Analytical and Statistical Properties of Tempered Stable Lévy Processes

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Workshop

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1 Introduction

Realvalued Lévy processes $(L_t, t \geq 0)$ are stochastic processes with independent stationary increments and cadlag trajectories. Examples are

- Brownian motion,
- (Compound) Poisson processes,
- Normal Inverse Gauss processes,
- (Generalized) Hyperbolic processes.

Every Lévy process $(L_t, t \geq 0)$ can be characterized by its *tripel of local characteristics* (γ, σ^2, ν), where

$$\gamma \in \mathbb{R}, \quad \sigma^2 \geq 0, \quad \text{and} \quad \int_{\mathbb{R}\setminus\{0\}} (x^2 \wedge 1)\nu(dx) < \infty. \tag{1}$$

a) $\nu = 0$: Brownian motion with drift γ and diffusion σ^2

b) $\sigma^2 = 0, \nu(I\!R) < \infty$: Compound Poisson process with drift γ, intensity $\nu(I\!R)$ and jump distribution $\frac{\nu(dx)}{\nu(I\!R)}$

c) $\nu(I\!R) = \infty$: infinitely active Lévy process

The one-dimensional distributions are in general not available, beside of Brownian motion, Poisson-process, Gamma-process and a few more examples.

L-processes are popular:

- turbulence: Hyperbolic processes, Generalized hyperbolic processes, (Barndorff-Nielsen)
- mathematical finance: Brownian motion, stable processes (Black-Scholes, Mandelbrodt)
- many explicite calculations are possible, using independent increments

$$d\nu(x) = \left(\frac{\alpha^-}{|x|^{1+\beta}}\mathbb{1}_{x<0} + \frac{\alpha^+}{x^{1+\beta}}\mathbb{1}_{x>0}\right)dx,$$

$$\alpha^-, \alpha^+ > 0, \beta \in (0,2)$$

Stable processes

No expectation if $\beta \leq 1$, always infinite variance

To avoid infinite moments, *damping of ν at the tails*:

$$\nu(dx) = \left[\frac{\alpha^-}{|x|^{1+\beta^-}}\mathrm{e}^{-\lambda^-|x|} \cdot \mathbb{1}_{x<0} + \frac{\alpha^+}{x^{1+\beta^+}}\mathrm{e}^{-\lambda^+ x} \cdot \mathbb{1}_{x>0}\right]dx$$

$$\alpha^\pm, \lambda^\pm > 0; \quad \beta^\pm \in [0,2).$$

Tempered stable processes

Special cases $\beta^\pm = 0$:

$$\nu(dx) = \left[\frac{\alpha^-}{|x|}\mathrm{e}^{-\lambda^-|x|} \cdot \mathbb{1}_{x<0} + \frac{\alpha^+}{x}\mathrm{e}^{-\lambda^+ x} \cdot \mathbb{1}_{x>0}\right]dx$$

$$\alpha^\pm, \lambda^\pm > 0$$

Bilateral Gamma processes

$$\nu(dx) = \left[\frac{\alpha^-}{|x|^{1+\beta^-}} \mathrm{e}^{-\lambda^-|x|} \cdot 1\!\!\mathrm{I}_{x<0} + \frac{\alpha^+}{x^{1+\beta^+}} \mathrm{e}^{-\lambda^+ x} \cdot 1\!\!\mathrm{I}_{x>0}\right] dx$$

$$L_1 \sim TS[\alpha^+, \lambda^+, \beta^+; \alpha^-, \lambda^-, \beta^-],$$

$$L_t \sim TS[\alpha^+ \cdot t, \lambda^+, \beta^+; \alpha^- \cdot t, \lambda^-, \beta^-]$$

In general: No explicite densities

Special cases:

- $\beta^{\pm} = 0$

  Bilateral Gamma processes
  $f_t(x) = f_t^{(\alpha^+,\lambda^+)} \star f_t^{(\alpha^-,\lambda^-)}$
  $= \frac{(\lambda^+)^{\alpha^+}(\lambda^-)^{\alpha^-}}{(\lambda^+ + \lambda^-)^{\alpha^-}\Gamma(\alpha^+)\Gamma(\alpha^-)} \mathrm{e}^{-\lambda + x} \int_0^\infty v^{\alpha^- - 1}\Big(x + \frac{v}{\lambda^+ + \lambda^-}\Big)^{\alpha^+ - 1} \mathrm{e}^{-v} dv$
  $= \frac{(\lambda^+)^{\alpha^+}(\lambda^-)^{\alpha^-}}{(\lambda^+ + \lambda^-)^{\frac{1}{2}(\alpha^+ + \alpha^-)}\Gamma(\alpha^+)} \, x^{\frac{1}{2}(\alpha^+ + \alpha^-) - 1} \mathrm{e}^{-\frac{x}{2}(\lambda^+ - \lambda^-)}$
  $\times W_{\frac{1}{2}(\alpha^+ - \alpha^-), \frac{1}{2}(\alpha^+ + \alpha^- - 1)}(x(\lambda^+ + \lambda^-)), \quad x > 0.$
  $W_{\lambda,\mu}(z)$: Whittaker function.

