

Analytical and Statistical Properties of Tempered Stable Lévy Processes

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Workshop

Asymptotical Statistics of Stochastic Processes VIII

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1 **H**

Realvalued Lévy processes $(L_t, t \geq 0)$ are stochastic processes with independent stationary increments and cadlag trajectories. Examples are

- Brownian motion,
- (Compound) Poisson processes,
- Normal Inverse Gauss processes,
- (Generalized) Hyperbolic processes.

Every Lévy process $(L_t, t \geq 0)$ can be characterized by its *triplet of local characteristics* (γ, σ^2, ν) , where

$$\gamma \in \mathbb{R}, \quad \sigma^2 \geq 0, \quad \text{and} \quad \int_{\mathbb{R} \setminus \{0\}} (x^2 \wedge 1) \nu(dx) < \infty. \quad (1)$$

- a) $\nu = 0$: Brownian motion with drift γ and diffusion σ^2
- b) $\sigma^2 = 0, \nu(\mathbb{R}) < \infty$: Compound Poisson process with drift γ , intensity $\nu(\mathbb{R})$ and jump distribution $\frac{\nu(dx)}{\nu(\mathbb{R})}$
- c) $\nu(\mathbb{R}) = \infty$: infinitely active Lévy process

The one-dimensional distributions are in general not available, beside of Brownian motion, Poisson-process, Gamma-process and a few more examples.

L-processes are popular:

- turbulence: Hyperbolic processes, Generalized hyperbolic processes, (Barndorff-Nielsen)
- mathematical finance: Brownian motion, stable processes (Black-Scholes, Mandelbrot)
- many explicit calculations are possible, using independent increments

$$d\nu(x) = \left(\frac{\alpha^-}{|x|^{1+\beta}} \mathbb{1}_{x<0} + \frac{\alpha^+}{x^{1+\beta}} \mathbb{1}_{x>0} \right) dx,$$

$$\alpha^-, \alpha^+ > 0, \beta \in (0, 2)$$

⚡

No expectation if $\beta \leq 1$, always infinite variance

To avoid infinite moments, *damping of ν at the tails*:

$$\nu(dx) = \left[\frac{\alpha^-}{|x|^{1+\beta^-}} e^{-\lambda^-|x|} \cdot \mathbb{1}_{x<0} + \frac{\alpha^+}{x^{1+\beta^+}} e^{-\lambda^+x} \cdot \mathbb{1}_{x>0} \right] dx$$

$$\alpha^\pm, \lambda^\pm > 0; \beta^\pm \in [0, 2).$$

⚡

Special cases $\beta^\pm = 0$:

$$\nu(dx) = \left[\frac{\alpha^-}{|x|} e^{-\lambda^-|x|} \cdot \mathbb{1}_{x<0} + \frac{\alpha^+}{x} e^{-\lambda^+x} \cdot \mathbb{1}_{x>0} \right] dx$$

$$\alpha^\pm, \lambda^\pm > 0$$

⚡

$$\nu(dx) = \left[\frac{\alpha^-}{|x|^{1+\beta^-}} e^{-\lambda^-|x|} \cdot \mathbb{1}_{x<0} + \frac{\alpha^+}{x^{1+\beta^+}} e^{-\lambda^+x} \cdot \mathbb{1}_{x>0} \right] dx$$

$$L_1 \sim TS[\alpha^+, \lambda^+, \beta^+; \alpha^-, \lambda^-, \beta^-],$$

$$L_t \sim TS[\alpha^+ \cdot t, \lambda^+, \beta^+; \alpha^- \cdot t, \lambda^-, \beta^-]$$

In general: No explicit densities

Special cases:

- $\beta^\pm = 0$

Bilateral Gamma processes

$$\begin{aligned} f_t(x) &= f_t^{(\alpha^+, \lambda^+)} \star f_t^{(\alpha^-, \lambda^-)} \\ &= \frac{(\lambda^+)^{\alpha^+} (\lambda^-)^{\alpha^-}}{(\lambda^+ + \lambda^-)^{\alpha^-} \Gamma(\alpha^+) \Gamma(\alpha^-)} e^{-\lambda^+ x} \int_0^\infty v^{\alpha^- - 1} \left(x + \frac{v}{\lambda^+ + \lambda^-} \right)^{\alpha^+ - 1} e^{-v} dv \\ &= \frac{(\lambda^+)^{\alpha^+} (\lambda^-)^{\alpha^-}}{(\lambda^+ + \lambda^-)^{\frac{1}{2}(\alpha^+ + \alpha^-)} \Gamma(\alpha^+)} x^{\frac{1}{2}(\alpha^+ + \alpha^-) - 1} e^{-\frac{x}{2}(\lambda^+ + \lambda^-)} \\ &\times W_{\frac{1}{2}(\alpha^+ - \alpha^-), \frac{1}{2}(\alpha^+ + \alpha^- - 1)}(x(\lambda^+ + \lambda^-)), \quad x > 0. \end{aligned}$$

