcal{L}\left(\left(\widetilde{W}^+_v \, , \, \widetilde{W}^-_v \right)_{v \geq 0} \right) \quad \text{on } C([0,\infty), \mathbb{R}^2)$$

and

$$\widetilde{P}_u \ := \begin{cases} \mathcal{L}\left(\left(\widetilde{W}^+_v + v \wedge u \, , \, \widetilde{W}^-_v \right)_{v \geq 0} \mid \widetilde{P}_0 \right) & \text{for } \ u > 0 \\ \mathcal{L}\left(\left(\widetilde{W}^+_v \, , \, \widetilde{W}^-_v + v \wedge |u| \right)_{v \geq 0} \mid \widetilde{P}_0 \right) & \text{for } \ u < 0 \end{cases}$$

then observation of the bivariate canonical process on $C([0,\infty), \mathbb{R}^2)$ with infinite time horizon $v = +\infty$ leads to likelihood ratios

$$\widetilde{L}^{u/0} \ = \ \frac{d\widetilde{P}_u}{d\widetilde{P}_0} \ = \ \exp\left\{ \widetilde{W}_u - \frac{1}{2}|u| \right\} \, , \quad u \in \mathbb{R}$$

in $\widetilde{\mathcal{E}}$ (use Liptser and Shiryaev 1981, Jacod and Shiryaev 1987) examples C+D : periodicity T fixed, known, and not depending on $\vartheta \in \Theta$

C : parametric families $\{S(\vartheta, \cdot) : \vartheta \in \Theta\} \subset C_{\mathrm{per}}([0, T])$ [H-Ku. SD 10] :

obtain LAN at ϑ with local scale $\delta_n(\vartheta) = n^{-1/2}$ provided parametrization is locally at ϑ smooth in $L^2([0, T], \lambda^{(\vartheta)})$ sense, for a measure $\lambda^{(\vartheta)}$ on $[0, T]$ related to $\sigma^{-2}(\cdot)$ and the invariant law of the T-segment chain $\mathbb{X}$ under ϑ

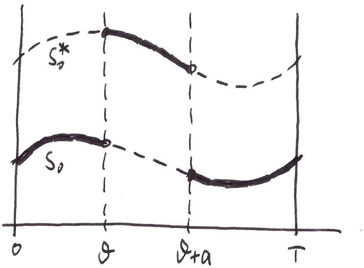
D : $\vartheta \in \Theta$ parametrizing points of discontinuity of $S(\vartheta, \cdot)$ [H-Ku. SISP 10] :

known functions $S_0 < S_0^* \in C_{\mathrm{per}}([0, T])$

known $0 < a < T$, put $\Theta := (0, T - a)$

the function $S(\vartheta, \cdot)$ sketched switches from S_0 at time ϑ to S_0^*, and back at time $\vartheta + a$

when $\sigma^2(\cdot)$ is bounded away from zero, with local scale $\delta_n(\vartheta) = n^{-1}$, obtain convergence of local models at ϑ to Ibragimov and Khasminskii limit exp. $\widetilde{\mathcal{E}}$



limit theorems in Markov processes with T-periodic semigroup

$(\xi_t)_{t\geq 0}$ a time inhomogeneous strong Markov process, càdlàg paths, $(P_{s,t})_{s<t}$ semigroup of transition probabilities, assume

$$\underline{T\text{-periodicity}}: \quad P_{s,t}(x,dy) \;=\; P_{s+kT,t+kT}(x,dy) \quad \text{for arbitrary } k\ .$$

need limit theorems as $t\to\infty$ for a large class of functionals of $(\xi_t)_{t\geq 0}$, of type

$$A=(A_t)_{t\geq 0} \quad , \quad A_t \;=\; \int_0^t H(s)\, f(s,\xi_s)\, \Lambda_T(ds)\ ,\ t\geq 0$$

for càdlàg nondecreasing $H\in\mathcal{RV}_\rho$ (regular variation at ∞ with index $\rho>0$), for functions $f(\cdot,\cdot)$ nonnegative and T-periodic in the time variable

$$f(s,x)=f(s+kT,x) \quad \text{for all } k$$

and for σ-finite measures $\Lambda_T(ds)$ which are T-periodic

$$\Lambda_T(B) \;=\; \Lambda_T(B+kT) \quad \text{for all } k$$

(a class by far larger than the class of additive functionals of $(\xi_t)_{t\geq 0}$)