  $$W_{\lambda,\mu}(z) = \frac{z^\lambda e^{-z/2}}{\Gamma(\mu - \lambda + 1/2)} \int_0^\infty t^{\mu - \lambda - 1/2} e^{-t} (1 + t/z)^{\mu + \lambda - 1/2} t, \quad \mu - \lambda > -1/2.$$

- $\beta^{\pm} = 0$ and moreover $\alpha^+ = \alpha^-$:

  Variance Gamma processes

- $\alpha^+ = \alpha^-, \beta^+ = \beta^- > 0$: CGMY processes, no explicite densities

A tempered stable process has finite variation trajectories if and only if $\beta^{\pm} \in [0,1)$ (i.e. $\int_{|x|<1} |x|\nu(dx) < \infty$)

$$L_t = \sum_{s \le t} \Delta L_s = \sum_{s \le t} \Delta L_s^+ - \sum_{s \le t} \Delta L_s^- = L_t^+ - L_t^-$$

.

$L_1^+ \sim TS[\alpha^+, \lambda^+, \beta^+; 0, 0, 0], L_1^- \sim TS[0, 0, 0; \alpha^-, \lambda^-, \beta^-].$

In the sequel we assume $\beta^{\pm} \in [0,1)$.

We will show, that this six-parameter family allows many explicit calculations,

present some proposals for parameter estimation

and invite for further studies.

2 Cumulants

Cumulant generating function

$$\psi(u) = \ln \mathrm{E} \exp(uL_1) = \int_{I\!R\setminus\{0\}} [\exp(ux) - 1]\nu(dx) =$$

$$\alpha^{-}\Gamma(-\beta^{-})\left\{(\lambda^{-} + u)^{\beta-} - (\lambda^{-})^{\beta^{-}}\right\}+\alpha^{+}\Gamma(-\beta^{+})\left\{(\lambda^{+} - u)^{\beta+} - (\lambda^{+})^{\beta^{+}}\right\},$$

$$u \in (-\lambda^{-}, \lambda^{+}), \beta^{\pm} \in (0, 1)$$

For the bilateral Gamma processes case $\beta^{\pm} = 0$ we have

$$\psi(u) = \alpha^{+} \ln \frac{\lambda^{+}}{\lambda^{+} - u} + \alpha^{-} \ln \frac{\lambda^{-}}{\lambda^{-} + u},$$

$$u \in (-\lambda^{-}, \lambda^{+})$$

This allows to calculate the cumulants:

The n–th order cumulant $\kappa_n = \dfrac{d^n\psi(u)}{du^n}|_{u=0}$ of L_1 :

$$\kappa_n = \Gamma(n-\beta^{+})\frac{\alpha^{+}}{(\lambda^{+})^{n-\beta^{+}}}+(-1)^n\ \Gamma(n-\beta^{-})\frac{\alpha^{-}}{(\lambda^{-})^{n-\beta^{-}}} \quad n \geq 1.$$

And for Bilateral Gamma processes:

$$\kappa_n = (n-1)!\Big(\frac{\alpha^{+}}{(\lambda^{+})^n} + (-1)^n\ \frac{\alpha^{-}}{(\lambda^{-})^n}\Big), \quad n \geq 1.$$

We have $\mu := EL_1 = \kappa_1, \sigma^2 := Var(L_1) = \kappa_2$.

3 Densities

$\beta^{\pm} \in (0,1)$: For the tempered stable processes no explicite 1-dimensional density is known, but

Unimodality:

$TS[\alpha^+, \lambda^+, \beta^+; \alpha^-, \lambda^-, \beta^-]$ has a density $f \in C^\infty(R)$,

$$\exists x_0 \in I\!R^1 : \begin{cases} f'(x) > 0 & : \quad x \in (-\infty, x_0) \\ f'(x_0) = 0, & : \\ f'(x) < 0 & : \quad x \in (x_0, \infty) \end{cases}$$