$W_{\lambda, \mu}(z)$: Whittaker function.

$$W_{\lambda, \mu}(z) = \frac{z^\lambda e^{-z/2}}{\Gamma(\mu - \lambda + 1/2)} \int_0^\infty t^{\mu - \lambda - 1/2} e^{-t} (1 + t/z)^{\mu + \lambda - 1/2} dt, \quad \mu - \lambda > -1/2.$$

- $\beta^\pm = 0$ and moreover $\alpha^+ = \alpha^-$:

Variance Gamma processes

- $\alpha^+ = \alpha^-, \beta^+ = \beta^- > 0$: CGMY processes, no explicit densities

A tempered stable process has finite variation trajectories if and only if $\beta^\pm \in [0, 1)$ (i.e. $\int_{|x|<1} |x| \nu(dx) < \infty$)

$$L_t = \sum_{s \leq t} \Delta L_s = \sum_{s \leq t} \Delta L_s^+ - \sum_{s \leq t} \Delta L_s^- = L_t^+ - L_t^-$$

.

$$L_1^+ \sim TS[\alpha^+, \lambda^+, \beta^+; 0, 0, 0], L_1^- \sim TS[0, 0, 0; \alpha^-, \lambda^-, \beta^-].$$

In the sequel we assume $\beta^\pm \in [0, 1)$.

We will show, that this six-parameter family allows many explicit calculations,

present some proposals for parameter estimation

and invite for further studies.

2

Cumulant generating function

$$\begin{aligned}\psi(u) &= \ln E \exp(uL_1) = \int_{\mathbb{R} \setminus \{0\}} [\exp(ux) - 1] \nu(dx) = \\ &\alpha^- \Gamma(-\beta^-) \{(\lambda^- + u)^{\beta^-} - (\lambda^-)^{\beta^-}\} + \alpha^+ \Gamma(-\beta^+) \{(\lambda^+ - u)^{\beta^+} - (\lambda^+)^{\beta^+}\}, \\ &u \in (-\lambda^-, \lambda^+), \beta^\pm \in (0, 1)\end{aligned}$$

For the bilateral Gamma processes case $\beta^\pm = 0$ we have

$$\begin{aligned}\psi(u) &= \alpha^+ \ln \frac{\lambda^+}{\lambda^+ - u} + \alpha^- \ln \frac{\lambda^-}{\lambda^- + u}, \\ &u \in (-\lambda^-, \lambda^+)\end{aligned}$$

This allows to calculate the cumulants:

The n -th order cumulant $\kappa_n = \frac{d^n \psi(u)}{du^n} \Big|_{u=0}$ of L_1 :

$$\kappa_n = \Gamma(n - \beta^+) \frac{\alpha^+}{(\lambda^+)^{n - \beta^+}} + (-1)^n \Gamma(n - \beta^-) \frac{\alpha^-}{(\lambda^-)^{n - \beta^-}} \quad n \geq 1.$$

And for Bilateral Gamma processes:

$$\kappa_n = (n - 1)! \left(\frac{\alpha^+}{(\lambda^+)^n} + (-1)^n \frac{\alpha^-}{(\lambda^-)^n} \right), \quad n \geq 1.$$

We have $\mu := EL_1 = \kappa_1, \sigma^2 := Var(L_1) = \kappa_2$.

3 **D**

$\beta^\pm \in (0, 1)$: For the tempered stable processes no explicit 1-dimensional density is known, but

E

$TS[\alpha^+, \lambda^+, \beta^+; \alpha^-, \lambda^-, \beta^-]$ has a density $f \in C^\infty(\mathbb{R})$,

$$\exists x_0 \in \mathbb{R}^1 : \begin{cases} f'(x) > 0 & : x \in (-\infty, x_0) \\ f'(x_0) = 0, & : \\ f'(x) < 0 & : x \in (x_0, \infty) \end{cases}$$

F

:

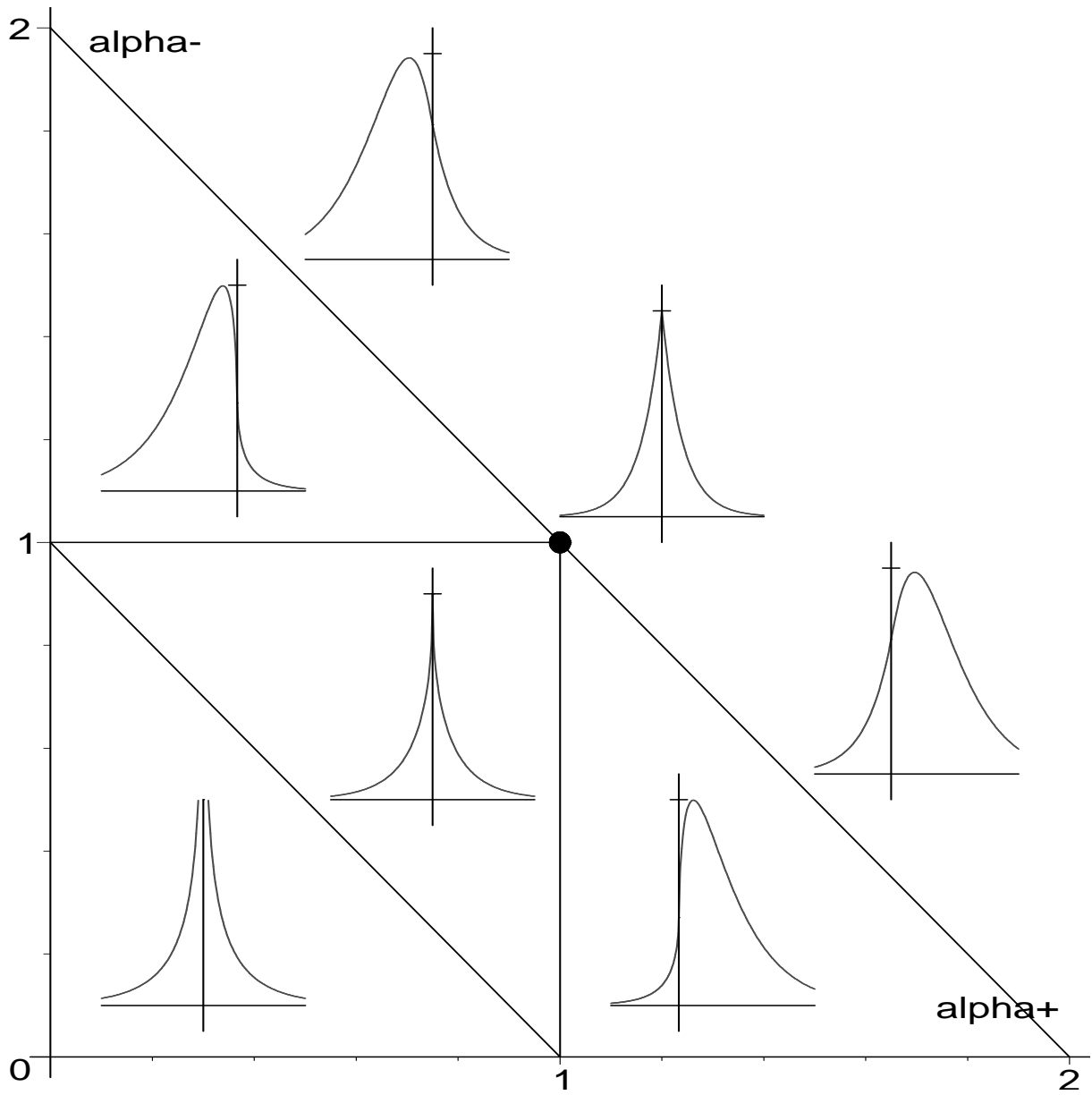
$$\lim_{x \rightarrow -\infty} \frac{\ln f(x)}{x} = \lambda^-, \quad \lim_{x \rightarrow \infty} \frac{\ln f(x)}{x} = -\lambda^+$$

G

:

The density f is infinitely divisible and selfdecomposable

Explicit expression for Bilateral Gamma processes $\beta^\pm = 0$ only.



4 ~~Non~~

The tempered stable distributions satisfy the following relations:

$$TS[\alpha_1^+, \lambda^+, \beta^+; \alpha_1^-, \lambda^-, \beta^-] \star TS[\alpha_2^+, \lambda^+, \beta^+; \alpha_2^-, \lambda^-, \beta^-] = \\ TS[\alpha_1^+ + \alpha_2^+, \lambda^+, \beta^+; \alpha_1^- + \alpha_2^-, \lambda^-, \beta^-]$$

This leads to

$$L_t \sim TS[\alpha^+ \cdot t, \lambda^+, \beta^+; \alpha^- \cdot t, \lambda^-, \beta^-]$$

Moreover, we have for every $c > 0$

$$cL_t \sim TS[\alpha^+ \cdot tc^{\beta^+}, \beta^+, \frac{\lambda^+}{c}; \alpha^- \cdot tc^{\beta^-}, \beta^-, \frac{\lambda^-}{c}]$$