for our statistical purposes, we will have to consider functionals like

$$A_t := \sum_{k\in\mathbb{N}_0, kT+r\le t} (kT+r)^\ell f(kT+r, \xi_{kT+r}) \ , \quad \Lambda_T(ds) := \sum_{k\in\mathbb{Z}} \epsilon_{kT+r}(ds)$$

for different $\ell \in \boldsymbol{N}_0$, or

$$A_t := \int_0^t s^\ell f(s,\xi_s)\, 1_{(r,r')}(i_T(s))\, ds \ , \quad \Lambda_T(ds) := 1_{(r,r')}(i_T(s))\, ds$$

where $0 \le r < r' < T$ is fixed; limit theorems for $A = (A_t)_{t\ge 0}$ as $t \to \infty$ will be obtained through the T-segment chain $\mathbb{X}$

$$\mathbb{X} = (\mathbb{X}_k)_k \ : \quad \mathbb{X}_k := (\xi_{kT+s})_{0\le s\le T} \ , \ k \in \mathbb{N}_0 \ , \quad D([0,T])\text{-valued}$$

(time-homogeneous by T-periodicity of the semigroup) and the T-grid chain

$$X = (X_k)_k \ : \quad X_k := \xi_{kT} \ , \ k \in \mathbb{N}_0$$

(time-homogeneous by T-periodicity of the semigroup of $(\xi_t)_{t\ge 0}$)

throughout, we shall work under the following periodic ergodicity condition :

(H) the T-grid chain X is positive Harris recurrent with invariant prob. μ

see [H-Löcherbach 11] for sufficient conditions implying (H) ;
see [H-Kutoyants SISP 10, theorem 2.1, MMS 11, lemma 2.2]
for proofs of the following two theorems

theorem 1 : the segment chain $\mathbb{X}$ on $D([0,T])$ is positive Harris recurrent iff (H) holds ; in this case, the invariant measure m of $\mathbb{X}$ is determined by

$$\left\{ \begin{array}{l} \text{for arbitrary } 0 = t_0 < t_1 < \ldots < t_\ell < t_{\ell+1} = T \text{ and arbitrary } A_i \, , \\ m\left(\{\pi_{t_i} \in A_i \, , \, 0 \leq i \leq \ell{+}1\}\right) \quad \text{is given by} \\ \int \ldots \int \mu(dx_0)\, 1_{A_0}(x_0) \prod_{i=0}^{\ell} P_{t_i, t_{i+1}}(x_i, dx_{i+1})\, 1_{A_{i+1}}(x_{i+1}) \, . \end{array} \right.$$

theorem 2 : for increasing processes $A = (A_t)_{t \geq 0}$ with the property

$$\begin{cases} \text{there is an } m\text{-integrable function } F : D([0,T]) \to [0,\infty) \\ \text{of form} \quad F : \alpha \longrightarrow \int_0^T \Lambda_T(ds)\, f(s,\alpha(s)) \\ \text{such that} \quad A_{kT} = \sum_{j=1}^k F(\mathbb{X}_j) \quad \text{for all } k = 1,2,\ldots \end{cases}$$

we have under (H) almost sure convergence as $t \to \infty$

$$\frac{1}{t} A_t \longrightarrow \frac{1}{T}\, m(F) = \frac{1}{T} \int_0^T \Lambda_T(ds) \int [\mu P_{0,s}](dy)\, f(s,y)$$

and obtain the same limit for every $H \in \mathcal{RV}_\rho$ with $\rho > 0$

$$\frac{1+\rho}{t\,H(t)} \int_0^t H(s)\, dA_s \longrightarrow \frac{1}{T}\, m(F) \quad \text{a.s. as } t \to \infty$$

(regular variation : see Bingham, Goldie and Teugels 1987)

convergence of local models : case $S(\vartheta, t) = S_0\left(\frac{1}{\vartheta}\, t\right)$ of example B

<u>assumptions :</u> • $S_0(\cdot)$ fixed and known, 1-periodic, piecewise continuous, having jumps of height ρ_j at times τ_j, $0 < \tau_1 < \ldots < \tau_\ell < 1$