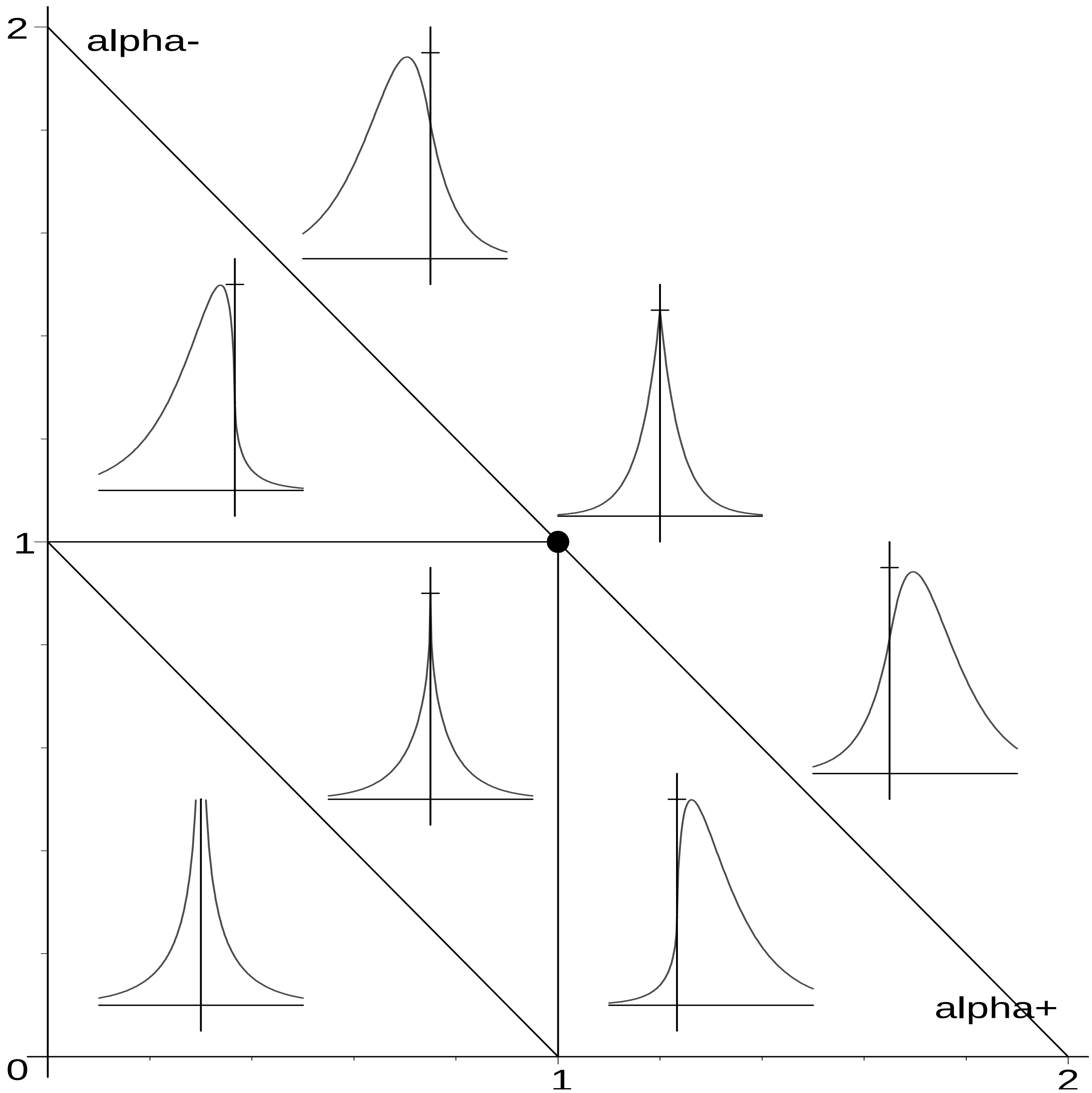
Tail behaviour:

$$\lim_{x \to -\infty} \frac{\ln f(x)}{x} = \lambda^-, \qquad \lim_{x \to \infty} \frac{\ln f(x)}{x} = -\lambda^+$$

Further properties:

The density f is infinitely divisible and selfdecomposable

Explicite expression for Bilateral Gamma processes $\beta^{\pm} = 0$ only.