(L_t) is a process with independent stationary increments, having finite moments of all order. Thus, for $t \rightarrow \infty$,

$$\frac{L_t - \mu \cdot t}{\sigma \cdot \sqrt{t}} \xrightarrow{d} N(0, 1)$$

This means

$$\frac{L_t}{\sigma \cdot \sqrt{t}} - \frac{\mu \sqrt{t}}{\sigma} \xrightarrow{d}$$

$$TS\left(\left(\frac{\sqrt{t^{2-\beta^+}}}{\sigma^{\beta^+}}\right)\alpha^+, \beta^+, \lambda^+ \sigma \sqrt{t}; \left(\frac{\sqrt{t^{2-\beta^-}}}{\sigma^{\beta^-}}\right)\alpha^-, \beta^-, \lambda^- \sigma \sqrt{t}\right) \star \delta_{\frac{-\mu \sqrt{t}}{\sigma}} \\ \sim N(0, 1)$$

5 ~~5.1~~

- Because L_t has finite moments of each order, we get

$$\lim_{t \rightarrow \infty} t^{-1} [L_t]^k = E[L_1]^k \quad P - a.s.$$

and

$$\hat{\mu}_k := n^{-1} \sum_{l=1}^n (\Delta L_{l\delta})^k \rightarrow \mu_k = E[L_1]^k.$$

These relations allow to estimate the moments μ_k of L_1 .

- Let $M(dt, dx)$ be the random measure associated with the jumps of $(L_t), t \geq 0$, i.e.

$$M([a, b] \times B) := \#\{s \in [a, b] | L_s - L_{s-} \in B\}$$

$M(dt, dx)$ is a Poisson measure with compensator

$$dt \otimes \nu(dx)$$

. We define the random variables

$$Z_n^+ := M((0, 1] \otimes (\varphi_{\beta^+}(n+1), \varphi_{\beta^+}(n)]), \quad n \geq 1,$$

and

$$Z_n^- := M((0, 1] \otimes (-\varphi_{\beta^-}(n), -\varphi_{\beta^-}(n+1)]), \quad n \geq 1,$$

where

$$\varphi_\beta(x) := (1 + \beta x)^{-\frac{1}{\beta}}, \quad \varphi_0(x) := e^{-x}.$$

The random variables $(Z_n^+)_{n \geq 1}$ and $(Z_n^-)_{n \geq 1}$ are mutually independent and Poisson distributed with parameter (for Z_n^+)

$$\alpha_n^+ := \nu((\varphi_{\beta^+}(n+1), \varphi_{\beta^+}(n)]) \rightarrow \alpha^+$$

.

Then we have by Kolmogorov's law of large numbers

$$n^{-1} \sum_{l=1}^n Z_l^+ \rightarrow \alpha^+$$

$$n^{-1} \sum_{l=1}^n Z_l^- \rightarrow \alpha^-$$

This gives the possibility to estimate α^\pm .

6 ~~En~~

Let $Q_T^{(\alpha^\pm, \lambda^\pm, \beta^\pm)}$ be the distribution of $(L_t, t \in [0, T])$.

There is another equivalent measure $Q_T^{(a^\pm, l^\pm, b^\pm)}$ making $(L_t, t \in [0, T])$ to a TS-process if and only if

$$\alpha^\pm = a^\pm \text{ and; } \beta^\pm = b^\pm.$$

In this case

$$\Lambda_T := \frac{dQ_T^{(\alpha^\pm, \lambda^\pm, \beta^\pm)}}{dQ_T^{(a^\pm, l^\pm, b^\pm)}} =$$

$$\exp[(\lambda^+ - l^+)L_T^+ - \psi^+(\lambda^+ - l^+) \cdot T] \times \exp[(\lambda^- - l^-)L_T^- - \psi^-(\lambda^- - l^-) \cdot T]$$

where

$$\psi^\pm(u) = \ln E \exp(uL_1^\pm) =$$

$$\frac{\alpha^\pm \Gamma(1 - \beta^\pm) \{(\lambda^\pm + u)^{\beta^\pm} - (\lambda^\pm)^{\beta^\pm}\}}{\beta^\pm}$$

In the limit case $\beta^\pm \rightarrow 0$ we get

$$\psi^\pm(u) = \alpha^\pm \cdot \left[\ln\left(\frac{\lambda^\pm + u}{\lambda^\pm}\right) \right]$$

Thus, with respect to the parameters λ^+ and λ^- the measures $Q_T^{(\alpha^\pm, \lambda^\pm, \beta^\pm)}$ form a *two-dimensional exponential family*, and the pair of random variable (L_T^+, L_T^-) is a *sufficient statistics* for the unknown parameter (λ^+, λ^-) .

7 ~~Step~~

To estimate the other parameters we come back to the first point: Assume we have observed the process at the moments $k\delta, k = 0, \dots, N$

$$\hat{\mu}_k := n^{-1} \sum_{l=1}^n (\Delta L_{l\delta})^k \rightarrow \mu_k = E[L_1]^k.$$

The method of moments

The moments μ_k are connected with the cumulants κ_k in a one to one way.