- $\sigma(\cdot) > 0$ Lipschitz, bounded, bounded away from 0 ; $b(\cdot)$ Lipschitz
- 'periodic ergodicity' condition (H) holds for solution of

$$(*) \qquad d\xi_t = \left[S_0\left(\frac{t}{\vartheta}\right) + b(\xi_t)\right] dt + \sigma(\xi_t) dW_t \quad , \quad t \geq 0$$

for every $\vartheta \in \Theta$, $\Theta = (0, \infty)$

<u>notations :</u> • P^ϑ law of $(*)$ on the canonical path space $(C, \mathcal{C}, \mathbb{F})$

- $\eta = (\eta_t)_{t \geq 0}$ canonical process on $(C, \mathcal{C}, \mathbb{F})$, $m^{(\vartheta)}$ its P^ϑ-martingale part
- $L^{\zeta/\vartheta}$ likelihood ratio process of P^ζ to P^ϑ relative to $\mathbb{F}$:

$$L^{\zeta/\vartheta} = \mathcal{E}_\vartheta\left(\int_0^\cdot \frac{S_0(\frac{t}{\zeta}) - S_0(\frac{t}{\vartheta})}{\sigma^2(\eta_t)}\, dm_t^{(\vartheta)}\right) = \mathcal{E}_\vartheta\left(\int_0^\cdot \frac{S_0(\frac{t}{\zeta}) - S_0(\frac{t}{\vartheta})}{\sigma(\eta_t)}\, dB_t^{(\vartheta)}\right)$$

with P^ϑ-Brownian motion $B^{(\vartheta)}$

localization : fix ϑ, for $n \geq 1$: local scale $\delta_n(\vartheta) = n^{-2}$, observation $(\eta_t)_{0 \leq t \leq n}$:

$$\left\{ P^{\vartheta + h/n^2} \mid \mathcal{F}_n : h \in \Theta_{\vartheta,n} \right\} \quad \text{parametrized by the local parameter } h$$

likelihoods $L_n^{(\vartheta+h/n^2)/\vartheta}$ of local parameter h to local parameter 0

theorem 3 : [H-Ku. MMS 11] as $n \to \infty$, have convergence of local models at ϑ

$$\left(L_n^{(\vartheta+h/n^2)\,/\,\vartheta} \right)_{h \in \Theta_{\vartheta,n}} \quad , \quad \Theta_{\vartheta,n} := \{ h : \vartheta + h/n^2 > 0 \}$$

in the sense of finite dimensional distributions to

$$\left(\widetilde{L}^{h/0} \right)_{h \in \mathbb{R}} \quad , \quad \widetilde{L}^{h/0} = \exp\left\{ \widetilde{W}(uJ) - \frac{1}{2}|uJ| \right\} \ , \ u \in \mathbb{R}$$

with some scaling factor $J := \frac{1}{2\vartheta^2} \sum_{j=1}^{\ell} \rho_j^2 \, [\mu^{(\vartheta)} P^{(\vartheta)}_{0,r_j\vartheta}](\frac{1}{\sigma^2})$
which depends on ϑ and on the sequence of jump times/heights of S_0

below, we give a sketch of the main steps of the proof, and will end with a stronger assertion which has the structure of a '2nd Le Cam lemma' consider $S_0(\cdot)$ piecewise constant, two jumps in $(0,1)$ at times $r_1 < r_2$:

$$S_0(t) = 1_{[r_1,r_2)}(i_1(t)) \quad \text{with } i_1(t) = t \text{ modulo } 1$$

then by 1-periodicity

$$S_0(t) = N_1(t) - N_2(t) \quad \text{where } \; N_j(t) := \sum_{k=0}^{\infty} 1_{[k+r_j,\infty)}(t)$$

in local models at ϑ, LRP $L_{\bullet n}^{(\vartheta+h/n^2)\,/\,\vartheta}$ is stochastic exponential under $P^{(\vartheta)}$ of

$$\int_0^{\bullet n} \frac{S_0(\frac{s}{\vartheta+h/n^2}) - S_0(\frac{s}{\vartheta})}{\sigma(\eta_s)} dB_s^{(\vartheta)} \;=\; \int_0^{\bullet n} j_\vartheta^{h,n}(s)\, \frac{1}{\sigma(\eta_s)} dB_s^{(\vartheta)}$$

where (here notation for case $h<0$, similiar in case $h>0$)