2
alpha-
1
0
1
2
alpha+

4 Normal approximation

The tempered stable distributions satisfy the following relations:

$$TS[\alpha_1^+, \lambda^+, \beta^+; \alpha_1^-, \lambda^-, \beta^-] \star TS[\alpha_2^+, \lambda^+, \beta^+; \alpha_2^-, \lambda^-, \beta^-] =$$

$$TS[\alpha_1^+ + \alpha_2^+, \lambda^+, \beta^+; \alpha_1^- + \alpha_2^-, \lambda^-, \beta^-]$$

This leads to

$$L_t \sim TS[\alpha^+ \cdot t, \lambda^+, \beta^+; \alpha^- \cdot t, \lambda^-, \beta^-]$$

Moreover, we have for every $c > 0$

$$cL_t \sim TS[\alpha^+ \cdot tc^{\beta^+}, \beta^+, \frac{\lambda^+}{c}; \alpha^- \cdot tc^{\beta^-}, \beta^-, \frac{\lambda^-}{c}]$$

(L_t) is a process with independent stationary increments, having finite moments of all order. Thus, for $t \to \infty$,

$$\frac{L_t - \mu \cdot t}{\sigma \cdot \sqrt{t}} \xrightarrow{d} N(0,1)$$

This means

$\frac{L_t}{\sigma \cdot \sqrt{t}} - \frac{\mu\sqrt{t}}{\sigma} \stackrel{d}{=}$

$$TS((\frac{\sqrt{t^{2-\beta^+}}}{\sigma^{\beta^+}})\alpha^+, \beta^+, \lambda^+\sigma\sqrt{t}; (\frac{\sqrt{t^{2-\beta^-}}}{\sigma^{\beta^-}})\alpha^-, \beta^-, \lambda^-\sigma\sqrt{t}) \star \delta_{\frac{-\mu\sqrt{t}}{\sigma}}$$

$$\sim N(0,1)$$

5 Laws of large numbers

- Because L_t has finite moments of each order, we get

$$\lim_{t\to\infty} t^{-1}[L_t]^k = E[L_1]^k \qquad P - a.s.$$

and

$$\hat{\mu}_k := n^{-1}\sum_{l=1}^{n}(\Delta L_{l\delta})^k \to \mu_k = E[L_1]^k.$$

These relations allow to estimate the moments μ_k of L_1.

- Let M(dt,dx) be the random measure associated with the jumps of $(L_t), t \geq 0$, i.e.

$$M([a,b]\times B) := \sharp\{s \in [a,b] | L_s - L_{s-} \in B\}$$

$M(dt,dx)$ is a Poisson measure with compensator

$$dt \otimes \nu(dx)$$

. We define the random variables

$$Z_n^+ := M((0,1]\otimes(\varphi_{\beta^+}(n+1), \varphi_{\beta^+}(n)]), \quad n \geq 1,$$

and

$$Z_n^- := M((0,1]\otimes(-\varphi_{\beta^-}(n), -\varphi_{\beta^-}(n+1)]), \quad n \geq 1,$$

where

$$\varphi_\beta(x) := (1+\beta x)^{-\frac{1}{\beta}}, \qquad \varphi_0(x) := \mathrm{e}^{-x}.$$

The random variables $(Z_n^+)_{n\geq 1}$ and $(Z_n^-)_{n\geq 1}$ are mutually independent and Poisson distributed with parameter (for Z_n^+)

$$\alpha_n^+ := \nu((\varphi_{\beta^+}(n+1), \varphi_{\beta^+}(n)]) \to \alpha^+$$

.

Then we have by Kolmogorovs law of large numbers

$$n^{-1}\sum_{l=1}^{n} Z_l^+ \to \alpha^+$$

$$n^{-1}\sum_{l=1}^{n} Z_l^- \to \alpha^-$$

This gives the possibility to etimate $\alpha^\pm$.

6 Exponential families

Let $Q_T^{(\alpha^\pm,\lambda^\pm,\beta^\pm)}$ be the distribution of $(L_t, t \in [0,T]$.

There is another equivalent measure $Q_T^{(a^\pm,l^\pm,b^\pm)}$ making $(L_t, t \in [0,T]$. to a TS-process if and only if

$$\alpha^\pm = a^\pm \text{ and; } \beta^\pm = b^\pm.$$

In this case

$$\Lambda_T := \frac{dQ_T^{(\alpha^\pm,\lambda^\pm,\beta^\pm)}}{dQ_T^{(a^\pm,l^\pm,b^\pm)}} =$$

$$\exp[(\lambda^+ - l^+)L_T^+ - \psi^+(\lambda^+ - l^+)\cdot T] \times \exp[(\lambda^- - l^-)L_T^- - \psi^-(\lambda^- - l^-)\cdot T]$$

where

$$\psi^\pm(u) = \ln \mathrm{E} \exp(uL_1^\pm) =$$

$$\frac{\alpha^\pm \Gamma(1-\beta^\pm)\left\{(\lambda^\pm + u)^{\beta^\pm} - (\lambda^\pm)^{\beta^\pm}\right\}}{\beta^\pm}$$

In the limit case $\beta^\pm \to 0$ we get

$$\psi^{\pm}(u) = \alpha^{\pm} \cdot [\ln(\frac{\lambda^{\pm} + u)}{\lambda^{\pm}}]$$

Thus, with respect to the parameters λ^+ and λ^- the measures $Q_T^{(\alpha^{\pm},\lambda^{\pm},\beta^{\pm})}$ form a *two-dimensional exponential family*, and the pair of random variable (L_T^+, L_T^-) is a *sufficient statistics* for the unknown parameter (λ^+, λ^-).