Here are the first four equations:

$$\kappa_1 = \mu_1$$

$$\kappa_2 = \mu_2 - \mu_1^2$$

$$\kappa_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3$$

$$\kappa_4 = \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4$$

Inserting the estimates $\hat{\mu}_k$ we get estimates $\hat{\kappa}_k$ for the cumulants.

Let us start with the case, that the parameters β^{pm} are known.

On the other side we know

$$\kappa_k = \Gamma(k - \beta^+) \frac{\alpha^+}{(\lambda^+)^{k - \beta^+}} + (-1)^k \Gamma(k - \beta^-) \frac{\alpha^-}{(\lambda^-)^{k - \beta^-}} \quad k \geq 1.$$

And for Bilateral Gamma processes:

$$\kappa_k = (k - 1)! \left(\frac{\alpha^+}{(\lambda^+)^k} + (-1)^k \frac{\alpha^-}{(\lambda^-)^k} \right), \quad k \geq 1.$$

The $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ satisfy the following system of equations

$$\alpha^+ \lambda^- - \alpha^- \lambda^+ - \kappa_1 \lambda^+ \lambda^- = 0,$$

$$\alpha^+ (\lambda^-)^2 + \alpha^- (\lambda^+)^2 - \kappa_2 (\lambda^+)^2 (\lambda^-)^2 = 0,$$

$$\alpha^+ (\lambda^-)^3 - \alpha^- (\lambda^+)^3 - \kappa_3 (\lambda^+)^3 (\lambda^-)^3 = 0,$$

$$\alpha^+ (\lambda^-)^4 + \alpha^- (\lambda^+)^4 - \kappa_4 (\lambda^+)^4 (\lambda^-)^4 = 0.$$

These equations define an implicate function

$$F(c, \theta) = 0, \tag{2}$$

where

$$c = (\kappa_1, \kappa_2, \kappa_3, \kappa_4) \text{ and } \theta = (\theta_1, \theta_2, \theta_3, \theta_4) = (\alpha^+, \lambda^+; \alpha^-, \lambda^-)$$

on $(\mathbb{R} \times (0, \infty))^2 \times (0, \infty)^4$.

Using the estimates $\hat{\kappa}_k$ of the cumulant for $k = 1, 2, 3, 4$, we get a nonlinear system of four equations for four parameters. No explicit solution seems to be possible. Defines the implicit function a bijective mapping?

An elementary but tedious calculation yields

$$\det \frac{\partial F}{\partial \theta}(c, \theta) > 0; \text{ on } (\mathbb{R} \times (0, \infty))^2 \times (0, \infty)^4.$$

Thus from the implicit function theorem it follows:

There exist an open $U \subset (\mathbb{R} \times (0, \infty))^2$ of κ ,

an open $V \subset (0, \infty)^4$ of θ

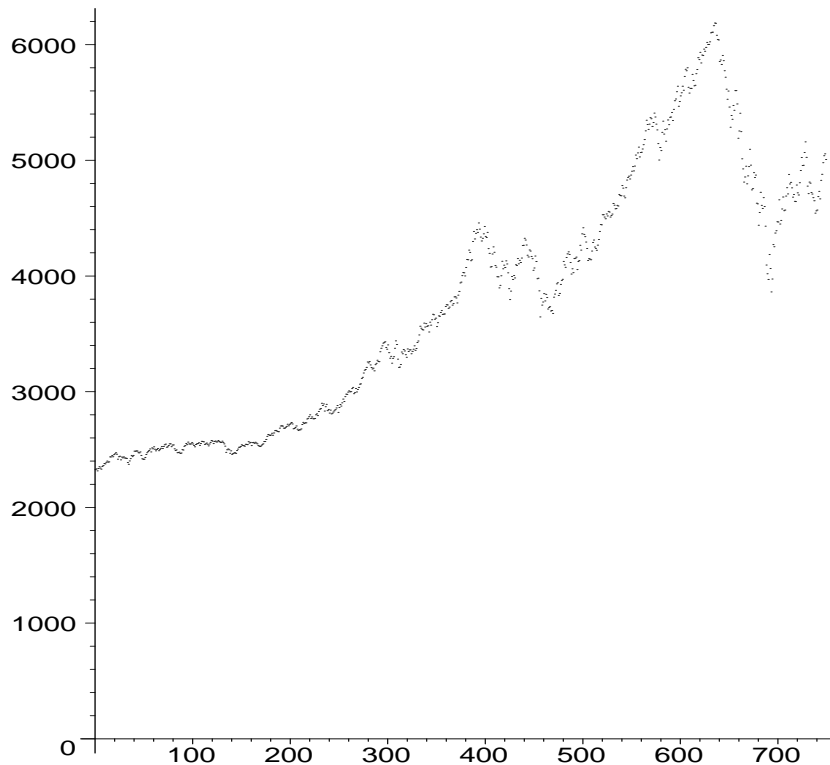
and a function $f \in C^1(U, V)$ such that $\theta = f(\kappa)$ and

$$F(c, f(c)) = 0, c \in U. \quad (3)$$

If the sample size n is large enough, then we get $\hat{\kappa} \in U$ and $\hat{\theta} = f(\hat{\kappa})$ is the unique V -valued solution of (2).