$$j_\vartheta^{h,n}(s) \;:=\; \left[N_1(\frac{s}{\vartheta+h/n^2}) - N_1(\frac{s}{\vartheta}) \right] - \left[N_2(\frac{s}{\vartheta+h/n^2}) - N_2(\frac{s}{\vartheta}) \right]$$

orthogonality : notation for case $h<0$:

$$\left[N_1\left(\frac{s}{\vartheta+h/n^2}\right)-N_1\left(\frac{s}{\vartheta}\right)\right] \quad := \quad \sum_{k=0}^{\infty} 1_{\underbrace{\left((\vartheta+h/n^2)(k+r_1),\,\vartheta(k+r_1)\right]}_{\text{length } \approx\, |h|\,\frac{k}{n^2}}}(s)$$

$$\left[N_2\left(\frac{s}{\vartheta+h/n^2}\right)-N_2\left(\frac{s}{\vartheta}\right)\right] \quad := \quad \sum_{k=0}^{\infty} 1_{\underbrace{\left((\vartheta+h/n^2)(k+r_2),\,\vartheta(k+r_2)\right]}_{\text{length } \approx\, |h|\,\frac{k}{n^2}}}(s)$$

in restriction to $s\in[0,n]$, all intervals occurring in the above sums

located left of $\vartheta(k+r_j)$, for $k\le O(\frac{n}{\vartheta})$ and $j=1,2$

are disjoint when n is large enough

same result in case $h>0$ where the intervals are located right of $\vartheta(k+r_j)$

thus intervals corresponding to positive and to negative values of $h\in\Theta_{\vartheta,n}$ automatically are disjoint in restriction to $[0,n]$ where n is large lemma 1 : for every t fixed, as $n \to \infty$, we have convergence in $P^{(\vartheta)}$-probability

$$\int_0^{tn} j_\vartheta^{h_1,n}(s)\, j_\vartheta^{h_2,n}(s)\, \frac{1}{\sigma^2(\eta_s)}\, ds \longrightarrow \begin{cases} (|h_1| \wedge |h_2|)\, t^2\, J & \text{if} \quad \operatorname{sgn}(h_1) = \operatorname{sgn}(h_2) \\ 0 & \text{if} \quad \operatorname{sgn}(h_1) \neq \operatorname{sgn}(h_2) \end{cases}$$

with scaling factor

$$J := \frac{1}{2\vartheta^2} \sum_{j=1}^{2} [\mu^{(\vartheta)} P^{(\vartheta)}_{0,r_j\vartheta}](\frac{1}{\sigma^2})$$

proof : 1) • $\operatorname{sgn}(h_1) = \operatorname{sgn}(h_2)$: in case $h_2 < h_1 < 0$ ($0 < h_1 < h_2$ similiar),

$$(+) \qquad \begin{cases} j_\vartheta^{h_1,n} \text{ concentrated on intervals } [(\vartheta + h_1/n^2)(k + r_j), \vartheta(k + r_j)) \\ j_\vartheta^{h_2,n} \text{ concentrated on intervals } [(\vartheta + h_2/n^2)(k + r_j), \vartheta(k + r_j)) \end{cases}$$

in restriction to $[0, Cn]$ (arbitrary constant $C < \infty$) when n is large, thus

$$j_\vartheta^{h_1,n}(s)\, j_\vartheta^{h_2,n}(s) \;=\; [j_\vartheta^{h_1,n}(s)]^2 \quad \text{on } [0, Cn] \text{ as } n \to \infty$$

where h_1 corresponds to $-(|h_1| \wedge |h_2|)$

• $\operatorname{sgn}(h_1) \neq \operatorname{sgn}(h_2)$: all intervals disjoint

2) continuing the proof, show (t fixed, $n \to \infty$) convergence in $P^{(\vartheta)}$-probability

$$\int_0^{tn} [j_\vartheta^{h_1,n}(s)]^2 \, \frac{1}{\sigma^2(\eta_s)} \, ds \quad \longrightarrow \quad |h_1| \, t^2 \, J \ :$$

• in restriction to $[0, Cn]$ for n large, functions $j_\vartheta^{h_1,n}(\cdot)$ take values $0, \pm 1$