7 Some other statistical properties

To estimate the other parameters we come back to the first point: Assume we have observed the process at the moments $k\delta, k = 0, \ldots, N$

$$\hat{\mu}_k := n^{-1} \sum_{l=1}^{n} (\Delta L_{l\delta})^k \to \mu_k = E[L_1]^k.$$

The method of moments

The moments μ_k are connected with the cumulants κ_k in a one to one way.

Here are the first four equations:

$$\kappa_1 = \mu_1$$
$$\kappa_2 = \mu_2 - \mu_1^2$$
$$\kappa_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_3^3$$
$$\kappa_4 = \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4$$

Inserting the estimates $\hat{\mu}_k$ we get estimates $\hat{\kappa}_k$ for the cumulants.

Let us start with the case, that the parameters β^{pm} are known.

On the other side we know

$$\kappa_k = \Gamma(k-\beta^+)\frac{\alpha^+}{(\lambda^+)^{k-\beta^+}} + (-1)^k\ \Gamma(k-\beta^-)\frac{\alpha^-}{(\lambda^-)^{k-\beta^-}} \quad k \geq 1.$$

And for Bilateral Gamma processes:

$$\kappa_k = (k-1)!\Big(\frac{\alpha^+}{(\lambda^+)^k} + (-1)^k\ \frac{\alpha^-}{(\lambda^-)^k}\Big), \quad k \geq 1.$$

The $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ satisfy the following system of equations

$$\alpha^+\lambda^- - \alpha^-\lambda^+ - \kappa_1\lambda^+\lambda^- = 0,$$

$$\alpha^+(\lambda^-)^2 + \alpha^-(\lambda^+)^2 - \kappa_2(\lambda^+)^2(\lambda^-)^2 = 0,$$

$$\alpha^+(\lambda^-)^3 - \alpha^-(\lambda^+)^3 - \kappa_3(\lambda^+)^3(\lambda^-)^3 = 0,$$

$$\alpha^+(\lambda^-)^4 + \alpha^-(\lambda^+)^4 - \kappa_4(\lambda^+)^4(\lambda^-)^4 = 0.$$

These equations define an implicite function

$$F(c, \theta) = 0, \tag{2}$$

where

$c = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ and $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) = (\alpha^+, \lambda^+; \alpha^-, \lambda^-)$

on $(\mathbb{R} \times (0, \infty))^2 \times (0, \infty)^4$.

Using the estimates $\hat{\kappa}_k$ of the cumulant for $k = 1, 2, 3, 4$, we get a nonlinear system of four equations for four parameters. No explicite solution seems to be possible. Defines the implicite function a bijective mapping?

An elementary but tedious calculation yields

$$\det \frac{\partial F}{\partial \theta}(c, \theta) > 0; \text{on } (I\!R \times (0, \infty))^2 \times (0, \infty)^4.$$

Thus from the implicite function theorem it follows:

There exist an open $U \subset (I\!R \times (0, \infty))^2$ of κ,

an open $V \subset (0, \infty)^4$ of θ

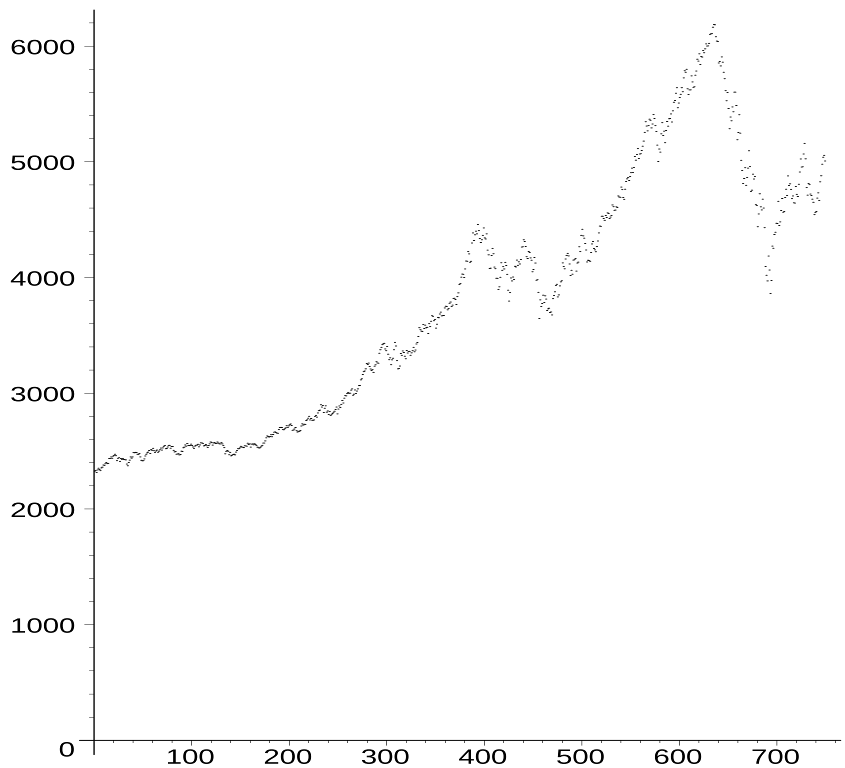
and a function $f \in C^1(U, V)$ such that $\theta = f(\kappa)$ and