Thus we have got an estimation of $\theta = (\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ by the method of moments.

Illustration



Application to a data set in the case $\beta^\pm = 0$: German stock index DAX, three years from 1996 to 1998

$$S_{k\delta} = \exp[L_{k\delta}], \quad k \leq 750$$

Assumption: L_δ verteilt nach $BG(\alpha^\pm\delta; \lambda^\pm)$.

Estimation by method of moments:

$$\hat{\theta} = (1.28, 0.78; 119.75, 80.82)$$

In the Bilateral Gamma case we have the densities and can therefore calculate the Maximum Likelihood function for discrete observations:

- The logarithm of the likelihood function for $\vartheta = (\alpha^+\delta, \alpha^-\delta, \lambda^+, \lambda^-)$ is given by

$$\begin{aligned} \ln \Lambda(\vartheta) &= \ln \prod_{i=1}^n f_{\delta}^{\vartheta}(x_i) = \\ & -n^+ \ln(\Gamma(\alpha^+\delta)) - n^- \ln(\Gamma(\alpha^-\delta)) \\ & + n(\alpha^+\delta \ln(\lambda^+) + \alpha^-\delta \ln(\lambda^-)) - \frac{\alpha^+\delta + \alpha^-\delta}{2} \ln(\lambda^+ + \lambda^-) \\ & + \left(\frac{\alpha^+\delta + \alpha^-\delta}{2} - 1 \right) \left(\sum_{i=1}^n \ln |x_i| \right) - \frac{\lambda^+ - \lambda^-}{2} \left(\sum_{i=1}^n x_i \right) \\ & + \sum_{i=1}^n \ln \left(W_{\frac{1}{2} \operatorname{sgn}(x_i)(\alpha^+\delta - \alpha^-\delta), \frac{1}{2}(\alpha^+\delta + \alpha^-\delta - 1)}(|x_i|(\lambda^+ + \lambda^-)) \right). \end{aligned}$$

- The first rough estimator $\hat{\vartheta}_0$ we get from the method of moments.
- With starting point $\hat{\vartheta}_0$

$$\text{maximize } \ln L(\vartheta), k \leq 4$$

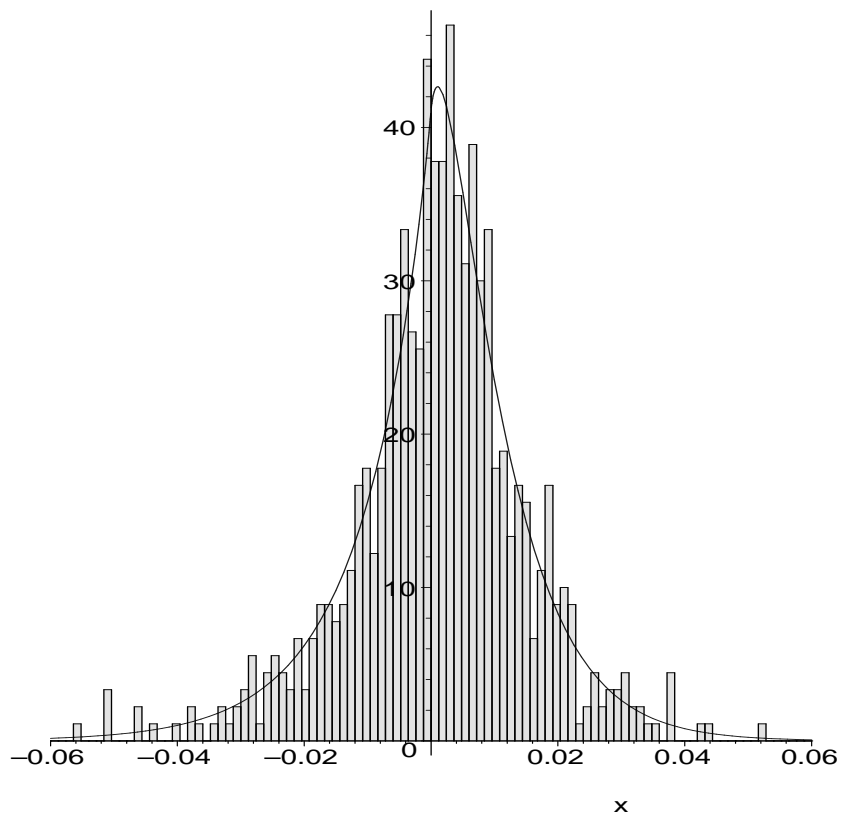
numerically to obtain $\hat{\vartheta}_{ML}$ (Hooke-Jeeves-algorithm).