• to deal with $\sigma^2(\eta_s)$ over small time intervals $(+)$ of length $O(\frac{k}{n^2})$,
we make use of exponential inequalities to control fluctuations in $t \to \eta_t$

$$\int_0^{tn} [j_\vartheta^{h_1,n}(s)]^2 \, \frac{1}{\sigma^2(\eta_s)} \, ds \ = \ \sum_{j=1,2} \sum_{k=0}^{O(\frac{tn}{\vartheta})} \int_{(\vartheta+h_1/n^2)(k+r_j)}^{\vartheta(k+r_j)} \frac{1}{\sigma^2(\eta_s)} \, ds$$

$$= \quad \sum_{j=1,2} \frac{|h_1|}{n^2} \sum_{k=0}^{O(\frac{tn}{\vartheta})} \frac{k}{\sigma^2(\vartheta(k+r_j))} \ + \ o_{P^{(\vartheta)}}(1)$$

$$\longrightarrow \quad |h_1| \, \frac{t^2}{2\vartheta^2} \sum_{j=1,2} [\mu^{(\vartheta)} P^{(\vartheta)}_{0,r_j\vartheta}](\frac{1}{\sigma^2}) \quad = \quad |h_1| \, t^2 \, J$$

by theorem 2, regular variation part, since for $F : \alpha \to \frac{1}{\sigma^2(\vartheta r_j)}$ on $C([0,\vartheta])$

$$\frac{1}{m} \sum_{k=0}^{m} \frac{1}{\sigma^2(\vartheta(k+r_j))} \ \longrightarrow \ \frac{1}{\vartheta} m^{(\vartheta)}(F) \ , \quad m \to \infty$$

□

taking $t = 1$: in lemma 1 appears the covariance kernel

$$(h_1, h_2) \longrightarrow \begin{cases} (|h_1| \wedge |h_2|) \, J & \text{if} \quad \mathrm{sgn}(h_1) = \mathrm{sgn}(h_2) \\ 0 & \text{if} \quad \mathrm{sgn}(h_1) \neq \mathrm{sgn}(h_2) \end{cases}$$

of double sided Brownian motion $(\widetilde{W}(hJ))_{h \in R}$ with time scaling constant J ;

so we can prove (for every finite collection $h_1, \ldots, h_r$, we consider r-dimensional martingales in time $t \geq 0$, use the martingale convergence theorem, and at the end return to $t = 1$)

lemma 2 : we have convergence of finite dimensional distributions

$$\left(\int_0^n j_\vartheta^{h,n}(s) \, \frac{1}{\sigma(\eta_s)} \, dB_s^{(\vartheta)} \right)_{h \in \Theta_{\vartheta,n}} \qquad \text{under } P^{(\vartheta)} \text{ as } n \to \infty$$

to double sided Brownian motion $(\widetilde{W}(hJ))_{h \in R}$ with time scaling constant J

as sketched along these main steps, we can prove the following theorem 4 which contains theorem 3, and in fact is much stronger :

theorem 4 : [H-Ku. MMS 11] under the above assumptions, we have a decomposition of log-likelihood ratios in local models at ϑ as follows :

- uniformly in compact h-intervals, differences

$$\left| \log L_n^{(\vartheta+h/n^2)\,/\,\vartheta} \;-\; \left\{ \int_0^n j_\vartheta^{h,n}(s)\, \frac{1}{\sigma(\eta_s)}\, dB_s^{(\vartheta)} - \frac{1}{2} \int_0^n [j_\vartheta^{h,n}(s)]^2\, \frac{1}{\sigma^2(\eta_s)}\, ds \right\} \right|$$

vanish in $P^{(\vartheta)}$-probability as $n \to \infty$

- we have finite dimensional convergence of

$$\left\{ \int_0^n j_\vartheta^{h,n}(s)\, \frac{1}{\sigma(\eta_s)}\, dB_s^{(\vartheta)} - \frac{1}{2} \int_0^n [j_\vartheta^{h,n}(s)]^2\, \frac{1}{\sigma^2(\eta_s)}\, ds \right\}_{h \in \Theta_{\vartheta,n}} \quad \text{under } P^{(\vartheta)}$$

as $n \to \infty$ to the likelihood function

$$\left(\widetilde{W}(hJ) - \frac{1}{2}\, |hJ| \right)_{h \in \mathbb{R}}$$

of the Ibragimov-Khasminskii limit experiment (scaling J as in theorem 3) this last result has the structure of a

'2nd Le Cam lemma for convergence of local models to the Ibragimov-Khasminskii limit experiment $\widetilde{\mathcal{E}}$*'*

end of my talk : thanks for your attention !

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