$$F(c, f(c)) = 0, c \in U. \tag{3}$$

If the sample size n is large enough, then we get $\hat{\kappa} \in U$ and $\hat{\theta} = f(\hat{\kappa})$ is the unique V-valued solution of (2).

Thus we have got an estimation of $\theta = (\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ by the method of moments.

Illustration



Application to a data set in the case $\beta^{\pm} = 0$: German stock index DAX, three years from 1996 to 1998

$$S_{k\delta} = \exp[L_{k\delta}], \qquad k \leq 750$$

Assumption: L_δ verteilt nach $BG(\alpha^{\pm}\delta; \lambda^{\pm})$.

Estimation by method of moments:

$$\hat{\theta} = (1.28, 0.78; 119.75, 80.82)$$

In the Bilateral Gamma case we have the densities and can therefore calculate the Maximum Likelihood function for discrete observations:

- The logarithm of the likelihood function for $\vartheta = (\alpha^+\delta, \alpha^-\delta, \lambda^+, \lambda^-)$ is given by

$$\ln \Lambda(\vartheta) = \ln \prod_{i=1}^{n} f_\delta^\vartheta(x_i) =$$

$$-n^+ \ln(\Gamma(\alpha^+\delta)) - n^- \ln(\Gamma(\alpha^-\delta))$$

$$+n(\alpha^+\delta \ln(\lambda^+) + \alpha^-\delta \ln(\lambda^-) - \frac{\alpha^+\delta + \alpha^-\delta}{2} \ln(\lambda^+ + \lambda^-))$$

$$+ \left(\frac{\alpha^+\delta + \alpha^-\delta}{2} - 1 \right) \left(\sum_{i=1}^{n} \ln |x_i| \right) - \frac{\lambda^+ - \lambda^-}{2} \left(\sum_{i=1}^{n} x_i \right)$$

$$+ \sum_{i=1}^{n} \ln \left(W_{\frac{1}{2}\mathrm{sgn}(x_i)(\alpha^+\delta - \alpha^-\delta), \frac{1}{2}(\alpha^+\delta + \alpha^-\delta - 1)}(|x_i|(\lambda^+ + \lambda^-)) \right).$$

- The first rough estimator $\hat{\vartheta}_0$ we get from the method of moments.
- With starting point $\hat{\vartheta}_0$

$$\text{maximize} \ln \ L(\vartheta), k \leq 4$$

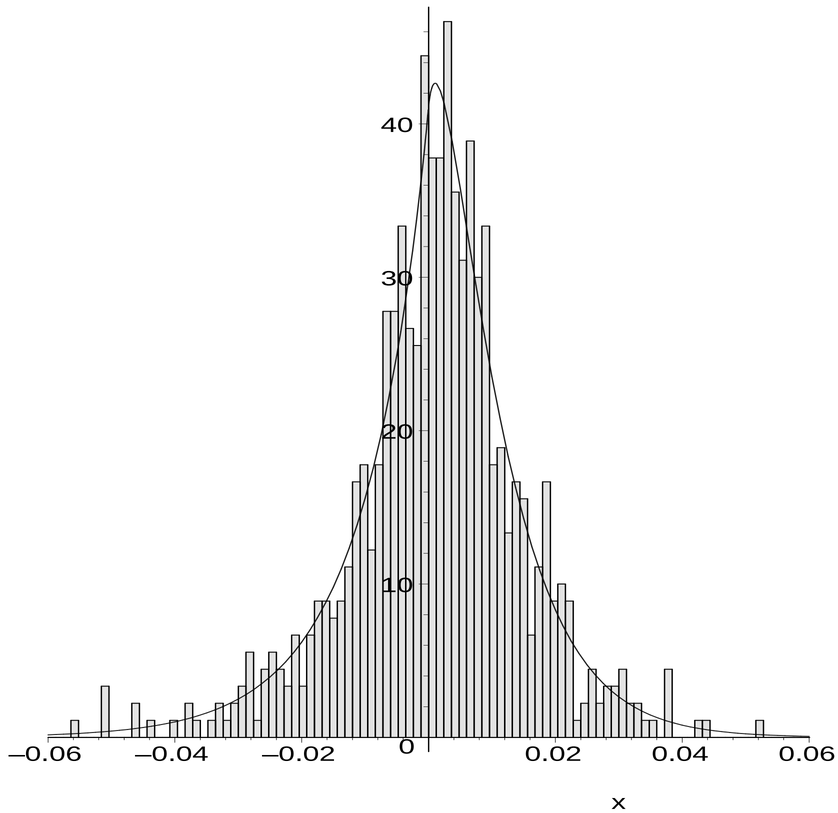
numerically to obtain $\hat{\vartheta}_{ML}$ (Hooke-Jeeves-algorithm).