In our example we obtain as the maximum likelihood estimation

$$\tilde{\theta}_{ML} = (1.55, 0.94; 133.96, 88.92)$$

Remember the estimation by method of moments:

$$\hat{\theta} = (1.28, 0.78; 119.75, 80.82)$$



If all six parameters are to be estimated, the same method works also.

- We estimate the moments $\mu_k, k \leq 6$,

by the empirical moments

$\hat{\mu}_k, k \leq 6$, with

$$\hat{\mu}_k := n^{-1} \sum_{l=1}^n (\Delta L_{l\delta})^k.$$

- Then we get an estimation of the first six cumulants through the one-to-one connection between cumulants and moments.

- and from the nonlinear system of six equations

$$\kappa_k = \Gamma(n - \beta^+) \frac{\alpha^+}{(\lambda^+)^{k - \beta^+}} + (-1)^k \Gamma(k - \beta^-) \frac{\alpha^-}{(\lambda^-)^{k - \beta^-}}, k = 1, \dots, 6$$

we calculate estimators for the unknown six parameters

$$\alpha^\pm, \lambda^\pm; \beta^\pm.$$

No Maximum Likelihood estimator is available.

The study of the corresponding implicate function

$$F(c, \theta) = 0$$

is not yet done.

8 ~~Flash~~

Assume $(L_t, t \in [0, T])$ is under P a

$TS[\alpha^+, \lambda^+, \beta^+; \alpha^-, \lambda^-, \beta^-]$ -Lévy process.

Define for $t \in [0, T]$

$$S_t := \exp[L_t], B_t := \exp[rt], \tilde{S}_t := \exp[L_t - rt].$$

For pricing derivatives one uses an equivalent measure Q , which makes $(\tilde{S}_t, t \in [0, T])$ to a martingale. ("Equivalent martingale measure")

Classical case: Black-Scholes-Theory, under P

$$S_t := \exp[\sigma W_t + \mu t], t \in [0, T].$$

There is one and only one equivalent martingale measure Q making $(\tilde{S}_t, t \in [0, T])$ to a martingale.

In our case of Lévy processes with jumps there are in general infinitely many equivalent martingale measures. One of them may exist within the exponential family

$$dQ_T^\vartheta = \exp[\vartheta L_t - \psi(\vartheta)t]dP_T$$

with

$$\begin{aligned} \psi(\vartheta) &= \ln \mathbb{E} \exp(\vartheta L_1) = \\ \alpha^- \Gamma(-\beta^-) \{(\lambda^- + \vartheta)^{\beta^-} - (\lambda^-)^{\beta^-}\} &+ \alpha^+ \Gamma(-\beta^+) \{(\lambda^+ - \vartheta)^{\beta^+} - (\lambda^+)^{\beta^+}\} \\ &\text{for some } \vartheta \in (-\lambda^-, \lambda^+). \end{aligned}$$

This family consists of all TS-processes with the distribution

$$L_1 \sim TS[\alpha^+, \lambda^+ - \vartheta, \beta^+; \alpha^-, \lambda^- + \vartheta, \beta^-].$$

FB.1 *Such an ϑ exists if and only if*

$$\lambda^+ + \lambda^- > 1 \text{ and } r \in (f(-\lambda^-), f(\lambda^+ - 1))$$

where

$$\begin{aligned} f(\vartheta) &:= f^+(\vartheta) + f^-(\vartheta), \\ f^+(\vartheta) &:= \alpha^+ \Gamma(-\beta^+) [(\lambda^+ - \vartheta - 1)^{\beta^+} - (\lambda^+ - \vartheta)^+], \\ f^-(\vartheta) &:= \alpha^- \Gamma(-\beta^-) [(\lambda^- - \vartheta - 1)^{\beta^-} - (\lambda^- - \vartheta)^-]. \end{aligned}$$

In this case, ϑ is the unique solution of

$$f(\vartheta) = r.$$

For Bilateral Gamma processes the formula for f simplifies to

$$f^\pm(\vartheta) := \alpha^\pm [\ln(\lambda^\pm - \vartheta) - \ln(\lambda^\pm \mp 1 \mp \vartheta)].$$