In our example we obtain as the maximum likelihood estimation

$$\tilde{\theta}_{ML} = (1.55, 0.94; 133.96, 88.92)$$

Remember the estimation by method of moments:

$$\hat{\theta} = (1.28, 0.78; 119.75, 80.82)$$



40
30
20
10
0
–0.06
–0.04
–0.02
0.02
0.04
0.06
x

If all six parameters are to be estimated, the same method works also.

- We estimate the moments $\mu_k, k \leq 6$,

  by the empirical moments

  $\hat{\mu}_k, k \leq 6$, with

  $\hat{\mu}_k := n^{-1} \Sigma_{l=1}^{n} (\Delta L_{l\delta})^k$.

- Then we get an estimation of the first six cumulants

  through the one-to-one connection between cumulants and

  moments.

- and from the nonlinear system of six equations

  $\kappa_k = \Gamma(n - \beta^+) \dfrac{\alpha^+}{(\lambda^+)^{k-\beta^+}} + (-1)^k \, \Gamma(k - \beta^-) \dfrac{\alpha^-}{(\lambda^-)^{k-\beta^-}}, k = 1, \cdots, 6$

  we calculate estimators for the unknown six parameters

$$\alpha^{\pm}, \lambda^{\pm}; \beta^{\pm}.$$

No Maximum Likelihood estimator is available.

The study of the corresponding implicite function

$$F(c, \theta) = 0$$

is not yet done.

8 Tempered stable stock models

Assume $(L_t, t \in [0.T])$ is under P a

$$TS[\alpha^+, \lambda^+, \beta^+; \alpha^-, \lambda^-, \beta^-]\text{-Lev\'y process.}$$

Define for $t \in [0.T]$

$$S_t := \exp[L_t], B_t := \exp[rt], \tilde{S}_t := \exp[L_t - rt].$$

For pricing derivatives one uses an equivalent measure Q, which makes $(\tilde{S}_t, t \in [0, T])$ to a martingale. ("Equivalent martingale measure ")

Classical case: Black-Scholes-Theory, under P

$$S_t := \exp[\sigma W_t + \mu t], \; t \in [0, T].$$

There is one and only one equivalent martingale measure Q making $(\tilde{S}_t, t \in [0, T])$ to a martingale.

In our case of Lévy processes with jumps there are in general infinitely many equivalent martingale measures. One of them may exist within the exponential family

$$dQ_T^{\vartheta} = \exp[\vartheta L_t - \psi(\vartheta)t]dP_T$$

with

$$\psi(\vartheta) = \ln \mathrm{E} \exp(\vartheta L_1) =$$

$$\alpha^{-}\Gamma(-\beta^{-})\left\{(\lambda^{-}+\vartheta)^{\beta^{-}} - (\lambda^{-})^{\beta^{-}}\right\} + \alpha^{+}\Gamma(-\beta^{+})\left\{(\lambda^{+}-\vartheta)^{\beta^{+}} - (\lambda^{+})^{\beta^{+}}\right\}$$

$$\text{for some } \vartheta \in (-\lambda^{-}, \lambda^{+}).$$

This family consists of all TS-processes with the distribution

$$L_1 \sim TS[\alpha^{+}, \lambda^{+}-\vartheta, \beta^{+}; \alpha^{-}, \lambda^{-}+\vartheta, \beta^{-}].$$

Proposition 8.1 *Such an ϑ exists if and only if*

$$\lambda^{+} + \lambda^{-} > 1 \text{ and } r \in (f(-\lambda^{-}), f(\lambda^{+}-1))$$

where

$$f(\vartheta) := f^{+}(\vartheta) + f^{-}(\vartheta),$$

$$f^{+}(\vartheta) := \alpha^{+}\Gamma(-\beta^{+})[(\lambda^{+}-\vartheta-1)^{\beta^{+}} - (\lambda^{+}-\vartheta)^{+},$$

$$f^{-}(\vartheta) := \alpha^{-}\Gamma(-\beta-)[(\lambda^{-}-\vartheta-1)^{\beta^{-}} - (\lambda^{-}-\vartheta)^{-}.$$

In this case, ϑ is the unique solution of

$$f(\vartheta) = r.$$

For Bilateral Gamma processes the formula for f simplifies to

$$f^{\pm}(\vartheta) := \alpha^{\pm}[\ln(\lambda^{\pm} - \vartheta) - \ln(\lambda^{\pm} \mp 1 \mp \vartheta)].$$

If we consider the two-dimensional exponential family from above

$$dQ_T^{(\vartheta^-,\vartheta^+)} = \exp[\vartheta^- L_T^- - \psi^-(\vartheta^-)\cdot T] \times \exp[\vartheta^+ L_T^+ - \psi^+(\vartheta^+)\cdot T],$$

where

$$\psi^{\pm}(u) = \ln \mathrm{E} \exp(uL_1^{\pm}) =$$

$$\frac{\alpha^{\pm}\Gamma(1-\beta^{\pm})\left\{(\lambda^{\pm}+u)^{\beta^{\pm}} - (\lambda^{\pm})^{\beta^{\pm}}\right\}}{\beta^{\pm}},$$

then we get,
$Q_T^{(\vartheta^-,\vartheta^+)}$ makes $(\tilde{S}_t), t \in [0,T]$ to a martingale if and only if

$$E^{(\vartheta^-,\vartheta^+)}\tilde{S}_t \equiv 1.$$

It is one condition for the two parameters $(\vartheta^-, \vartheta^+)$, and we expect to get an one-parametric family of equivalent martingale measures within the two-dimensional exponential family. Indeed, we obtain

Proposition 8.2 • *If*

$$-\alpha^{+}\Gamma(-\beta^{+}) \leq r, \textit{ then no pair}$$

$$(\vartheta^{-}, \vartheta^{+}) \in (-\infty, \lambda^{+})$$

exists with $Q_T^{(\vartheta^{-},\vartheta^{+})}$ *as an equivalent martingale measure.*

- *If*

$$-\alpha^{+}\Gamma(-\beta^{+}) > r, \tag{4}$$

then there exist an interval $(\theta_1^{+}, \theta_2^{+}) \subseteq [-\infty, \lambda^{+} - 1]$ *and a continuous, strictly increasing, bijective map*

$$\Phi|(\theta_1^{+}, \theta_2^{+}) \rightarrow (-\infty, \lambda^{-}),$$

such that

a) $\forall\, \vartheta^{+} \in (\theta_1^{+}, \theta_2^{+})\ \exists\, \vartheta^{-} \in (-\infty, \lambda^{-}) : Q_T^{(\vartheta^{-},\vartheta^{+})}$ *is an equivalent martingale measure for* $(\tilde{S}_t), t \in [0, T]$.

b) $\vartheta^{-} = \Phi(\vartheta^{+})$

c) $\Phi(\vartheta^{+}) := (f^{-})^{-1}(r - f^{+}(\vartheta^{+}))$.

To choose an appropriate equivalent martingale measure, one often tries to minimize the *relative entropy*:

$$H(Q|P) := E_P[\frac{dQ}{dP} \ln \frac{dQ}{dP}].$$

Define $M_P :=$

$\{(\vartheta^+, \vartheta^-) \in (-\infty, \lambda^+)\times(-\infty, \lambda^-)|P^{(\vartheta^+,\vartheta^-)}\text{is a mart. measure}\}$

Then we have$M_P = \{(\vartheta, \Phi(\vartheta) \in I\!R^2 | \vartheta \in (\vartheta_1^+, \vartheta_2^+)\}.$

Proposition 8.3 *If (4) holds, then there exists*

$$\theta^+ \in (-\infty, \lambda^+) \textit{ and } \theta^- \in (-\infty, \lambda^-),$$

such that

$$H(P^{(\theta^+,\theta^-)}|P) = \min_{(\vartheta^+,\vartheta^-)\in M_P} H(P^{(\vartheta^+,\vartheta^-)}|P).$$

Using Esche,Schweizer (2005) and Hubalek,Sgarra (2006) one can show, that there exists an equivalent martingale measure, for which the relative entropy attains its minimum, and that this measure makes $(\tilde{S}_t)$ again to a Lévy process. But this Lévy process has completely other local characteristics, see also Krol,Küchler (2009).

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