If we consider the two-dimensional exponential family from above

$$dQ_T^{(\vartheta^-, \vartheta^+)} = \exp[\vartheta^- L_T^- - \psi^-(\vartheta^-) \cdot T] \times \exp[\vartheta^+ L_T^+ - \psi^+(\vartheta^+) \cdot T],$$

where

$$\psi^\pm(u) = \ln E \exp(uL_1^\pm) =$$

$$\frac{\alpha^\pm \Gamma(1 - \beta^\pm) \{(\lambda^\pm + u)^{\beta^\pm} - (\lambda^\pm)^{\beta^\pm}\}}{\beta^\pm},$$

then we get,

$Q_T^{(\vartheta^-, \vartheta^+)}$ makes $(\tilde{S}_t), t \in [0, T]$ to a martingale if and only if

$$E^{(\vartheta^-, \vartheta^+)} \tilde{S}_t \equiv 1.$$

It is one condition for the two parameters $(\vartheta^-, \vartheta^+)$, and we expect to get an one-parametric family of equivalent martingale measures within the two-dimensional exponential family.

Indeed, we obtain

B.2

• *If*

$-\alpha^+\Gamma(-\beta^+) \leq r$, *then no pair*

$$(\vartheta^-, \vartheta^+) \in (-\infty, \lambda^+)$$

exists with $Q_T^{(\vartheta^-, \vartheta^+)}$ as an equivalent martingale measure.

• *If*

$$-\alpha^+\Gamma(-\beta^+) > r, \tag{4}$$

then there exist an interval $(\theta_1^+, \theta_2^+) \subseteq [-\infty, \lambda^+ - 1]$ and a continuous, strictly increasing, bijective map

$$\Phi|_{(\theta_1^+, \theta_2^+)} \rightarrow (-\infty, \lambda^-),$$

such that

a) $\forall \vartheta^+ \in (\theta_1^+, \theta_2^+) \exists \vartheta^- \in (-\infty, \lambda^-) : Q_T^{(\vartheta^-, \vartheta^+)}$

is an equivalent martingale measure for $(\tilde{S}_t), t \in [0, T]$.

b) $\vartheta^- = \Phi(\vartheta^+)$

c) $\Phi(\vartheta^+) := (f^-)^{-1}(r - f^+(\vartheta^+))$.

To choose an appropriate equivalent martingale measure, one often tries to minimize the *relative entropy*:

$$H(Q|P) := E_P\left[\frac{dQ}{dP} \ln \frac{dQ}{dP}\right].$$

Define $M_P :=$

$$\{(\vartheta^+, \vartheta^-) \in (-\infty, \lambda^+) \times (-\infty, \lambda^-) \mid P^{(\vartheta^+, \vartheta^-)} \text{ is a mart. measure}\}$$

Then we have $M_P = \{(\vartheta, \Phi(\vartheta) \in \mathbb{R}^2 \mid \vartheta \in (\vartheta_1^+, \vartheta_2^+)\}$.

FB.3

If (4) holds, then there exists

$$\theta^+ \in (-\infty, \lambda^+) \text{ and } \theta^- \in (-\infty, \lambda^-),$$

such that

$$H(P^{(\theta^+, \theta^-)}|P) = \min_{(\vartheta^+, \vartheta^-) \in M_P} H(P^{(\vartheta^+, \vartheta^-)}|P).$$

Using Esche, Schweizer (2005) and Hubalek, Sgarra (2006) one can show, that there exists an equivalent martingale measure, for which the relative entropy attains its minimum, and that this measure makes (\tilde{S}_t) again to a Lévy process. But this Lévy process has completely other local characteristics, see also Krol, Küchler (2009).

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Cont, R. and Tankov, P. (2004) *Financial modelling with jump processes*. Chapman and Hall / CRC Press, London.

Esche, F. and Schweizer, M. (2005) Minimal entropy preserves the Lévy property: How and why. *Stochastic Processes and Their Applications* **115**(2), 299–327.

Hubalek, F. and Sgarra, C. (2006) Esscher transforms and the minimal entropy martingale measure for exponential Lévy models. *Quantitative Finance* **6**(2), 125–145.

Krol, K. and Küchler, U. (2009) Minimal entropy martingale measure for Lévy processes. Preprint 9, Institut für Mathematik, Humboldt-Universität zu Berlin (<http://www.mathematik.hu-berlin.de>)

Küchler, U. and Tappe, S. (2008) Bilateral Gamma distributions and processes in financial mathematics. *Stochastic Processes and their Applications* **118**(2), 261–283.

Küchler, U. and Tappe, S. (2008) On the shapes of bilateral Gamma densities. *Statistics and Probability Letters* **78**, 2478–2484.

Küchler, U. and Tappe, S. (2011) Tempered stable distributions and applications to financial mathematics Manuscript

Sato, K. (1999) *Lévy processes and infinitely divisible distributions*. Cambridge studies in advanced mathematics, Cambridge.